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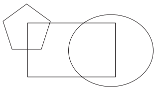

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Recherches sur les déformations no. 2

pp. 7–15

László Kozma, Julian Ławrynowicz, and Lajos Tamássy

OBITUARY: ROMAN STANISŁAW INGARDEN (1920–2011)

Summary

Professor Roman Stansław Ingarden (* October 1, 1920), the outstanding scientist and our good friend has passed away on July 12, 2011. We remember him as the founder of the Toruń school of quantum information theory and of two worldwide known journals, who invented several important ideas and results in stochastical thermodynamics, its geometrical characterization including the electromagnetism, quantum information theory, and theory of open (dissipative) systems. Roman contributed with 19 papers to our journal; it makes us great honour.

Meeting Professor Ingarden

We have known Professor Roman Stansław Ingarden (* 1920, Zakopane, † 2011, Kraków) for several decades. He often visited Debrecen and Łódź, and also we had the opportunity to meet him several times in Toruń. We were fortunate enough to cooperate with him. Every meeting was not only fruitful, but also a pleasant event. We admired his wide knowledge and interest both in physics, language, and in any kind of culture. Roman was the founder of two worldwide known journals: *Report on Mathematical Physics* and *Open Systems and Information Dynamics*. He published 19 papers in our journal. He will be sadly missed for his extraordinarily inspiring intellect. His passing away is a considerable loss.

The concept of Ingarden space

Roman had discovered [2, 10] that a Finsler space (in the most physical cases it is enough to consider a Randers space) can serve as a very precise mathematical model for the electromagnetic field – the *Ingarden electromagnetic space*, if we consider the triple (space, metric, connection), where we take into account the Cartan or Lorentz nonlinear connection. In fact, the E. Cartan's approach is rather complicated, but using the variational problem for the arc length in a Finsler space:

$$\ell(\gamma) = \int_0^1 F\left(x, \frac{d}{dt}x\right) dt,$$

where F stands for the Finsler metric, and taking the parametrization of the curve Γ by the natural parameter:

$$s = \int_0^1 \alpha\left(x, \frac{d}{d\tau}dx\right) d\tau,$$

we obtain, for the extremal curves the Lorentz equations

$$\frac{d^2x^j}{ds^2} + \gamma^j_{k\ell}\left(x, \frac{d}{ds}x\right)\frac{dx^k}{ds}\frac{dx^\ell}{ds} = \alpha F^j_k\left(x, \frac{d}{ds}x\right)\frac{d}{ds}x^k$$

and, consequently, we determine a *Lorentz nonlinear connection* N [12–15]:

$$N^j_k = \gamma^j_{k\ell}y^k - \alpha F^j_k.$$

Another specification leads to the *Ingarden thermodynamic space*, where all thermodynamical processes appear to be automatically irreversible.

Some electromagnetical applications including open systems

As far as electromagnetical applications are concerned, including those to open systems, we mention the following [1, 5, 9]: additional interactions between gravitational and electromagnetic forces, properties of electromagnetic "lenses" including the torsion of electron trajectories, generalizations of the Helmholtz-Lagrange law for an electric "lens", importance of combining the focal length calculation with a non-Riemannian geometry, immersion electromagnetic "lenses" in practice and in the constructed Randers-type Ingarden electromagnetic space, torsion-depending deformations within the electromagnetic spaces, electromagnetic space of an electromagnetic microscope, deformations of potentials in an electromagnetic space with the help of generating functions, "lens"-thickness depending deformations in relation with the scanning microscope, explicit formula for the focal length depending on the electromagnetic "lens" thickness, and potentials generating functions dependence vs. immersion electromagnetic "lenses" dependence.

Some thermodynamical applications including open systems

As far as thermodynamical application are concerned, including those to open systems, we mention the following [1, 3, 5, 9, 16]: thermodynamical interpretation of the

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平成22年7月22日
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Matsumoto spaces and generalized Matsumoto spaces, principles of thermodynamics including the electromagnetic effects, Randers and Ingarden spaces vs. openness and dissipativity of the system; hyperbolicity, Minkowskian spaces, and parabolicity in thermodynamic geometry, thermodynamic parameters and geometry in presence of the electromagnetic field, and advantages of a statistical and a stochastical thermodynamics.

Relationship with noncommutative associative and commutative nonassociative Finsler geometries

In this direction, following [1, 3, 5, 6, 8, 9], we mention the following areas of interest and results of Roman: solitons in the Randersian-Ingarden physics, complex Randersian physics vs. isospectral deformations, complex gauge connections of interacting fields, self-duality equations for gauge theories, homogeneity vs. gauge theories of the second order, forward and backward metrics in general relativity, coincidence of the manifold topology with that generated by forward metric balls, the clocks synchronization, the inertia tensor vs. non-inertial frames; SU(2)-based non-abelian physical models, generalizations of the Lagrangian and its embedding in the electroweak model, a concept of the five-dimensional model of nonlinear electrodynamics, ferroelectric crystals in a Finsler geometry, the Finsler-geometrical counterpart of the sine-Gordon equation for the surface, and simplifying the external field in terms of the metric and connection.

We summarize the latter three sections with the following remark by Roman, published in [6], p. 9:

> *The Lorentz connection has been defined in such a way that in a consequence a generalization of the Maxwell equations of the electromagnetic field followed, as was the primary aim of Ingarden in his doctor thesis. Such was the surprising and amazing result of the Miron's paper. The most interesting is, however, that prof. Miron also obtained a special form of interaction between the gravitational field and the electromagnetic field. Up to now no such interaction was known in physics. It would be interesting to study the physical consequences of such interaction. We shall try to do it in our book, joint with J. Ławrynowicz, L. Kozma, and L. Tamássy.*

At the moment the book is nearly finished. It contains a realization of the physical programme described in the three latter sections and a necessary exposition of some mathematical foundations. The tentative title reads: *Finsler Geometry and Physics. From Algebraic Foundations to Applications.* It is written in cooperation with Mihai Anastasiei and Radu Miron (Iaşi), Hideo Shimada and Sorin Vasile Sabau (Sapporo).

Quantum information theory and foundation of the Toruń School of that theory

In 1976 Roman started to use the generalized quantum mechanics of open systems and the generalized concept of observable to construct a *quantum information theory* being a straightforward generalization of Shanon's theory (1948) [4]. Before he had observed that in the usual quantum mechanics of closed systems there is no place for a general concept of joint and conditional probability. Together with his younger Toruń colleagues A. Kossakowski and A. Jamiołkowski he introduced the concept of quantum dynamical semigroups, superoperators which preserve semipositivity, and *entanglement*, understood as the manifestation of quantum correlations between the constituent parts in the system as opposed to much weaker classical correlations which can well be encoded in a product or, in alternative terminology, a separate state [11].

An important method for the detection of entangled states is based on the so-called etanglement witnesses. Let $\mathcal{H}$ be a Hilbert space of a composite quantum system: $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. An Hermitian operator $W \in L(\mathcal{H}_1 \times \mathcal{H}_2)$ is an *entanglement withness* whenever

$$((x \otimes y, Wx \otimes y)) \geq 0 \quad \text{for any} \quad x \in \mathcal{H}_1 \quad \text{and} \quad y \in \mathcal{H}_2,$$

and

$$((\eta, W\eta)) < 0 \quad \text{for } \textit{some} \quad \eta \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

An entanglement witness has a negative eigenvalue, but it is positive on separable pure states, i.e. it is block positive. From an experimentalist's point of view, entanglement witness is a nonlocal observable whose expectation value, when measured at a state ρ, i.e. the quantity $\mathrm{Tr}(\rho W)$, can serve as a direct indicator of the entanglement present in ρ [7]. Entanglement proves to be particularly useful in quantum cryptography, communication and information processing. Roman initiated the famous Toruń group of researchers dealing with quantum information theory, known as the Toruń School in this subject.

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on September 29, 2011

WSPOMNIENIE O ROMANIE STANISŁAWIE INGARDENIE (1920–2011)

S t r e s z c z e n i e

Profesor Roman Stanisław Ingarden (* 1. października 1920 r.), znakomity naukowiec i nasz serdeczny przyjaciel, odszedł do wieczności 12. lipca 2011 roku. Pamiętamy o Nim jako o twórcy Toruńskiej Szkoły Kwantowej Teorii Informacji i dwóch znanych na całym świecie czasopism naukowych, który wniósł do nauki szereg ważnych idei i wyników w zakresie termodynamiki stochastycznej, jej geometrycznej charakteryzacji z włączeniem elektromagnetyzmu, kwantowej teorii informacji oraz teorii układów otwartych (dysypatywnych). Roman opublikował w naszym czasopiśmie 19 prac, co stanowi dla nas wielki honor.

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Recherches sur les déformations no. 2

pp. 17–34

In memory of
Professor Roman Stanisław Ingarden

Massimo Vaccaro

BASICS OF LINEAR PARA-QUATERNIONIC GEOMETRY II

DECOMPOSITION OF A GENERIC SUBSPACE OF A PARA-QUATERNIONIC HERMITIAN VECTOR SPACE

Summary

In the present Part II of the paper we investigate relevant classes of subspaces of a para-quaternionic Hermitian vector space, in particular the decomposition of a generic subspace. After showing in [8] some fundamental results concerning (Hermitian) para-type structures on a real vector space, we consider here the tensorial presentation $(H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$ of a para-quaternionic Hermitian vector space $(V, \widetilde{Q}, g)$ and characterize some classes of *special subspaces* defined in terms of their behaviour with respect to the endomorphisms of the para-quaternionic structure $\widetilde{Q} = \mathfrak{sl}(H)$ and in some cases also in terms of the Hermitian metric $g = \omega^H \otimes \omega^E$. We will prove that a generic subspace of a para-quaternionic Hermitian vector space is the direct sum of such fundamental bricks (see Proposition 2.9).

The tensorial presentation

$$(V, \widetilde{Q}, g) \simeq (H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$$

turns out to be quite convenient to characterize such special subspaces. After proving that the *para-quaternionic subspaces coincide with the products* $H \otimes E'$, $E' \subseteq E$ (Proposition 1.4), the basic tool consists in restricting to *pure* subspaces, containing no nontrivial para-quaternionic subspaces, and by showing that *pure special subspaces are* $U^{F,T}$ *subspaces* defined by means of tensor product and depending on a given subspace $F \subset E$ and a linear operator $T : F \to E$ (see Definition 1.1). Viceversa we also give the precise *conditions for a* $U^{F,T}$ *subspace to be a special subspace of any given type.*

This presentation is also useful from the metrical point of view to determine, for each subspace, the signature of the induced metric. We give then the conditions for the above special subspaces to be g-nondegenerate. In this article we report and develop some results of [7] providing, in many cases, additional details and proofs.

1. Relevant classes of subspaces of a para-quaternionic Hermitian vector space

In the following, V will be the standard para-quaternionic Hermitian vector space $(H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$ that has been defined in [8]. There it has been shown that any para-hypercomplex admissible basis (I, J, K) of $\widetilde{Q} = \mathfrak{sl}(H)$ is a standard para-hypercomplex Hermitian structure which corresponds to a symplectic basis (h_1, h_2) of H such that $\omega^H = h_1^* \wedge h_2^*$ and with respect to which it is $I = \mathcal{I} \otimes \mathrm{Id},\ J = \mathcal{J} \otimes \mathrm{Id},\ K = \mathcal{K} \otimes \mathrm{Id}$ where

$$\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

We recall that, as an $\widetilde{\mathbb{H}}$-module, where by $\widetilde{\mathbb{H}}$ we denote the algebra of para-quaternions, on a para-hypercomplex Hermitian vector space $(V^{4n}, \{I, J, K\}, g)$ we define the ($\widetilde{\mathbb{H}}$-valued)-Hermitian product $(\cdot) = (\cdot)_{\{I,J,K\}}$ by

$$\begin{aligned} (\cdot) : V \times V &\to \widetilde{\mathbb{H}}, \\ (X, Y) &\mapsto X \cdot Y = g(X, Y) + ig(X, IY) - jg(X, JY) - kg(X, KY). \end{aligned} \tag{2}$$

When considering a para-quaternionic Hermitian vector space we observe that two Hermitian products $(\,\cdot\,)_{\{I,J,K\}}$, $(\,\cdot\,)_{\{I',J',K'\}}$, referred to different admissible bases, are related by an inner automorphism of $\widetilde{\mathbb{H}}$.

In the next subsection we consider some classes of subspaces defined only in terms of the tensor product structure of V. Their role will be clear when, in the following subsections, we shall introduce some special classes of subspaces which is natural to consider in a para-quaternionic Hermitian vector space.

1.1. Special subspaces of $H^2 \otimes E^n$

Fixed a symplectic basis (h_1, h_2) of H, any $X \in H \otimes E$ can be written in a unique way

$$X = h_1 \otimes e + h_2 \otimes e', \qquad e, e' \in E. \tag{3}$$

Let denote by $p_i : H \otimes E \to E,\ i = 1, 2$, the natural linear projections defined by

$$p_1(X) = e; \quad p_2(X) = e'. \tag{4}$$

If U is a subspace then $p_1(U) = E_1$, $p_2(U) = E_2$ are subspaces of E, depending on the chosen symplectic basis (h_1, h_2) in H. Notice that the sum $p_1(U) + p_2(U)$ is invariant.

With respect to the tensor product structure, the following subspaces of V can be defined. First of all there are the *product subspaces* $H' \otimes E'$, with $H' \subseteq H$ and $E' \subseteq E$ any given subspaces. Referring to the dimension of the non trivial factor in H, only two classes of such subspaces are to be considered.

A nonzero product subspace $U = h \otimes E' \subset H \otimes E$ where h is a fixed element in H and $E' \subset E$ a subspace, will be called a *decomposable subspace* (meaning that all elements in U are decomposable vectors; subspaces $H \otimes e', e' \in E$ will not be considered under such terminology). With respect to the metric g, any decomposable subspace is totally isotropic.

We introduce another important family of subspaces that we denote by $U^{F,T}$.

Definition 1.1. Let (h_1, h_2) be a symplectic basis of H, $F \subseteq E$ a subspace, and let $T : F \to E$ be a linear map. We define the following subspace of $H \otimes E$:

$$U^{F,T} := \{h_1 \otimes f + h_2 \otimes Tf,\ f \in F\}. \tag{5}$$

Note that the map

$$\begin{array}{rrl} \phi : & F \to & U^{F,T}, \\ & f \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array} \tag{6}$$

is an isomorphism of real vector spaces. By saying that a subspace $U \subset H \otimes E$ is a $\mathbf{U^{F,T}}$ *subspace*, we will mean that it admits the form (5) with respect to some symplectic basis (h_1, h_2) of H.

As a first example of subspaces admitting the $U^{F,T}$ form we have the decomposable subspaces $U = h \otimes E',\ h \in H,\ E' \subseteq E$: in any basis $(h_1 = h, h_2)$ let $F = E'$ and $T \equiv 0$. Also, in a basis (h_1, h_2) with $h_1, h_2 \neq h$, one has

$$F = p_1(U) = p_2(U) = E' \quad \text{and} \quad T = \lambda \mathrm{Id}, \quad \text{where} \quad \lambda = \frac{\beta}{\alpha} \quad \text{for} \quad h = \alpha h_1 + \beta h_2.$$

On the other hand, U does not admit the form (1.1) with respect to any basis $(h_1, h_2 \equiv \alpha h),\ \alpha \in \mathbb{R}$.

It is immediate to prove the following

Proposition 1.1.

a) *A subspace U is a $U^{F,T}$ subspace iff there exists $h \neq 0$ in H such that $(h \otimes E) \cap U = \{0\}$.*

b) *With respect to the symplectic basis (h_1, h_2), the map T for the subspace $U = U^{F,T}$ is injective iff $(h_i \otimes E) \cap U = \{0\}$, $i = 1, 2$.*

Observe that if $U = U^{F,T}$ with respect to h_1, h_2 and also $U = U^{F',T'}$ with respect to h_1', h_2', where

$$h_1 = \alpha h_1' + \beta h_2', \qquad \text{and} \qquad h_2 = \gamma h_1' + \delta h_2',$$

then

$$F' = (\alpha \mathrm{Id} + \gamma T)F \qquad \text{and} \qquad T' = (\alpha \mathrm{Id} + \gamma T)^{-1}(\beta \mathrm{Id} + \delta T).$$

Proposition 1.2. *A $U^{F,T}$ subspace can always be written as $U^{F',T'}$ with T' injective by performing a suitable change of basis in H.*

Proof. The proof follows from the fact that a subspace

$$U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ f \in F\}$$

of dimension m contains at most m distinct nonzero decomposable vectors $k_i \otimes f_i$, $i = 1, \dots, t$, if the $k_i \in H$ are pairwise independent.

Remark that, from the isomorphism (6), the decomposable subspaces contained in a $U = U^{F,T}$ subspace are direct addends in U.

In general, a subspace $U \subset V$ does not admit the form $U^{F,T}$: an example is given by any product subspace $H \otimes E'$, $E' \subset E$. On the other hand, any subspace can be written as direct sum of some $U^{F,T}$ subspaces. In fact, we have the following proposition whose proof can be found in [7]:

Proposition 1.3. *Any subspace U can be written in the forms*

1) $(h \otimes F') \oplus U^{F'',T''}$ *for some $h \in H$ and $U^{F'',T''}$ of maximal dimension with respect to all subspaces of the form $U^{F,T}$ contained in U.*

2) $k_1 \otimes F_1 \oplus \ldots \oplus k_s \otimes F_s \oplus U^{\widetilde{F},\widetilde{T}}$ *with the $k_i \in H, i = 1, \dots, s$, pairwise independent and $U^{\widetilde{F},\widetilde{T}}$ of maximal dimension with respect to all subspaces of the form $U^{F,T}$ containing no decomposable subspace.*

Concerning the unicity of the presentation of the form $U^{F,T}$ we state the following lemma whose proof is straightforward (see proof of Proposition 1.2):

Lemma 1.1. *Given a subspace $U \subset H \otimes E$ the following conditions are equivalent:*

1) *U contains no nonzero decomposable vectors;*

2) *$U = U^{F,T}$ with respect to* ***any*** *symplectic basis $\mathcal{B} = (h_1, h_2)$, $F = F(\mathcal{B}) \subset E$ and $T = T(\mathcal{B})$ injective;*

3) *there exists a basis (h_1, h_2) such that $U = U^{F,T}$ for some subspace $F \subset E$ and some linear injective map T with no invariant line (i. e. if $Tf = \lambda f$, $\lambda \in \mathbb{R}$, $f \in F$ then $f = 0$).*

A necessary condition for 1), 2), 3) *to hold is that* $\dim U \leq \dim E$.

From the metrical point of view we have easily the following

Lemma 1.2. *Let $U = U^{F,T}$ be a subspace and ϕ the isomorphism* (6). *Let $g_F = \phi^* g_U$ be the pullback of the (possibly degenerate) restriction of g to U. Then*

$$g_F(f, f') = -2(\omega^E \circ T)^{sym}(f, f') = -[\omega^E(Tf, f') + \omega^E(Tf', f)]. \tag{7}$$

1.2. Para-quaternionic subspaces

Definition 1.2. A subspace $U \subset V$ is called *para-quaternionic* if it is $\widetilde{Q}$-invariant or, equivalently, for one and hence for any para-hypercomplex basis (I, J, K) of $\widetilde{Q}$ one has $IU \subset U$, $JU \subset U$, $KU \subset U$.

The sum and the intersection of para-quaternionic subspaces is para-quaternionic.

Proposition 1.4. *Let $(E')^k \subset E$ be any subspace. Then*

$$U^{2k} = H \otimes E' \tag{8}$$

is a para-quaternionic subspace of dimension $2k$. Viceversa any para-quaternionic subspace of V has this form. Moreover U is para-quaternionic Hermitian (with neutral metric) iff E' is ω^E-symplectic.

Proof. The subspace $U^{2k} = H^2 \otimes (E')^k \subset H \otimes E$ is clearly $\widetilde{Q}$-invariant. Viceversa, it is immediate to verify that, for any basis (h_1, h_2) in H, the subspaces $E' = p_1(U)$ and $p_2(U)$ coincide. Moreover, using the invariance with respect to a para-hypercomplex basis, we deduce that the decomposable vectors $h \otimes e'$ are in U for any $e \in E'$ and $h \in H$. Therefore $U = H \otimes E'$.

From the metrical point of view, the subspaces $h_1 \otimes E'$ and $h_2 \otimes E'$ are totally isotropic and the metric on U, with respect to the decomposition $U = h_1 \otimes E' \oplus h_2 \otimes E'$, is given by

$$g|_U = \begin{pmatrix} 0 & \omega^E|_{E'} \\ (\omega^E|_{E'})^t & 0 \end{pmatrix} \tag{9}$$

and has neutral signature $(r, 2(k-r), r)$ with $r = rk(\omega^E|_{E'})$ or equivalently $(k-s, 2s, k-s)$ with $s = \dim\ \ker(\omega^E|_{E'})$ (see Lemma (2.4) in [8]). Then U is Hermitian para-quaternionic if and only if E' is $\omega^E|_{E'}$-symplectic, which implies that any Hermitian para-quaternionic subspace has necessarily dimension $4m$ and neutral signature $(2m, 2m)$.

Remark 1.1. Referring to the decompositions 1), 2) given in Proposition (1.3), notice that a para-quaternionic subspace $U = H \otimes E'$ decomposes respectively as

$$\begin{aligned} &1) \quad U = h_1 \otimes E' \oplus h_2 \otimes E', \\ &2) \quad U = h_1 \otimes E' \oplus \{h_1 \otimes e' + h_2 \otimes Te',\ e' \in E'\} \end{aligned}$$

with respect to any basis (h_1, h_2), with T any automorphism of E' with no real eigenvalues. In this case then, the dimensions of the maximal $U^{F,T}$ subspaces in $U = H \otimes E'$ in the decompositions 1) and 2) coincide and equal the dimension of E'.

Any subspace U of V contains a (possibly zero) *maximal para-quaternionic subspace*

$$U_0 = U \bigcap_{A \in \widetilde{Q}} A(U).$$

Equivalently, $U_0 = U \cap IU \cap JU \cap KU$ for any admissible basis (I, J, K) of $\widetilde{Q}$.

Definition 1.3. A subspace $U \subset V$ is called *pure* if $U_0 = \{0\}$, i.e. it contains no nonzero para-quaternionic subspaces.

Clearly, from Proposition (1.1a), *any $U^{F,T}$ subspace is pure.*

The viceversa is *not true*. For example, let consider the 3-dimensional subspace

$$U =< h_1 \otimes e_1, h_2 \otimes e_2, h_3 \otimes e_3 >$$

where $< h_1, h_2, h_3 >= H$ and $e_3 \in< e_1, e_2 >$. It is straightforward to verify that U is pure and, for any $h \in H$, the decomposable subspace $h \otimes \mathbb{R}e \subset U$ for some $e \in< e_1, e_2 >$. By Proposition (1.1), the pure subspace U is not a $U^{F,T}$ subspace.

1.3. Complex subspaces

Definition 1.4. A subspace $U \subset V$ is called *complex* if there exists a compatible complex structure $I \in \widetilde{Q}$ such that U is I-invariant, i.e. $IU \subset U$. We denote it by (U, I).

We shall include I into an admissible basis (I, J, K) of $\widetilde{Q}$. Such a basis will be called *adapted* to the subspace (U, I). Adapted bases are defined up to a rotation in the real plane spanned by J and K.

The following statements have been proved in [7].

Lemma 1.3.

(1) *The complex structure I is unique up to its sign unless U is para-quaternionic.*

(2) *A complex subspace $(U, I) \subset (V, \widetilde{Q})$ is pure if and only if there exists a para-complex structure $J \in \widetilde{Q}$ such that $IJ = -JI,\ JU \cap U = \{0\}$.*

Lemma 1.4. *If (U, I) is pure complex, then, for any $A \in \widetilde{Q},\ A \neq \pm aI,\ a \in \mathbb{R}$, one has $AU \cap U = \{0\}$.*

Proof. In fact, let (I, J, K) be an adapted basis such that $JU \cap U = \{0\}$. Let $A = aI + bJ + cK$ and suppose there exists a nonzero $X \in U$ such that $AX \in U$. $AX \neq 0$ would imply that $Y = bJX + cKX \in U$ and $JY \in U$; hence a contradiction if $(b, c) \neq (0, 0)$.

Adapted bases of a pure complex subspace are then defined up to a rotation in the plane $< J, K >$.

Considering now also the metric structure of V we have the following special class of pure complex subspaces.

Definition 1.5. An Hermitian complex subspace (U, I) of V is called *totally complex* if there exists an adapted hypercomplex basis (I, J, K) such that $JU \perp U$ ($\Leftrightarrow KU \perp U$) with respect to the (nondegenerate) induced metric g or equivalently iff, with respect to the adapted basis (I, J, K), the restriction to U of the Hermitian product (2) has complex values. In fact, following the terminology of [3], a totally complex subspace could be called a *subspace with complex Hermitian product.*

Note that the hypothesis of being Hermitian is necessary to ensure that any totally complex subspace is pure.

Let (U, I) be a totally complex subspace and (I, J, K) an adapted basis such that $JU \perp U$. Any $A = aI + bJ + cK \in \widetilde{Q}$, satisfies $AU \perp U$ if and only if $a = 0$. Then again, adapted bases are defined up to a rotation in the plane $< J, K >$.

Proposition 1.5. *Any complex subspace (U, I) is a direct sum of the maximal para-quaternionic subspace U_0 and a pure I-complex subspace (U', I), i.e. $U = U_0 \oplus U'$. If (U'', I) is another pure I-complex subspace complementary to U_0, then U' and U'' are isomorphic as I-complex spaces.*

Proof. Let (U, I) be a complex subspace and U_0 its maximal para-quaternionic subspace. We prove the existence of a pure complex supplementary to U_0 in U. Let $X_1 \notin U_0$. Clearly the vector $IX_1 \notin U_0$. Moreover for the I-complex 2-plane $U_1 =< X_1, IX_1 >$ we have $U_0 \cap U_1 = \{0\}$. Consider now $X_2 \notin U_0 \oplus U_1$. Then $U_2 =< X_2, IX_2 >$ has trivial intersection with $U_0 \oplus U_1$. By carrying on this procedure we build up an I-invariant supplement

$$U' = U_1 \oplus \ldots \oplus U_{m-s} =< X_1, IX_1, \ldots , X_{m-s}, IX_{m-s} >$$

to U_0 in U. Moreover suppose there exists $X \in U'$ such that $JX \in U'$ where J is a compatible para-complex structure anticommuting with I. Then

$$< X, IX, JX, IJX = KX >$$

is a para-quaternionic subspace in U'. Contradiction.

If U'' is another pure I-complex complementary to U_0, then U', U'' are isomorphic as I-complex subspaces. In fact, let $\phi : U' \to U''$ be the isomorphism that, for any $U \ni X = X_0 + X_1 = \tilde{X}_0 + X_2$ maps X_1 to X_2. Then $\phi : IX_1 \mapsto IX_2$, i.e. $\phi(IX_1) = I\phi(X_1)$.

Hence *the description of complex subspaces reduces to the description of pure complex subspaces.*

Let I be a compatible complex structure and (h_1, h_2) a symplectic basis of H with respect to which $I = \mathcal{I} \otimes \mathrm{Id}$ with $\mathcal{I}$ as in (1).

Theorem 1.1. *With respect to (h_1, h_2), a subspace $U \subseteq V$ is pure I-complex iff $U = U^{F,T}$ with T a complex structure on $F = p_1(U)$. Then the map*

$$\begin{array}{llll} \phi : & (F, T^{-1}) & \to & (U^{F,T}, I), \\ & f & \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array}$$

is an isomorphism of complex vector spaces. The signature of the metric on U is of type $(2p, 2s, 2q)$, $2s = \dim\ker g|_U$, and U is Hermitian if and only if F is g_F-nondegenerate. In this case $\phi : (F, T^{-1}, g_F) \to (U, I, g|_U)$ is an isomorphism of Hermitian spaces. In particular T^{-1} is g_F-skew symmetric. The Kaehler form of (U, I) is given by

$$\phi^*(g|_U \circ I) = g_F \circ T^{-1} = -(\omega^E|_F + \omega^E|_F(T\,\cdot, T\,\cdot)).$$

The subspace (U, I) *is totally complex if and only if* F *is* ω^E*-symplectic and* T *is* $\omega^E|_F$*-skew-symmetric* ($\Longleftrightarrow$ *if* T *preserves the form* $\omega^E|_F$*, i.e.*

$$\omega^E|_F(f, f') = \omega^E|_F(Tf, Tf') \qquad \forall f, f' \in F) \tag{10}$$

or, equivalently, $g_F = -2\omega^E|_F \circ T$.

Proof. Since (U, I) is pure, no decomposable vector is in U. From Lemma (1.1), $U = U^{F,T}$ with respect to the symplectic basis $(h_1, h_2 = \mathcal{I}h_1)$; it is then straightforward to verify that $T^2 = -Id$. Viceversa, it is immediate to verify that the pure subspace

$$U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ T^2 = -\mathrm{Id}\}$$

is I-invariant. The statements about the isomorphism ϕ are straightforward to verify. The expression of the Kaehler form follows from a direct calculation. The signature of the metric on U is $(2p, 2s, 2q)$, $2s = \dim\ker g|_U$ (see Proposition (2.12) in [8]) and clearly equals the signature of g_F on F. Consequently U is Hermitian pure complex if and only if F is g_F-nondegenerate.

It is easy to verify that $U \perp JU$ if and only if, for any $f, f' \in F$,

$$0 = g(h_1 \otimes f + h_2 \otimes Tf, h_1 \otimes f' - h_2 \otimes Tf') = -\omega^E(f, Tf') - \omega^E(Tf, f'),$$

that is $\omega^E(f, Tf') = -\omega^E(Tf, f')$ which is equivalent to (10). The metric on U verifies

$$g(h_1 \otimes f + h_2 \otimes Tf, h_1 \otimes f' + h_2 \otimes Tf') = g_F(f, f') = 2\omega^E(f, Tf').$$

Therefore, the nondegeneracy of U implies that F is ω^E-symplectic.

Observe that by a change of symplectic basis in H, the subspace F remains the same whereas the complex structure T changes into a diagonalizable automorphism of F with only a pair of complex (conjugated) eigenvalues. Only if the change of the basis in H is represented by an orthogonal (besides symplectic) matrix then T is (the same) complex structure.

This theorem reduces classification of pure complex subspaces to the classification of pairs (F, T) *with* $F \subset E$ *and* T *being a complex structure on* F. In particular, in the classification of totally complex subspaces, F is, in addition, $\omega^E|_F$-symplectic and T preserves $\omega^E|_F$.

1.4. Para-complex subspaces

Definition 1.6. A subspace $U \subset V$ is called *weakly para-complex* (*or product*) if there exists a para-complex structure $K \in \widetilde{Q}$ such that U is K-invariant, i.e. $KU \subset U$. We denote such subspace by (U, K). A *para-complex* subspace (U, K) is a weakly para-complex subspace such that dim $U_K^+ =$ dim U_K^-.

The eigenspaces V_K^+, V_K^- of a given para-complex structure $K \in \widetilde{Q}$ are decomposable subspaces (then totally isotropic) of V, i.e.

$$V_K^+ = h' \otimes E', \quad V_K^- = h'' \otimes E'' \quad \text{and} \quad E' \oplus E'' = E.$$

As a first consequence any weakly para-complex subspace which is not para-complex is degenerate. The presence of decomposable vectors produces a difference when passing from the complex to weakly para-complex case but, as we shall see, a common treatment of both cases is still possible.

The following statements have been proved in [7].

Lemma 1.5. (1) *A weakly para-complex subspace (U, K) of V is pure iff there exists a complex structure $I \in \widetilde{Q}$ anti-commuting with K such that $IU \cap U = \{0\}$.*

(2) *If (U, K) is pure weakly para-complex, then for any compatible complex structure $\tilde{I}$, one has $\tilde{I}U \cap U = \{0\}$. Then the admissible bases are defined up to a pseudo-rotation in the plane $< I, J >_{\mathbb{R}}$.*

(3) *Any weakly para-complex subspace (U, K) is a direct sum $U = U_0 \oplus \tilde{U}$ of the maximal para-quaternionic subspace and of a pure weakly K para-complex subspace $\tilde{U}$. If $\tilde{U}'$ is another K pure weakly para-complex complementary subspace, then $\tilde{U}$ and $\tilde{U}'$ are isomorphic as weakly K-para-complex spaces.*

Assume $\tilde{U} \neq \{0\}$. If $\tilde{U} \not\subseteq U_K^{\pm}$, then the para-complex structure $K \in \widetilde{Q}$ is unique up to its sign. Otherwise the family of para-complex structures

$$\tilde{K}_a = aI + aJ \pm K \quad \textit{if} \quad \tilde{U} \subset U_K{}^{+} \quad (\tilde{K}_a = aI - aJ \pm K, \quad \textit{if} \quad \tilde{U} \subset U_K{}^{-})$$

preserves U for any adapted basis (I, J, K).

Proof. We prove only the existence of a K-invariant supplementary to U_0 in (U, K) (first part of point 3) since the presence of eigenvectors of K leads to some differences with respect to the analogous proof of Proposition (1.5). Let $U = U_K^+ \oplus U_K^-$ be the decomposition of U into the eigenspaces of K. Let U_0 be the maximal para-quaternionic subspace in U and $U_0 = U_0^+ \oplus U_0^-$, the decomposition of U_0 into the eigenspaces of K. Then, for any pair of complementary U_1 and U_2 to U_0^+ and U_0^- in U_K^+ and U_K^-, the subspace $U_1 \oplus U_2$ is clearly K-invariant. Moreover it is pure for the maximality of U_0.

Hence also *the description of weakly para-complex subspaces reduces to that one of pure subspaces.* In this case nevertheless there exists a difference regarding the uniqueness of the para-complex structure. The reason for such a difference is a consequence of the results in the next subsection.

Definition 1.7. Let (U, K) be a K-Hermitian para-complex subspace. Then U is called *totally para-complex* if there exists a complex structure $I \in \widetilde{Q}$ anticommuting with K such that $IU \perp U$ with respect to the induced metric $g|_U$ or equivalently iff, with respect to the adapted basis (I, J, K), the restriction to U of the Hermitian product (2) has para-complex values.

Note that, as in the complex case, the hypothesis of being Hermitian is necessary to ensure that any totally para-complex subspace is pure. It is straightforward to verify that adapted bases are defined up to a rotation in the plane $< I, J >$.

Let J be a compatible para-complex structure and (h_1, h_2) a symplectic basis of H such that $J = \mathcal{J} \otimes \mathrm{Id}$ with $\mathcal{J}$ as in (1).

Theorem 1.2. *With respect to (h_1, h_2), a subspace $U \subseteq V$ is pure weakly J-para-complex iff $U = U^{F,T}$ with T a product structure on $F = p_1(U)$. Then the map*

$$\begin{array}{cccc} \phi: & (F,T) & \to & (U^{F,T}, J), \\ & f & \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array}$$

is an isomorphism of weakly para-complex vector spaces. The subspace U is J-Hermitian if and only if F is g_F-nondegenerate and hence necessarily para-complex. In this case, the signature of $g|_U$ is always neutral and $\phi : (F, T, g_F) \to (U, J, g|_U)$ is an isomorphism of Hermitian para-complex spaces. In particular T is g_F-skew symmetric. The para-Kaehler form is given by

$$\phi^*(g|_U \circ J) = g_F \circ T = -(\omega|_F - \omega|_F(T\,\cdot, T\,\cdot)).$$

The para-complex subspace (U, J) is totally para-complex if and only if T is $\omega^E|_F$-skew-symmetric $\iff$ the form $\omega^E|_F$ is skew-invariant with respect to T, i.e.

$$\omega^E|_F(f, f') = -\omega^E|_F(Tf, Tf') \qquad \forall f, f' \in F$$

or, equivalently, $g_F = -2\omega^E|_F \circ T$, and F is $\omega^E|_F$-symplectic.

Proof. Let (U^k, J) be a pure weakly para-complex subspace in $H \otimes E$, $(h_1, \mathcal{J}h_1 = h_2)$ be a symplectic basis and (I, J, K) an adapted basis. Clearly $h_i \otimes E \cap U = \{0\}$, $i = 1, 2$ since U is pure. Then, from Proposition (1.1), $U = U^{F,T}$. In particular, with respect to a symplectic basis $(h_1, \mathcal{J}h_1 = h_2)$, it is straightforward to verify that $T^2 = \mathrm{Id}$. Viceversa the pure subspace $U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ T^2 = \mathrm{Id}\}$ is clearly J-invariant.

The eigenspaces V_J^+ and V_J^+ are decomposable subspaces and consequently they are totally isotropic. Then, (see Lemma (2.4) in [8]), the signature of the induced metric on U (which equals the signature of g_F on F) is $(m, k - 2m, m)$, where $m = rk\, g(V_J^+ \times V_J^-)$. By (7), U if J-Hermitian with neutral signature iff F is g_F nondegenerate.

Consider now a J-Hermitian para-complex subspace $U = \{h_1 \otimes f + h_2 \otimes Tf, f \in F\}$. Then

$$IU = \{h_1 \otimes -Tf + h_2 \otimes f, f \in F\}.$$

Imposing $U \perp IU$ it follows that the condition for (U, J) to be totally para-complex is given by

$$\omega|_F(f, f') = -\omega|_F(Tf, Tf') \tag{11}$$

($\Leftrightarrow T$ is $\omega^E|_F$-skew-symmetric). The Hermitianity hypothesis on F implies that F if ω^E-symplectic. Then the decomposition $F = E_1 \oplus E_2$ into into ± 1-eigenspaces of T is a Lagrangian decomposition (i.e. $\omega^E|_{E_1} \equiv 0, \quad \omega^E|_{E_2} \equiv 0$) of the symplectic space F.

By an admissible change of symplectic basis in H such that the correspondence $p_1(U) \mapsto p_2(U)$ is still injective, F remains the same whereas the para-complex structure T turns into a diagonalizable automorphism of F with only a pair of real eigenvalues of opposite sign. Only through a change of basis represented by an orthogonal matrix, T remains the same para-complex structure on F.

The latter theorem reduces classification of weakly pure para-complex subspaces to that one of pairs (F, T) with $F \subseteq E$ and T a product structure on F.

Differently from the pure complex case, where (U, I) admits the form $U^{F,T}$ with respect to all symplectic bases of H, in the pure weakly para-complex case, the presence of decomposable vectors in (U, J) and Lemma (1.1) allow for some special presentations of (U, J) different from the $U^{F,T}$ form. In particular, using the decomposition of (U, J) into the ± 1 eigenspaces of J on U, we have the following

Proposition 1.6. *Let (U, J) be a pure weakly para-complex subspace with $(h_1, h_2 = \mathcal{J}h_1)$ a symplectic basis. Let moreover (I, J, K) be an adapted basis. The pure weakly para-complex subspace decomposes as*

$$(U, J) = (h'_1 \otimes E_1) \oplus (h'_2 \otimes E_2),$$

where $E_1 \oplus E_2 = F$ is the T ± 1-eigenspaces decomposition of F,

$$h'_1 = -\frac{1}{\sqrt{2}}(h_1 + h_2),\ h'_2 = \frac{1}{\sqrt{2}}(h_1 - h_2)$$

is the symplectic basis of eigenvectors of $\mathcal{J}$, and

$$h'_1 \otimes E_1 = U_J^+ \quad \text{and} \quad h'_2 \otimes E_2 = U_J^-$$

are the eigenspaces of $J|_U$.

1.5. Nilpotent subspaces

Definition 1.8. A subspace $U \neq \{0\} \subset H \otimes E$ is called *nilpotent* if there exists a nonzero nilpotent endomorphism $A \in \widetilde{Q}$ such that $AU \subset U$.

The nilpotent subspace U will be called also A-*nilpotent* even if, as we will see later, such a nilpotent endomorphism is never unique.

If U is nilpotent we call *degree of nilpotency of* U the minimum integer n such that $A^n U = \{0\}$, $A \in \widetilde{Q}$. Clearly, since $A^2 = 0$, the degree of nilpotency of U is at most 2, and equal to 1 if $U \subset \ker A$.

Proposition 1.7. *A subspace U is nilpotent of degree 1 iff it is a decomposable subspace $h \otimes F$, $h \in H$, $F \subset E$. More generally, let $A \in \widetilde{Q}$ be a nilpotent endomorphism and* $\ker A = h \otimes E$. *A subspace U is A-nilpotent iff one has*

$$h \otimes p_2(U) \subset U.$$

Proof. We first observe that the subspace $p_2(U)$ is invariant under any change of symplectic basis $(h_1, h_2) \mapsto (h_1, h_2')$. The first statement is straightforward. Let U be a A-nilpotent subspace where $A \in \widetilde{Q}$ with $\ker A = h \otimes E$. Let $(h_1 \equiv h, h_2)$ be a symplectic basis of H (then $A(h_2 \otimes E) = h_1 \otimes E$). For any $X = h_1 \otimes e_1 + h_2 \otimes e_2$ with $e_2 \neq 0$ in U the vector $AX \in U$ implies that $h_1 \otimes e_2 \in U$. So, being $E_1 = p_1(U),\ E_2 = p_2(U)$, the A-invariance of U implies that $h_1 \otimes E_2 \subset U$ ($\Rightarrow E_2 \subseteq E_1$). Viceversa, let U be a subspace. If the subspace $h_1 \otimes p_2(U) \subseteq U$, then, for any $A \in \widetilde{Q},\ \|A\|^2 = 0$ with $\ker A = h_1 \otimes E$, the subspace U is clearly A-nilpotent.

Obviously, all para-quaternionic subspaces are nilpotent of degree 2 with respect to any nilpotent structure in $\widetilde{Q}$.

Proposition 1.8. *Any nilpotent subspace (U, A) is a direct sum of the maximal para-quaternionic subspace U_0 and a pure A-invariant subspace (U', A), i.e. $U = U_0 \oplus U'$. If (U'', A) is another pure A-invariant subspace complementary to U_0, then U' and U'' are isomorphic as A-nilpotent spaces.*

Proof. Let (U, A) be a nilpotent subspace and $U_0 \subset U$ be the maximal para-quaternionic subspace. Let $\ker A|_U = \ker A|_{U_0} \oplus U_1$ with U_1 any complementary.

For every decomposition $U = U_0 \oplus U_1 \oplus U_2$ the complementary

$$U_2 = \{h_1 \otimes f + h_2 \otimes f',\ f \in F,\ f' \in F'\}$$

contains no decomposable vectors. In fact, if

$$X = h \otimes \bar{f} = h_1 \otimes \alpha \bar{f} + h_2 \otimes \beta \bar{f} \in U_2,$$

then by the A-invariance $AX = h_1 \otimes \beta \bar{f} \in \mathrm{Ker} A|_U$. Thus the vector $h_2 \otimes \beta \bar{f} \in U$ which implies that $X \in U_0$. Contradiction.

By Proposition (1.1) we have

$$U_2 = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ f \in F\}$$

with T injective and $TF = F'$.

Moreover $(h_1 \otimes F) \cap \ker A|_U = \{0\}$ or, equivalently, $F \cap E_1' = \{0\}$. In fact, suppose $f \in F \cap E_1'$. Then $h_1 \otimes f \in \ker A|_U$ and $h_2 \otimes Tf \in U_0$ which leads to the contradiction with $h_1 \otimes f + h_2 \otimes Tf \in U_0 \oplus U_1$.

Finally, $AU_2 \subset U_1$. In fact, let $X = h_1 \otimes f + h_2 \otimes Tf \in U_2$ and suppose

$$AX = h_1 \otimes Tf = h_1 \otimes e_o + h_1 \otimes e_1 \in U_0 \oplus U_1.$$

Then, since $h_2 \otimes e_0 \in U_0$, we get $h_1 \otimes f + h_2 \otimes e_1 \in U$. Since $F \cap E_1' = \{0\}$, then $h_1 \otimes f + h_2 \otimes e_1 \in U_2$ which, by the injectivity of T leads to a contradiction again. The subspace $U' = U_1 \oplus U_2$ is then an A-invariant complement of U_0 in U. If (U'', A) is another A-invariant complement of U_0 in U, let us define the bijection

$$\begin{array}{rccc} \phi : & U' & \to & U'', \\ & X' & \mapsto & Y'' \end{array}$$

for each

$$U \ni Z = X_0 + X' = Y_0 + Y'' \quad \text{with} \quad X_0, Y_0 \in U_0, \quad X' \in U', \quad Y'' \in U''.$$

Then $\phi AX' = AY'' = A\phi X'$, i.e. ϕ is an isomorphism of pure A-nilpotent subspaces.

Remark that any pure A-nilpotent subspace U contains one and only one decomposable subspace that is $\ker A|_U$. In fact, if $h \otimes e' \in U$, then, by the A-invariance, we have $H \otimes e' \subset U$.

From the previous proposition, we get the following characterization of nilpotent subspaces with respect to Proposition (1.3):

Theorem 1.3. *Let $A \in \widetilde{Q}$ be a nilpotent endomorphism such that* $\ker A = h_1 \otimes E$ *and let (h_1, h_2) be a symplectic basis. The subspace*

$$U = (h_1 \otimes E') \oplus \{h_1 \otimes f + h_2 \otimes Tf, \quad f \in F\} \tag{12}$$

with $F \cap E' = \{0\}$, $TF \subset E'$ and T injective is pure A-nilpotent of the form $U = U^{F',T'}$ with $F' = E' \oplus F$ and $T' = 0 \oplus T : E' \oplus F \to E'$. Viceversa, any pure A nilpotent subspace can be written in the form (12).

Moreover, a sufficient condition for U to be nondegenerate is that $p_2(U)$ is ω^E-symplectic.

Proof. The subspace U in (12) is clearly A-nilpotent with respect to all $A \in \widetilde{Q}$ such that $\ker A = h_1 \otimes E$. Moreover it is pure since it is a $U^{F,T}$ subspace. Viceversa, let U be a pure A-nilpotent subspace with $\ker A|_U = h_1 \otimes E'$. Fix a symplectic basis (h_1, h_2). Then $U = (h_1 \otimes E') \oplus U'$ with

$$U' = \{h_1 \otimes f + h_2 \otimes \tilde{f}, \ f \in F, \ \tilde{f} \in \tilde{F}\}$$

being any complementary. Since U is nilpotent, from Proposition (1.7) we get $\tilde{F} \subset E'$. Moreover, since U is pure, no decomposable vectors exist in U' which implies $F \cap E' = \{0\}$. Then

$$U' = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf, \ f \in F\} \quad \text{with} \quad T : F \to \tilde{F} \quad \text{injective.}$$

The sufficient condition for U to be nondegenerate is straightforward.

In the next section the subspace $U' = \{h_1 \otimes f + h_2 \otimes Tf, \ f \in F\}$ with $F \cap TF = \{0\}$ and $T : F \to TF$ an isomorphism given in (12), will be called *real.* Therefore *any pure A-nilpotent subspace is the direct sum of a degree 1 A-nilpotent subspace and a* (*possibly trivial*) *real subspace.*

We have seen that a non trivial decomposable subspace is a degree 1 nilpotent subspace and a pure weakly para-complex subspace not para-complex. From (12) we have then that every nontrivial pure nilpotent subspace contains a nontrivial pure weakly para-complex subspace not para-complex. Moreover, any nontrivial pure weakly para-complex subspace is the direct sum of a pure para-complex subspace and a degree 1 nilpotent subspace.

1.6. Real subspaces

Definition 1.9. A subspace $U \subset V$ is called *real* if $AU \cap U = \{0\}, \quad \forall A \in \widetilde{Q}$. Equivalently, U contains neither a nontrivial complex nor weakly para-complex subspace.

Let us prove the above equivalence. If $AU \cap U = \{0\}$, $\forall A \in \widetilde{Q}$, clearly no nontrivial complex or weakly para-complex subspaces are in U. Viceversa, let U contain no nontrivial complex or weakly para-complex subspaces. Then, as remarked in the previous section, it contains no nontrivial nilpotent subspaces as well.

A real subspace U is pure. Also, $\dim U \leq \frac{1}{2} \dim V$.

Definition 1.10. A non degenerate real subspace $U \subset V$ is called *totally real* if for one and hence for any para-hypercomplex basis (I, J, K) of $\widetilde{Q}$,

$$IU \perp U, \qquad JU \perp U, \quad KU \perp U$$

or, equivalently, if the Hermitian product (2) has real values for any admissible basis (I, J, K) of $\widetilde{Q}$. In fact, in [3] such subspace is called a *subspace with real Hermitian product.*

The implication in the first statement is straightforward to verify. In this case $\dim U \leq \frac{1}{4} \dim V$.

Theorem 1.4. *A subspace $U \subseteq V$ is real iff $U = U^{F,T}$ with respect to a symplectic basis (h_1, h_2), where the linear map*

$$T : F = E_1 = p_1(U) \to p_2(U)$$

is an isomorphism such that, for any non trivial subspace $W \subset F \cap TF$, we have $TW \nsubseteq W$.

The subspace U is nondegenerate if and only F is g_F-nondegenerate.

Let $E_2 = TE_1$. The real subspace U is totally real iff

$$\omega^E|_{E_1} = \omega^E|_{E_2} \equiv 0 \quad \textit{and} \quad T|_{E_1} \quad \textit{is} \quad \omega^E|_F - \textit{skew-symmetric}, \tag{13}$$

which implies $E_1 \cap E_2 = \{0\}$.

Proof. Let U be a real subspace. As in the case of pure complex subspaces, also the real subspace U contains no decomposable vectors (since it contains no non trivial weakly para-complex subspace). From Lemma (1.1), fixed any symplectic basis (h_1, h_2) of H, we can write

$$U = U^{F,T} = \{h_1 \otimes e + h_2 \otimes Te, \ e \in F = E_1 = p_1(U)\}.$$

Suppose $T\tilde{F} \subset \tilde{F}$ for some subspace $\tilde{F} \subset W = E_1 \cap E_2$. Then $\tilde{F}$ has to be an even-dimensional subspace direct sum of 2-dimensional T-invariant real subspaces $\tilde{F}_i$. We show that necessarily $\tilde{F} = \{0\}$.

Let then $\tilde{F} \supseteq \tilde{F}_i =< e, Te >_{\mathbb{R}}$ be a T-invariant plane with $T(Te) = \lambda e + \mu Te$. It is easy to verify that neither μ nor λ can be zero. Moreover, let us consider the non-null vector $X = h_1 \otimes e + h_2 \otimes Te \in U$. For any map $A \in \widetilde{Q}, \ A = \alpha I + \beta J + \gamma K$, where (I, J, K) is the para-hypercomplex basis associated to the chosen basis (h_1, h_2) of H and with

$$\alpha(\gamma) = \frac{\gamma}{\mu}(\lambda - 1), \qquad \beta(\gamma) = \frac{\gamma}{\mu}(1 + \lambda),$$

the vector $AX \in U$ and this gives a contradiction.

Viceversa, let $U = U^{F,T}$ with respect to the symplectic basis (h_1, h_2); denote $E_1 = F,\ E_2 = TF$, and assume that $T : E_1 \to T(E_1) = E_2$ is an isomorphism such that for any non trivial subspace $W \subset E_1 \cap E_2$ we have $TW \subsetneq W$. Let

$$A = \alpha I + \beta J + \gamma K, \in \widetilde{Q}.$$

Suppose there exists a non null vector $X = h_1 \otimes e + h_2 \otimes Te \in U$, such that

$$AX = h_1 \otimes (-\gamma e + (\beta - \alpha)Te) + h_2 \otimes ((\alpha + \beta)e + \gamma Te) \neq 0 \quad \text{in} \quad U.$$

This implies that $T^2 e \in < e, Te > \subset (E_1 \cap E_2)$, which gives a contradiction.

By (7), the subspace U is nondegenerate if and only F is g_F-nondegenerate.

Let $U = \{X = h_1 \otimes e + h_2 \otimes Te,\ e \in E_1\}$ be a totally real subspace in V. Then calculating $IU,\ JU,\ KU$ and imposing orthogonality conditions $IU \perp U,\ JU \perp U,\ KU \perp U$, we obtain

$$\omega^E|_{E_1} = \omega^E|_{E_2} \equiv 0,$$

from 1) and 2), and

$$\omega^E(e, Te') + \omega^E(Te, e') = 0, \quad \forall e, e' \in E_1$$

from 3).

Viceversa, given a pure real subspace $U = U^{F,T}$, from (13) we obtain $IU \perp U,\ JU \perp U,\ KU \perp U$. For any

$$X = h_1 \otimes e + h_2 \otimes Te \quad \text{and} \quad Y = h_1 \otimes e' + h_2 \otimes Te',$$

we get

$$g(X, Y) = \omega^E(e, Te') - \omega^E(Te, e') = 2\omega^E(e, Te'). \tag{14}$$

Since U is nondegenerate, $\omega^E(E_1 \times E_2)$ is nondegenerate as well, and hence $E_1 \cap E_2 = \{0\}$. We have denoted by $\omega^E(E_1 \times E_2)$ the restriction of the symplectic form ω^E to the subspace $E_1 \times E_2$ of $E \times E$. Moreover, by saying that $\omega^E(E_1 \times E_2)$ is nondegenerate, we mean that

$$\ker\, \omega^E(E_1 \times E_2) = \{b \in E_2 \mid \omega^E(e_1, e_2) = 0,\ \forall e_1 \in E_1\} = \{0\}.$$

2. Decomposition of a generic subspace

Let $U \subset V$ be a subspace of the para-quaternionic Hermitian vector space

$$V = (H \otimes E, \widetilde{Q} = \mathfrak{sl}_2(\mathbb{R}) \otimes \mathrm{Id}, g = \omega^H \otimes \omega^E).$$

For any $A \in \widetilde{Q}$ we denote by U_A the maximal A-invariant subspace in U.

The following proposition shows that, by using para-quaternionic, pure complex, weakly pure para-complex and real subspaces as building blocks, we can construct any subspace $U \subset V$.

Proposition 2.9. *Let U be a subspace in V and U_0 be its maximal para-quaternionic subspace. Then U admits a direct sum decomposition of the form*

$$U = U_0 \oplus U'$$

with

$$U' = U^1_{I_1} \oplus \ldots \oplus U^p_{I_p} \oplus U^{p+1}_{J_1} \oplus \ldots \oplus U^{p+q}_{J_q} \oplus U^R,$$

where $U^i_{I_i}$, $i = 1, \ldots, p$, are pure I_i-complex subspaces, whereas $U^j_{J_j}$, $i = 1, \ldots, q$, are J_j-pure weakly para-complex subspaces and U^R is real.

Proof. We denote by U^1 a complement to U_0 in U and choose a complex structure I_1 so that $U^1_{I_1} \neq \{0\}$; then we can write $U = U_0 \oplus U^1_{I_1} \oplus U^2$, where U^2 is a complement to $U_0 \oplus U^1_{I_1}$.

Let us choose now a complex structure I_2 so that $U^2_{I_2} \neq \{0\}$. Then $U = U_0 \oplus U^1_{I_1} \oplus U^2_{I_2} \oplus U^3$, where U^3 is a complement to $U_0 \oplus U^1_{I_1} \oplus U^2_{I_2}$.

Denote by $p+1$ the step in which U^{p+1} has no invariant pure complex subspace. Then choose a para-complex structure J_1 so that $U^{p+1}_{J_1} \neq \{0\}$. Then

$$U = U_0 \oplus U^1_{I_1} \oplus \ldots \oplus U^p_{I_p} \oplus U^{p+1}_{J_1} \oplus U^{p+2},$$

where U^{p+2} is a complement to $U_0 \oplus U^1_{I_1} \oplus \ldots \oplus U^p_{I_p} \oplus U^{p+1}_{J_1}$. By carrying on this construction, we arrive at a complementary U^{p+q+1} which has neither pure complex nor pure weakly para complex invariant subspaces. In this case $U^{p+q+1} = U^R$ is real.

Observe now that, by using as building blocks para-complex subspaces instead of weakly para-complex ones, we necessarily need to use also nilpotent subspaces; for example, let think of a decomposable subspace $U = h \otimes F$.

The decomposition of Proposition (2.9) is clearly not unique. The first reason depends obviously on the nonuniqueness of the complement at each step. Moreover the decomposition depends on the chosen *order of types of subspaces*, i.e. if we first consider pure complex subspaces and then pure weakly para-complex or the other way round.

As an example, let (I, J, K) be a hypercomplex basis of $\widetilde{Q}$ and $X \in V$ a non-decomposable vector in $H \otimes E$ Consider $U^3 =< X, IX, JX >\subset \widetilde{Q}X$ (observe that vectors X, IX, JX are mutually orthogonal). Then clearly, according to the chosen order, $U = U_1 \oplus U_2$ with $(U_1, I) =< X, IX >$ a pure complex subspace and $U_2 =< JX >$ a real subspace. Yet, also $U = U_1 \oplus U_2$ with $(U_1, J) =< X, JX >$ a pure para-complex subspace and $U_2 =< IX >$ a real subspace.

To overtake this pair of problems, we may fix the order of types of subspaces, for example first complex (resp. para-complex), and select, among all U^p_A, $A \in \widetilde{Q}$, $A^2 = -\mathrm{Id}$, (resp. $A^2 = \mathrm{Id}$), the one with maximal dimension. Yet, as appears clearly from the above example, in the second case we could have also $U = U_1 \oplus U_2$ with $(U_1, K = IJ) =< IX, JX >$ being a pure para-complex subspace and $U_2 =< X >$ a real subspace. Moreover, looking for a canonical *metrical* decomposition, we clearly need to consider also the signature of the metric induced on each addend.

Further investigations are needed to verify the existence of some criteria to define a canonical affine or metrical decomposition.

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PODSTAWY LINIOWEJ GEOMETRII PARA-KWATERNIONOWEJ II

ROZKŁAD PRZESTRZENI GENERUJĄCEJ PARA-KWATERNIONOWEJ HERMITOWSKIEJ PRZESTRZENI WEKTOROWEJ

S t r e s z c z e n i e

W obecnej części II pracy badamy stosowne klasy podprzestrzeni para-kwaternionowych hermitowskich przestrzeni wektorowych, a w szczególności rozkład przestrzeni generującej.

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In memory of
Professor Roman Stanisław Ingarden

Leszek Wojtczak

A REMARK ON SURFACE PHENOMENA

Summary

The paper contains a review of some surface phenomena from the point of view of the geometrical surface phenomena. First of all, the relation between the anharmonicity and the disorder is underlined.

An original remark concerning the development of the geometrical model to the Finsler, Ingarden and Randers geometries is suggested.

The surface topography plays an essential role in the description and explanation of surface phenomena in the context of relations between various kinds of them.

In the present note we consider the model in which the topography is reflected by the boundary conditions which are responsible for the behaviour of surface scattering and its relation to the spectroscopic and thermodynamic properties including spatial distributions of the order parameters at the superficial area.

First of all, the most important role of the surface is to consider the stability conditions for crystals with restricted dimensions. The ideal surface cut along a regular crystallographic lattice plane does not stabilise a crystal and the real situation is always connected with some deformation of the structure at the surface leading to its reconstruction. In this case, one of the conventional models which describe a deviation from the ideal crystallographic structure is the model of topological correlations between occupied lattice sites reflecting topological disorder at the surface in the approximation of the nearest neighbours [e.g. 1]. The stability of the sample is then determined by the structure of the surface whose distance from the successive lattice plane in its superficial area is different than the lattice constant of a bulk material and the surface disorder effect is achieved by the vacancies in the superficial layer.

In this way, all the surface characteristics determined self consistently differ from their values for infinite samples.

At the level of a quantum mechanics approach the stability conditions are determined by the geometrical characteristics of the considered sample.

The spectroscopy can be described by the boundary conditions for the wave functions of created quasi-particles in the sample, first of all, phonons, but also quasi-electrons, magnons or other elementary quasi-excitations.

The formulation of the thermodynamics in terms of quasi-phonons leads to the definition of the mean square displacement of atoms from their equilibrium positions. This factor allows us to consider the stability of a sample with restricted dimensions in the sense of the Lindemann criterion [2] applied to the surface models when the distribution of lattice vibrations perpendicular to the surface is taken into account [3, 4]. The illustrative example in this case is the geometry of thin films where the superficial layers and internal layers are of different parametrization of the two-dimensional geometries distributed in the direction of the film thickness.

The above model of lattice vibrations applied to the melting phenomenon description, as well as the surface melting influence discussion seem to be most natural examples for the consideration concerning the stability conditions. Two aspects should be then taken into account [5]. First of all, the influence of anharmonic effects which are connected with the distinction between the phase transition temperature represented by the melting point and the solid state instability temperature determining the interval in which the existence of a sample with the surface can be considered.

In this case, usually, the renormalizing approach based on the procedure within Green's function technique allows us to derive the maximal value of the mean square displacement of atoms which corresponds to the Lindemann's parameter determining the critical size of the lattice constant at the melting temperature. On the other hand, the temperature for the maximum of the mean square displacement of atoms can be interpreted as the solid phase instability point.

In the above sense, the instability temperature must be overestimated in reference to the melting criterion. This situation does not appear, however, in the case of Valenta's theory [5] in which the considered parameter is of harmonic type potential and then it is not confined from above.

Next, we can see that the topological correction equivalent to the influence of anharmonicity, does not introduce any inconvenience distinguishing between the phase transition point and the point confirming the phase from above.

The thickness of the superfacial layer increases and the thickness of the solid layer decreases in the kinetic process when the temperature is increasing. Thus, we can always consider the part of a sample as a thin film for some step of the process.

The second important question connected with the role of the surface in the physical models of surface phenomena is the damping at the surface.

The nature of damping seems to be clarified [6]. namely, the damping is connected with the interaction between a body moving in an effective field and the field itself. The different origins of these fields seem to be equivalent.

The idea taken into consideration by Ford, Kac and Mazur [7] that the damping is provided by a heat bath of harmonic oscillations has its own equivalent form in the approach proposed by Zwiek and Uruh [8] who based their modes on the coupling between a damped body and a scalar field which can be quantized in the form of harmonic oscillators.

Another description shows that a damping can be evaluated as a conservative system replacing its dissipative element by a string whose vibrations can be quantized. In this model Yurke [9] discussed various concepts in classical as well as statistical mechanics. The classical and quantum case can be distinguished by means of the appearance od stochastic forces corresponding to heat bath fluctuations. We can see that the nature of damping in the considered case can be explained from the physical point of view in terms of topological correlations at the surface [cf. 10]. This means the damping is induced by a disorder [6].

In analogy to the discussion introduced by Rowlands [11] in connection with the damping of acoustic modes in one-dimensional chains with the topological disorder we can conclude that the damping can be interpreted as a result of phase mixing between the individual modes. The solution reported by Rowlands [11] can be treated as a reference solution of the equation of motion including topological disorder.

In both the considered cases, i.e. anharmonicity and disorder, we can see that the solutions are of self-consistent type. This means that the conditions which stabilize a sample lead to the damping at the same time.

This conclusion is of particular meaning for the spectroscopic properties. It suggests a mutual relation between the spectroscopy and the dissipative forces introduced by the damping description in terms of the surface correlations. From the geometrical point of view the model of surface topography can be constructed on the basis of different metrics of the space inside the sample and in its surface area.

In this case the convenient description of the model can be expected in terms of Finsler geometry in its particular construction using a Randers and an Ingarden-Randers type space.

A remark on the surface phenomena description is of particular interest in the present note devoted to the memory of Professor Roman Ingarden in connection with the suggestion of applicability of his theories [12] to thin film inhomogeneities.

It is worth-while remembering here that the structure of a film is considered usually as the composition of two-dimensional sublattices with a distribution of thermodynamic characteristics across a film. Each of the sublattices is described by characteristic parameters. For when the surface deformation is determined in terms the geometrical model connected with the space curvature the surface energy contribution is related to the energy minimum principle which determines the curvature radius at the surface in the proper structure [13]. In the case of Minkowski metric, which is homogeneous due to the light velocity to be constant, the surface correlations do not depend directly on the space properties. However, when we consider the problem self-consistently, we should obtain the distribution of the correlations as well as the distribution of the light velocity, mutually dependent. In this case even the Minkowski metric is linear [14].

Ingarden discovered that a Finsler space can serve as a very precise mathematical model which can be used in the most physical situations [13]. One of these situations seems to be connected with the calculations of the surface correlations which are then described self consequently by the parameters of Ingarden or Randers metrics [12].

The important property of the Ingarden space, on which a thin film is embedded, is connected with its flat space character corresponding to a two-dimensional layer dependent on the local velocity of light; in consequence, it can be different inside a sample and on its surface. Then the superficial correlations can be used for the description of the inhomogeneous metric at the surface. The correlations are of electromagnetic nature [13]. First of all, the Ingarden space is of an electromagnetic origin. However, another specialization leads to the Ingarden thermodynamic space when all thermodynamic processes can appear to be automatically irreversible. Some thermodynamical applications including open systems can be considered by means of the above mentioned interpretation in the case of the Randers and Ingarden spaces in dependence on the open and dissipative character of the system, i.e. hyperbolicity in statistical or/and stochastical thermodynamics [13].

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PRZYCZYNEK DO TEORII ZJAWISK POWIERZCHNIOWYCH

S t r e s z c z e n i e

Artykuł zawiera przegląd pewnych zjawisk powierzchniowych z punktu widzenia powierzchniowej struktury geometrycznej. Przede wszystkim została podkreślona relacja pomiędzy deformacją anharmoniczną oraz nieporządkiem.

Oryginalną uwagę podaną w ninejszej nocie stanowi rozszerzenie modelu geometrycznego i jego zastosowanie w ujęciu geometrii Finslera oraz szczególnych przypadkach tej geometrii, a mianowicie, geometrii Randersa i geometrii Ingardena.

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THE GENERALIZED FOURIER COEFFICIENTS AND EXTREMAL QUASICONFORMAL EXTENSION OF A QUASISYMMETRIC AUTOMORPHISM OF THE UNIT CIRCLE

Summary

The *generalized Fourier coefficients* $\hat{\gamma}(m,n)$ of a homeomorphic self-mapping γ of the unit circle $\mathbb{T}$ are defined by the formula

$$\hat{\gamma}(m,n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(\mathrm{e}^{\mathrm{i}t})^m \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\ , \quad m,n \in \mathbb{Z}\ .$$

In the paper [12] the following inequalities were proved:

$$\frac{1}{K}\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 \le \sum_{n=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty}\sqrt{|n|}\hat{\gamma}(m,n)\lambda_m\right|^2 \le K\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2\ ,$$

provided γ admits a K-quasiconformal extension to the unit disk $\mathbb{D}$ and $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ is a sequence such that $\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 < +\infty$. Note that they look similarly to the Grunsky inequalities for holomorphic functions in the classes $\Sigma(k)$, $0 \le k \le 1$. This paper provides an answer to the question about the equality in these inequalities.

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Introduction

Given a function $f : \mathbb{T} \to \mathbb{C}$ measurable on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $m, n \in \mathbb{Z}$ we define

$$\hat{f}(m,n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})^m \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{T}} f(z)^m z^{-n}|\mathrm{d}z| \ , \tag{0.1}$$

provided the respective functions are integrable on $\mathbb{T}$. If $m = 1$ then (0.1) takes the form of

$$\hat{f}(1,n) = \hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{T}} f(z) z^{-n}|\mathrm{d}z| \ , \tag{0.2}$$

and so $\hat{f}(1,n)$ is just a n-th Fourier coefficient of the function f. This justifies to call $\hat{f}(m,n)$ the (m,n)-*generalized Fourier coefficient* of the function f. If f satisfies the following condition

$$0 < \operatorname*{ess\,inf}_{z\in\mathbb{T}}|f(z)| \le \operatorname*{ess\,sup}_{z\in\mathbb{T}}|f(z)| < +\infty \ ,$$

then $\hat{f}(m,n)$ is well defined for any $m, n \in \mathbb{Z}$. Let $\mathrm{Hom}^+(\mathbb{T})$ denote the family of all sense-preserving homeomorphic self-mappings of $\mathbb{T}$. If $f \in \mathrm{Hom}^+(\mathbb{T})$, then all generalized Fourier coefficients $\hat{f}(m,n)$, $m, n \in \mathbb{Z}$, are well defined. For $K \ge 1$ let $\mathrm{Q}(\mathbb{T}; K)$ be the class of all $\gamma \in \mathrm{Hom}^+(\mathbb{T})$ which admit a K-quasiconformal extension to the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Homeomorphisms belonging to the class $\mathrm{Q}(\mathbb{T}) := \bigcup_{K\ge 1} \mathrm{Q}(\mathbb{T}; K)$ were called by Krzyż as *quasisymmetric automorphisms of the unit circle*; cf. [4] and [5]. He noticed that each $f \in \mathrm{Q}(\mathbb{T})$ can be described by a similar condition to the well-known Beurling-Ahlfors quasisymmetricity condition; cf. [2]. For another characterizations of the class $\mathrm{Q}(\mathbb{T})$ see [16] and [13].

In the paper [12] the following result was proved.

Theorem A. *Given $K \ge 1$ let $\gamma \in \mathrm{Q}(\mathbb{T}, K)$. If $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ is a sequence such that*

$$\sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 < +\infty \ , \tag{0.3}$$

then for each $n \in \mathbb{Z}$ the sequence $\mathbb{N} \ni p \mapsto \sum\limits_{m=-p}^{p} \sqrt{|n|}\hat{\gamma}(m,n)\lambda_m$ is convergent as $p \to \infty$ and

$$\frac{1}{K}\sum_{n=-\infty}^{\infty} |n||\lambda_n|^2 \le \sum_{n=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty}\sqrt{|n|}\hat{\gamma}(m,n)\lambda_m\right|^2 \le K\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 \ . \tag{0.4}$$

Note that the inequalities (0.4) look similarly to the Grunsky inequalities for holomorphic functions in the classes $\Sigma(k)$, $0 \le k \le 1$; cf. [14, Sect. 3.1 and 9.4]. By the works of R. Kühnau [6–8] and Y. Shen [15] we know that the inequalities (0.4) can be improved in general, because the quasiconformality constant K can be replaced by a better one for a certain γ. In this paper we prove that the second (resp. first) equality in (0.4) is possible for a nonzero sequence $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ iff γ (resp. γ^{-1}) admits the extremal regular quasiconformal Teichmüller extension φ to $\mathbb{D}$ with the complex dilatation

$$\frac{\bar{\partial}\varphi}{\partial\varphi} = \frac{K-1}{K+1}\frac{\overline{F'}}{F'} \quad \text{a.e. on } \mathbb{D}\ , \tag{0.5}$$

where $F : \mathbb{D} \to \mathbb{C}$ is a non-constant holomorphic function with finite Dirichlet integral. This completes the considerations from [12]. Here and later on we abbreviate *almost everywhere* and *almost every* to *a.e.*

1. On a modification of the Poisson integral

Let $\mathrm{L}^0(\mathbb{T})$ be the class of all Lebesgue's measurable functions $f : \mathbb{T} \to \mathbb{C}$. We adopt the usual notation $\mathrm{L}^1(\mathbb{T})$ for the class of all functions $f \in \mathrm{L}^0(\mathbb{T})$ which are integrable on $\mathbb{T}$ with respect to the Lebesgue arc-length measure, i.e. $\int_{\mathbb{T}} |f(z)||\mathrm{d}z| < +\infty$. Let $\mathrm{P}[f]$ be the Poisson integral of a function $f \in \mathrm{L}^1(\mathbb{T})$, i.e.

$$\begin{aligned} \mathrm{P}[f](z) :&= \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|\mathrm{d}u| \\ &= \hat{f}(0) + \sum_{n=1}^{\infty}\hat{f}(n)z^n + \sum_{n=1}^{\infty}\hat{f}(-n)\overline{z}^n\ , \quad z \in \mathbb{D}\ . \end{aligned} \tag{1.1}$$

It is well known that $\mathrm{P}[f]$ is a complex-valued harmonic function on $\mathbb{D}$. Moreover, if the function f is continuous, then the function $\mathrm{P}[f]$ is the unique solution to the Dirichlet problem for f which means that for every $z \in \mathbb{T}$, $\mathrm{P}[f](\zeta) \to f(z)$ as $\mathbb{D} \ni \zeta \to z$. Let $\mathrm{D}[F]$ denote the Dirichlet integral of a function $F : \mathbb{D} \to \mathbb{C}$, a.e. differentiable on $\mathbb{D}$, i.e.

$$\mathrm{D}[F] := \int_{\mathbb{D}}\left(\left|\frac{\partial F}{\partial x}\right|^2 + \left|\frac{\partial F}{\partial y}\right|^2\right)\mathrm{d}x\mathrm{d}y = 2\int_{\mathbb{D}}(|\partial F|^2 + |\bar{\partial}F|^2)\mathrm{d}x\mathrm{d}y\ , \tag{1.2}$$

where

$$\partial F := \frac{1}{2}\left(\frac{\partial F}{\partial x} - \mathrm{i}\frac{\partial F}{\partial y}\right)\ , \quad \bar{\partial}F := \frac{1}{2}\left(\frac{\partial F}{\partial x} + \mathrm{i}\frac{\partial F}{\partial y}\right)$$

are so-called the formal derivatives of F. If $F : \mathbb{D} \to \mathbb{C}$ is a harmonic mapping in $\mathbb{D}$ given by the series expansion

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\overline{z}^n\ , \quad z \in \mathbb{D}\ ,$$

with coefficients $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$, then integrating by substitution we obtain

$$\begin{aligned}\int_{\mathbb{D}}(|\partial F|^2+|\bar\partial F|^2)\mathrm{d}x\mathrm{d}y &= \lim_{R\to 1^-}\lim_{p\to\infty}\int_0^R\int_0^{2\pi}\Big(\Big|\sum_{n=1}^{p} na_n r^{n-1}\mathrm{e}^{\mathrm{i}(n-1)t}\Big|^2 \\ &\quad + \Big|\sum_{n=1}^{p} na_{-n} r^{n-1}\mathrm{e}^{-\mathrm{i}(n-1)t}\Big|^2\Big) r\mathrm{d}t\mathrm{d}r \\ &= \lim_{R\to 1^-}\int_0^R 2\pi\Big(\sum_{n=1}^{\infty} n^2|a_n|^2 r^{2n-1}+\sum_{n=1}^{\infty} n^2|a_{-n}|^2 r^{2n-1}\Big)\mathrm{d}r \\ &= \pi\lim_{R\to 1^-}\sum_{n=-\infty}^{\infty}|n||a_n|^2R^{2n} = \pi\sum_{n=-\infty}^{\infty}|n||a_n|^2\ ,\end{aligned}$$

and consequently,

$$\mathrm{D}[F] = 2\pi\sum_{n=-\infty}^{\infty}|n||a_n|^2\ ; \tag{1.3}$$

cf. [12, (1.2)]. By Theorem A we can modify the Poisson integral $\mathrm{P}[f]$ as follows. Setting $\mathrm{H}^{1/2} := \{h \in \mathrm{L}^1(\mathbb{T}) : \mathrm{D}[\mathrm{P}[h]] < +\infty\}$ we conclude from (1.3) that

$$2\pi\sum_{n=-\infty}^{\infty}|n||\hat f(n)|^2 = \mathrm{D}[\mathrm{P}[f]] < +\infty\ , \quad f\in\mathrm{H}^{1/2}\ . \tag{1.4}$$

Applying now Theorem A for an arbitrary fixed $K \ge 1$, $\gamma \in \mathrm{Q}(\mathbb{T}; K)$ and the sequence $\mathbb{Z} \ni n \mapsto \lambda_n := \hat f(n)$ we know that for each $n \in \mathbb{Z}$ the sequence $\mathbb{N} \ni p \mapsto \sum\limits_{m=-p}^{p}\sqrt{|n|}\hat\gamma(m,n)\lambda_m$ is convergent as $p\to\infty$ and we may define

$$\hat f(0;\gamma) := \hat f(0) \quad\text{and}\quad \hat f(n;\gamma) := \lim_{p\to\infty}\sum_{m=-p}^{p}\hat\gamma(m,n)\hat f(m)\ , \quad n\in\mathbb{Z}\setminus\{0\}\ . \tag{1.5}$$

Moreover, by the inequalities (0.4),

$$\frac{1}{K}\sum_{n=-\infty}^{\infty}|n||\hat f(n)|^2 \le \sum_{n=-\infty}^{\infty}|n||\hat f(n;\gamma)|^2 \le K\sum_{n=-\infty}^{\infty}|n||\hat f(n)|^2\ . \tag{1.6}$$

This means, by (1.4), that the *modified Poisson integral operator*

$$\mathrm{H}^{1/2}\ni f\mapsto \mathrm{P}_\gamma[f] := \sum_{n=0}^{\infty}\hat f(n;\gamma)z^n + \sum_{n=1}^{\infty}\hat f(-n;\gamma)\overline{z}^n\ , \quad z\in\mathbb{D}\ , \tag{1.7}$$

is well defined for all $f \in \mathrm{H}^{1/2}$ and $\gamma \in \mathrm{Q}(\mathbb{T})$; cf. [9]. Combining (1.7) with (1.3) and (1.4) we can rewrite the inequalities (1.6) in the following shorter form

$$\frac{1}{K}\,\mathrm{D}[\mathrm{P}[f]] \le \mathrm{D}[\mathrm{P}_\gamma[f]] \le K\,\mathrm{D}[\mathrm{P}[f]]\ , \tag{1.8}$$

where $K \ge 1$, $\gamma \in \mathrm{Q}(\mathbb{T}, K)$ and $f \in \mathrm{H}^{1/2}$; cf. [9, Corollary 1.1]. As a matter of fact the inequalities in (1.8) are equivalent to the ones in (0.4).

By the definition of the operator P_γ the following its properties can be inferred; cf. [9, Remark 1.2].

Theorem B. *For any $\gamma \in \mathrm{Q}(\mathbb{T})$ the following properties hold:*

(i) *If γ is the identity mapping on $\mathbb{T}$, then $\mathrm{P}_\gamma = \mathrm{P}$, i.e. the mapping $(f, \gamma) \mapsto \mathrm{P}_\gamma[f]$ generalizes the Poisson operator* P;

(ii) $\mathrm{P}_\gamma[\mu f + \nu g] = \mu\, \mathrm{P}_\gamma[f] + \nu\, \mathrm{P}_\gamma[g]$ *as $\mu, \nu \in \mathbb{C}$ and $f, g \in \mathrm{H}^{1/2}$, i.e. the operator P_γ is linear;*

(iii) $\mathrm{P}_\gamma[\overline{f}] = \overline{\mathrm{P}_\gamma[f]}$ *as $f \in \mathrm{H}^{1/2}$;*

(iv) $\mathrm{P}_\gamma[\operatorname{Re} f] = \operatorname{Re} \mathrm{P}_\gamma[f]$ *and* $\mathrm{P}_\gamma[\operatorname{Im} f] = \operatorname{Im} \mathrm{P}_\gamma[f]$ *as $f \in \mathrm{H}^{1/2}$.*

In [11] and [10] the operator $\boldsymbol{B}_\gamma$ was assign to every $\gamma \in \mathrm{Q}(\mathbb{T})$. We recall now its construction. For all $f, g \in \mathrm{L}^0(\mathbb{T})$ the notation $f \doteqdot g$ means that $f - g$ equals a constant function a.e. on $\mathbb{T}$. It is clear that $\doteqdot$ is an equivalence relation in the class $\mathrm{L}^0(\mathbb{T})$. Let $[f/\doteqdot]$ stands for the abstract class of $f \in \mathrm{L}^0(\mathbb{T})$ with respect to $\doteqdot$. Consider the class

$$\boldsymbol{H} := \{[f/\doteqdot] : f \in \operatorname{Re} \mathrm{L}^1(\mathbb{T}) \text{ and } \mathrm{D}[\mathrm{P}[f]] < +\infty\}\ . \tag{1.9}$$

Here and subsequently, we set $\operatorname{Re} X := \{\operatorname{Re} f : f \in X\}$ for any family X of complex-valued functions. It can be verified in the standard way that $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ is a real Hilbert space, where

$$\big\|[f/\doteqdot]\big\|_{\boldsymbol{H}} := \sqrt{\frac{1}{2}\mathrm{D}[\mathrm{P}[f]]}\ , \quad f \in \operatorname{Re} \mathrm{H}^{1/2}\ ; \tag{1.10}$$

cf. [11, Sect. 2.4]. We adopt the usual notation $\mathrm{C}(\mathbb{T})$ for the class of all complex-valued continuous functions on $\mathbb{T}$. From (1.9), (1.10) and (1.4) it follows that the set $\{[f/\doteqdot\ ] : f \in \operatorname{Re} \mathrm{C}(\mathbb{T})\} \cap \boldsymbol{H}$ is dense in $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$. Moreover, it may be concluded from [11, (2.5.1) and Theorems 2.5.3 and 2.4.3] that the inequalities

$$\frac{1}{K}\mathrm{D}[\mathrm{P}[f]] \le \mathrm{D}[\mathrm{P}[f \circ \gamma]] \le K\, \mathrm{D}[\mathrm{P}[f]]$$

hold for all $K \ge 1$, $f \in \mathrm{C}(\mathbb{T})$ and $\gamma \in \mathrm{Q}(\mathbb{T}, K)$. Then there exists the unique linear continuous operator $\boldsymbol{B}_\gamma : \boldsymbol{H} \to \boldsymbol{H}$ in $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ satisfying

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [f \circ \gamma/\doteqdot]\ , \quad f \in \operatorname{Re} \mathrm{C}(\mathbb{T}) \cap \mathrm{H}^{1/2}\ . \tag{1.11}$$

As a matter of fact $\boldsymbol{B}_\gamma$ is a linear homeomorphism of the space $(\boldsymbol{H}, \|\cdot\|_{\boldsymbol{H}})$ onto itself; cf. [11, Corollary 2.5.4]. Note, that the operator $\boldsymbol{B}_\gamma$ is defined implicitly by the condition (1.11). From the famous Beurling-Ahlfors result [2] we know that a quasisymmetric automorphism γ of $\mathbb{T}$ does not have to be absolutely continuous function. Moreover, γ can be even purely singular. Therefore in such a case the composite mapping $f \circ \gamma$ is not Lebesgue's measurable function in general. In consequence, $f \circ \gamma \notin \mathrm{L}^0(\mathbb{T})$ for a certain $f \in \mathrm{H}^{1/2}$, and so the family $\operatorname{Re} \mathrm{C}(\mathbb{T}) \cap \mathrm{H}^{1/2}$

can not be replaced by $\operatorname{Re}\mathrm{H}^{1/2}$ in (1.11). This means that defining the operator $\boldsymbol{B}_\gamma$ directly by composition of functions fails for a singular γ. This problem was overcome in [11, Sect. 2.5] where the following result was stated:

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}[f]]\circ\gamma/\doteqdot]\ , \quad f\in\operatorname{Re}\mathrm{H}^{1/2}\ ; \tag{1.12}$$

cf. [11, (2.5.8)]. Here and later on the symbol $\mathrm{Tr}[F]$ denotes the radial limiting valued function of a function $F:\mathbb{D}\to\mathbb{C}$, i.e. for every $z\in\mathbb{T}$,

$$\mathrm{Tr}[F](z) := \lim_{t\to 1^-} F(tz)$$

as the limit exists, while $\mathrm{Tr}[F](z) := 0$ otherwise. It is well known that $\mathrm{Tr}[\mathrm{P}[f]] = f$ a.e. on $\mathbb{T}$ for every $f\in L^1(\mathbb{T})$; cf. [3, Sect. 1.2]. By (1.8),

$$\mathrm{Tr}[\mathrm{P}_\gamma[f]]\in\mathrm{H}^{1/2}\ , \quad f\in\mathrm{H}^{1/2}\,, \gamma\in\mathrm{Q}(\mathbb{T})\ . \tag{1.13}$$

From this and Theorem B we see that $\mathrm{Tr}\circ\mathrm{P}_\gamma:\mathrm{H}^{1/2}\to\mathrm{H}^{1/2}$ is a linear operator as $\gamma\in\mathrm{Q}(\mathbb{T})$. The operators $\boldsymbol{B}_\gamma$ and $\boldsymbol{H}\ni\boldsymbol{f}\mapsto\|\boldsymbol{B}_\gamma(\boldsymbol{f})\|_{\boldsymbol{H}}$ can be described by means of the operator $\mathrm{Tr}\circ P_\gamma$ as follows.

Theorem C. [9, Theorem 2.1 and Corollary 2.2] *For every* $\gamma\in\mathrm{Q}(\mathbb{T})$*,*

$$\boldsymbol{B}_\gamma([f/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}_\gamma[f]]/\doteqdot]\ , \quad f\in\operatorname{Re}\mathrm{H}^{1/2}\ , \tag{1.14}$$

as well as

$$\|\boldsymbol{B}_\gamma([f/\doteqdot])\|^2_{\boldsymbol{H}} = \frac{1}{2}\,\mathrm{D}[\mathrm{P}_\gamma[f]]\ , \quad f\in\operatorname{Re}\mathrm{H}^{1/2}\ . \tag{1.15}$$

2. Auxiliary properties of the Dirichlet integral

It is well known that the Dirichlet integral is quasi-invariant; cf. [1, p. 18]. In particular, it means that for every harmonic function $F:\mathbb{D}\to\mathbb{C}$ and every K-quasiconformal self-mapping φ of $\mathbb{D}$,

$$\mathrm{D}[F\circ\varphi]\le K\,\mathrm{D}[F]\ . \tag{2.1}$$

We need the inequality (2.1) in a little bit more stronger version. Let us recall that the *signum function* sgn is defined by the formula

$$\mathbb{C}\ni z\mapsto \operatorname{sgn}(z) := \begin{cases} z/|z| & \text{as } z\neq 0\ , \\ 0 & \text{as } z=0\ . \end{cases} \tag{2.2}$$

Lemma 2.1 *If* $K\ge 1$ *and* φ *is a quasiconformal self-mapping of* $\mathbb{D}$*, then for every harmonic function* $F:\mathbb{D}\to\mathbb{C}$ *with* $\mathrm{D}[F]<+\infty$ *the following inequality holds*

$$\mathrm{D}[F\circ\varphi] = K\,\mathrm{D}[F] - \mathrm{Q}_\varphi[F] - \mathrm{R}_\varphi[F]\ , \tag{2.3}$$

where

$$\mathrm{Q}_\varphi[F] := 4\int_{\mathbb{D}} |\partial\varphi\partial\overline{\varphi}|\big|(\partial F\circ\varphi)\operatorname{sgn}(\partial\varphi) - (\bar\partial F\circ\varphi)\operatorname{sgn}(\partial\overline{\varphi})\big|^2 \mathrm{d}x\mathrm{d}y\ ,$$

$$\mathrm{R}_\varphi[F] := 2\int_{\mathbb{D}} (|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)\Big(K - \frac{|\partial\varphi| + |\bar\partial\varphi|}{|\partial\varphi| - |\bar\partial\varphi|}\Big)(|\partial\varphi|^2 - |\bar\partial\varphi|^2)\mathrm{d}x\mathrm{d}y\ .$$

Proof. Fix a harmonic function $F:\mathbb{D}\to\mathbb{C}$ such that $\mathrm{D}[F] < +\infty$. Let us observe first that for any $a_1, a_2 \in \mathbb{C}$ and $b_1, b_2 \in \mathbb{C}\setminus\{0\}$

$$\begin{aligned}(|a_1|^2 + |a_2|^2)(|b_1| + |b_2|)^2 - |a_1b_1 + a_2b_2|^2 - |a_1\overline{b_2} + a_2\overline{b_1}|^2&\\ = 2|a_1|^2|b_1b_2| + 2|a_2|^2|b_1b_2| - 4\operatorname{Re}\big(a_1b_1\overline{a_2b_2}\big)&\\ = 2|b_1b_2|\left(\left|a_1\frac{b_1}{|b_1|}\right|^2 + \left|a_2\frac{b_2}{|b_2|}\right|^2 - 2\operatorname{Re}\Big(a_1\frac{b_1}{|b_1|}\overline{a_2\frac{b_2}{|b_2|}}\Big)\right)&\\ = 2|b_1b_2|\left|a_1\frac{b_1}{|b_1|} - a_2\frac{b_2}{|b_2|}\right|^2 .&\end{aligned}$$

Hence for all $a_1, a_2, b_1, b_2 \in \mathbb{C}$,

$$\begin{aligned}(2.4)\quad |a_1b_1 + a_2b_2|^2 + |a_1\overline{b_2} + a_2\overline{b_1}|^2&\\ = (|a_1|^2 + |a_2|^2)(|b_1| + |b_2|)^2 - 2|b_1b_2|\big|a_1\operatorname{sgn}(b_1) - a_2\operatorname{sgn}(b_2)\big|^2 .&\end{aligned}$$

Differentiating the composed mapping $F\circ\varphi$ we see that a.e. in $\mathbb{D}$,

$$\partial(F\circ\varphi) = (\partial F\circ\varphi)\partial\varphi + (\bar\partial F\circ\varphi)\partial\overline{\varphi}\ ,$$
$$\bar\partial(F\circ\varphi) = (\partial F\circ\varphi)\bar\partial\varphi + (\bar\partial F\circ\varphi)\bar\partial\overline{\varphi}\ .$$

Applying now the identity (2.4) and the change of variables formula we obtain

$$\begin{aligned}\mathrm{D}[F\circ\varphi] &= 2\int_{\mathbb{D}}(|\partial(F\circ\varphi)|^2 + |\bar\partial(F\circ\varphi)|^2)\mathrm{d}x\mathrm{d}y\\ &= 2\int_{\mathbb{D}}(|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi| + |\bar\partial\varphi|)^2\mathrm{d}x\mathrm{d}y\\ &\quad - 4\int_{\mathbb{D}}|\partial\varphi\partial\overline{\varphi}|\big|(\partial F\circ\varphi)\operatorname{sgn}(\partial\varphi) - (\bar\partial F\circ\varphi)\operatorname{sgn}(\partial\overline{\varphi})\big|^2\mathrm{d}x\mathrm{d}y\\ &= 2\int_{\mathbb{D}}(|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi| + |\bar\partial\varphi|)^2\mathrm{d}x\mathrm{d}y - \mathrm{Q}_\varphi[F]\end{aligned}$$

as well as

$$\begin{aligned}2\int_{\mathbb{D}}&(|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi| + |\bar\partial\varphi|)^2\mathrm{d}x\mathrm{d}y\\ &= 2\int_{\mathbb{D}}(|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)\frac{|\partial\varphi| + |\bar\partial\varphi|}{|\partial\varphi| - |\bar\partial\varphi|}(|\partial\varphi|^2 - |\bar\partial\varphi|^2)\mathrm{d}x\mathrm{d}y\\ &= 2K\int_{\mathbb{D}}(|\partial F\circ\varphi|^2 + |\bar\partial F\circ\varphi|^2)(|\partial\varphi|^2 - |\bar\partial\varphi|^2)\mathrm{d}x\mathrm{d}y - \mathrm{R}_\varphi[F]\\ &= 2K\int_{\mathbb{D}}(|\partial F|^2 + |\bar\partial F|^2)\mathrm{d}x\mathrm{d}y - \mathrm{R}_\varphi[F] = K\,\mathrm{D}[F] - \mathrm{R}_\varphi[F]\ .\end{aligned}$$

These equalities yield the equality (2.3), which proves the lemma. □

Lemma 2.2 *If $\gamma \in \mathrm{Q}(\mathbb{T})$ and φ is its quasiconformal extension to $\mathbb{D}$, then*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = \mathrm{D}[\mathrm{P}[f]\circ\varphi] - \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]]\ , \quad f \in \mathrm{H}^{1/2}\ . \tag{2.5}$$

Proof. Given $\gamma \in \mathrm{Q}(\mathbb{T})$ suppose that φ is its quasiconformal extension to $\mathbb{D}$. From [10, Theorem 1.2] and Theorem C it follows that

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = \mathrm{D}[\mathrm{P}[f]\circ\varphi] - \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]]\ , \quad f \in \mathrm{Re}\,\mathrm{H}^{1/2}\ . \tag{2.6}$$

Let $f \in \mathrm{H}^{1/2}$. Then $f \in L^1(\mathbb{T})$ and $\mathrm{D}[\mathrm{P}[f]] < +\infty$. Hence $\mathrm{Re}\, f, \mathrm{Im}\, f \in L^1(\mathbb{T})$. By (1.1), we get

$$\mathrm{P}[f] = \mathrm{P}[\mathrm{Re}\, f + \mathrm{i}\,\mathrm{Im}\, f] = \mathrm{P}[\mathrm{Re}\, f] + \mathrm{i}\,\mathrm{P}[\mathrm{Im}\, f]\ . \tag{2.7}$$

Combining this with (1.2) we see that

$$\mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]] + \mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]] = \mathrm{D}[\mathrm{P}[f]] < +\infty\ .$$

Therefore $\mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]] < +\infty$ and $\mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]] < +\infty$ which means that $\mathrm{Re}\, f, \mathrm{Im}\, f \in \mathrm{H}^{1/2}$. Applying now (2.6) we obtain

$$\begin{aligned} \mathrm{D}[\mathrm{P}_\gamma[\mathrm{Re}\, f]] &= \mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]\circ\varphi] - \mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Re}\, f]]\ ,\\ \mathrm{D}[\mathrm{P}_\gamma[\mathrm{Im}\, f]] &= \mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]\circ\varphi] - \mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Im}\, f]]\ . \end{aligned} \tag{2.8}$$

From Theorem B it follows that

$$\mathrm{P}_\gamma[f] = \mathrm{P}_\gamma[\mathrm{Re}\, f] + \mathrm{i}\,\mathrm{P}_\gamma[\mathrm{Im}\, f]\ . \tag{2.9}$$

By (1.2) we see that for every function $F : \mathbb{D} \to \mathbb{C}$ differentiable a.e. on $\mathbb{D}$,

$$\mathrm{D}[F] = \mathrm{D}[\mathrm{Re}\, F] + \mathrm{D}[\mathrm{Im}\, F]\ .$$

Combining this with (2.9) and (2.7) we conclude that

$$\begin{aligned} \mathrm{D}[\mathrm{P}_\gamma[f]] &= \mathrm{D}[\mathrm{P}_\gamma[\mathrm{Re}\, f] + \mathrm{i}\,\mathrm{P}_\gamma[\mathrm{Im}\, f]] = \mathrm{D}[\mathrm{P}_\gamma[\mathrm{Re}\, f]] + \mathrm{D}[\mathrm{P}_\gamma[\mathrm{Im}\, f]]\ ,\\ \mathrm{D}[\mathrm{P}[f]\circ\varphi] &= \mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]\circ\varphi + \mathrm{i}\,\mathrm{P}[\mathrm{Im}\, f]\circ\varphi] = \mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]\circ\varphi] + \mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]\circ\varphi] \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]] &= \mathrm{D}[(\mathrm{P}[\mathrm{Re}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Re}\, f]) + \mathrm{i}(\mathrm{P}[\mathrm{Im}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Im}\, f])]\\ &= \mathrm{D}[\mathrm{P}[\mathrm{Re}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Re}\, f]] + \mathrm{D}[\mathrm{P}[\mathrm{Im}\, f]\circ\varphi - \mathrm{P}_\gamma[\mathrm{Im}\, f]]\ . \end{aligned}$$

These equalities together with the equalities (2.8) yield the equality in (2.5) for any $f \in \mathrm{H}^{1/2}$, which completes the proof. □

3. The main results

We recall that a quasiconformal self-mapping φ of $\mathbb{D}$ is said to be a regular Teichmüller mapping if there exists a non-zero holomorphic function F in $\mathbb{D}$ and a constant k, $0 \le k < 1$, such that the complex dilatation of φ is of the form

$$\frac{\bar\partial\varphi}{\partial\varphi} = k\frac{\overline{F}}{|F|} \quad \text{a.e. on } \mathbb{D}\ .$$

Theorem 3.1 *Suppose that* $K > 1$, $\gamma \in \mathrm{Q}(\mathbb{T};K)$ *and* $f \in \mathrm{H}^{1/2}$ *satisfies* $\mathrm{D}[\mathrm{P}[f]] > 0$. *Then*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = K\,\mathrm{D}[\mathrm{P}[f]] \tag{3.1}$$

iff there exist $\alpha, c \in \mathbb{R}$ *such that* $\mathrm{e}^{\mathrm{i}\alpha} f(z) - c\mathrm{i} \in \mathbb{R}$ *for a.e.* $z \in \mathbb{T}$ *and* γ *admits a regular quasiconformal Teichmüller extension* φ *to* $\mathbb{D}$ *with the complex dilatation*

$$\frac{\bar\partial\varphi}{\partial\varphi} = \mathrm{e}^{-2\mathrm{i}\alpha}\frac{K-1}{K+1}\frac{\overline{\partial\,\mathrm{P}_\gamma[f]}}{\partial\,\mathrm{P}_\gamma[f]} \quad \textit{a.e. in } \mathbb{D}\ . \tag{3.2}$$

Proof. Fix $K > 1$, $\gamma \in \mathrm{Q}(\mathbb{T};K)$ and $f \in \mathrm{H}^{1/2}$ such that $\mathrm{D}[\mathrm{P}[f]] > 0$. Suppose first that the equality (3.1) holds. Since $\gamma \in \mathrm{Q}(\mathbb{T},K)$ there exists a K-quasiconformal extension φ of γ to $\mathbb{D}$. Applying now Lemmas 2.2 and 2.1 we have

$$\begin{aligned}\mathrm{D}[\mathrm{P}_\gamma[f]] &= \mathrm{D}[\mathrm{P}[f]\circ\varphi] - \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]]\\ &= K\,\mathrm{D}[\mathrm{P}[f]] - \mathrm{Q}_\varphi[\mathrm{P}[f]] - \mathrm{R}_\varphi[\mathrm{P}[f]] - \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]]\ .\end{aligned}$$

Combining this with (3.1) we see that the following equalities

$$\mathrm{Q}_\varphi[\mathrm{P}[f]] = 0\ ,\quad \mathrm{R}_\varphi[\mathrm{P}[f]] = 0 \quad\text{and}\quad \mathrm{D}[\mathrm{P}[f]\circ\varphi - \mathrm{P}_\gamma[f]] = 0$$

hold. They respectively imply

$$(\partial\,\mathrm{P}[f]\circ\varphi)\partial\varphi|\partial\,\overline{\varphi}| = (\bar\partial\,\mathrm{P}[f]\circ\varphi)\partial\,\overline{\varphi}|\partial\varphi| \quad \text{a.e. in } \mathbb{D} \tag{3.3}$$

$$|\bar\partial\varphi| = \frac{K-1}{K+1}|\partial\varphi| \quad \text{a.e. in } \mathbb{D} \tag{3.4}$$

as well as

$$\partial(\mathrm{P}[f]\circ\varphi) = \partial\,\mathrm{P}_\gamma[f] \quad\text{and}\quad \bar\partial(\mathrm{P}[f]\circ\varphi) = \bar\partial\,\mathrm{P}_\gamma[f] \quad \text{a.e. in } \mathbb{D}\ . \tag{3.5}$$

Since $K > 1$, (3.4) shows that $|\bar\partial\varphi| > 0$ a.e. in $\mathbb{D}$. Then (3.3) yields $|\partial\,\mathrm{P}[f]| = |\overline{\bar\partial\,\mathrm{P}[f]}|$ a.e. in $\mathbb{D}$. By the assumption, $\mathrm{D}[\mathrm{P}[f]] > 0$. Since both the functions $\partial\,\mathrm{P}[f]$ and $\overline{\bar\partial\,\mathrm{P}[f]}$ are holomorphic in $\mathbb{D}$, we see that $\mathrm{P}[f]$ is not a constant function and

$$\big|\overline{\bar\partial\,\mathrm{P}[f]}/\partial\,\mathrm{P}[f]\big| = 1 \quad \text{a.e. in } \mathbb{D}\ .$$

Then the maximum principle for holomorphic functions shows that for a certain $\alpha \in \mathbb{R}$,

$$\overline{\bar\partial\,\mathrm{P}[f](z)} = \mathrm{e}^{2\mathrm{i}\alpha}\partial\,\mathrm{P}[f](z)\ ,\quad z \in \mathbb{D}\ . \tag{3.6}$$

Hence for every $z \in \mathbb{D}$,

$$\begin{aligned}0 &= \mathrm{e}^{\mathrm{i}\alpha}\partial\,\mathrm{P}[f](z) - \overline{\mathrm{e}^{\mathrm{i}\alpha}\bar\partial\,\mathrm{P}[f](z)} = \partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z) - \overline{\bar\partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z)}\\ &= \partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z) - \partial\,\mathrm{P}[\overline{\mathrm{e}^{\mathrm{i}\alpha}f}](z) = \partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f - \overline{\mathrm{e}^{\mathrm{i}\alpha}f}](z) = 2\mathrm{i}\partial\,\mathrm{P}[\mathrm{Im}(\mathrm{e}^{\mathrm{i}\alpha}f)](z)\end{aligned}$$

and

$$\begin{aligned}0 &= \mathrm{e}^{\mathrm{i}\alpha}\bar\partial\,\mathrm{P}[f](z) - \overline{\mathrm{e}^{\mathrm{i}\alpha}\partial\,\mathrm{P}[f](z)} = \bar\partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z) - \overline{\partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z)}\\ &= \bar\partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f](z) - \bar\partial\,\mathrm{P}[\overline{\mathrm{e}^{\mathrm{i}\alpha}f}](z) = \bar\partial\,\mathrm{P}[\mathrm{e}^{\mathrm{i}\alpha}f - \overline{\mathrm{e}^{\mathrm{i}\alpha}f}](z) = 2\mathrm{i}\bar\partial\,\mathrm{P}[\mathrm{Im}(\mathrm{e}^{\mathrm{i}\alpha}f)](z).\end{aligned}$$

Therefore there exists $c \in \mathbb{R}$ such that $\operatorname{Im}(e^{i\alpha} f)(z) = c$ for a.e. $z \in \mathbb{T}$, and so $e^{i\alpha} f(z) - ci \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$. Combining (3.3) and (3.4) we conclude that

$$(K-1)(\partial\,\mathrm{P}[f]\circ\varphi)\partial\varphi = (K+1)(\bar\partial\,\mathrm{P}[f]\circ\varphi)\partial\,\overline{\varphi} \quad \text{a.e. in } \mathbb{D}\ . \tag{3.7}$$

Differentiating the composed mapping $\mathrm{P}[f]\circ\varphi$ we conclude from the first equality in (3.5) that

$$\partial\,\mathrm{P}_\gamma[f] = (\partial\,\mathrm{P}[f]\circ\varphi)\partial\varphi + (\bar\partial\,\mathrm{P}[f]\circ\varphi)\partial\,\overline{\varphi} \quad \text{a.e. in } \mathbb{D}\ .$$

Hence and by (3.7) we see that a.e. in $\mathbb{D}$,

$$\begin{aligned}(K-1)\partial\,\mathrm{P}_\gamma[f] &= 2K(\bar\partial\,\mathrm{P}[f]\circ\varphi)\partial\,\overline{\varphi}\\ (K+1)\partial\,\mathrm{P}_\gamma[f] &= 2K(\partial\,\mathrm{P}[f]\circ\varphi)\partial\varphi\ .\end{aligned}$$

Combining this with (3.6) we obtain

$$\frac{\bar\partial\varphi}{\partial\varphi} = \frac{K-1}{K+1}\frac{\partial\,\mathrm{P}[f]\circ\varphi}{\overline{\bar\partial\,\mathrm{P}[f]}\circ\varphi}\frac{\overline{\partial\,\mathrm{P}_\gamma[f]}}{\partial\,\mathrm{P}_\gamma[f]} = e^{-2i\alpha}\frac{K-1}{K+1}\frac{\overline{\partial\,\mathrm{P}_\gamma[f]}}{\partial\,\mathrm{P}_\gamma[f]} \quad \text{a.e. on } \mathbb{D}\ ,$$

and consequently (3.2) holds. This completes the proof in the direction ($\Rightarrow$).

Conversely, assume now that there exist $\alpha, c \in \mathbb{R}$ such that $e^{i\alpha} f(z) - ci \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$ and γ admits a regular quasiconformal Teichmüller extension φ to $\mathbb{D}$ with the complex dilatation satisfying the equality (3.2). Setting $f_\alpha := e^{i\alpha} f - ci$ we see that $f_\alpha \in \operatorname{Re}\mathrm{H}^{1/2}$. Then by (3.2) and Theorem B,

$$\frac{\bar\partial\varphi}{\partial\varphi} = \frac{K-1}{K+1}\frac{\overline{\partial\,\mathrm{P}_\gamma[f_\alpha]}}{\partial\,\mathrm{P}_\gamma[f_\alpha]} \quad \text{a.e. in } \mathbb{D}\ . \tag{3.8}$$

Since the function f_α is real-valued, $\overline{f_\alpha} = f_\alpha$. By (1.13), $g := \mathrm{Tr}[\mathrm{P}_\gamma[f_\alpha]] \in \mathrm{H}^{1/2} \subset \mathrm{L}^1(\mathbb{T})$, and we may consider the Schwarz integral of g, i.e. the function $G : \mathbb{D} \to \mathbb{C}$ defined by the following formula

$$G(z) := \frac{1}{2\pi}\int_{\mathbb{T}} g(u)\frac{u+z}{u-z}|\mathrm{d}u| = \hat g(0) + 2\sum_{n=1}^{\infty}\hat g(n) z^n\ , \quad z \in \mathbb{D}\ . \tag{3.9}$$

By (3.9) and (1.1), G is a holomorphic function in $\mathbb{D}$ and

$$\operatorname{Re} G(z) = \mathrm{P}[g](z) = \mathrm{P}_\gamma[f_\alpha](z)\ , \quad z \in \mathbb{D}\ , \tag{3.10}$$

as well as

$$2\partial\,\mathrm{P}_\gamma[f_\alpha](z) = G'(z) \quad \text{and} \quad 2\overline{\partial\,\mathrm{P}_\gamma[f_\alpha](z)} = \overline{G'(z)}\ , \quad z \in \mathbb{D}\ . \tag{3.11}$$

Let $F : \mathbb{D} \to \mathbb{C}$ be the function satisfying the equation

$$2KF\circ\varphi(z) = (K+1)G(z) + (K-1)\overline{G(z)}\ , \quad z \in \mathbb{D}\ . \tag{3.12}$$

Differentiating both sides of this equality we see that a.e. in $\mathbb{D}$,

$$\begin{aligned}2K(\partial F\circ\varphi)\partial\varphi + 2K(\bar\partial F\circ\varphi)\partial\,\overline{\varphi} &= (K+1)G'\ ,\\ 2K(\partial F\circ\varphi)\bar\partial\varphi + 2K(\bar\partial F\circ\varphi)\bar\partial\,\overline{\varphi} &= (K-1)\overline{G'}\ .\end{aligned} \tag{3.13}$$

Since a.e. in $\mathbb{D}$,

$$\partial\varphi\,\bar\partial\,\overline{\varphi} - \bar\partial\varphi\,\partial\,\overline{\varphi} = \partial\varphi\,\overline{\partial\varphi} - \bar\partial\varphi\,\overline{\bar\partial\varphi} = |\partial\varphi|^2 - |\bar\partial\varphi|^2 > 0\ ,$$

we conclude from (3.13), (3.8) and (3.11) that $\bar{\partial}F = 0$ a.e. in $\mathbb{D}$. In this way the function F is holomorphic in $\mathbb{D}$; cf. [1, Sect. II.B]. Hence and by (3.13),

$$2K(F' \circ \varphi)\partial\varphi = (K+1)G' \quad \text{and} \quad 2K(F' \circ \varphi)\bar{\partial}\varphi = (K-1)\overline{G'} \quad \text{a.e. in } \mathbb{D} \ .$$

Applying now the change of variables formula we obtain

$$\begin{aligned}
&(K+1)^2 \int_{\mathbb{D}} |G'|^2 \mathrm{d}x\mathrm{d}y - (K-1)^2 \int_{\mathbb{D}} |\overline{G'}|^2 \mathrm{d}x\mathrm{d}y \\
&\quad = 4K^2 \int_{\mathbb{D}} |(F' \circ \varphi)\partial\varphi|^2 \mathrm{d}x\mathrm{d}y - 4K^2 \int_{\mathbb{D}} |(F' \circ \varphi)\bar{\partial}\varphi|^2 \mathrm{d}x\mathrm{d}y \\
&\quad = 4K^2 \int_{\mathbb{D}} |F' \circ \varphi|^2 (|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) \mathrm{d}x\mathrm{d}y \\
&\quad = 4K^2 \int_{\mathbb{D}} |F'|^2 \mathrm{d}x\mathrm{d}y \ .
\end{aligned}$$

Hence and by (3.11)

$$\mathrm{D}[\mathrm{P}_\gamma[f_\alpha]] = \int_{\mathbb{D}} |G'|^2 \mathrm{d}x\mathrm{d}y = K \int_{\mathbb{D}} |F'|^2 \mathrm{d}x\mathrm{d}y = K\, \mathrm{D}[\mathrm{Re}\, F] \ . \tag{3.14}$$

Since $f_\alpha \in \mathrm{H}^{1/2}$, we deduce from (3.14) and (1.8) that $\tilde{f}_\alpha := \mathrm{Tr}[\mathrm{Re}\, F] \in \mathrm{H}^{1/2}$ and

$$\mathrm{D}[\mathrm{P}_\gamma[f_\alpha]] = K\, \mathrm{D}[\mathrm{P}[\tilde{f}_\alpha]] \ . \tag{3.15}$$

From (3.12) and (3.10) it follows that

$$\mathrm{P}[\tilde{f}_\alpha] \circ \varphi(z) = \mathrm{Re}\, F \circ \varphi(z) = \mathrm{Re}\, G(z) = \mathrm{P}_\gamma[f_\alpha](z) \ , \quad z \in \mathbb{D} \ .$$

Applying now [11, Theorem 2.5.2] we have

$$\mathrm{Tr}[\mathrm{P}[\tilde{f}_\alpha]] \circ \gamma(z) = \mathrm{Tr}[\mathrm{P}[\tilde{f}_\alpha] \circ \varphi](z) = \mathrm{Tr}[\mathrm{P}_\gamma[f_\alpha]](z) \ , \quad z \in \mathbb{T} \ . \tag{3.16}$$

By Theorem C and (1.12) we obtain respectively

$$\boldsymbol{B}_\gamma([f_\alpha/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}_\gamma[f_\alpha]]/\doteqdot] \quad \text{and} \quad \boldsymbol{B}_\gamma([\tilde{f}_\alpha/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}[\tilde{f}_\alpha]] \circ \gamma/\doteqdot] \ .$$

Combining these equalities with (3.16) we have

$$\boldsymbol{B}_\gamma([f_\alpha/\doteqdot]) = \boldsymbol{B}_\gamma([\tilde{f}_\alpha/\doteqdot]) \ .$$

Therefore $[f_\alpha/\doteqdot] = [\tilde{f}_\alpha/\doteqdot]$, because the operator $\boldsymbol{B}_\gamma$ is injective; cf. [11, Corollary 2.5.4]. Then (3.15) shows that

$$K\, \mathrm{D}[\mathrm{P}[f_\alpha]] = K\, \mathrm{D}[\mathrm{P}[\tilde{f}_\alpha]] = \mathrm{D}[\mathrm{P}_\gamma[f_\alpha]] \ .$$

Since $f_\alpha = \mathrm{e}^{\mathrm{i}\alpha} f(z) - c\mathrm{i}$, we infer from this and Theorem B the equality (3.1). This completes the proof in the direction ($\Leftarrow$).

Both the implications yield the equivalence, which is our claim. $\square$

Corollary 3.2 *Suppose that* $K > 1$, $\gamma \in \mathrm{Q}(\mathbb{T}; K)$ *and* $f \in \mathrm{H}^{1/2}$ *satisfies* $\mathrm{D}[\mathrm{P}[f]] > 0$. *Then*

$$\mathrm{D}[\mathrm{P}_\gamma[f]] = \frac{1}{K}\, \mathrm{D}[\mathrm{P}[f]] \tag{3.17}$$

iff there exist $\alpha, c \in \mathbb{R}$ *such that* $\mathrm{e}^{\mathrm{i}\alpha} f(z) - c\mathrm{i} \in \mathbb{R}$ *for a.e.* $z \in \mathbb{T}$ *and* γ^{-1} *admits a regular quasiconformal Teichmüller extension* φ *to* $\mathbb{D}$ *with the complex dilatation*

$$\frac{\bar\partial\varphi}{\partial\varphi} = \mathrm{e}^{-2\mathrm{i}\alpha}\frac{K-1}{K+1}\frac{\overline{\partial\,\mathrm{P}[f]}}{\partial\,\mathrm{P}[f]} \quad \textit{a.e. in } \mathbb{D}\ . \tag{3.18}$$

Proof. Fix $K > 1$, $\gamma \in \mathrm{Q}(\mathbb{T}; K)$ and $f \in \mathrm{H}^{1/2}$ such that $\mathrm{D}[\mathrm{P}[f]] > 0$. Then also $\gamma^{-1} \in \mathrm{Q}(\mathbb{T}; K)$. By the definition of the class $\mathrm{H}^{1/2}$ we have $f_1 := \operatorname{Re} f \in \operatorname{Re}\mathrm{H}^{1/2}$ and $f_2 := \operatorname{Im} f \in \operatorname{Re}\mathrm{H}^{1/2}$. Then by Theorem B and (1.8) we can see that $g_k := \mathrm{Tr}[\mathrm{P}_\gamma[f_k]] \in \operatorname{Re}\mathrm{H}^{1/2}$ as $k = 1, 2$. From Theorem C it follows that

$$\boldsymbol{B}_\gamma([f_k/\doteqdot]) = [\mathrm{Tr}[\mathrm{P}_\gamma[f_k]]/\doteqdot] = [g_k/\doteqdot]\ , \quad k = 1, 2\ , \tag{3.19}$$

and consequently,

$$[\mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[g_k]]/\doteqdot] = \boldsymbol{B}_{\gamma^{-1}}([g_k/\doteqdot]) = [f_k/\doteqdot]\ , \quad k = 1, 2\ , \tag{3.20}$$

because the operator $\boldsymbol{B}_\gamma$ is injective and $\boldsymbol{B}_\gamma^{-1} = \boldsymbol{B}_{\gamma^{-1}}$; cf. [11, Corollary 2.5.4]. Applying Theorem C once more we conclude from (3.19) and (3.20) that

$$\mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g_k]] = \mathrm{D}[\mathrm{P}[f_k]] \quad \text{and} \quad \mathrm{D}[\mathrm{P}_\gamma[f_k]] = \mathrm{D}[\mathrm{P}[g_k]]\ , \quad k = 1, 2\ . \tag{3.21}$$

Thus setting $g := \mathrm{Tr}[\mathrm{P}_\gamma[f]]$ we deduce from Theorem B that

$$\begin{aligned}\mathrm{D}[\mathrm{P}[g]] &= \mathrm{D}[\mathrm{P}[g_1] + \mathrm{i}\,\mathrm{P}[g_2]] = \mathrm{D}[\mathrm{P}[g_1]] + \mathrm{D}[\mathrm{P}[g_2]] = \mathrm{D}[\mathrm{P}_\gamma[f_1]] + \mathrm{D}[\mathrm{P}_\gamma[f_2]]\\ &= \mathrm{D}[\mathrm{P}_\gamma[f_1] + \mathrm{i}\,\mathrm{P}_\gamma[f_2]] = \mathrm{D}[\mathrm{P}_\gamma[f_1 + \mathrm{i}f_2]] = \mathrm{D}[\mathrm{P}_\gamma[f]]\end{aligned} \tag{3.22}$$

as well as

$$\begin{aligned}\mathrm{D}[\mathrm{P}[f]] &= \mathrm{D}[\mathrm{P}[f_1] + \mathrm{i}\,\mathrm{P}[f_2]] = \mathrm{D}[\mathrm{P}[f_1]] + \mathrm{D}[\mathrm{P}[f_2]]\\ &= \mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g_1]] + \mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g_2]] = \mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g_1] + \mathrm{i}\,\mathrm{P}_{\gamma^{-1}}[g_2]]\\ &= \mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g_1 + \mathrm{i}g_2]] = \mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g]]\ .\end{aligned} \tag{3.23}$$

From (3.20) and Theorem B it follows that

$$\mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[g]] = \mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[g_1]] + \mathrm{i}\,\mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[g_2]] \doteqdot f_1 + \mathrm{i}f_2 = f\ ,$$

from which

$$\partial\,\mathrm{P}_{\gamma^{-1}}[g] = \partial\,\mathrm{P}[f]\ . \tag{3.24}$$

Suppose now that the equality (3.17) holds. Then by (3.22), (3.23) and the assumption $\mathrm{D}[\mathrm{P}[f]] > 0$ we have

$$\mathrm{D}[\mathrm{P}_{\gamma^{-1}}[g]] = K\,\mathrm{D}[\mathrm{P}[g]] > 0\ . \tag{3.25}$$

Applying Theorem 3.1 with γ and f replaced respectively by γ^{-1} and g we see that there exist $\alpha, c_0 \in \mathbb{R}$ such that $\mathrm{e}^{\mathrm{i}\alpha} g(z) - c_0\mathrm{i} \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$ and γ^{-1} admits a regular quasiconformal Teichmüller extension φ to $\mathbb{D}$ with the complex dilatation

$$\frac{\bar\partial\varphi}{\partial\varphi} = \mathrm{e}^{-2\mathrm{i}\alpha}\frac{K-1}{K+1}\frac{\overline{\partial\,\mathrm{P}_{\gamma^{-1}}[g]}}{\partial\,\mathrm{P}_{\gamma^{-1}}[g]} \quad \text{a.e. in } \mathbb{D}\ . \tag{3.26}$$

This together with (3.24) yields (3.18). Moreover,

$$e^{i\alpha} f \doteqdot e^{i\alpha}\,\mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[g]] - c_0 i = \mathrm{Tr}[\mathrm{P}_{\gamma^{-1}}[e^{i\alpha} g - c_0 i]]\ .$$

Hence there exists $c \in \mathbb{R}$ such that $e^{i\alpha} f(z) - ci \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$. In this way the corollary was proved in the direction ($\Rightarrow$).

Conversely, assume now that there exist $\alpha, c \in \mathbb{R}$ such that $e^{i\alpha} f(z) - ci \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$ and γ^{-1} admits a regular quasiconformal Teichmüller extension φ to $\mathbb{D}$ with the complex dilatation satisfying the equality (3.18). Combining (3.18) with (3.24) we get the equality (3.26). Moreover, by Theorem B,

$$e^{i\alpha} g \doteqdot e^{i\alpha}\,\mathrm{Tr}[\mathrm{P}_{\gamma}[f]] - ci = \mathrm{Tr}[\mathrm{P}_{\gamma}[e^{i\alpha} f - ci]]\ .$$

Hence there exists $c_0 \in \mathbb{R}$ such that $e^{i\alpha} g(z) - c_0 i \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$. Applying now Theorem 3.1 with γ and f replaced respectively by γ^{-1} and g we conclude that the equality in (3.25) holds. Then by (3.22) and (3.23) we derive the equality (3.17), and this completes the proof in the direction ($\Leftarrow$).

Both the implications yield the equivalence, which is our claim. □

We are now in a position to answer to the question about a possible equality in (0.4).

Remark 3.3 *Given $K \geq 1$ let $\gamma \in \mathrm{Q}(\mathbb{T}, K)$. Suppose that $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ is a sequence satisfying the condition* (0.3)*. Then by* (1.3)*, $f := \mathrm{Tr}[F] \in \mathrm{H}^{1/2}$, where F is the function defined by*

$$F(z) := \sum_{n=0}^{\infty} \lambda_n z^n + \sum_{n=1}^{\infty} \lambda_{-n} \overline{z}^n\ , \quad z \in \mathbb{D}\ . \tag{3.27}$$

Hence for every $\alpha \in \mathbb{R}$,

$$\begin{aligned}\mathrm{Im}\big(e^{i\alpha} F(z)\big) &= \frac{1}{2i}\big(e^{i\alpha} F(z) - \overline{e^{i\alpha} F(z)}\big) \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}\big(e^{i\alpha}\lambda_n - \overline{e^{i\alpha}\lambda_{-n}}\big) z^n + \frac{1}{2i}\sum_{n=1}^{\infty}\big(e^{i\alpha}\lambda_{-n} - \overline{e^{i\alpha}\lambda_n}\big)\overline{z}^n \\ &= \sum_{n=0}^{\infty}\mathrm{Im}\big(e^{i\alpha}\lambda_n\big) z^n + \sum_{n=1}^{\infty}\mathrm{Im}\big(e^{i\alpha}\lambda_{-n}\big)\overline{z}^n\ , \quad z \in \mathbb{D}\ .\end{aligned}$$

Therefore for all $\alpha, c \in \mathbb{R}$, $e^{i\alpha} f(z) - ci \in \mathbb{R}$ for a.e. $z \in \mathbb{T}$ iff $\mathrm{Im}\big(e^{i\alpha}\lambda_0\big) = c$ and

$$\mathrm{Im}\big(e^{i\alpha}\lambda_n\big) = 0\ , \quad n \in \mathbb{Z}\setminus\{0\}\ . \tag{3.28}$$

By (3.27)*, $\hat{f}(n) = \lambda_n$ for $n \in \mathbb{Z}$. Combining this with* (1.5) *and* (1.7) *we have*

$$\partial\,\mathrm{P}_{\gamma}[f](z) = \sum_{n=1}^{\infty} n\hat{f}(n;\gamma) z^{n-1} = \sum_{n=1}^{\infty} n\Big(\sum_{m=-\infty}^{\infty}\hat{\gamma}(m,n)\lambda_m\Big) z^{n-1}\ , \quad z \in \mathbb{D}\ . \tag{3.29}$$

Moreover, by (1.1)*,*

$$\partial\,\mathrm{P}[f](z) = \sum_{n=1}^{\infty} n\hat{f}(n) z^{n-1} = \sum_{n=1}^{\infty} n\lambda_n z^{n-1}\ , \quad z \in \mathbb{D}\ . \tag{3.30}$$

By the equalities (1.4), (1.5) *and* (1.7) *we know that the equality*

$$\sum_{n=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty}\sqrt{|n|}\hat{\gamma}(m,n)\lambda_m\right|^2 = K\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 \tag{3.31}$$

holds iff the equality (3.1) *holds. Then by Theorem 3.1 and* (3.29)*, the equality* (3.31) *holds iff there exists* $\alpha\in\mathbb{R}$ *satisfying the condition* (3.28) *and* γ *admits a regular quasiconformal Teichmüller extension* φ *to* $\mathbb{D}$ *such that for a.e.* $z\in\mathbb{D}$*,*

$$\begin{aligned}(K+1)\bar{\partial}\varphi(z)\mathrm{e}^{\mathrm{i}\alpha}\sum_{n=1}^{\infty}n\Big(\sum_{m=-\infty}^{\infty}\hat{\gamma}(m,n)\lambda_m\Big)z^{n-1}&\\ =(K-1)\partial\varphi(z)\mathrm{e}^{-\mathrm{i}\alpha}\sum_{n=1}^{\infty}n\Big(\sum_{m=-\infty}^{\infty}\overline{\hat{\gamma}(m,n)\lambda_m}\Big)\overline{z}^{n-1}&\ .\end{aligned} \tag{3.32}$$

On the other hand side, from the equalities (1.4), (1.5) *and* (1.7) *it follows that the equality*

$$\sum_{n=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty}\sqrt{|n|}\hat{\gamma}(m,n)\lambda_m\right|^2 = \frac{1}{K}\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 \tag{3.33}$$

holds iff the equality (3.17) *holds. Then by Corollary 3.2 and* (3.30)*, the equality* (3.33) *holds iff there exists* $\alpha\in\mathbb{R}$ *satisfying the condition* (3.28) *and* γ^{-1} *admits a regular quasiconformal Teichmüller extension* φ *to* $\mathbb{D}$ *such that for a.e.* $z\in\mathbb{D}$*,*

$$(K+1)\bar{\partial}\varphi(z)\mathrm{e}^{\mathrm{i}\alpha}\sum_{n=1}^{\infty}n\lambda_n z^{n-1} = (K-1)\partial\varphi(z)\mathrm{e}^{-\mathrm{i}\alpha}\sum_{n=1}^{\infty}n\overline{\lambda_n}\overline{z}^{n-1}\ . \tag{3.34}$$

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UOGÓLNIONE WSPÓŁCZYNNIKI FOURIERA I EKSTREMALNE ROZSZERZENIE QUASIKONFOREMNE QUASISYMETRYCZNEGO AUTOMORFIZMU OKRĘGU JEDNOSTKOWEGO

S t r e s z c z e n i e

Uogólnione współczynniki Fouriera $\hat{\gamma}(m,n)$ homeomorfizmu γ okręgu jednostkowego $\mathbb{T}$ na siebie są określone formułą

$$\hat{\gamma}(m,n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(\mathrm{e}^{\mathrm{i}t})^m \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\ , \quad m,n \in \mathbb{Z}\ .$$

W pracy [12] zostały udowodnione następujące nierówności

$$\frac{1}{K}\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 \le \sum_{n=-\infty}^{\infty}\left|\sum_{m=-\infty}^{\infty}\sqrt{|n|}\hat{\gamma}(m,n)\lambda_m\right|^2 \le K\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2\ ,$$

o ile γ dopuszcza K-quasikonforemne rozszerzenie na koło jednostkowe $\mathbb{D}$ i $\mathbb{Z} \ni n \mapsto \lambda_n \in \mathbb{C}$ jest ciągiem spełniającym warunek

$$\sum_{n=-\infty}^{\infty}|n||\lambda_n|^2 < +\infty.$$

Zauważmy, że wyglądają one podobnie jak nierówności Grunsky'ego dla funkcji holomorficznych w klasach $\Sigma(k)$, $0 \le k \le 1$. Niniejsza praca dostarcza odpowiedzi na pytanie kiedy w tych nierównościach zachodzi równość.

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In memory of
Professor Roman Stanisław Ingarden

Janusz Garecki

IS TORSION NEEDED IN A THEORY OF GRAVITY? A REAPPRAISAL I

MOTIVATION FOR INTRODUCING AND LACK OF EXPERIMENTAL EVIDENCE

Summary

It is known that General Relativity (**GR**) uses a Lorentzian Manifold $(M_4; g)$ as a geometrical model of the physical spacetime. The metric g is required to satisfy Einstein's equations. Since the 1960s many authors have tried to generalize this geometrical model of the physical space-time by introducing torsion. In the first part of the paper we discuss the present status of torsion in a theory of gravity, motivation for introducing, and lack of experimental evidence.

1. Introduction

In past we were enthusiast of torsion, mainly under influence of excellent papers given by F. W. Hehl and A. Trautman. But studying *Poincare' field theories of gravity* (**PGT**) one can easily see that torsion leads to serious complications, especially calculational.

About twenty years ago we have observed that the our idea of the *superenergy and supermomentum tensors* (very effective in general relativity) fails in a spacetime having torsion. So, our interest to torsion has diminished.

In the meantime we have read many papers by C. M. Will, G. Esposito-Farese, T. Damour, S. Kopeikin, S. G. Thuryshev and others devoted recent experiments on gravity. As we understood all these experiments confirmed standard general relativity (**GR**) with a very high precision and excluded torsion, at least in vacuum.

Besides, during the last three decades there was given many interesting papers on *universality* of the **GR** equations. So, in consequence, we have decided to analyze status of torsion in gravitational physics. From this analysis the review has originated.

Of course, *we do not prove* that torsion is not admissible at all. Rather, we only give short information about recent gravitational experiments and collect problems which arise when one introduces torsion as a part of the geometrical structure of the physical spacetime. But, as you see, we will finish review with the conclusion (based mainly on *Ockham razor*):

1. Torsion in needn't in a theory of gravity:
2. The Levi-Civita connection is sufficient for the all physical applications. This the most simple connection is exactly just what we need.

The paper is organized as follows. In Section 2 we remind a general definition of torsion and in Section 3 we consider motivations to introduce torsion into geometrical model of the physical spacetime. We will see that these motivations *are not convincing*. In Section 4 we very shortly discuss experimental evidence for torsion and Section 5 we present arguments against torsion in a theory of gravity. We will conclude in Section 6 (from the facts given in the two previous Sections) that torsion rather *should not be introduced into a geometrical model of the physical spacetime.*

2. Torsion of a linear connection $\omega^i{}_k$ on $L(M)$

We confine to the metric-compatible connection which satisfies

$$Dg_{ik} = dg_{ik} - \omega^p{}_i g_{pk} - \omega^p{}_k g_{ip} = 0$$

because we do not see any reasons to consider more general connection. Here, and in the following, D means exterior covariant derivative and d is the ordinary exterior derivative.

One can give the following, general definition of torsion Θ^i [31, 32] of a linear connection

$$\Theta^i := D\theta^i = d\theta^i + \omega^i{}_k \wedge \theta^k =: \frac{1}{2} Q^i{}_{kl} \theta^k \wedge \theta^l. \tag{1}$$

Here θ^i are *canonical 1-forms* (or *soldering 1-forms*) on the principial bundle of the linear frames $L[M, GL(n;R), \pi]$ ($L(M)$ in short) over a manifold M, and $Q^i{}_{kl}$ denote components of the torsion tensor.

After pulling back by local section $\sigma : U \to L(M)$; $U \subset M$, one gets on the base M

$$\tilde{\Theta}^i = d\vartheta^i + \tilde{\omega}^i{}_k \wedge \vartheta^k = \frac{1}{2} \tilde{Q}^i{}_{kl} \vartheta^k \wedge \vartheta^l. \tag{2}$$

$\tilde{\omega}^i{}_k := \sigma_* \omega^i{}_k$ are pull-back of $\omega^i{}_k$ and ϑ^i are pull-back of θ^i. $\vartheta^l := \vartheta^0,\ \vartheta^1,\ \vartheta^2,\ \vartheta^3$ form a Lorentzian coframe on M.

In a coordinate (= holonomic) frame $\{\partial_i\}$ and dual coframe $\{dx^k\}$ on M one has $\tilde{\omega}^i{}_k = \Gamma^i{}_{lk} dx^l$, and $\tilde{Q}^i{}_{kl} = \Gamma^i{}_{kl} - \Gamma^i{}_{lk}$.

3. Motivation to introduce of torsion into gravity

In the 1960s–1970s some researchers introduced torsion into the theory of gravity. We omit here older attempts to introduce torsion because they have only historical meaning [1–6]. The main motives (only theoretical) were the following:

1. Studies on geometric theory of dislocations (Theory of a generalized Cosserat continuum) led, following Günter, Hehl, Kondo, and Kröner, to heuristic arguments for a metric spacetime with torsion, i.e., to Riemann–Cartan spacetime.
2. Investigations of spinning matter in **GR** resulted in conclusion that the canonical energy-momentum tensor of matter ${}_cT_i{}^k$ can be source of curvature and the canonical intrinsic spin density tensor ${}_cS^{ikl} = (-)_c S^{kil}$ can be source of torsion of the underlying spacetime. From this Einstein-Cartan-Sciama-Kibble (**ECSK**) theory and its generalizations originated.
3. Attemts to formulate gravity as a gauge theory for Lorentz group L or for Poincare' group P by using *Palatini's formalism* led to a space-time endowed with a metric-compatible connection which might have (but not necessarily) non–vanishing torsion, i.e., again one was led to Riemann-Cartan space-time [7–11].

Some remarks are in order concernig 3.

1. If we admit a metric-compatible connection with torsion when "gauging" groups L or P by using Palatini's approach and Ehreshmann theory of connection, then we will end up with strange situation, different then in ordinary gauge fields: we get a "gauge theory" which has two gauge potentials

$$\vartheta^i \ - \ translational \ (= pseudoorthonormal \ coframe), \tag{3}$$

$$\omega^i{}_k \ - \ rotational \ (= metric - compatible \ linear \ connection), \tag{4}$$

and two gauge strengths

$$\Theta^i = D\vartheta^i \ - \ translational \ (torsion), \tag{5}$$

$$\Omega^i{}_k = D\omega^i{}_k \ - \ rotational \ (curvature). \tag{6}$$

Notice that ϑ^i *do not transform* like gauge potentials and contribute to

$$\omega^i{}_k = {}_{LC}\omega^i{}_k + K^i{}_k;$$

besides, the *gauge strengths*

$$\Theta^i = K^i{}_k \wedge \vartheta^k$$

contribute to

$$\omega^i{}_k = {}_{LC}\omega^i{}_k + K^i{}_k \quad \text{and also to} \quad \Omega^i{}_k = d\omega^i{}_k + \omega^i{}_p \wedge \Omega^p{}_k.$$

Here ${}_{LC}\omega^i{}_k$ denotes the *Levi-Civita Connection* and $K^i{}_k$ is the contortion. So, the gauge potentials and gauge strengths *are not independent* in the case. This is not satisfactory and suggests *other approach to "gauging" gravity.*

2. Besides, the action integrals in these trials to gauge gravity *didn't have* forms like an action integral for a gauge field, $\int tr(F \wedge \star F)$, and led to very complicated field equations of 3rd order, different from **GR** equations. These field equations contain many arbitrary parameters (10 apart from Λ in the case of the so-called Poincare' Gravity Theories, **PGT**). Here $\star$ means Hodge duality operator.
   There exist many serious problems connected with these field equations: tachyons, ghosts, instability of their solutions, ill–posedness Cauchy problem, etc., (see, e.g., [71]).
   We would like to emphasize that there exists an old approach to "gauge" gravity proposed by Yang [69] which has action typical for a gauge field: $\int \Omega^i{}_k \wedge \star\Omega^k{}_i$. But, unfortunately, this approach leads to incorrect theory of gravity.

The above theoretical motives *are not convincing*. For example, the often used argument for torsion (following from study of spinning matter in **GR**) based on the (non-homogeneous) holonomy theorem [4, 11] (this theorem says that torsion gives translations, and curvature gives Lorentz rotations in tangent spaces of a Riemann-Cartan manifold induced by (Cartan) displacement along loops) holds only if one uses *Cartan displacement* which displaces vectors and contactpoints [12]. Ordinary parallell displacement (which displaces only vectors) gives only Lorentz rotations (= homogeneous holonomy group) even in a Riemann-Cartan space-time [12]. Moreover, there are other geometrical interpretations of torsion, e.g., Bompiani [13] *connects torsion with rotations in tangent spaces*, not with translations.

We also needn't to generalize **GR** in order to get a gauge theory with L or P as a gauge group [14, 48, 70]. The most convincing argument in this field is given by *Cartan's approach* to connection and geometry [70].

Roughly speaking, in Cartan's approach (for details, see [70]) one combines the linear Ehreshmann connection form ω and coframe field θ into one connection $A = \omega \oplus \theta$ valued in a larger Lie algebra $\mathbf{g}$ (In our case ω is the Ehresmann connection on principal bundle of the orthonormal frames $O[M, L, \pi]$ and θ is the soldering form on this bundle. $\mathbf{g}$ is the algebra of the Poincare' group P or de Sitter group).

In consequence, one has only one gauge potential $A = \omega \oplus \theta$ and one gauge strength $\hat{F} = \Omega - \frac{\Lambda}{3}\theta \wedge \theta$ (Λ is the cosmological constant) for gravity.

Using Cartan's approach to connection one can write the ordinary **GR** action with Λ

$$S_g = \int \sqrt{|g|}(R - \Lambda)d^4x,$$

in the form

$$S_g = (-)\frac{3}{2G\Lambda} \int tr(\hat{F} \wedge \star\hat{F}),$$

i.e., *exactly* in the form of the action of a gauge field.

Thus, the Cartan's (not Ehreshmann) approach to connection and geometry suits to correct "gauging" of **GR**. The Ehreshmann theory suits to ordinary gauge fields.

There exists also an other approach to **GR** as a gauge theory developed by A. Ashtekar, C. Rovelli, J. Lewandowski and covorkers (Ashtekar's variables) [15, 16, 18, 74]. In this approach **GR** is also very akin to a Yang–Mills theory.

In resuming, one can say that *we needn't generalize or modify* **GR** in order to obtain a gauge theory of gravity.

4. Experimental evidence for torsion

Up to now we *have no experimental evidence* for existence torsion in Nature (see, e.g., [75]). There exist only very *stringent constraints* on torsion components obtained in a speculative, purely theoretical, methods (see, eg., [72, 75]).

To the contrary, all gravitational experiments confirmed with a very high precision ($\sim 10^{-14}$) Einstein's Equivalence Principle (**EEP**) and, with a smaller precision (up to $0,003\%$ in Solar System , i.e., in weak field, and up to $0,05\%$ in binary pulsars, i.e., in strong gravitational fields) the General Relativity (**GR**) equations [19–25]. Here by **EEP** we mean a formulation of this Principle given by C. W. Will [19]. In this formulation (this constructive formulation of the Principle can be experimentally tested) the **EEP** states:

1. The Weak Equivalence Principle (**WEP**) is valid. This means that the trajectory of a freely falling spherical test body (one not acted upon by non-gravitational forces and being too small to be affected by tidal forces) *is independent of its internal structure and material composition.*
2. Local Lorentz Invariance (**LLI**) is valid. This means that the outcome of any local non-gravitational experiment *is independent of the velocity of the freely-falling and non-rotating reference frame in which it is performed.*
3. Local Position Invariance (**LPI**) is valid. This means that the outcome of any local non-gravitational experiment performed in a freely-falling and non-rotating reference frame *is independent of where and when in the Universe it is performed.*

Following C. M. Will, the only theories of gravity that can embody **EEP** in the above constructive formulation of the Principle are those that satisfy the postulates of *metric theories of gravity* [19], which are:

1. Spacetime is endowed with a symmetric metric.
2. The trajectories of freely falling spherical test bodies are geodesics of that metric.
3. In local freely-falling and non-rotating reference frames the non-gravitational laws of physics are those written in the language of Special Relativity (**SRT**).

C. M. Will called the **EEP** "heart and soul of **GR**".

The **EEP** implies a *universal pure metric coupling between matter and gravity.* It admits **GR**, of course, and, at most, some of the so-called *scalar-tensor theories* (these, which respect **EEP**) [19, 21, 22] without torsion.

So, torsion seems to be *excluded* in vacuum or at least *very strongly constrained* in vacuum by the latest gravitational experiments, i.e., at least propagating torsion is excluded or very strongly constrained by these experiments which have confirmed **EEP** with very high precision. Torsion is excluded or very strongly constrained at least in vacuum because if we neglect a cosmological background, then the all gravitational experiments were performed in vacuum. This means that **ECSK** theory can survive since this theory is identical in vacuum with **GR**. Of course, the same is true for other gravity theories which in vacuum reduce to **GR**. But the gravity theories of such a kind do not admit propagating free torsion. As a consequence, at least freely propagating torsion still seems to be *purely hypothetical.*

We would like to emphasize that T. Damour already concluded in past [22]: "Einstein was right at least 99.9999999999% concerning **EEP** and 99, 9% concerning Lagrangian and field equations".

Thus, from the experimental point of view, up to now, torsion *is not needed* in a theory of gravity.

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CZY TORSJA JEST POTRZEBNA W TEORII GRAWITACJI? NOWE SPOJRZENIE I

MOTYWACJA WPROWADZENIA I BRAK EKSPERYMENTALNEGO UZASADNIENIA

S t r e s z c z e n i e

W pracy pokazano, że wprowadzenie skręcenia do modelu matematycznego fizycznej czasoprzestrzeni nie jest *ani konieczne, ani wskazane.*

W pierwszej części pracy omawiamy motywację ewentualnego wprowadzenia skręcenia i brak eksperymentalnego uzasadnienia.

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In memory of
Professor Roman Stanisław Ingarden

Stanisław Bednarek and Jerzy Krysiak

APPLICATION OF THE CYLINDRICAL LENESES IN EDUCATIONAL PHYSICAL EXPERIMENTS

Summary

The purpose of this paper is to present the properties of cylindrical lenses and provide some examples of their use in performing easy school physics experiments. Such experiments could be successfully conducted in the process of science education, in fun experiments that teach physics and science fair projects, or used to entertain an audience by staging tricks, effects or illusions of seemingly impossible or supernatural feats.

1. Simplest cylindrical lens

A cylindrical lens is a piece of transparent substance, commonly glass or plastic, bounded by one or two cylindrical surfaces of different radii of curvature, with their axes being parallel to each other. In its simplest form, a lens can be a rod of transparent material of circular cross section. The optimal diameter of such a rod cannot exceed more than a few centimeters. Since a rod of appropriate diameter can be difficult to find at home situation or in the school physics laboratory, any clear plastic bottle with plane sides can be utilized as a substitute for a lens. The bottle must be filled with water or other substance, such as glycerin. The thickness of the bottle wall must be substantially less than its diameter so as to avoid the deviation of a beam of light passing through the wall. Therefore, glass bottles with walls several millimeters thick are unsuitable for the purpose.

It is important that the bottle should be filled up with water, which means that there should not be any air bubbles in it, otherwise this would lead to undesirable changes in direction of a beam of light passing through the bottle. In order to avoid the production of air bubbles, the water must be first boiled and then cooled down to the temperature of not lower than $4\,^{\circ}\mathrm{C}$. The heating allows for the removal of gases that have been dissolved in the water. The easiest way of filling up the bottle without producing bubbles is to submerge it vertically in an appropriately deep water-filled vessel. After the bottle has been filled up, it should be maintained for a while in the same vertical position underwater and then corked. While holding the bottle it should not be squeezed, since on releasing the squeeze an empty space would be created within it. When the bottle is removed from the vessel and its content is warmed to room temperature, its volume increases and the bottle is filled to full capacity with water.

2. Set of words

After the bottle has been filled up with water, an appropriate list of words should be prepared. For this purpose, the following twenty words in capitals should be typed in a column on a sheet of paper: BED, BEE, BECK, BEDECK, BOB, BOOK, BOOKED, COB, COD, CODE, CODED, COED, DIOXIDE, EBB, ECHO, ICE, KICK, OH, OX, OXIDE. They will constitute the first column of words. Next to it, in another column there are twenty other words: BAD, BELL, BLUR, BOSS, BILL, BOY, BLUE, CAB, CABLE, CABIN, CADET, CADRE, DREAM, ENTER, ENTRE, IDOL, KERNEL, OAR, OAK, OAT, CAKE. To type these words a vector graphics editor CorelDraw has been used, applying Tw Cen MT Condensed Extra Bold font. This clean cut, sans serif with geometric shapes is most appropriate for display in headlines. Besides, the diagonal bar in the letter **K**, for instance, crosses exactly at the midpoint of the vertical bar. Both rows of words should be typed on a single piece of paper.

3. Magical experiment

With the bottle and the sheet of paper ready, we can set to performing experiments. The paper is laid on top of the table. The water-filled bottle is held over the list of words in such a way that the longitudinal axis is directed horizontally and parallel towards the list. We try to move the bottle perpendicular to the surface of the sheet and look at the words through the bottle. At some distance from the bottle to the sheet we can notice that the images of the words are of the same size as the real words on the sheet of paper. Moreover, we can see that the images of the words in the first column below the word BED are not inverted, whereas the images of the words in the adjacent column have been reversed and can be seen "upside down". Why this is so? Could the water-filled bottle be really capable of deciphering the meanings of these words and reverse and invert the images of some of them and not others.

The observed effect can be explained as follows. The water-filled bottle acts as a cylindrical focusing lens. If the distance of the axis from the words is double that of its focal length, the lens creates a real image that is inverted. The size of the image is equal to the size of the object. The bottle produces an inverted image of all the words in both columns [1]. The secret behind the effect is that the printed words in the first column are made up of letters that have a horizontal axis of symmetry. That is why their reversed images look exactly the same as the words on the sheet of paper. In other words, the words are invariable, that is, unchanging, after they have been transformed by symmetry to their elongated axis. Those seeing the words at first glance fail to notice the symmetry of letters and therefore are surprised by the effect. This is exactly what enables us to make use of a cylindrical lens to demonstrate an interesting magic trick that consists of inverting only some selected words. For this reason, such a lens will be referred to as the magic lens in the further part of this paper.

4. Different cylindrical lens

To facilitate performing the experiment, distance rings of appropriate diameters are mounted at both ends of the transparent rod or water-filled bottle. The ring diameter should be such that the axial distance of the rod or bottle from the lettered surface is equal to double focal length. Such a lens looks like a spool with transparent axis (Photo 1). Thus, to observe the said effect, it is suffice to move such a lens across the sheet with lettering on it.

The magic lens can also be made in another way, as by fastening a transparent rod at an appropriate distance above the surface along which the lettered sheet is moved. This has been demonstrated in Photo 2. It is a lens of a large dimension, 40 cm in length and 6 cm in diameter designed for use in the Exploratorium, interactive laboratory in Lodz. A thin-walled glass tube filled with water is used in place of lens (Fig. 1). Since the lens should work within a relatively wide range of temperatures, as temperatures change throughout the year, thermal expansion compensator has been applied. At both ends of the tube there are two corked thin-walled containers, facing upward, which are filled with water and air. Water capacity has been calculated in such a way that when the temperature is close to the near limit at which the lens ceases to work, water fills the entire tube volume and no air bubbles are produced. On the other hand, after the temperature rises to the upper point where the lens works, there is still some air left in the containers. Since the trapped air is contractible in volume, it prevents exerting excessive pressure and bursting of the tube. The tube-ends, which are equipped with containers, are concealed from the spectators by means of lens fixing brackets.

The magic lens does not have to be made from an entirely clear composition. A segment of a transparent cylinder bounded by a cylindrical surface and a plane parallel to the cylinder axis can serve the purpose. Such a lens has a longer focal length than a full cylinder, for the focusing capacity of a cylinder section is smaller than that of a full cylinder. Moreover, the section that has been obtained from the cylinder of great radius can be used, which expands the focal length. Since a lens of this kind is far thinner than a full rod, it has fewer optical flaws, especially chromatic aberration.

To facilitate the positioning of the lens in question at an appropriate distance from the words, rectangular mounting brackets can be attached to both flat ends of the cylinder section. Their height should be such that the lens distance from the lettered surface on which the lens is placed is double that of the focal length. To help stiffen the whole structure the bottom brackets should be connected with a plane clear plate. The lens of this kind is shown on Fig. 2 and Photo 3. Different configurations of words are scanned by the lens by moving its bracket across the lettered surface. In place of the cylinder section, a Fresnel lens can be attached to the bracket. The lens can be made from a conventional cylindrical lens by dividing/cutting it into a set of concentric sections, each one having a convex surface of the same curvature as the corresponding section, and mounting them on a transparent plate (Fig. 3).

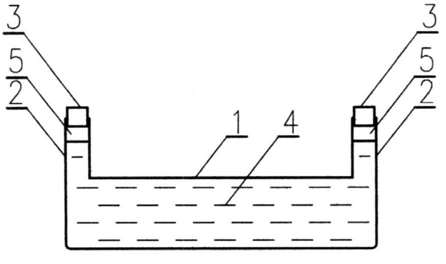

Fig. 1: The cylindrical lens structure with a heat expansion compensator: 1 – glass tube, 2 – container 3 – cork, 4 – water, 5 – air.

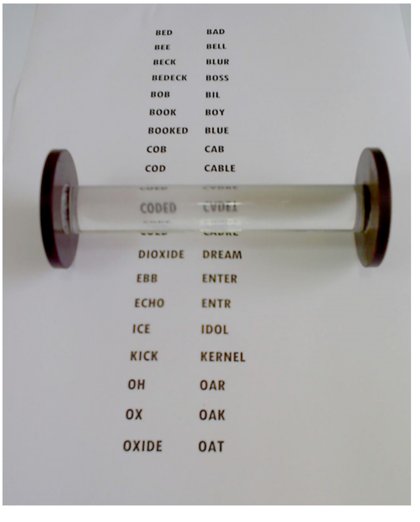

Photo 1: The magic spool functioning as a cylindrical lens.

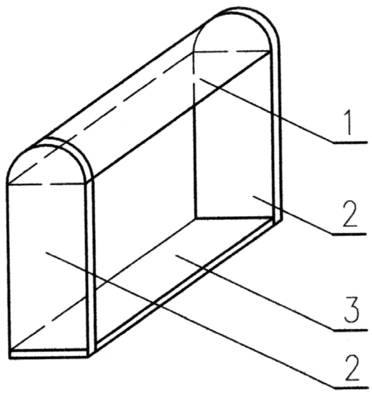

Fig. 2: The magic frame structure; 1 – lens made from segment of a transparent cylinder, 2 – mount bracket, 3 – plain transparent plate.

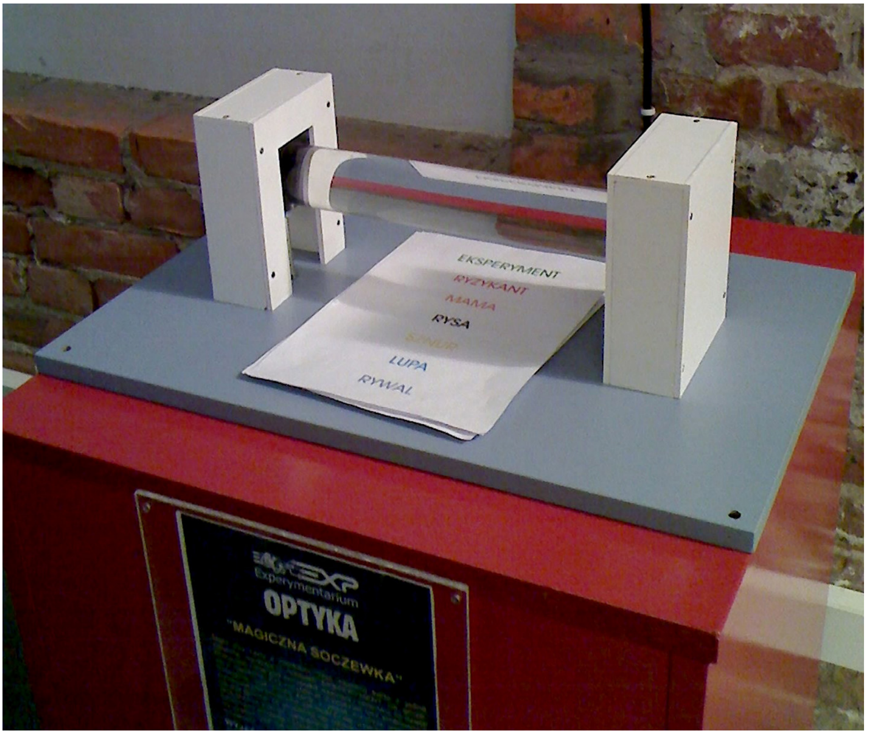

Photo 2: The magic spool of large size with a heat expansion compensator as used in the Exploratorium in Łódź.

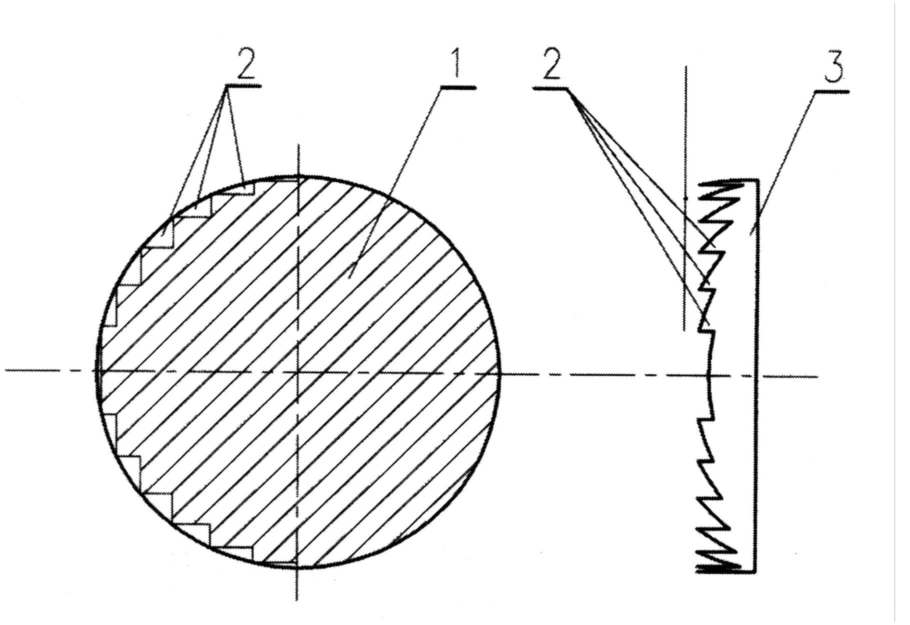

Fig. 3: The principal of constructing a cylindrical Fresnel lens; 1 – the rejected segment of the cylinder, the retained cylinder sections of the convex curvature, 3 – the parallel plane portion of a Fresnel lens.

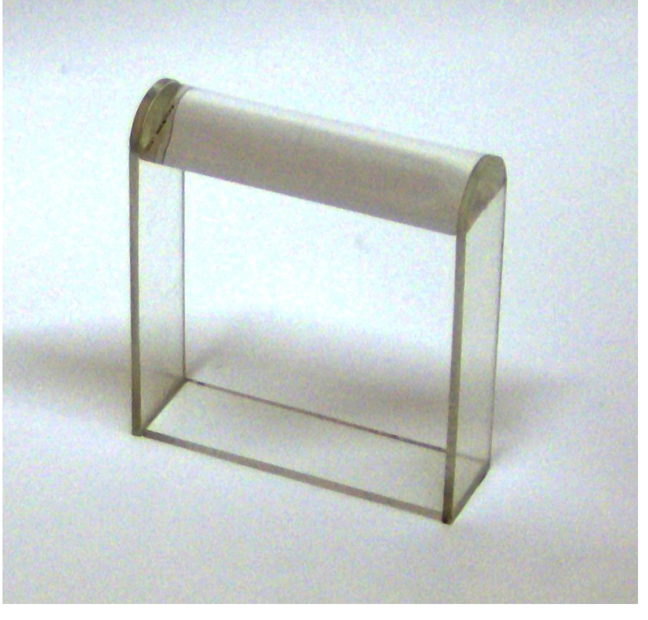

Photo 3: The magic spoon made from a segment of the transparent cylinder.

Fresnel lenses are extremely useful for the illusionist arts, for the magic tricks to be shown to large circus audiences or during a science fair. The lenses can have the shape of elongated laths that resemble magic wands. The magician holds one end of such a lens horizontally in front of a perpendicular chart at a distance that is double its focal length. On the chart contains some typed/printed words, some of which having a horizontal axis of symmetry, and some not. The spectators are in front of the lens, also at a distance double the focal length. As the magician moves the lens in a perpendicular plane, the spectators can see that some words are inverted, and others remain unchanged. Long focal lengths of such lenses allow for seating the viewers at a relatively great distance from the lettered chart.

5. Conclusions

Finally, it is worth adding that different words from those that have been provided may be included in the list. It is important that some of them should be formed only with letters having a horizontal axis of symmetry. Any pictures can be used instead of words, yet some of them should have a horizontal axis of symmetry. The more inquiring reader may, at this point, note that when a focusing spherical lens is placed at a distance double that of its focal length from an object, it also gives a real image, turned and of the same size as the object. This is true, but a spherical lens inverts the image so the object's reflection is not only upside down (up is down), but also its right side becomes the left side [2]. The experiments that have been described here show how some scientific principles and laws of physics may be used to create magic illusions.

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ZASTOSOWANIE SOCZEWEK CYLIDRYCZNYCH W EDUKACYJNYCH DOŚWAIDCZENIACH FIZYCZNYCH

S t r e s z c z e n i e

Celem tego artykułu jest przedstawienie własności i przykładów zastosowań wykonanych w różny sposób soczewek cylindrycznych do przeprowadzenia edukacyjnych doświadczeń z fizyki. Te interesujące doświadczenia można łatwo i skutecznie przeprowadzić w procesie nauczania fizyki, albo wykonać podczas pokazów naukowych popularyzujcych fizykę dla licznych grup widzów. Niektóre z tych doświadczeń wykazują również cechy zadziwiających sztuk iluzjonistycznych.

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In memory of
Professor Roman Stanisław Ingarden

Andrzej Polka

SPACE MODELLING WITH MULTIDIMENSIONAL VECTOR PRODUCTS

Summary

In the present work a multiproduct of vectors has been introduced and described; i.e. means the product of vectors of a number greater than two, and a number of examples of their uses in different kinds of cases have been presented, especially in transformations of vectors between spaces.

1. Introduction

The vector calculus is a good tool for describing many complicated problems, both in mechanics and other areas of science and technology. It can be used for building spatial mathematical models describing problems of a high degree of complexity in which numerous aspects of these questions are taken into consideration. Its advantage is that it allows one to describe multi-parameter problems in any space, both three- and multi-dimensional.

The building of a space is an open question, its dimension and character is a question of choice, and can be changed during a modelling process, depending on needs, possibilities or a subjective choice. One thing, however, does not change – any ortho-Cartesian space is formed as a vector (a Cartesian) product of lower-order spaces, hence the expansion or the reduction of the space dimension consists in defining the next vector product or abandoning one of the previously defined products. In addition, the orientation of the space selected is determined by the assumed, exemplary order of lower spaces in the product of a higher order.

In turn, a vector in any space is a scalar product of two matrices: the matrix of coordinates of the vector in the space adopted and the matrix of the space basis (the matrix of versors of the coordinate system). A scalar product of versors determines the reciprocal relationship of two different spaces (vectors) and also allows the vector to be transformed between the spaces. In addition, it can be used to present different forms occurring in a space; for example, an equation of a plane in a three-dimensional space or of a hyperplane in a space of any dimension.

Any vector which is the product of a scalar value and a versor lies in a one-dimensional space and can be described in any – one or several – spaces of different dimensions by means of transformation matrices, which are a form of a scalar product of versors of the appropriate axes. The same vector subjected to the operation of an operator (differentiation with respect to any variable, integration, multiplication by an even number of other vectors or using it in a vector product) creates a new vector in another one-dimensional space, which can be transformed into the same or other spaces. It results from the above that the calculus of vectors makes use of vector products of versors for creating new spaces and scalar products of versors for determining the reciprocal relationships between these spaces. In this sense, we can say that the modelling of a problem in any n-dimensional space is modelling by means of products. In particular, in spaces higher than a three-dimensional one, when both the space and the vector have an abstract character we can speak of *modelling by means of the product*.

The concepts of basic products of two vectors in a three-dimensional space – the scalar product and the vector product – are defined as early as in the secondary school teaching programme. The vector product, among other things, allows one to determine vectors of such parameters of motion as the speed and acceleration of a point of a body, and in addition, vectors of the moment of force or angular momentum with respect to any pole. Special kinds of even scalar products in a three-dimensional space are moments of inertia of solid bodies, mass moments in dynamics and cross-sectional moments in the mechanics of materials.

In this paper the same vector can be written in three different ways:

- in the classical form, which is a geometrical sum of projections of a vector onto the axes of the system,
- in the form of a column matrix whose terms contain versors of the axis,
- in the form of a column matrix whose terms do not contain versors of the axis, but only the values of the coordinates.

Thus, the following notations of the vector have been assumed:

$\vec{\boldsymbol{a}}$ the vector in the classical notation constitutes a geometrical sum of projections of the vector onto the system axes,

$\hat{\boldsymbol{a}}$ the vector in the form of a column matrix. The terms of the matrix are the versors or vectors of projections of the vector $\vec{\boldsymbol{a}}$ onto the system axes,

$\boldsymbol{a}$ the vector in the form of a column matrix. The terms of the matrix are scalars, coordinates of the vector $\vec{\boldsymbol{a}}$ on the system axes.

In the text the following designations of the matrix have been assumed:

- one-dimensional matrices (column or verse ones) have been denoted by small letters in bold, $(\boldsymbol{a}, \boldsymbol{a}^T, \hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}^T)$,
- rectangular matrices have been denoted by capital letters in bold, $(\boldsymbol{A}, \hat{\boldsymbol{A}})$.

Rectangular matrices, depending on whether their terms are scalars or vectors, have been designated as follows:

$\hat{\boldsymbol{A}}, \hat{\boldsymbol{P}}$ the rectangular matrix whose terms are vectors,

$\boldsymbol{A}, \boldsymbol{P}$ the rectangular matrix whose terms are scalars.

A scalar product of two vectors in the classical notation is denoted as $\vec{\boldsymbol{a}}\vec{\boldsymbol{c}}$, to distinguish it from the notation of the corresponding product of the matrix, e.g. $\boldsymbol{a}^T\boldsymbol{c}$.

2. Multiproducts of vectors

The calculus of vectors, in which the concept of a vector in a three- and n-dimensional space and basic operations on vectors – namely, their sum and product – have been defined, is the basic tool of description of many physical phenomena and thereby, constitutes an elementary problem of classical mechanics. Operations on vectors are described by means of vector equations and have a well-known geometrical interpretation in one-, two- and three-dimensional spaces. Although, the sum of vectors is defined for two vectors, this definition can easily be extended to comprise any number of them. Vector products, however, are subject to quantitative limitation, since the calculus of vectors defines two kinds of products:

– the scalar products $\vec{\boldsymbol{a}}\vec{\boldsymbol{c}}$ only for two vectors in any given space, and

– the vector product of $n-1$ vectors in an n dimensional space [1], in particular, the vector product of two vectors in a three-dimensional space.

The following two situations can be a starting-off point for considerations [2]:

• when the axes (versors) in a problem are related only to the vectors functioning in it, which means that spaces K^1 of successive vectors are not contained in any coordinate system.

• when the vectors being multiplied lie in a Euclidean space E^n, or in a n dimensional Cartesian space K^n, determined by systems of n axes, and then they are described in these systems of axes by coordinates. In a situation of a system of orthogonal axes, spaces K^1 of successive vectors lie in a space K^n.

In the latter case, the problem is that the scalar multiplication of vectors is performed on the axes of an orthogonal system as the product of the versors of the axes (positive unit vectors lying on the system axes); at the same time, the product of the versors of the same axis is equal to 1, whereas the product of the versors of different axes is equal to 0. This results from the definition of a scalar product of the versors of two axes $\vec{\boldsymbol{e}}_k$ and $\vec{\boldsymbol{e}}_m$, forming an angle φ_{km}, $\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = \cos\varphi_{km}$. This simple model functions well when two versors are multiplied, but when three or more versors are multiplied, it loses its uniqueness. This is so because in the case of the product of many vectors, the result of multiplication depends on a random order of multiplication of versors and can be equal to 0, 1 or to one of the versors [2]. This ambiguity can be avoided if two possible preferences of the choice of multiplication are distinguished and defined and – as a result of this assumption – four basic kinds of products of versors and – consequently, four basic kinds of products of vectors are defined. Such multiplication of vectors is an unambiguous operation, independent of the order of the vectors in the product. As a result, *multiproducts* (or plural products) of the versors in an orthogonal space, which are invariants of the product of these vectors.

In the first case, when the axes and their versors are associated with the vectors lying randomly in a space, i.e., they are not orthogonal axes, the multiproduct of these vectors depends on the order of their multiplication. This is demonstrated by means of an example of the product of three vectors $\vec{\boldsymbol{a}}\vec{\boldsymbol{g}}\vec{\boldsymbol{c}}$, lying on any three axes in a space and described as $\vec{\boldsymbol{a}} = a\vec{\boldsymbol{e}}_a$, $\vec{\boldsymbol{g}} = g\vec{\boldsymbol{e}}_g$, $\vec{\boldsymbol{c}} = c\vec{\boldsymbol{e}}_c$.The product of vectors $\vec{\boldsymbol{a}}\vec{\boldsymbol{g}}\vec{\boldsymbol{c}}$ depends on the order of their multiplication. Multiplying these vectors in a different order, we obtain three different resultant vectors, of different lengths, lying on different direction.

$$\vec{\boldsymbol{a}}\vec{\boldsymbol{g}}\vec{\boldsymbol{c}} = agc\,\vec{\boldsymbol{e}}_a\vec{\boldsymbol{e}}_g\vec{\boldsymbol{e}}_c = \begin{cases} agc\cos\phi_{gc}\vec{\boldsymbol{e}}_a = (\vec{\boldsymbol{g}}\vec{\boldsymbol{c}})\vec{\boldsymbol{a}} \\ agc\cos\phi_{ac}\vec{\boldsymbol{e}}_g = (\vec{\boldsymbol{c}}\vec{\boldsymbol{a}})\vec{\boldsymbol{g}} \\ agc\cos\phi_{ag}\vec{\boldsymbol{e}}_c = (\vec{\boldsymbol{a}}\vec{\boldsymbol{g}})\vec{\boldsymbol{c}} \end{cases}$$

where

$$(\vec{\boldsymbol{g}}\vec{\boldsymbol{c}})\vec{\boldsymbol{a}} \neq (\vec{\boldsymbol{c}}\vec{\boldsymbol{a}})\vec{\boldsymbol{g}} \neq (\vec{\boldsymbol{a}}\vec{\boldsymbol{g}})\vec{\boldsymbol{c}}.$$

The constant value for the product of the three vectors $\vec{\boldsymbol{a}}\vec{\boldsymbol{g}}\vec{\boldsymbol{c}}$, independent of the order of their multiplication – that is to say, an invariant of this product - is the vector sum of the right hand side of the products, $\vec{\boldsymbol{p}}^3 = (\vec{\boldsymbol{g}}\vec{\boldsymbol{c}})\vec{\boldsymbol{a}} + (\vec{\boldsymbol{c}}\vec{\boldsymbol{a}})\vec{\boldsymbol{g}} + (\vec{\boldsymbol{a}}\vec{\boldsymbol{g}})\vec{\boldsymbol{c}}$. This vector in has been denoted as $\vec{\boldsymbol{p}}^3$ and called a *vector of the sum of the products of three vectors.*

3. Kinds of multiproducts of versors

The scalar product of versors of two axes assuming the orthogonality of the axes, i.e., when the angle between the axes $\varphi_{km} = 1/2\pi$, assumes one of the two values:

$$\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = 0 \quad \text{if} \quad k \neq m, \qquad \vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = 1 \quad \text{if} \quad k = m.$$

In the case of the multiproduct of m vectors, products of n versors, belonging to different axes of the system, are obtained. The result of such a multiproduct of versors of different axes, occurring in an n dimensional space, depends on two factors [2].

The first factor is the number of the versors multiplied.

• When an even number of v versors are multiplied, a *scalar value* equal to 0 or ± 1 is obtained. Such a product can be called an *even product of v versors.*

• When an odd number of d versors are multiplied, a *zero vector or a vector equal to one of the versors* is obtained. Such a product can be called an *odd product of d versors.*

The other factor determining the result of the product is the preference of the multiplication adopted, i.e., the selected priority of the product in a situation when it is necessary to multiply a number of versors belonging to different or the same axes. In such a situation it should be decided whether to give priority of the same versors ($\vec{\boldsymbol{e}}_j\vec{\boldsymbol{e}}_j = 1$) or different versors ($\vec{\boldsymbol{e}}_j\vec{\boldsymbol{e}}_k = 0$).

• A product in which priority is given to the multiplication of versors of the same indices ($\vec{\boldsymbol{e}}_j\vec{\boldsymbol{e}}_j = 1$) can be called *a product of the first kind or a homoproduct of versors* and designated as

$f^v_{[e]}$ – if it is an even product and $\vec{\boldsymbol{f}}^{\,d}_{[e]}$ – if it is an odd product.

• A product in which priority is given to the multiplication of versors of different indices ($\vec{\boldsymbol{e}}_j\vec{\boldsymbol{e}}_k = 0$) can be called *a product of the second kind* or *a heteroproduct of versors* and denoted as

$s^v_{[e]}$ – if it is an even product and $\vec{\boldsymbol{s}}^{\,d}_{[e]}$ – if it is an odd product .

And so, for example:

$$f^4[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_k] = (\boldsymbol{e}_l\boldsymbol{e}_l)\,(\boldsymbol{e}_k\boldsymbol{e}_k) = 1, \quad s^4[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_k] = (\boldsymbol{e}_l\boldsymbol{e}_k)\,(\boldsymbol{e}_l\boldsymbol{e}_k) = 0,$$
$$f^4[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_u] = (\boldsymbol{e}_l\boldsymbol{e}_l)\,(\boldsymbol{e}_k\boldsymbol{e}_u) = 0, \quad s^4[\boldsymbol{e}_l\boldsymbol{e}_l\boldsymbol{e}_l\boldsymbol{e}_l] = (\boldsymbol{e}_l\boldsymbol{e}_l)\,(\boldsymbol{e}_l\boldsymbol{e}_l) = 1.$$

Even products of the first and the second kind of four versors can be can be equal to zero or one.

$$\vec{\boldsymbol{f}}^{\,3}[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l] = (\boldsymbol{e}_l\boldsymbol{e}_l)\,\boldsymbol{e}_k = \boldsymbol{e}_k, \quad \vec{\boldsymbol{s}}^{\,3}[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l] = (\boldsymbol{e}_l\boldsymbol{e}_k)\,\boldsymbol{e}_l = \boldsymbol{0},$$
$$\vec{\boldsymbol{f}}^{\,3}[\boldsymbol{e}_l\boldsymbol{e}_k\boldsymbol{e}_l] = (\boldsymbol{e}_l\boldsymbol{e}_l)\,\boldsymbol{e}_k = \boldsymbol{e}_k, \quad \vec{\boldsymbol{s}}^{\,3}[\boldsymbol{e}_l\boldsymbol{e}_l\boldsymbol{e}_l] = \boldsymbol{e}_l\boldsymbol{e}_l\boldsymbol{e}_l = \boldsymbol{e}_l.$$

Odd numbers of the first and second kind of three versors, on the other hand, disappear or are the versors of the axes.

4. A general case of multiproducts of versors of the axis and vectors in an n-dimensional space

Let us now consider a general case, a problem of a product of m vectors in an orthogonal n-dimensional space [3]. Let $\vec{\boldsymbol{a}}_i$, ($i = 1, \ldots, m$) be the i-th vector and simultaneously, let its j-th coordinate in a Cartesian n-dimensional space have the value a_{ij}, ($j = 1, \ldots, n$). After denoting the versor of the j-th axis of the coordinate system as $\vec{\boldsymbol{e}}_j$, we can write a projection of the i-th vector onto the j-th axis of the system $\vec{\boldsymbol{a}}_{ij}$ and the vector itself $\vec{\boldsymbol{a}}_i$,

$$(1) \qquad \vec{\boldsymbol{a}}_{ij} = a_{ij}\vec{e}_j; \quad \vec{\boldsymbol{a}}_i = \sum_{j=1}^{n} \vec{\boldsymbol{a}}_{ij} = \sum_{j=1}^{n} a_{ij}\vec{\boldsymbol{e}}_j$$

where the versor of the j-th

$$\vec{\boldsymbol{e}}_j = [0, 0, 0 \ldots, j = 1, \ldots, 0, 0, 0].$$

The product of m vectors defined in an n-dimensional space by coordinates a_{ij}, can be written in the following way

$$\vec{\boldsymbol{a}}_1 \vec{\boldsymbol{a}}_2 \ldots \vec{\boldsymbol{a}}_m = \prod_{i=1}^{m} \vec{\boldsymbol{a}}_i = \prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \vec{\boldsymbol{e}}_j. \tag{2}$$

The following matrix notation can be an illustration of equation (2). Let us introduce a vector of the coordinate system $\vec{\boldsymbol{n}}$, the vector whose projections onto the particular axes of the system will be the versors of these axes. This vector can be written in classical notation or as a matrix

$$\vec{\boldsymbol{n}} = \vec{\boldsymbol{e}}_1 + \vec{\boldsymbol{e}}_2 + \ldots + \vec{\boldsymbol{e}}_n = \sum_{j=1}^{n} \vec{\boldsymbol{e}}_j, \qquad \boldsymbol{n}^T = |1, 1, 1, \ldots, 1|. \tag{3}$$

The vector of the coordinate system $\vec{\boldsymbol{n}}$ can also be written in the form of a column matrix $\widehat{\boldsymbol{n}}$, containing the versors of the system axes, such that $\widehat{\boldsymbol{n}}^T = |\vec{\boldsymbol{e}}_1 \ \vec{\boldsymbol{e}}_2 \ldots \ \vec{\boldsymbol{e}}_n|$

Let us now construct a matrix $\boldsymbol{A}$, whose terms a_{ij} are the coordinates of the vectors multiplied, $\boldsymbol{A} = [a_{ij}]$, $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ and a matrix $\widehat{\boldsymbol{A}}$, whose terms are the vectors of projections of the successive vectors $\vec{\boldsymbol{a}}_i$ onto the system axes, written as products of corresponding coordinates and versors of the axes.

The matrix $\widehat{\boldsymbol{A}}$ has the form $\widehat{\boldsymbol{A}} = |a_{ij} \vec{\boldsymbol{e}}_j|$, $(i = 1, \ldots, m;\ j = 1, \ldots, n)$.

$$\boldsymbol{A} = \left| \begin{matrix} a_{11} \ a_{12} \ \ldots \ a_{1n} \\ a_{21} \ a_{22} \ \ldots \ a_{2n} \\ \ldots\ldots \\ a_{m1} \ a_{m2} \ \ldots \ a_{mn} \end{matrix} \right|, \qquad \widehat{\boldsymbol{A}} = \left| \begin{matrix} a_{11}\vec{\boldsymbol{e}}_1 \ a_{12}\vec{\boldsymbol{e}}_2 \ \ldots \ a_{1n}\vec{\boldsymbol{e}}_n \\ a_{21}\vec{\boldsymbol{e}}_1 \ a_{22}\vec{\boldsymbol{e}}_2 \ \ldots \ a_{2n}\vec{\boldsymbol{e}}_n \\ \ldots\ldots \\ a_{m1}\vec{\boldsymbol{e}}_1 \ a_{m2}\vec{\boldsymbol{e}}_2 \ \ldots \ a_{mn}\vec{\boldsymbol{e}}_n \end{matrix} \right|. \tag{4}$$

Let us also introduce a column matrix of the multiplied vectors $\widehat{\boldsymbol{a}}$, containing the multiplied vectors

$$\vec{\boldsymbol{a}}_j \ (i = 1, \ldots, m), \qquad \widehat{\boldsymbol{a}}^T = |\vec{\boldsymbol{a}}_1 \ \vec{\boldsymbol{a}}_2 \ \ldots \ \vec{\boldsymbol{a}}_{\boldsymbol{m}}|. \tag{5}$$

The matrix of the multiplied vectors $\widehat{\boldsymbol{a}}$ is a product of the matrix $\boldsymbol{A}$ and the matrix of the vector of the coordinate system $\vec{\boldsymbol{n}}$ or a product of the matrix $\widehat{\boldsymbol{A}}$ and the matrix $\boldsymbol{n}$.

$$\widehat{\boldsymbol{a}} = \boldsymbol{A}\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{A}}\boldsymbol{n}. \tag{6}$$

The left hand side of equation (2) is a product of the terms of matrix (6), whereas its right hand side contains all possible products of all the rows of the matrix $\widehat{\boldsymbol{A}}$.

5. Solution

The expansion of the right hand side of equation (2), multiplication and arrangement of its successive terms causes it to be transformed into a sum products of m elements, each containing one element of every row of the matrix $\widehat{\boldsymbol{A}}$ (4). Simply speaking, every element of every row of the matrix $\widehat{\boldsymbol{A}}$ is multiplied by every element of every other row. In this way, a sum n^m of products is obtained [3].

Each of them consists of a product of m scalar values, which are different coordinates of successive vectors $i = 1, 2, \ldots, m$, and a product of versors of different axes. The sequence of the indices of the versors of the axes and, at the same time, successive coordinates in each of these products constitutes a variation with repetitions u, containing m elements and made up of natural numbers $1, 2, \ldots, n$, which are numbers of the system axes. Thus, after the multiplication and arrangement, product (2) can be written as a sum containing n^m of products of the coordinates a_{iu} and the versors $\vec{\boldsymbol{e}}_u$

$$\prod_{i=1}^{m} \vec{\boldsymbol{a}}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu} \vec{\boldsymbol{e}}_u \tag{7}$$

where $u = u_{ki} (u \in N)$ is the i-th term of the k-th variation with repetitions u_k, consisting of m elements, constructed on an n-element set of natural numbers $\{1, 2, \ldots, n\}$ and n^m is the number of m-element variations with repetitions, constructed on an n-element set.

The elements of the matrix $\boldsymbol{U}$ of dimensions (n^m, m) are the indices of product (7). The matrix $\boldsymbol{U}$, containing the terms u_{ki}, in the successive rows contains all possible m-element variations with the repetitions $u_k, (k = 1, 2, \ldots, n^m)$ of natural numbers from the n element set. Successive elements $i = 1, 2, \ldots, m$ belonging to the k-th row of this matrix are the indices $u = u_{ki}$ of a successive, k-th product $a_{iu} \vec{\boldsymbol{e}}_u$

$$\boldsymbol{U} = \begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1m} \\ u_{21} & u_{22} & \ldots & u_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n^m 1} & u_{n^m 2} & \ldots & u_{n^m m} \end{vmatrix}. \tag{8}$$

We can now see that expression (7) is a sum of products of all the elements of any given column of the matrix of projections of the vectors $\widehat{\boldsymbol{A}}$ (4) when a complete permutation of the elements in all the rows is performed.

The expansion of the right hand side of equation (7) leads us to conclude that it is composed of a sum of n^m products, each of which consists of m^2 elements:

- the scalar coordinates a_{ij}, making up this product, constitute the first m elements,
- the versors of the axis $\vec{\boldsymbol{e}}_j$, accompanying these coordinates, constitute the next m elements.

Thus, product (7) can be written in the form

$$\prod_{i-1}^{m} \vec{\boldsymbol{a}}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu} \vec{\boldsymbol{e}}_u = \sum_{k=1}^{n^m} \left(\prod_{k=1}^{m} a_{iu} \prod_{i=1}^{m} \vec{\boldsymbol{e}}_u \right). \tag{9}$$

The calculation of a product of m scalar coordinates a_{iu} is not a problem. On the other hand, an attempt to multiply m versors by $\vec{\boldsymbol{e}}_u$ one another, belonging to n

different axes of the coordinate system, makes us realize that there is a need to define products of the versors of the axes in a general case, i.e. when we deal with the number of n^m variations with repetitions of the indices of the axes. This is so because the products $\prod_{i=1}^{m} \vec{\boldsymbol{e}}_u$ constitute m element variations with repetitions, constructed on an n element set of the versors of the coordinate system.

6. Multiproducts of versors of the axes

We make use of the definition of a scalar product of versors of any two axes of an orthogonal system $\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = 0$ if $k \neq m$, $\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = 1$ if $k = m$. When multiplying a greater number of versors belonging to several different axes, we notice that the result depends on two factors.

The first factor is the number of the versors being multiplied.

• If we deal with an even number of versors (e.g. $m = v$), then, as a result of multiplication, we obtain a scalar value, equal to 0 or 1. We can call such a product an *even product of v versors.*

• If we multiply an odd number of versors (e.g. $m = d$), then we obtain a *zero vector* or a vector equal to one of the versors. We will call such a product an *odd product of d versors.*

The other factor determining the result of a product is the preference of multiplication adopted, i.e. the priority of a product chosen in the situation when we deal with the multiplication of several versors belonging to different or the same axes. Then, the priority can be given to a product of the same ($\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_k = 1$) or different ($\vec{\boldsymbol{e}}_k\vec{\boldsymbol{e}}_m = 0$) versors.

• A product in which the priority is given to the multiplication of versors of the same indices will be called a *product of the first kind* or a *homo-product of versors* and denoted as:

$$f^{v}_{[e]} \ - \ \text{if it is an even product,} \qquad \vec{\boldsymbol{f}}^{d}_{[e]} \ - \ \text{if it is an odd product,}$$

• A product in which the priority is given to the multiplication of versors of different indices will be called a *product of the second kind* or a *hetero-product of versors* and denoted as:

$$s^{v}_{[e]} \ - \ \text{if it is an even product,} \qquad \vec{\boldsymbol{s}}^{v}_{[e]} \ - \ \text{if it is an odd product,}$$

In formula (9) there appears a multiproduct of m versors belonging to n different axes of the system. This multiproduct can be denoted as δ_k (k-th multiproduct $a_{iu}\vec{\boldsymbol{e}}_u$) and written in the following form

$$\delta_k = \prod_{i=1}^{m} \vec{\boldsymbol{e}}_u = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{\,b_{kj}} \quad \text{where} \quad \sum_{j=1}^{n} b_{kj} = m. \tag{10}$$

A multiproduct of m versors (10) is a product of versors of all the axes occurring in the problem ($j = 1, 2, \dots, n$); at the same time, each of them is raised to a natural power b_{kj} indicating how many times this versors appears in the k-th product.

Note also that in any given k-th multiproduct δ_k any j-th versor $\vec{\boldsymbol{e}}_j$ may not occur, which means that $b_{kj} = 0$. Then, for this versor $\vec{\boldsymbol{e}}_j^0 = 1$. Therefore we can deduce, that the product (5.12) does not depend on any versor disappearing.

We will now present a method of determination of k-th multiproduct δ_k for m versors. Depending on the values that are adopted by a sequence b_{kj} of power exponents for successive versors of the axis $\vec{\boldsymbol{e}}_j$, the multiproduct δ_k comes into the category of a product of the first kind (homo-product) or the second kind (hetero-product) discussed earlier. At the same time, we take into account whether we deal with an even ($m = v$) or an odd ($m = d$) multiproduct of the versors of the axis.

For every k-th multiproduct $\delta_k = \prod_{i=1}^{m} \vec{\boldsymbol{e}}_u$ (10), corresponding to the k-th variation u_k, we construct a zero-one matrix $\boldsymbol{E}_k$ forming this product of versors. The matrix $\boldsymbol{E}_k$ has dimensions (m,n). The rows of this matrix are successive versors $\vec{\boldsymbol{e}}_u = |0, 0, \ldots, u = 1, \ldots, 0, 0|$ making up the k-th product. We also create a column matrix $\boldsymbol{p}$, containing m elements whose all elements are numbers 1, so $\boldsymbol{p}^T = |1, 1, \ldots, 1|$.

We multiply the matrix

$$\boldsymbol{E}_k^T \boldsymbol{p} = \boldsymbol{b}_k \tag{11}$$

and and obtain a column matrix $\boldsymbol{b}_k$ containing n elements such that

$$\boldsymbol{b}_k^T = |b_{k1} b_{k2} \ldots b_{kn}| \quad \text{and} \quad \sum_{j=1}^{n} b_{kj} = m.$$

The elements of the matrix $\boldsymbol{b}_k$ are numbers $b_{kj} \in N$, which are exponents of powers of the multiproduct of the versors δ_k described by formula (10).

Making use of equation (10), we defined four basic, described earlier, kinds of multiproducts of versors:

- an even homo-product $f_{[e]}^v$, i.e. an *even product $m = v$ versors of the first kind*, when $m = v$, $v \in \{4, 6, 8, \ldots\}$. It will appear when all the exponents of the powers b_{kj} are even numbers; it will disappear even if only one of the exponents is an odd number. Then, the multiproduct of the versors δ_k will be introduced as $\delta_f^v = f_{[e]}^v$.

$$f_{[e]}^v = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{b_j} \overset{f}{=} \begin{cases} 1 & \text{when} \quad \underset{j}{\forall}\, b_j \in \{0, 2, 4, \ldots, v\} \\[2ex] 0 & \text{when} \quad \underset{j}{\exists}\, b_j \in \{1, 3, 5, \ldots, v-1\} \end{cases} \tag{12}$$

- an even hetero-product $S_{[e]}^v$, i.e. an *even product $m = v$ versors of the second kind*, when $m = v$, $v \in \{4, 6, 8, \ldots\}$. It will appear only if one of the exponents of the power b_{kj} is equal to an even number of the versors v being multiplied, whereas the remaining ones will be equal to zero. Then, δ_k will be replaced by $\delta_s^v = s_{[e]}^v$

$$s_{[e]}^v = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{b_j} \overset{s}{=} \begin{cases} 1 & \text{when} \quad \underset{j}{\exists}\, b_j = v \\[2ex] 0 & \text{when} \quad \underset{j}{\forall}\, b_j < v \end{cases} \tag{13}$$

• a homo-product $\vec{\boldsymbol{f}}^{d}_{[e]}$, i.e. an *odd product $m = d$ versors of the first kind*, when $m = d$, $d \in \{3, 5, 7, \dots\}$. It will appear when one of the exponents is an odd number; all the remaining exponents of the powers b_{kj} will be even numbers. It disappears when more than one of the exponents is an odd number. In this case δ_k will be replaced by $\delta^d_f = \vec{\boldsymbol{f}}^{d}_{[e]}$

$$(14)\qquad \vec{\boldsymbol{f}}^{d}_{[e]} = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{\,b_j} \stackrel{f}{=}$$

$$\stackrel{f}{=} \begin{cases} \vec{\boldsymbol{e}}_k \text{ when } \underset{k\in\{1,2,\dots,n\}}{\exists} b_k \in \{1,3,5,\dots,d\} \wedge \underset{j\neq k}{\forall} b_j \in \{0,2,4,\dots,d-1\} \\ \\ \mathbf{0} \text{ when } \underset{k,l\in\{1,2,\dots,n\}}{\exists} b_k \in \{1,3,5,\dots,d\} \wedge b_l \in \{1,3,5,\dots,d\} \end{cases}$$

• an odd hetero-product $\vec{\boldsymbol{s}}^{v}_{[e]}$, i.e. an *odd product $m = d$ versors of the second kind*, when $m = d$, $d \in \{3, 5, 7, \dots\}$. It will appear only if one of the exponents of the power b_{kj} is equal to an odd number of the versors d being multiplied; the remaining ones will be equal to zero. Then δ_k will be replaced by $\delta^d_s = \vec{\boldsymbol{s}}^{v}_{[e]}$

$$(15)\qquad \vec{\boldsymbol{s}}^{v}_{[e]} = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{\,b_j} \stackrel{s}{=} \begin{cases} \vec{\boldsymbol{e}}_k & \text{when} \quad \underset{k\in\{1,2,\dots,n\}}{\exists} b_k = d \\ \\ \mathbf{0} & \text{when} \quad \underset{j}{\forall}\, b_j < d \end{cases}$$

The analysis of formulae (12–15) allows us to conclude that a product of the second kind is contained in a product of the first kind, and this relationship holds both for even and odd products. Hence, we can introduce a third kind of product of m versors, which is a difference of a homo-product and a hetero-product, and designate it as

$t^v_{[e]} = f^v_{[e]} - s^v_{[e]}$ for even products and $\vec{\boldsymbol{t}}^{v}_{[e]} = \vec{\boldsymbol{f}}^{v}_{[e]} - \vec{\boldsymbol{s}}^{v}_{[e]}$ for odd products.

Two products of the third kind can be defined as:

• *an even product of the third kind* $v \in \{4, 6, 8, \dots\}$. It will appear when all the exponents of the powers b_{kj} are even numbers, but smaller than v. It disappears when even one of the exponents is an odd number or equal to v. Then δ_k will be replaced by $\delta^v_t = t^v_{[e]}$

$$(16)\qquad t^v_{[e]} = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{\,b_j} \stackrel{t}{=} \begin{cases} 1 & \text{when} \quad \underset{j}{\forall}\, b_j \in \{0,2,4,\dots,v-2\} \\ \\ 0 & \text{when} \quad \underset{j}{\exists}\, b_j \in \{1,3,5,\dots,v-1\} \vee \underset{j}{\exists}\, b_j = v \end{cases}$$

• *an odd product of the third kind* $d \in \{3, 5, 7, \dots\}$. It will appear when one of the exponents of the powers b_{kj} is an odd number but smaller than d, and all the remaining exponents of the powers b_{kj} are even numbers. It disappears when more than one of the exponents is an odd number or when one of the exponents is equal to d. Then δ_k will be replaced by $\delta^v_t = \vec{\boldsymbol{t}}^{d}_{[e]}$

$$(17) \qquad \vec{\boldsymbol{t}}^{d}_{[e]} = \prod_{j=1}^{n} \vec{\boldsymbol{e}}_j^{b_j} \stackrel{t}{=}$$

$$\stackrel{t}{=} \begin{cases} \vec{\boldsymbol{e}}_k \ \text{ when } \underset{k\in\{1,2,\dots,n\}}{\exists} b_k \in \{1,3,5,\dots,d-2\} \wedge \underset{j\neq k}{\forall} b_j \in \{0,2,4,\dots,d-1\} \\ \\ \mathbf{0} \ \text{ when } \underset{k,l\in\{1,2,\dots,n\}}{\exists} b_k \in \{1,3,5,\dots,d\} \wedge b_l \in \{1,3,5,\dots,d\} \end{cases}$$

As with multiproducts of versors of the first and second kind, an even product of the third kind $t^{v}_{[e]}$ is a scalar, whereas an odd product of the third kind $\vec{\boldsymbol{t}}^{d}_{[e]}$ is a vector.

7. Multiproducts of vectors in an n dimensional space

After introducing a multiproduct of the versors of the axis δ_k in the form of equation (10) and defining four basic (12–15) and two derivative (16–17) multiproducts of the versors of the axis, we can return to the product of m vectors. After taking into consideration (10) in (9) we have

$$(18) \qquad \prod_{i=1}^{m} \vec{\boldsymbol{a}}_i = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu} \vec{\boldsymbol{e}}_u = \sum_{k=1}^{n^m} \prod_{i=1}^{m} a_{iu} \delta_k .$$

In formula (18) expression (10) is a multiproduct of the versors δ_k formed for the k-th variation with repetitions u_k, described in matrix (8).

8. Scalar products of m vectors

Let us return to a starting point and calculate a product of m vectors determined by means of simple versors on which these vectors lie, i.e. the form

$$(19) \qquad \vec{\boldsymbol{a}}_i = a_i \vec{\boldsymbol{e}}_i, \qquad \vec{\boldsymbol{a}}_1 \vec{\boldsymbol{a}}_2 \dots \vec{\boldsymbol{a}}_m = \prod_{i=1}^{m} \vec{\boldsymbol{a}}_i = \prod_{i=1}^{m} a_i \vec{\boldsymbol{e}}_i .$$

As proved using an example of a product of three and four vectors the result of multiplication depends on the order of selection of pairs of vectors, for which a definition of a scalar product of two versors is used. Thus, we must separately consider a product of an even number ($m = v$) of vectors and separately a product of an odd ($m = d$) number of vectors.

9. Even product. Sum of the products

It is obvious that the result of multiplication of an even number of vectors will be a scalar, since we deal with a multiplicity of a scalar product of two vectors. It is also obvious that if formulae for a scalar product of two vectors are used, the result of the operation will depend on the randomly adopted order of multiplication of pairs of vectors, since, for example

$$\text{(a)} \qquad (\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_2)(\vec{\boldsymbol{a}}_3\vec{\boldsymbol{a}}_4)\dots(\vec{\boldsymbol{a}}_{v-1}\vec{\boldsymbol{a}}_v) \neq (\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_3)(\vec{\boldsymbol{a}}_2\vec{\boldsymbol{a}}_4)\dots(\vec{\boldsymbol{a}}_{v-1}\vec{\boldsymbol{a}}_v) \neq \dots$$

Thus, multiplying the same vectors in different orders we will obtain many values of the same product

$$\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_2\dots\vec{\boldsymbol{a}}_v = \prod_{i=1}^{v}\vec{\boldsymbol{a}}_i.$$

There will be as many results as there are ways to create two-element scalar products within a v element set. Two of these possibilities, are given by way of example in inequality (a). The number of such products, composed of $v/2$ pairs of vectors forming scalar products in a v element set is c_v

$$\text{(20)} \qquad c_v = (v-1)(v-3)\dots[v-(v-1)] = \prod_{n=1}^{v/2}[v-(2n-1)].$$

As demonstrated, there is a certain constant value of products, attainable when a product of m vectors is multiplied in different ways, both in the case of even products and odd products.

For even products, it is a sum of products of all the scalar products formed by the pairs of the vectors which are two-element combinations without repetitions, created with the use of the whole set being multiplied $\{\vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_2, \dots, \vec{\boldsymbol{a}}_v\}$. The number of these products is equal to c_v and determined by dependence (20). We will denote this sum of products as p^v and call it an invariant of an even product of v vectors. For an even product of v vectors the *sum of products* p^v is a scalar value equal to

$$p^v = (\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_2)(\vec{\boldsymbol{a}}_3\vec{\boldsymbol{a}}_4)\dots(\vec{\boldsymbol{a}}_{v-1}\vec{\boldsymbol{a}}_v) + (\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_3)(\vec{\boldsymbol{a}}_2\vec{\boldsymbol{a}}_4)\dots(\vec{\boldsymbol{a}}_{v-1}\vec{\boldsymbol{a}}_v) + \dots$$

$$\text{(21)} \qquad p^v = \sum_{i=1}^{c_v}\prod_{j=1}^{v/2} s_{ij}; \qquad s_{ij} = (\vec{\boldsymbol{a}}_k\vec{\boldsymbol{a}}_l).$$

Indices $\{k,l\} \in N$ form successive, j-th two-term elements of a combination without repetitions, formed on the v element set of indices. The set of v natural numbers is divided into $v/2$ pairs, forming an i-th combination of numbers. By changing the composition of the pairs forming a group $\{k,l\}$ of indices we obtain the number c_v (5.26) of all combinations of the indices.

It can be proved that for a fixed, even number of vectors v there is a relationship between the sum of products p^v and the even multiproducts of the vectors of the first (f^v), second (s^v) and third (t^v) kind described earlier, of the form

$$\text{(22)} \qquad p^v = f^v + (c_v - 1)s^v.$$

Since $f^v = s^v + t^v$ formula (22) can be transformed into the form

$$\text{(23)} \qquad p^v = c_v s^v + t^v.$$

The equations above can be used to calculate the invariant of the odd multiproduct of d vectors, which is defined below as the vector of the sum of products $\vec{\boldsymbol{p}}^{\,d}$.

10. Odd products. Vector of the sum of the product

It is obvious that the result of multiplication of an odd number of d vectors will be a vector, since we deal with a situation in which every d term product of vectors can be expressed as a product of any selected k-th vector and an even number $d-1$ of the remaining vectors

$$\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_2\ldots\vec{\boldsymbol{a}}_d = \prod_{i=1}^{d}\vec{\boldsymbol{a}}_i = \vec{\boldsymbol{a}}_k\prod_{\substack{i=1\\ i\neq k}}^{d}\vec{\boldsymbol{a}}_i \tag{24}$$

Product (24) is a vector lying on the direction of the vector $\vec{\boldsymbol{a}}_k$, which means that, depending on the choice of the index k, we obtain d of different vectors lying on d different directions and, additionally, having a different length, depending on the way of multiplication of the even product of d-1 vectors. It result from the above that if formula (24) is used, the result of the operation will depend on the randomly adopted order of multiplication of vectors.

As in the case of an even product, we can define an invariant of the odd multiproduct of d vectors as a *vector of the sum of products* $\vec{\boldsymbol{p}}^{\,d}$, which is a vector sum of all possible vectors (24) times the corresponding invariants of the even products of $v = d-1$ vectors. Thus, let us denote an invariant of the even product $v = d-1$ vectors, containing the products of the vectors $\{\vec{\boldsymbol{a}}_1\vec{\boldsymbol{a}}_2\ldots\vec{\boldsymbol{a}}_{i-1}\vec{\boldsymbol{a}}_{i+1}\ldots\vec{\boldsymbol{a}}_v\}$ as p_{-i}^{d-1}. Then, the vector of the sum of products can be written in the form

$$\vec{\boldsymbol{p}}^{d} = \sum_{i=1}^{d} p_{-i}^{d-1}\vec{\boldsymbol{a}}_i. \tag{25}$$

The value of direction of the vector $\vec{\boldsymbol{p}}^{d}$ is, for a definite group of the vectors being multiplied, constant and independent of the order of their multiplication.

If, in the matrix $\widehat{\boldsymbol{A}}$ (4), containing successive multiplied vectors in successive m rows (in the case of the odd product $m = d$), we draw an i-th vector, we will obtain a matrix $\widehat{\boldsymbol{A}}_{-i}^{d-1}$ (26) containing an even number of $v = d-1$ vectors, for which we calculate an invariable sum of products p^v.

$$\widehat{\boldsymbol{A}}^{\,d} = \left|\begin{array}{cccc} a_{11}\vec{\boldsymbol{e}}_1 & a_{12}\vec{\boldsymbol{e}}_2 & \ldots & a_{1n}\vec{\boldsymbol{e}}_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1}\vec{\boldsymbol{e}}_1 & a_{i2}\vec{\boldsymbol{e}}_2 & \ldots & a_{in}\vec{\boldsymbol{e}}_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{d1}\vec{\boldsymbol{e}}_1 & a_{d2}\vec{\boldsymbol{e}}_2 & \ldots & a_{dn}\vec{\boldsymbol{e}}_n \end{array}\right|, \qquad \widehat{\boldsymbol{A}}_{-i}^{\,d-1} = \left|\begin{array}{cccc} a_{11}\vec{\boldsymbol{e}}_1 & a_{12}\vec{\boldsymbol{e}}_2 & \ldots & a_{1n}\vec{\boldsymbol{e}}_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1}\vec{\boldsymbol{e}}_1 & a_{i2}\vec{\boldsymbol{e}}_2 & \ldots & a_{in}\vec{\boldsymbol{e}}_n \\ \hline \ldots & \ldots & \ldots & \ldots \\ a_{d1}\vec{\boldsymbol{e}}_1 & a_{d2}\vec{\boldsymbol{e}}_2 & \ldots & a_{dn}\vec{\boldsymbol{e}}_n \end{array}\right| \tag{26}$$

The matrix $\widehat{\boldsymbol{A}}_{-i}^{\,d-1}$ of dimensions $(d-1, n)$ is an i-th complement of the matrix $\widehat{\boldsymbol{A}}^{\,d}$, corresponding to the i-th vector of the multiproduct $\vec{\boldsymbol{a}}_i$.

11. Definition of a dyad

A dyad or, in other words, *the external product of two vectors*, is a rectangle matrix of dimensions $n \times m$, of the form $\boldsymbol{P}_{n\times m}$, containing products of the coordinates of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ (in matrix form $\boldsymbol{a}^T = [a_1 a_2 \dots a_n]$ and $\boldsymbol{b}^T = [b_1 b_2 \dots b_m]$), in a general case, of the vectors defined in different-dimension spaces $n \times m$ [11]. The terms of a dyad are products of the coordinates of vectors equal to, respectively:

$$\boldsymbol{P}_{ab} = |p_{ij}|; \quad p_{ij} = a_i b_j, \qquad (i = 1, 2, \dots, n;\ j = 1, 2, \dots, m). \tag{27}$$

From notation (27) it results that a dyad is sensitive to the order of its elements – the vectors that it is composed of. A change in the order of the elements leads to a transpose matrix. In addition, the product of the dyad $\boldsymbol{P}_{ab}$ and its transpose matrix $\boldsymbol{P}_{ab}^T$ are equal to

$$\boldsymbol{P}_{ab} = \boldsymbol{P}_{ba}^T \quad \text{and} \quad \boldsymbol{P}_{ab}\boldsymbol{P}_{ba}^T = b^2 \boldsymbol{P}_{aa}. \tag{28}$$

The matrix $\boldsymbol{P}_{ab}$ corresponds to the product of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ determined in the form of different-dimension matrices (29); at the same time, to the product written in such a form (in such an order) that the multiplication of the matrix is impossible. Therefore, if

$$\boldsymbol{a} = \begin{vmatrix} a_1 \\ \dots \\ a_n \end{vmatrix} \quad \text{and} \quad \boldsymbol{b}^T = |b_1 \dots b_m|$$

then the notation of the product of the matrix

$$\boldsymbol{a}\boldsymbol{b}^T = \begin{vmatrix} a_1 \\ \dots \\ a_n \end{vmatrix} |b_1 \dots b_m| \tag{29}$$

is an equivalent of the dyad $\boldsymbol{P}_{ab}$ in the multiproduct. The matrix $\boldsymbol{P}_{ab}$ can be used for the notation of multiproducts of vectors of any order, with the use of any pairs of scalar products.

Thus, the dyad, which can also be referred to as *a matrix of the product of two vectors* $\boldsymbol{P}_{ab}$, has the form

$$\boldsymbol{P}_{ab} = \begin{vmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_m \\ \dots & \dots & \dots & \dots \\ a_n b_1 & a_n b_2 & \dots & a_n b_m \end{vmatrix} \tag{30}$$

and – which is easy to find out – contains appropriate products of the terms of the coordinate of both vectors $\boldsymbol{a}$ and $\boldsymbol{b}^T$. A good illustration of the possibilities offered by the use of a dyad in the notation of any multiprodutcs are the simplest odd and even products which – after a dyad has been introduced – can be formed identically in many ways; at the same time, the vectors can be reordered in different possible ways in scalar products.The correctness of the identities cited can be checked by performing appropriate transformations. For example, in an odd product of the third order $(\vec{a}\vec{b})\vec{c}$

$$(31)\qquad \begin{gathered} (\boldsymbol{a}^T \underline{\boldsymbol{b})\boldsymbol{c}^T} = \boldsymbol{a}^T \boldsymbol{P}_{bc} \quad \text{or} \quad (\boldsymbol{b}^T \underline{\boldsymbol{a})\boldsymbol{c}^T} = \boldsymbol{b}^T \boldsymbol{P}_{ac}, \\ \underline{\boldsymbol{c}(\boldsymbol{a}^T}\boldsymbol{b}) = \boldsymbol{P}_{ca}\boldsymbol{b} \quad \text{or} \quad \underline{\boldsymbol{c}(\boldsymbol{b}^T}\boldsymbol{a}) = \boldsymbol{P}_{cb}\boldsymbol{a}, \end{gathered}$$

as well as in an even product of the fourth order $(\vec{a}\vec{b})(\vec{c}\vec{d})$

$$(32)\qquad \begin{gathered} (\boldsymbol{a}^T \underline{\boldsymbol{b})(\boldsymbol{c}^T}\boldsymbol{d}) = \boldsymbol{a}^T \boldsymbol{P}_{bc}\boldsymbol{d} \quad \text{or} \quad (\boldsymbol{b}^T \underline{\boldsymbol{a})(\boldsymbol{c}^T}\boldsymbol{d}) = \boldsymbol{b}^T \boldsymbol{P}_{ac}\boldsymbol{d}, \\ (\boldsymbol{a}^T \underline{\boldsymbol{b})(\boldsymbol{d}^T}\boldsymbol{c}) = \boldsymbol{a}^T \boldsymbol{P}_{bd}\boldsymbol{c} \quad \text{or} \quad (\boldsymbol{b}^T \underline{\boldsymbol{a})(\boldsymbol{d}^T}\boldsymbol{c}) = \boldsymbol{b}^T \boldsymbol{P}_{ad}\boldsymbol{c}. \end{gathered}$$

Identities (31) and (32) show the manner of writing a dyad into a multiproduct and, incidentally, explain its other name: an external product of two vectors. It is worth noting that all the expressions (31) describe the same vector written by means of four different dyads and expressed twice in the form of a horizontal (transpose) matrix, and twice in the form of a vertical matrix. A similar observation can be made about products (32). This is the same scalar value $(\vec{a}\vec{b})(\vec{c}\vec{d})$ written in four different ways. In multiproducts of higher orders a dyad can be used any possible number of times; taking into account identical elements (31) and (32) in the notation of a multiproduct. For example:

$$\begin{aligned} &(\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{d})(\boldsymbol{e}^T\boldsymbol{f})(\boldsymbol{g}^T\boldsymbol{h})(\ )\ldots(\ ) = (\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{d})(\ )\ldots(\ )\boldsymbol{e}^T\boldsymbol{P}_{fg}\boldsymbol{h} = \\ =\ &(\ )\ldots(\ )(\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{g}^T\boldsymbol{h})\boldsymbol{c}^T\boldsymbol{P}_{de}\boldsymbol{f} = (\ )\ldots(\ )(\boldsymbol{e}^T\boldsymbol{f})\boldsymbol{c}^T\boldsymbol{P}_{dg}\boldsymbol{P}_{ha}\boldsymbol{b} = \\ =\ &(\ )\ldots(\ )\boldsymbol{f}^T\boldsymbol{P}_{ec}\boldsymbol{P}_{dg}\boldsymbol{P}_{ha}\boldsymbol{b} = \text{etc.} \end{aligned}$$

The example above show just one possibility of using dyads in the notation of scalar vector multiproducts in any orthogonal space.

12. Dyad of scalar products of the versors of the axes

The versors of the axes can also be the terms of the matrix of the external product of two vectors. Then, a dyad contains scalar products of the versors of the axes of the system. In a general case, they are products of the versors of the axes of two different coordinate systems, whereas in a special case – as shown below – the dyad $\widehat{\boldsymbol{P}}$ contains products of the versors of the same coordinate system.

$$\widehat{\boldsymbol{P}}^{e}_{nm} = |p_{ij}|, \quad p_{ij} = \vec{e}_i\vec{e}_j, \quad (i,j = 1,2,\ldots,n)$$

$$(33)\qquad \widehat{\boldsymbol{P}}^{e}_{nm} = \begin{vmatrix} \vec{e}_1\vec{e}_1 & \vec{e}_1\vec{e}_2 & \ldots & \vec{e}_1\vec{e}_n \\ \vec{e}_2\vec{e}_1 & \vec{e}_2\vec{e}_2 & \ldots & \vec{e}_2\vec{e}_n \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{e}_1 & \vec{e}_n\vec{e}_2 & \ldots & \vec{e}_n\vec{e}_n \end{vmatrix} = \begin{vmatrix} 1 & \vec{e}_1\vec{e}_2 & \ldots & \vec{e}_1\vec{e}_n \\ \vec{e}_2\vec{e}_1 & 1 & \ldots & \vec{e}_2\vec{e}_n \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{e}_1 & \vec{e}_n\vec{e}_2 & \ldots & 1 \end{vmatrix}$$

The dyad $\widehat{\boldsymbol{P}}^{e}_{nm}$ is a square, n dimensional and symmetrical matrix.

If a system of axes is an orthogonal system, the terms of the matrix (33) are the values of Kronecker's delta δ_{ij}; in that case, the dyad $\widehat{\boldsymbol{P}}^{e}_{nm}$ becomes a unit matrix of a dimension n, i.e. a matrix $\boldsymbol{I}_n$. Then $\widehat{\boldsymbol{P}}^{e}_{nm} = |\delta_{ij}| = \boldsymbol{I}_n$.

This form a dyad – the unit matrix $\boldsymbol{I}_n$ – occurs with a scalar product of two vectors $\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}$, written in the same space in classical form,

$$\vec{\boldsymbol{a}}\vec{\boldsymbol{b}} = (a_1\vec{\boldsymbol{e}}_1 + \ldots + a_n\vec{\boldsymbol{e}}_{\boldsymbol{n}})\,(b_1\vec{\boldsymbol{e}}_1 + \ldots + b_n\vec{\boldsymbol{e}}_n),$$

when this form of a product is reduced to the product of a matrix

$$\vec{\boldsymbol{a}}\vec{\boldsymbol{b}} = (\boldsymbol{a}^T\widehat{\boldsymbol{e}}_n)\,(\boldsymbol{b}^T\widehat{\boldsymbol{e}}_n) = (\boldsymbol{a}^T\underline{\widehat{\boldsymbol{e}}_n)\,(\widehat{\boldsymbol{e}}_n^T}\boldsymbol{b}) = \boldsymbol{a}^T\widehat{\boldsymbol{P}}^{e}_{nm}\boldsymbol{b} = \boldsymbol{a}^T\boldsymbol{I}_n\boldsymbol{b} = \boldsymbol{a}^T\boldsymbol{b}, \tag{34}$$

where $\widehat{\boldsymbol{e}}_n(\widehat{\boldsymbol{e}}_n^T = |\vec{\boldsymbol{e}}_1\vec{\boldsymbol{e}}_2\ldots\vec{\boldsymbol{e}}_n|)$ is a matrix of versors of the coordinate system axes.

13. Dyad as a matrix of transformation of orthogonal systems

A dyad looks different when – in a general case – is a matrix containing products of the versors of the axes of two different orthogonal reference systems. Then, a dyad is made up of scalar products c_{ij} of the form

$$\widehat{\boldsymbol{P}}^{e}_{ab} = |p_{ij}|, \quad p_{ij} = \vec{\boldsymbol{e}}_{ai}\vec{\boldsymbol{e}}_{bj}, \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m) \tag{35}$$

where $\vec{\boldsymbol{e}}_{ai}$ is the versors of the first coordinate system of an n-dimensional orthogonal space and $\vec{\boldsymbol{e}}_{bj}$ is the versors of the axes of the second coordinate system of an m-dimensional orthogonal space. In other words, the matrix $\widehat{\boldsymbol{e}}_a(\widehat{\boldsymbol{e}}_a^T = |\vec{\boldsymbol{e}}_{a1}\vec{\boldsymbol{e}}_{a2}\ldots\vec{\boldsymbol{e}}_{an}|)$ is a matrix of the versors $\vec{\boldsymbol{e}}_{ai}$ of the base of the vector $\vec{\boldsymbol{a}}$ in an n dimensional space and $\widehat{\boldsymbol{e}}_b(\widehat{\boldsymbol{e}}_b^T = |\vec{\boldsymbol{e}}_{b1}\vec{\boldsymbol{e}}_{b2}\ldots\vec{\boldsymbol{e}}_{bm}|)$ is a matrix of the versors $\vec{\boldsymbol{e}}_{bj}$ of the base of the vector $\vec{\boldsymbol{b}}$ in an m dimensional space. Then, the dyad has the following form:

$$\widehat{\boldsymbol{P}}^{e}_{ab} = \begin{vmatrix} \vec{\boldsymbol{e}}_{a1}\vec{\boldsymbol{e}}_{b1} & \vec{\boldsymbol{e}}_{a1}\vec{\boldsymbol{e}}_{b2} & \ldots & \vec{\boldsymbol{e}}_{a1}\vec{\boldsymbol{e}}_{bm} \\ \vec{\boldsymbol{e}}_{a2}\vec{\boldsymbol{e}}_{b1} & \vec{\boldsymbol{e}}_{a2}\vec{\boldsymbol{e}}_{b2} & \ldots & \vec{\boldsymbol{e}}_{a2}\vec{\boldsymbol{e}}_{bm} \\ \ldots & \ldots & \ldots & \ldots \\ \vec{\boldsymbol{e}}_{an}\vec{\boldsymbol{e}}_{b1} & \vec{\boldsymbol{e}}_{an}\vec{\boldsymbol{e}}_{b2} & \ldots & \vec{\boldsymbol{e}}_{an}\vec{\boldsymbol{e}}_{bm} \end{vmatrix} = \begin{vmatrix} c_{a1b1} & c_{a1b2} & \ldots & c_{a1bm} \\ c_{a2b1} & c_{a2b2} & \ldots & c_{a2bm} \\ \ldots & \ldots & \ldots & \ldots \\ c_{anb1} & c_{anb2} & \ldots & c_{anbm} \end{vmatrix} \tag{36}$$

where $c_{aibj} = \cos\varphi_{aibj}$ is a scalar product of the corresponding versors of the base of two orthogonal systems of axes. The dyad $\widehat{\boldsymbol{P}}^{e}_{ab}$ is a matrix of transformation of any vector $\vec{\boldsymbol{a}}$ from an n dimensional space to the vector $\vec{\boldsymbol{b}}$ in an m dimensional space. Let us assume that the same vector $\vec{\boldsymbol{u}} = u\vec{\boldsymbol{e}}_u$, lying in a one-dimensional space, is a vector $\vec{\boldsymbol{a}}$ in an orthogonal n dimensional space, while it is a vector $\vec{\boldsymbol{b}}$ in another orthogonal m dimensional space. Then, we have

$$\vec{\boldsymbol{u}} = u\vec{\boldsymbol{e}}_u = \boldsymbol{u}\widehat{\boldsymbol{e}}_u$$

(in matrix notation $\boldsymbol{u}$ and $\widehat{\boldsymbol{e}}_u$ is a one-term matrix),

$$\vec{\boldsymbol{a}} = a_1\vec{\boldsymbol{e}}_{a1} + \ldots + a_n\vec{\boldsymbol{e}}_{an} = \boldsymbol{a}^T\widehat{\boldsymbol{e}}_a \quad \text{and} \quad \vec{\boldsymbol{b}} = b_1\vec{\boldsymbol{b}}_{b1} + \ldots + b_n\vec{\boldsymbol{e}}_{bn} = \boldsymbol{b}^T\widehat{\boldsymbol{e}}_b. \tag{37}$$

As this is the same vector, then

$$\boldsymbol{u}^T \widehat{\boldsymbol{e}}_u = \boldsymbol{a}^T \widehat{\boldsymbol{e}}_a = \boldsymbol{b}^T \widehat{\boldsymbol{e}}_b. \tag{38}$$

The property used here states that every vector equation (including a matrix one) after the multiplication of both sides by any vector (matrix) will hold true. The right-handed multiplication of equation (38) by a one-term matrix $\widehat{\boldsymbol{e}}_u^T$ leads to Chasles's equation in the form of transpose matrices:

$$\boldsymbol{u}^T = \boldsymbol{a}^T \widehat{\boldsymbol{e}}_a \widehat{\boldsymbol{e}}_u^T = \boldsymbol{a}^T \widehat{\boldsymbol{P}}^e_{au}, \qquad \boldsymbol{u}^T = \boldsymbol{b}^T \widehat{\boldsymbol{e}}_b \widehat{\boldsymbol{e}}_u^T = \boldsymbol{b}^T \widehat{\boldsymbol{P}}^T_{bu} \tag{39}$$

where the dyads $\widehat{\boldsymbol{P}}^e_{au}$ and $\widehat{\boldsymbol{P}}^e_{bu}$ are one-row matrices of transformation of the coordinates of the vectors; of the vector $\vec{\boldsymbol{a}}$ from the n dimensional space and the vector $\vec{\boldsymbol{b}}$ from the m dimensional space, respectively, to a one-term matrix $\boldsymbol{u}^T$, of the coordinate u of the vector $\vec{\boldsymbol{u}}$ in the one-dimensional space.

On the other hand, after the right-handed multiplication of equation (38) by the matrix $\widehat{\boldsymbol{e}}_a^T$ we obtained equations (40) of transformation of the coordinates of the vectors $\vec{\boldsymbol{u}}$ (from the one-dimensional space) and $\vec{\boldsymbol{b}}$ (from the m dimensional space) to the coordinates of the vector $\vec{\boldsymbol{a}}$ in the n dimensional space,

$$\boldsymbol{a}^T = \boldsymbol{u}^T \widehat{\boldsymbol{e}}_u \widehat{\boldsymbol{e}}_a^T = \boldsymbol{u}^T \widehat{\boldsymbol{P}}^e_{ua}, \quad \boldsymbol{a}^{\boldsymbol{T}} = \boldsymbol{b}^{\boldsymbol{T}} \widehat{\boldsymbol{e}}_b \widehat{\boldsymbol{e}}_a^T = \boldsymbol{b}^{\boldsymbol{T}} \widehat{\boldsymbol{P}}^{\boldsymbol{e}}_{ba}. \tag{40}$$

In equations (40) the dyad $\widehat{\boldsymbol{P}}^e_{ua}$ is a one-column, n row matrix of transformation of the one-dimensional vector $\vec{\boldsymbol{u}}$ to n coordinates of the vector $\vec{\boldsymbol{a}}$, whereas the dyad $\widehat{\boldsymbol{P}}^e_{ba}$ of dimensions $m \times n$ is a matrix of transformation of the vector $\vec{\boldsymbol{b}}$ form the m dimensional space to the vector $\vec{\boldsymbol{a}}$ in an n dimensional space. If the row matrix $\boldsymbol{a}^T$ is to be turned into a column one $\boldsymbol{a}$, we transpose expressions (40) and obtain

$$\boldsymbol{a} = \left[\boldsymbol{u}^T \widehat{\boldsymbol{P}}^e_{ua}\right]^T = \widehat{\boldsymbol{P}}^e_{au}\boldsymbol{u}, \qquad \boldsymbol{a} = \left[\boldsymbol{b}^T \widehat{\boldsymbol{P}}^e_{ba}\right]^T = \widehat{\boldsymbol{P}}^e_{ab}\boldsymbol{b}. \tag{41}$$

In an orthogonal space coordinates of any vector $\vec{\boldsymbol{a}}$ on the system axes are equal to scalar values of the rectangular projections of the vector onto these axes. This means that successive coordinates of the vector $\vec{\boldsymbol{a}}$, ($\boldsymbol{a}^T = [a_1 a_2 \ldots a_n]$) have the values $a_i = \vec{\boldsymbol{a}}\vec{\boldsymbol{e}}_i$ respectively, i.e.

$$\boldsymbol{a}^T = [\vec{\boldsymbol{a}}\vec{\boldsymbol{e}}_1\, \vec{\boldsymbol{a}}\vec{\boldsymbol{e}}_2 \ldots \vec{\boldsymbol{a}}\vec{\boldsymbol{e}}_n]\,.$$

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MODELOWANIE PRZESTRZENI ZA POMOCĄ WIELOWYMIAROWYCH MULTIILOCZYNÓW WEKTORÓW

S t r e s z c z e n i e

W pracy opisano i zdefiniowano multiiloczyny wektorów i wersorów w przestrzeni n-wymiarowej, przy czym jako multiiloczyn potraktowano iloczyn wektorów w ilości większej niż dwa. Podano pewną ilość przykładów ich zastosowań w różnego rodzaju przypadkach, zwłaszcza w transformacjach wektorów między przestrzeniami. Do zapisu multiiloczynów użyto diady, czyli macierzy iloczynu zewnętrznego dwóch wektorów, która przy transformacjach wektorów spełnia rolę macierzy transformacji.

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In memory of
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THE RELATIONSHIP AMONG THE KIRCHHOFF EQUATIONS FOR THE LOOP OF ELECTRICAL CIRCUITS

Summary

We show what follows. For an electrical circuit, assume that Kirchhoff equations are satisfied for (large) loops given by all permutations of all nodes. Then the Kirchhoff equations are satisfied in any loop. Precisely, for permutations this result is given as some theorem for an antisymmetric matrix.

1. Main result and its interpretation

The aim of this paper is the following characterisation of antisymmetric matrices related to the Kirchhoff equations.

Theorem 1. *Let* $(a_{ij})_{1\le i,j\le n}$, $a_{ij}\in\mathbb{R}$, $n>3$ *be a matrix such that*

$$a_{ij}+a_{ji}=0 \quad \textit{for } i\neq j,$$

and for any permutation π *of the set* $\{1,\dots,n\}$ *we have*

$$a_{\pi(n)\pi(1)}+\sum_{i=1}^{n-1} a_{\pi(i)\pi(i+1)}=0.$$

Then
(i) *for some* $V_i\in\mathbb{R}$, $i=1,\dots,n$,

$$a_{ij}=V_j-V_i,\quad i\neq j. \tag{1}$$

(ii) *for any* $m \in \{1, \dots, n-2\}$ *and for any permutation* π *of the set* $\{1, \dots, n\}$,

$$a_{\pi(n-m)\pi(1)} + \sum_{i=1}^{n-m-1} a_{\pi(i)\pi(i+1)} = 0.$$

Theorem 1 provides a generalized solution of the question posed in the preprint by A. Paszkiewicz, *Ogólne własności informacji* (in Polish; *General properties of information*).

Theorem 1 has obvious interpretation in terms of the Kirchhoff equations. For a given (oriented) edge joining points i and j of electrical circuits, let $\mathcal{E}_{ij}$ be electromotive force and U_{ij} be voltage drop for this edge. Put

$$a_{ij} = \mathcal{E}_{ij} - U_{ij}. \tag{2}$$

Kirchhoff's second law ([Wr] p. 167) says that *in any loop of circuit sum of voltage drops for suitable segments of the circuit is equal to the sum of electromotive forces occurring in the circuit.* We can this write the formula as

$$U_{\pi(n)\pi(1)} + \sum_{i=1}^{n-1} U_{\pi(i)\pi(i+1)} = \mathcal{E}_{\pi(n)\pi(1)} + \sum_{i=1}^{n-1} \mathcal{E}_{\pi(i)\pi(i+1)},$$

where π is any permutation of the set $\{1, 2, \dots, n\}$. Transforming the above expression we have

$$\mathcal{E}_{\pi(n)\pi(1)} - U_{\pi(n)\pi(1)} + \sum_{i=1}^{n-1} \left(\mathcal{E}_{\pi(i)\pi(i+1)} - U_{\pi(i)\pi(i+1)}\right) = 0,$$

and using (2) we have

$$a_{\pi(n)\pi(1)} + \sum_{i=1}^{n-1} a_{\pi(i)\pi(i+1)} = 0.$$

It is wellknown, that the above equations follow from the simplest equations $a_{ij} + a_{jk} + a_{ki} = 0$, for pairwise different i, j, k ($1 \le i, j, k \le n$). Moreover, these equations determine existence of potentials V_i for points i ($1 \le i \le n$) of electrical circuit satisfying (1).

At present we are going to show that equations for long cycles determine existence of potentials and all equations of Kirchhoff for shorter cycles.

2. Auxiliary lemma

In this section we will give usefull corollaries 1 and 2 below. These Corollaries immediatelly follow from the following

Lemma 1. *Let* $x_1, \dots, x_n, y_1, \dots, y_{n-1} \in \mathbb{R}$, $n \in \mathbb{N}$ *and* $n \ge 2$. *Then*

$$\det\begin{bmatrix} y_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ y_2 & 1 & 1 & \dots & 0 & 0 & 0 \\ y_3 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ y_{n-2} & 0 & 0 & \dots & 1 & 1 & 0 \\ y_{n-1} & 0 & 0 & \dots & 0 & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{n-2} & x_{n-1} & x_n \end{bmatrix}$$
$$= (-1)^{n+1}\big[x_1 + \sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{i+j}x_iy_j\big].$$

Proof. The proof will be carried out by induction with respect n. For $n = 2$ the assertion follows from

$$\det\begin{bmatrix} y_1 & 1 \\ x_1 & x_2 \end{bmatrix} = x_2y_1 - x_1,$$

and

$$(-1)^3(x_1 + (-1)^3x_2y_1) = x_2y_1 - x_1.$$

Assume that the assertion holds for n. We show that it is true for $n+1$. Using the Laplace expansion with respect to the $(n+1)$th column and the induction hypothesis we get

$$\det\begin{bmatrix} y_1 & 1 & 0 & \dots & 0 & 0 \\ y_2 & 1 & 1 & \dots & 0 & 0 \\ y_3 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_n & 0 & 0 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_n & x_{n+1} \end{bmatrix}$$
$$= (-1)^{2n+1}\det\begin{bmatrix} y_1 & 1 & 0 & \dots & 0 & 0 \\ y_2 & 1 & 1 & \dots & 0 & 0 \\ y_3 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{n-1} & 0 & 0 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{bmatrix}$$
$$+(-1)^{2n+2}x_{n+1}\det\begin{bmatrix} y_1 & 1 & 0 & \dots & 0 & 0 \\ y_2 & 1 & 1 & \dots & 0 & 0 \\ y_3 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{n-1} & 0 & 0 & \dots & 1 & 1 \\ y_n & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= (-1)^{2n+1}(-1)^{n+1}\big[x_1 + \sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{i+j}x_iy_j\big]$$

$$+(-1)^{2n+2}x_{n+1}(-1)^{n+1}\big[y_n + \sum_{j=1}^{n-1}(-1)^{n+j}y_j\big]$$

$$= (-1)^{n+2}\big[(-1)^{2n}\big(x_1 + \sum_{i=2}^{n}\sum_{j=1}^{i-1}(-1)^{i+j}x_iy_j\big) + (-1)^{2n}\sum_{j=1}^{n}(-1)^{n+1+j}x_{n+1}y_j\big]$$

$$= (-1)^{n+2}\big[x_1 + \sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}x_iy_j\big].$$

This completes the proof. $\square$

From the above lemma we immediately obtain the following two corollaries.

Corollary 1. *Let $c_1, \ldots, c_n \in \mathbb{R}$, $n \in \mathbb{N}$ and $n \geq 2$. Then*

$$\det\begin{bmatrix} c_1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ c_2 & 1 & 1 & \ldots & 0 & 0 & 0 \\ c_3 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{n-2} & 0 & 0 & \ldots & 1 & 1 & 0 \\ c_{n-1} & 0 & 0 & \ldots & 0 & 1 & 1 \\ c_n & (-1)^3 & (-1)^4 & \ldots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \end{bmatrix}$$

$$= (-1)^{n+1}\big[c_n + \sum_{i=1}^{n-1}(-1)^{i+1}(n-i)c_i\big].$$

Corollary 2. *For the n-dimensional matrix*

$$B = \begin{bmatrix} 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ (-1)^2 & (-1)^3 & (-1)^4 & \ldots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \end{bmatrix}$$

we have

$$\det B = (-1)^{n+1}n.$$

3. Proof of Theorem 1

Consider the following representation:

$$
\begin{aligned}
a_{12} &= V_2 - V_1,\\
a_{23} &= V_3 - V_2,\\
a_{34} &= V_4 - V_3,\\
&\vdots\\
a_{n-1,n} &= V_n - V_{n-1},
\end{aligned}
\tag{3}
$$

where $V_i \in \mathbb{R}$. Obviously the numbers V_i exist. From the assumption

$$a_{n,1} + \sum_{i=1}^{n-1} a_{i,i+1} = 0.$$

Thus we easily get

$$a_{n,1} = V_1 - V_n.$$

Since the matrix is antisymmetric, then

$$
\begin{aligned}
a_{21} &= V_1 - V_2,\\
a_{32} &= V_2 - V_3,\\
a_{43} &= V_3 - V_4,\\
&\vdots\\
a_{n,n-1} &= V_{n-1} - V_n,\\
a_{1,n} &= V_n - V_1.
\end{aligned}
$$

It remains to show that

$$a_{ij} = V_j - V_i,$$

where $1 < |i - j| < n - 1$.

Consider any (i, j) such that $|i - j| = 2$, that is to say:

$$(1,3), (2,4), (3,5), (4,6), \ldots, (n-2, n). \tag{4}$$

To antisymmetricity of matrix A above pairs exhaust any possibilities. Now we choose $n - 3$ permutations (transpositions) binding any two adjacent pairs of (4) as follows:

$$
\begin{aligned}
(1,3), (2,4) &\longmapsto \pi_1 = (1, 3, 2, 4, 5, 6, \ldots, n),\\
(2,4), (3,5) &\longmapsto \pi_2 = (1, 2, 4, 3, 5, 6, 7, \ldots, n),\\
(3,5), (4,6) &\longmapsto \pi_3 = (1, 2, 3, 5, 4, 6, 7, 8 \ldots, n),\\
&\vdots\\
(n-3, n-1), (n-2, n) &\longmapsto \pi_{n-3} = (1, 2, 3, 4, \ldots, n-3, n-1, n-2, n),
\end{aligned}
$$

and one permutation of all pairs of binding (4):

$$\pi_{n-2} = (1, 3, 5, \ldots, n-2, n, n-1, n-3, \ldots, 6, 4, 2) \quad \text{when } n \in 2\mathbb{N}-1,$$

or

$$\pi_{n-2} = (1, 3, 5, \ldots, n-3, n-1, n, n-2, \ldots, 6, 4, 2) \quad \text{when } n \in 2\mathbb{N},$$

where $2\mathbb{N}$ denote the set of even natural numbers and $2\mathbb{N}-1$ denote the set of odd natural numbers.

Let us consider the following system of $n-2$ equations with $n-2$ unknowns x_{ij}:

$$\tag{5} \begin{cases} x_{13} + x_{24} = c_1, \\ x_{24} + x_{35} = c_2, \\ x_{35} + x_{46} = c_3, \\ \vdots \\ x_{n-3,n-1} + x_{n-2,n} = c_{n-3}, \\ (-1)^2 x_{13} + (-1)^3 x_{24} + \cdots + (-1)^{n-1} x_{n-2,n} = c_{n-2}, \end{cases}$$

where constants on the right-hand side are equal:

$$c_1 = -\Big(\sum_{\substack{i \neq 1 \\ i \neq 3}} a_{\pi_1(i)\pi_1(i+1)}\Big) - a_{\pi_1(n)\pi_1(1)},$$

$$c_2 = -\Big(\sum_{\substack{i \neq 2 \\ i \neq 4}} a_{\pi_2(i)\pi_2(i+1)}\Big) - a_{\pi_2(n)\pi_2(1)},$$

$$c_3 = -\Big(\sum_{\substack{i \neq 3 \\ i \neq 5}} a_{\pi_3(i)\pi_3(i+1)}\Big) - a_{\pi_3(n)\pi_3(1)},$$

$$\vdots$$

$$c_{n-3} = -\Big(\sum_{\substack{i \neq n-3 \\ i \neq n-1}} a_{\pi_{n-3}(i)\pi_{n-3}(i+1)}\Big) - a_{\pi_{n-3}(n)\pi_{n-3}(1)},$$

$$c_{n-2} = -a_{21} - (-1)^{n-1} a_{n,n-1}$$

for $i \neq n-3$, $i \neq n-1$ $\pi_{n-1}(i+1) = \pi_{n-1}(i) + 1$. Obviously a_{ij} satisfies this system.

From (3) and antisymmetricity of the matrix A we have

$$\begin{aligned} c_1 &= \sum_{\substack{i \neq 1 \\ i \neq 3}} a_{\pi_1(i+1)\pi_1(i)} + a_{\pi_1(1)\pi_1(n)} \\ &= a_{23} + a_{54} + a_{65} + \cdots + a_{1,n} \\ &= V_3 - V_2 + V_4 - V_5 + V_5 - V_6 + \cdots + V_n - V_1 = V_3 - V_1 + V_4 - V_2, \end{aligned}$$

$$
\begin{aligned}
c_2 &= \sum_{\substack{i\neq 2\\ i\neq 4}} a_{\pi_2(i+1)\pi_2(i)} + a_{\pi_2(1)\pi_2(n)}\\
&= a_{21} + a_{34} + a_{65} + a_{76} + \cdots + a_{1,n}\\
&= V_1 - V_2 + V_4 - V_3 + V_5 - V_6 + \cdots + V_n - V_1 = V_4 - V_2 + V_5 - V_3,
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
c_{n-3} = &\sum_{\substack{i\neq n-3\\ i\neq n-1}} a_{\pi_{n-3}(i+1)\pi_{n-3}(i)} + a_{\pi_{n-3}(1)\pi_{n-3}(n)}\\
=&a_{21} + a_{32} + a_{43} + \cdots + a_{n-3,n-2} + a_{n-2,n-1} + a_{1,n}\\
=&V_1 - V_2 + V_2 - V_3 + V_3 - V_4 + \cdots + V_{n-2} - V_{n-3} + V_{n-1} - V_{n-2}\\
&+ V_n - V_1\\
=&V_{n-1} - V_{n-3} + V_n - V_{n-2},
\end{aligned}
$$

$$c_{n-2} = a_{12} + (-1)^{n-1}a_{n-1,n} = V_2 - V_1 + (-1)^{n-1}(V_n - V_{n-1}).$$

Note that the main matrix of the system (5) has the form

$$
C = \begin{bmatrix}
1 & 1 & 0 & \dots & 0 & 0 & 0\\
0 & 1 & 1 & \dots & 0 & 0 & 0\\
0 & 0 & 1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & 1 & 1 & 0\\
0 & 0 & 0 & \dots & 0 & 1 & 1\\
(-1)^2 & (-1)^3 & (-1)^4 & \dots & (-1)^{n-3} & (-1)^{n-2} & (-1)^{n-1}
\end{bmatrix},
$$

so, by Corollary 2, $\det C = (-1)^{n-1}(n-2)$. Obviously $\det C \neq 0$. Moreover, the matrix in which the first column was replaced by the column of free terms is of the form

$$
C_1 = \begin{bmatrix}
c_1 & 1 & 0 & \dots & 0 & 0 & 0\\
c_2 & 1 & 1 & \dots & 0 & 0 & 0\\
c_3 & 0 & 1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
c_{n-4} & 0 & 0 & \dots & 1 & 1 & 0\\
c_{n-3} & 0 & 0 & \dots & 0 & 1 & 1\\
c_{n-2} & (-1)^3 & (-1)^4 & \dots & (-1)^{n-3} & (-1)^{n-2} & (-1)^{n-1}
\end{bmatrix}.
$$

By Corollary 1

$$\begin{aligned}\det C_1 =&(-1)^{n-1}\big[c_{n-2}+\sum_{i=1}^{n-3}(-1)^{i+1}(n-2-i)c_i\big]\\ =&(-1)^{n-1}\big[V_2-V_1+(-1)^{n-1}(V_n-V_{n-1})\\ &+\sum_{i=1}^{n-3}(-1)^{i+1}(n-2-i)(V_{i+2}-V_i+V_{i+3}-V_{i+1})\big]\\ =&(-1)^{n-1}[V_2-V_1+(-1)^{n-1}(V_n-V_{n-1})\\ &+(-1)^2(n-3)(V_3-V_1+V_4-V_2)\\ &+(-1)^3(n-4)(V_4-V_2+V_5-V_3)\\ &+(-1)^4(n-5)(V_5-V_3+V_6-V_4)\\ &+\cdots+(-1)^{n-2}(V_{n-1}-V_{n-3}+V_n-V_{n-2})]\\ =&(-1)^{n-1}(n-2)(V_3-V_1).\end{aligned}$$

Thus, using Cramer's theorem ([Ko] p. 111) we obtain

$$x_{13}=\frac{\det C_1}{\det C}=\frac{(-1)^{n-1}(n-2)(V_3-V_1)}{(-1)^{n-1}(n-2)}=V_3-V_1.$$

By uniqueness of solutions of the system (5) we obtain

$$a_{13}=V_3-V_1.$$

Moreover, directly from the system (5), uniqueness of the solutions of this system and antisymmetricity of matrix A we have

$$a_{ij}=V_j-V_i \quad \text{for } |i-j|=2.$$

Now we show that

$$a_{ij}=V_j-V_i \quad \text{for } 2<|i-j|<n-1.$$

For this aim, consider the following cases.

1° $i,j,n\in 2\mathbb{N}$ or $i\in 2\mathbb{N}$, $j,n\in 2\mathbb{N}-1$. Choose the permutations

$$\begin{aligned}\tau_1 =&(i,j,j+2,j+4,\dots,n-2,n,n-1,n-3,\dots,j+1,j-1,j-2,\\ &\dots,i+2,i+1,i-1,i-3,\dots,3,1,2,4,6,\dots,i-2)\end{aligned}$$

with obvious conventions for $i=2$ and $j=n$. For the permutation τ_1, by assumptions and the proved part of the theorem we obtain

$$\begin{aligned}a_{ij} =&V_j-V_{j+2}+V_{j+2}-V_{j+4}+\cdots+V_{n-2}-V_n+V_n-V_{n-1}\\ &+V_{n-1}-V_{n-3}+\cdots+V_{j+1}-V_{j-1}+V_{j-1}-V_{j-2}+\cdots+V_{i+2}-V_{i+1}\\ &+V_{i+1}-V_{i-1}+V_{i-1}-V_{i-3}+\cdots+V_3-V_1+V_1-V_2+V_2-V_4\\ &+V_4-V_6+\cdots+V_{i-2}-V_i\\ =&V_j-V_i.\end{aligned}$$

In the further cases we arrange our calculations analogously as above, so we write only permutations and the resulting equations:

2° $i, j, n \in 2\mathbb{N}-1$ or $i \in 2\mathbb{N}-1,\ j, n \in 2\mathbb{N}$,

$$\begin{aligned}\tau_2 =&(i, j, j+2, j+4, \dots, n-2, n, n-1, n-3, \dots, j+1, j-1, j-2,\\ &\dots, i+2, i+1, i-1, i-3, \dots, 4, 2, 1, 3, 5, \dots, i-2),\end{aligned}$$

$$\begin{aligned}a_{ij} =&V_j - V_{j+2} + V_{j+2} - V_{j+4} + \cdots + V_{n-2} - V_n + V_n - V_{n-1}\\ &+ V_{n-1} - V_{n-3} + \cdots + V_{j+1} - V_{j-1} + V_{j-1} - V_{j-2} + \cdots + V_{i+2} - V_{i+1}\\ &+ V_{i+1} - V_{i-1} + V_{i-1} - V_{i-3} + \cdots + V_4 - V_2 + V_2 - V_1 + V_1 - V_3\\ &+ V_3 - V_5 + \cdots + V_{i-2} - V_i\\ =&V_j - V_i.\end{aligned}$$

3° $i, n \in 2\mathbb{N}, j \in 2\mathbb{N}-1$ or $i, j \in 2\mathbb{N},\ n \in 2\mathbb{N}-1$,

$$\begin{aligned}\tau_3 =&(i, j, j+2, j+4, \dots, n-3, n-1, n, n-2, \dots, j+1, j-1, j-2,\\ &\dots, i+2, i+1, i-1, i-3, \dots, 3, 1, 2, 4, 6, \dots, i-2),\end{aligned}$$

$$\begin{aligned}a_{ij} =&V_j - V_{j+2} + V_{j+2} - V_{j+4} + \cdots + V_{n-3} - V_{n-1} + V_{n-1} - V_n\\ &+ V_n - V_{n-2} + \cdots + V_{j+1} - V_{j-1} + V_{j-1} - V_{j-2} + \cdots + V_{i+2} - V_{i+1}\\ &+ V_{i+1} - V_{i-1} + V_{i-1} - V_{i-3} + \cdots + V_3 - V_1 + V_1 - V_2 + V_2 - V_4\\ &+ V_4 - V_6 + \cdots + V_{i-2} - V_i\\ =&V_j - V_i.\end{aligned}$$

4° $n \in 2\mathbb{N}, i, j \in 2\mathbb{N}-1$ or $j \in 2\mathbb{N},\ i, n \in 2\mathbb{N}-1$,

$$\begin{aligned}\tau_4 =&(i, j, j+2, j+4, \dots, n-3, n-1, n, n-2, \dots, j+1, j-1, j-2,\\ &\dots, i+2, i+1, i-1, i-3, \dots, 4, 2, 1, 3, 5, \dots, i-2),\end{aligned}$$

$$\begin{aligned}a_{ij} =&V_j - V_{j+2} + V_{j+2} - V_{j+4} + \cdots + V_{n-3} - V_{n-1} + V_{n-1} - V_n\\ &+ V_n - V_{n-2} + \cdots + V_{j+1} - V_{j-1} + V_{j-1} - V_{j-2} + \cdots + V_{i+2} - V_{i+1}\\ &+ V_{i+1} - V_{i-1} + V_{i-1} - V_{i-3} + \cdots + V_4 - V_2 + V_2 - V_1 + V_1 - V_3\\ &+ V_3 - V_5 + \cdots + V_{i-2} - V_i\\ =&V_j - V_i.\end{aligned}$$

This gives (i). The part (ii) immediately follows from (i). □

Remark 1. Is easy to see that in the proof of Theorem 1 we used only certain $\frac{1}{2}(n^2 - 3n + 2)$ permutations with all $n!$ possible; thus the assumption of Theorem 1 can be restricted to those permutations.

Remark 2. Since for every complex matrix A

$$A = \mathrm{Re}A + i\mathrm{Im}A$$

so, Theorem 1 holds for matrixes with complex coefficients.

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ZWIĄZKI MIĘDZY RÓWNANIAMI KIRCHHOFFA DLA PĘTLI OBWODÓW ELEKTRYCZNYCH

S t r e s z c z e n i e

Wykazujemy co następuje. Dla elektrycznego obwodu, załóżmy, że równania Kirchoffa są spełnione dla (dużych) pętli określonych przez wszystkie permutacje wszystkich punktów węzłowych. Wówczas równania Kirchoffa są spełnione dla każdej pętli. Dokładnie, dla permutacji, wynik ten podajemy jako pewne twierdzenie dla macierzy antysymetrycznej.

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SURFACE SEGREGATION IN BINARY ALLOY THIN FILMS IN VALENTA-SUKIENNICKI MODEL VS. THE EXPERIMENTAL DATA

Summary

The study of thin films has been very intense during the last decades. It is observed, both theoretically and experimentally that in thin alloy films the concentrations of atoms differ between inner and outer layers in broad range of temperature. This effect, called surface segregation is chosen and discussed in the article although we also mention the other classical surface phenomena: relaxation, adsorption and reconstruction.

We apply, one of many, the so called Valenta-Sukiennicki model [13] considering the pairwise interactions between atoms and originally used only for stoichiometric alloys. Based on our previous considerations [39, 40] we decide to use the extended version of this model, which describes binary alloys of arbitrary concentrations of atoms. In the study we shall consider $\mathrm{Cu}_x\mathrm{Ni}_{1-x}$ fcc alloys. We present the calculations concerning the segregation effect in 10 layers thin films of this alloy.

1. Introduction

It is observed, both theoretically and experimentally that in thin alloy films the concentrations of atoms differ between inner and outer layers in broad range of temperatures. This effect, called surface segregation is considered to be one of the classical surface phenomena. The other often considered phenomena are: relaxation – meaning the change of interlayers' distances between the outer layers, meaning usually their contraction, adsorption of other atoms to the surface, and reconstruction – meaning the change of ordering pattern in the surface layer(s) compared to the internal ones.

It is easy, within the Valenta-Sukiennicki model to include the effect of relaxation, but the contraction between layers distance is about a couple of percent for most atoms [21], therefore it seems that the effect is relatively weak and it might be neglected. The reconstruction effect cannot be fully observed within the model as we introduce only the long range order parametres. As for the adsorption effect, it probably also effects the segregation effect weakly so this also might be neglected in the considerations concerning surface segregation. The inclusion of the adsorption phenomenon in the Valenta-Sukiennicki model is still an awaiting challenge.

2. Description of the Valenta-Sukiennicki model

For the sake of simplicity let us consider a stoichiometric binary alloy AB_3 thin film of fcc lattice and (111) surface orientation. We divide our system into n atomic layers parallel to the surface. Each layer consists of N atoms and the number N is big enough to assume the surface of the layer infinite. The lattice of the alloy consists of two sublattices, α and β. The relative number of α sites (β sites) is equal to F_α (F_β). We have $F_\alpha + F_\beta = 1$. In a stoichiometric alloy we additionally have $F_A = F_\alpha$ and $F_B = F_\beta$, where F_A, F_B denote the concentration of A and B atoms in the sample, respectively.

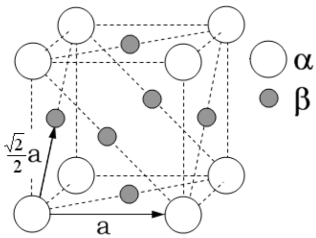

Fig. 1: AB_3 alloy with fcc lattice and (111) orientation of the surface

Denote by $p_X^\sigma(i)$ the probability that the site σ in i-th layer is occupied by atom X, $\sigma \in \{\alpha, \beta\}$, $X \in \{A, B\}$, $i = 1, 2, \ldots, n$. In a completely disordered state we have:

$$p_A^\alpha(i) = p_A^\beta(i), \quad p_B^\alpha(i) = p_B^\beta(i), \quad i = 1, 2, \ldots, n.$$

We always have:

$$p_A^\alpha(i) + p_B^\alpha(i) = 1 \quad \text{and} \quad p_A^\beta(i) + p_B^\beta(i) = 1. \tag{1}$$

The concentration of atoms A (atoms B) in i-th layer is given by:

$$z_A(i) = F_\alpha p_A^\alpha(i) + F_\beta p_A^\beta(i) \quad \left(z_B(i) = F_\alpha p_B^\alpha(i) + F_\beta p_B^\beta(i) \right).$$

Obviously,

$$\sum_{i=1}^{n} z_A(i) = nF_\alpha \quad \text{and} \quad \sum_{i=1}^{n} z_B(i) = nF_\beta. \tag{2}$$

and

$$z_A(i) = 1 - z_B(i), \quad i = 1, 2, \ldots, n.$$

We define a long-range order parameter $t(i)$ as

$$t_i = \frac{p_A^\alpha(i) - z_A(i)}{1 - F_\alpha}, \quad i = 1, 2, \ldots, n. \tag{3}$$

In a completely disordered state $t(i) = 0$, while in a completely ordered state $t(i) = 1$ for $i = 1, 2, \ldots, n$.

The free energy of the system is given by:

$$F = U - TS, \tag{4}$$

where U denotes the internal energy of the system, T absolute temperature, S entropy. The equilibrium of the system is attained when the free energy of the system is minimized.

Internal energy. In Bragg-Williams approximation the internal energy is given as an average over the energies corresponding to given long-range order.

Let R denote the smallest distance between atoms in the lattice. The number of pairs of nearest neighbours of atoms X and Y situated at the distance of R, and such that X is in i-th layer, Y is in j-th layer ($j = i, i \pm 1, i \pm 2$) equals

$$\begin{aligned} \langle XY \rangle^{ij} = \frac{1}{2} N \Big(F_\alpha \left(p_X^\alpha(i) r_{\alpha\alpha} p_Y^\alpha(j) + p_X^\alpha(i) r_{\alpha\beta} p_Y^\beta(j) \right) \\ + F_\beta \left(p_X^\beta(i) r_{\beta\alpha} p_Y^\alpha(j) + p_X^\beta(i) r_{\beta\beta} p_Y^\beta(j) \right) \Big), \end{aligned} \tag{5}$$

for $X, Y \in \{A, B\}$, $i = 1, 2, \ldots, n$, where $r_{\sigma\tau}$ denotes coefficients characteristic for the lattice of the alloy, namely the number of neighbours of an atom occupying a σ site in i-th layer which are situated in τ site in $i + j$-th layer [38, 60]. The approach including only the first neighbours is called the first neighbours approach. We have shown that in case of binary fcc alloys including only the first neighbours gives a good enough approximation of the internal energy of the system [38, 42].

Denoting the interaction energies between atoms X and Y situated at the smallest distance R as $-v_{XY}$, the internal energy of the film consisiting of n layers is given by

$$U = - \sum_{\substack{i, j = 1, 2, \ldots, n \\ X, Y \in \{A, B\}}} \langle XY \rangle^{ij} v_{XY}. \tag{6}$$

Entropy. Entropy of binary alloy film is calculated according to the formula

$$S_\mathrm{B} = k_\mathrm{B} \ln g, \tag{7}$$

where g denotes the number of configurations of a given state for atoms concetrations in layers and long-range order parameters [13, 38].

3. Non-stoichiometric alloys

Non-stoichiometric alloys are more difficult to describe than stoichiometric ones although, for practical reasons, they are more interesting than the stoichiometric ones [48]. It has been observed that tiny deviations from stoichiometry in the bulk composition of the NiPt-L1(0) ordered alloy have a great impact on the atomic configuration of the (111) surface [46].

Non-stoichiometric alloys cannot be described by Valenta-Sukiennicki model easily, as we no longer have equality between the number of sites and the number of the corresponding lattice, which makes formula (5) invalid. In order to overcome this problem we assume that some kind of order exists in the alloy [24], although we do not know it. This alows us to calculate the mean approximate value of the coefficients $r_{\sigma\tau}^{R_s}$ in (5) as in [41].

4. Problems with experimental data

As our aim is to obtain numerical results which could be compared with the experimental data a couple of remarks should be made before.

Firstly, many models, including the Valenta-Sukiennicki model assume that the system stabilizes at the lowest level of free energy. Some experimental data show, however, that in case of alloys of gold there are problems in obtaining the state of minimal energy despite annealing [45].

Substantial amount of work has been devoted to Cu-Ni alloys of different concentrations due to the catalytic properties of this alloy. There is strong consensus that Cu has tendency to segregate to the surface in broad range of temperatures and at all concentrations of Cu in the bulk. It is, however, very interesting to know what is the shape of the profile of the segregation – it might be oscillatory or monotonic, say, exponential [59]. Some authors claim that the segregation might occur only on the first one or two layers of the alloys [53]. It is also observed that the clean equilibrated surfaces of the Cu-rich polycrystalline alloys consist almost entirely of Cu atoms [11]. There are controversies and uncertainties concerning the reliability of the experimental data themselves [52, 54]. One of the reasons is difficulty in obtaining reliable experimental data, the other is applying many simplifying assumptions in calculations which in effect lead to unreliable results. For example, some researchers even assume the monotonic segregation profile in order to interpret the results of their experiments [62].

Another problem is stoichiometry and its connection with surface segregation. It has been observed that even tiny deviations from stoichiometry in the bulk composition of the NiPt-L1(0) ordered alloy have a great impact on the atomic configuration of the (111) surface [4, 46].

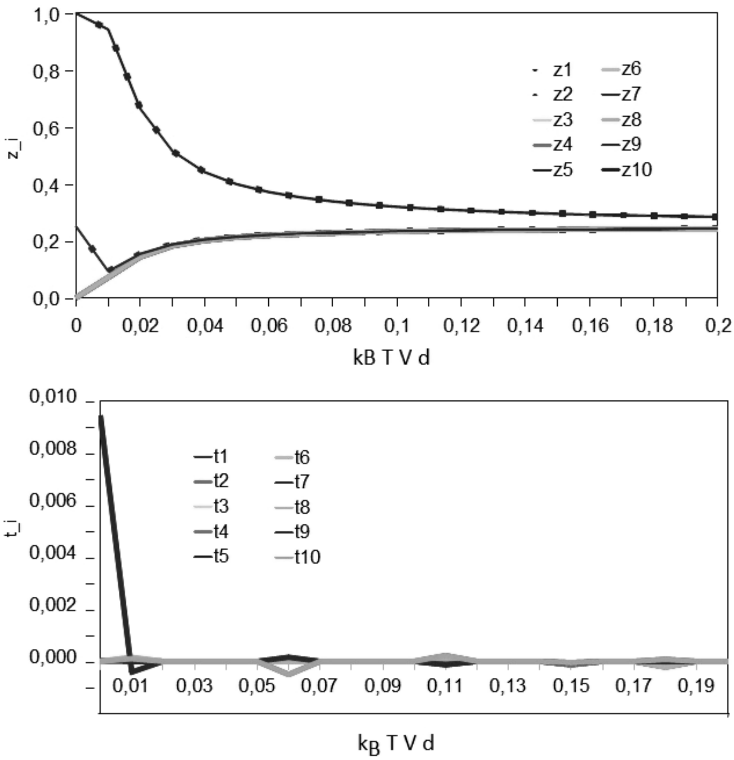

Fig. 2: Concentrations of Cu atoms in layers 1–10 in a 10 layers sample of $CuNi_3$ alloy of orientation (111) in dependence of temperature (d stands for an unknown positive constant) – upper graph. The lower graph shows the long range order parameters values for this sample.

5. Theoretical and experimental data for Cu_xNi_{1-x} alloys

Cu_xNi_{1-x} alloys are, for some reasons under special interest of researchers. One of the reasons is the similar Cu and Ni atom sizes, which also makes them interesting from the point of view of Valenta-Sukiennicki model.

It is known that Cu_xNi_{1-x} alloys have fcc built. In order to make the computations we have to substitute the values of interactions between Cu-Cu, Ni-Ni and Cu-Ni pairs of atoms to the formula (6). We assume that the values of interactions between nearest neighbours atoms are

$$v_{\mathrm{Cu-Cu}} = 0.42\mathrm{eV}, \quad v_{\mathrm{Ni-Ni}} = 0.476\mathrm{eV}, \tag{8}$$

and

$$v_{\mathrm{Cu-Ni}} = \sqrt{v_{\mathrm{Cu-Cu}} \cdot v_{\mathrm{Ni-Ni}}} \approx 0.447\mathrm{eV}, \tag{9}$$

where the latest have been calculated according to one of the mixing rules commonly used in molecular dynamics numerical programms [1]. It seems that the previously

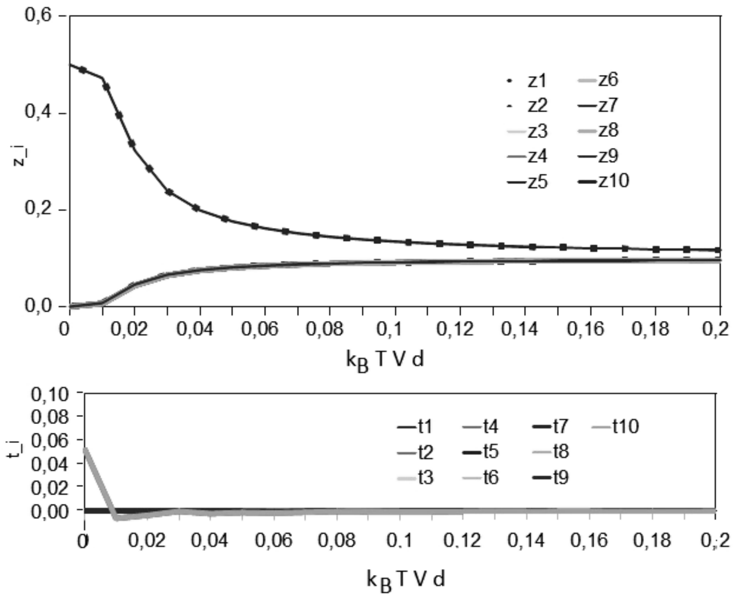

Fig. 3: Concentrations of Cu atoms in layers 1–10 in a 10 layers sample of $Cu_{0.1}Ni_{0.9}$ alloy of orientation (111) in dependence of temperature (d stands for an unknown positive constant) – upper graph. The lower graph shows the long range order parameters values for this sample.

used V parametre [38, 41] equal to $V = v_{AB} - (v_{AA} + v_{AB})/2$ will not be suitable this time as it is negative. Let us define the new parametre V

$$V := \frac{v_{Cu-Cu} - v_{Ni-Ni}}{v_{Cu-Ni}} \tag{10}$$

expressing the relative difference between Cu-Cu and Ni-Ni interactions. In case of Cu-Ni alloys we obtain $V = 0.125$. As both in the case of stoichiometric and non-stoichiometric case we define α sites as those occupied by Cu atoms and β sites as those occupied by Ni atoms and so the relative number of sites is equal to the relative number of corresponding atoms we can assume the classical long-range order parametres (3).

The results of the calculations of concentrations of Cu atoms in layers in a sample containing 10 layers and also the long range order parametres are given in the figures 2 and 3. The concentration of Cu atoms in the first sample is 0.25, while in the second it is 0.1. The horizontal axis in each case represents increasing values of temperature, namely $k_B \cdot T \cdot V \cdot d$, where k_B denotes Boltzmann constant, T the absolute temperature, V is defined by (10) and d denotes a positive constant.

Firstly, it can be observed that the segregation effect can be seen mostly in the first two external layers. In both samples Cu segregates to the surface and this is observed in the whole range of temperatures. Practically, in case of $CuNi_3$ alloy at low temperatures the external surface consist of Cu atoms exclusively, which confirms the results of [11]. With the rise of temperature the segregation effect in both samples becomes smaller. As for the long range order parametres in both samples they are always very close to zero.

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SEGREGACJA POWIERZCHNIOWA W CIENKICH WARSTWACH STOPÓW DWUSKŁADNIKOWYCH W MODELU VALENTY-SUKIENNICKIEGO A DANE EKSPERYMENTALNE

S t r e s z c z e n i e

Studium cienkich warstw było bardzo intensywne w ostatnich dzisięcioleciach. Stwierdzono, zarówno teoretycznie jak i doświadczalnie, że w cienkich próbkach koncentracje atomów między powierzchniową a wewnętrzną warstwą jednoatomową różnią się między sobą w szerokim zakresie temperatur. Efekt ten, zwany segregacją powierzchniową jest wyróżniony i dyskutowany w artykule, ale wspominamy też o innych klasycznych zjawiskach powierzchniowych: relaksacji, adsorpcji i rekonstrukcji.

Spośród różnych modeli wybieramy model Valenty-Sukiennickiego [13], skoncentrowany na rozważaniu par zespołów atomów i oryginalnie odnoszący się jedynie do stopów stoichiometrycznych. W oparciu o nasze poprzednie badania [39, 40] decydujemy się, by użyć rozszerzonej wersji modelu, która opisuje stopy dwuskładnikowe o dowolnej koncentracji atomów. W szczególności rozważamy stopy $\mathrm{Cu}_x\mathrm{Ni}_{1-x}$. Przedstawiamy wyliczenia dotyczące efektu segregacji dla 10 warstw jednoatomowych takiego stopu.

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In memory of
Professor Roman Stanisław Ingarden

Krzysztof Pomorski and Przemysław Prokopow

NUMERICAL SOLUTIONS OF TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS FOR VARIOUS SUPERCONDUCTING STRUCTURES

Summary

The use of relaxation method in solving static and time dependent Ginzburg-Landau (GL) equations is described by a few instructive examples. The main focus of interest is the solution of GL equations applied to unconventional Josephson junction made by putting non-superconducting strip on the top of superconducting strip for s, d and p-wave superconductor. Certain solutions of Ginzburg-Landau equation are obtained in the case of placement of Josephson junction in time dependent temperature gradient, time dependent or time independent external magnetic field or when given junction is polarized by dependent or time independent superconducting current. Also the solutions of GL equations for simple geometries as cylinder, rectangular, torus are presented. Certain perspectives of extension of used GL model into non-equilibrium Green function picture are drawn.

1. Motivation

Studying superconducting structures is important both for fundamental and applied science. There are many description levels of superconducting or superfluid phase as by use of phenomenological or microscopic models. Because of technical complication one usually starts from phenomenological level and then moves to more microscopic and fundamental description. Therefore in this paper we will mainly use Ginzburg-Landau model. Because of application perspectives as in THz electronics, superconducting qubit [1, 4], particular attention is paid to unconventional

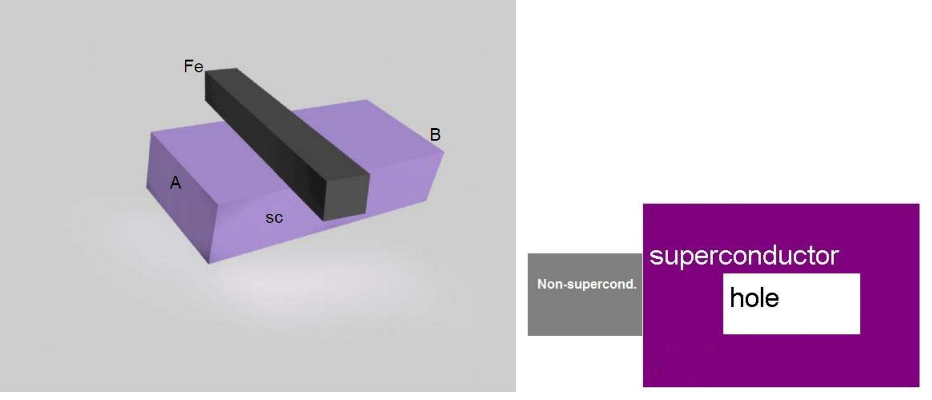

Fig. 1: Scheme of unconventional Josephson junction made by putting non-superconducting strip on the top of superconductor strip (left) and SQUID made from such structure (right).

Josephson junction made by putting non-superconducting, ferromagnetic, antiferromagnetic or ferroelectric strip on the top of superconducting strip as depicted in the Fig. 1. When the ferromagnetic or ferroelectric material is placed on the top of unconventional Josephson junction then such system is regarded as field induced Josephson junction.

In unconventional Josephson junction the Cooper pairs from superconductor diffuse into non-superconducting element and therefore the superconducting order parameter inside superconductor is decreased. Also unpaired electrons from non-superconducting element diffuse into superconductor what brings further reduction of superconducting order parameter. If ferromagnetic material with non-zero magnetization is placed on the top of superconductor then the magnetic field breaks the Cooper pairs and lowers more the superconducting order parameter. Also electric field can be used to modulate the superconducting properties of superconductor as presented in [11]. Therefore we can replace ferromagnet with ferroelectric material and should obtain similar properties.

Having certain geometry of non-superconducting element placed on the top of thin superconductor it is possible to obtain the Josephson junction. This is because after placement of the non-superconducting element on the top of superconductor, one Cooper pair reservoir (superconductor) in terms of superconducting order parameter will be effectively separated into 2 superconducting reservoirs as described in [5, 4]. The interaction between reservoirs is the origin of the Josephson effect. Such approach is quite similar to the approach presented in [12]. With such defined Josephson junction, we can build superconducting devices as the Josephson junction array, SQUID (as depicted in Fig. 12 and 16), current limiter and other elements. Many of these devices can be made in analogy to weak-link Josephson junctions presented by K. Likharev [9] and others.

2. Computational model

There are various methods, which can be used to solve the Ginzburg-Landau equations as the finite difference method, spectral methods, annealing methods (as by [18]) and many others. Because of simplicity and numerical stability even for the case of complex set of nonlinear equations the relaxation method is used. Deriving Ginzburg-Landau equations we look for the case of functional derivative of free energy functional F set to the zero with respect to the physical fields upon which it depends.

Then we obtain the following equations:

$$\frac{\delta}{\delta\psi}F[\psi,\vec{A},\vec{M},\vec{E}]=0,\ \frac{\delta}{\delta\vec{A}}F[\psi,\vec{A},\vec{M},\vec{E}]=0, \tag{1}$$

$$\frac{\delta}{\delta\vec{M}}F[\psi,\vec{A},\vec{M},\vec{E}]=0,\ \frac{\delta}{\delta\vec{E}}F[\psi,\vec{A},\vec{M},\vec{E}]=0, \tag{2}$$

where $\vec{A}$ is vector potential, $\vec{M}$ is the magnetization, ψ is the superconducting order parameter(s) and $\vec{E}$ is the electric field.

To approach the solutions given as the configuration of the ($|\psi|$,$\vec{M}$,$\vec{A}$,$\vec{E}$) fields we need to make the initial guess of physical fields configuration and order parameter in the given space using certain physical intuition. The initial guess should be not too far from the solution. Having the initial guess we perform the calculation of fields change on the given lattice with each iteration step virtual time δt according to the scheme:

$$\frac{\delta}{\delta\psi}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_1\frac{\delta\psi}{\delta t},\ \frac{\delta}{\delta\vec{A}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_2\frac{\delta\vec{A}}{\delta t}, \tag{3}$$

$$\frac{\delta}{\delta\vec{M}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_3\frac{\delta\vec{M}}{\delta t},\ \frac{\delta}{\delta\vec{E}}F[\psi,\vec{A},\vec{M},\vec{E}]=-\eta_4\frac{\delta\vec{E}}{\delta t} \tag{4}$$

Here $\eta_1,\eta_2,\eta_3,\eta_4$ are phenomenological constants. The δt cannot have too big value since it might bring the numerical instability in the simulation. If δt has very small value the arriving to the solution is long. One of the signature of approaching the solution is the minimization of free energy functional. Then one can observe the characteristic plateua in the free energy as the function of iteration (virtual time) is observed.

It should be underlined that the relaxation method applied here is in the framework of the Ginzburg-Landau formalism. Much more detailed knowledge about superconductor properties and in the whole range of superconductor properties as from 0 to critical temperature is given by propagator formalism as: Usadel, Eilenberger, Gorkov or non-equilibrium Keldysh formalism. In those formalisms there occurs self-consistency equations for the superconducting order parameter. In order to avoid the difficulties it is always good to plug the superconducting order parameter from the Ginzburg-Landau equation and then start to use propagators as it is depicted in Fig. 2. Many applications of relaxation method as for many types of gauge fields are pointed by Adler [17].

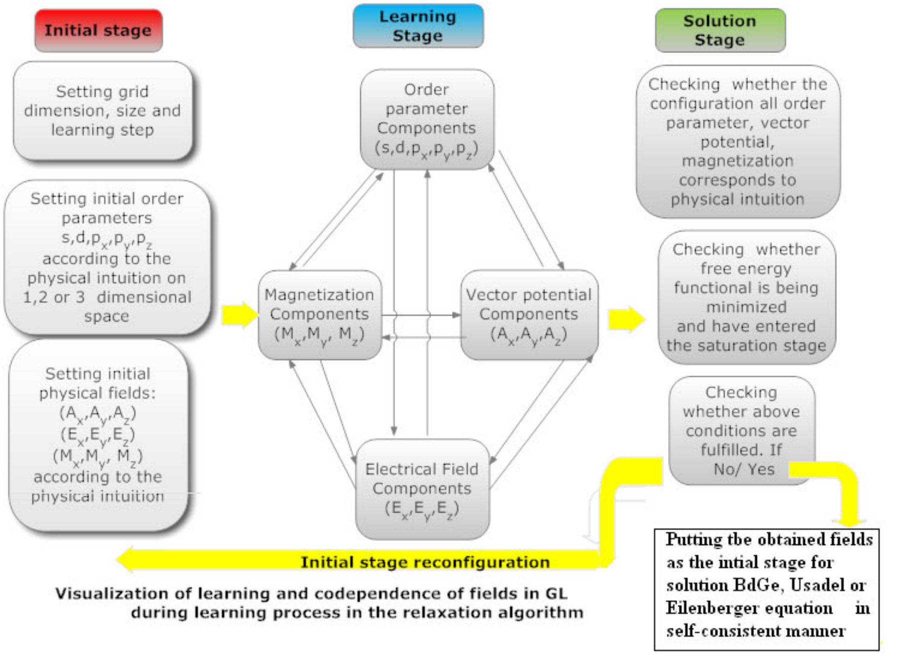

Fig. 2: Schematic illustration of generalized relaxation method.

3. S-wave superconducting structure in time-dependent temperature gradient

For s-wave superconducting structure in time-dependent temperature gradient we can write Ginzburg-Landau equation of the following form

$$\gamma\frac{d}{dt}\psi(x,y,t) = \alpha(x,y,t)\psi(x,y,t) + \beta\psi(x,y,t)|\psi(x,y,t)|^2 + \frac{1}{2m}((\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x(x,y))^2$$
$$+(\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y(x,y)) + (\frac{\hbar}{i}\frac{d}{dz} - \frac{2e}{c}A_z(x,y))^2)\psi(x,y,t)$$

with

$$\psi(x,y,t) = |\psi(x,y,t)|exp(i\phi(x,y,z,t)) \tag{5}$$

$$\phi = \frac{2ei}{\hbar c}(\int_{z_a}^{z_b}(A_z(x,y,t)dz + \int_{x_a}^{x_b} A_x(x,y,t)dx + \int_{y_a}^{y_b} A_y(x,y,t)dy)) \tag{6}$$

where $\alpha(x,y,t)$ incorporates the existing gradient of temperatures and temperature field across the sample and the total electric current flowing via the sample is the sum of superconducting current and normal component of the form

$$j_z = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dz}\psi - h.c. - \frac{e^*}{c}A_z|\psi|^2) + \frac{dA_z}{dt}\sigma_n \tag{7}$$

$$j_y = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dy}\psi - h.c. - \frac{e^*}{c}A_y|\psi|^2) + \frac{dA_y}{dt}\sigma_n \tag{8}$$

$$j_x = \frac{j\hbar e^*}{2m^*}(\psi^\dagger \frac{d}{dx}\psi - h.c. - \frac{e^*}{c}A_x|\psi|^2) + \frac{dA_x}{dt}\sigma_n \tag{9}$$

The normal current component are proportional to the derivative of vector potential with time and brings the dissipation that heats the studied system locally as from Drude model. It shall be underlined that the direct control of $A_z(x,y,t)$ vector potential in laboratory conditions is not possible, but we can control the integral

$$\int\int j_z(x,y,t)dxdy = I(t) \tag{10}$$

as by setting current source as to be of the certain function of time $I(t)$. The condition of total current flowing via the system is incorporated inside the relaxation algorithm. The second controllable integral is given by external magnetic field as can be fixed to be at the point $B(x_1, y_1, t)$, which imposes conditions on A_x and A_y. It gives another constrain $\oint \vec{A} \circ \vec{dr} = 2\pi n$, where n is the integer number.

The additionary boundary conditions comes from normal to the surface superconductor-vacuum derivatives given as

$$(\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x(x,y,t))\psi(x,y,t) = 0, (\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y(x,y,t))\psi(x,y,t) = 0 \tag{11}$$

and for superconductor-normal metal interface we have

$$\frac{1}{b}\psi(x,y,t) = (\frac{\hbar}{i}\frac{d}{dy} - \frac{2e}{c}A_y)\psi(x,y,t) \tag{12}$$

The constant b can be determined from microscopic model as given by [10]. Let us consider the SQUID as depicted in Fig. 1. In first numerical computations we set A to be zero what means that there is no electric current flow and magnetic field in the system. We incorporate the temperature gradient into GL equations by keeping γ coefficient to be constant and by setting $\alpha(x,y,t) = \alpha_0 + a(x-x_0)(t-t_0)$. We set $t_0 = x_0 = 0$.

Then we obtain the following α fields as depicted in the Fig. 3, 4 and 5. The situation when there is no temperature gradient in the sample we name as zero temperature gradient. Then temperature of sample is linearly time dependent so $\alpha(x,y,t) = \alpha_0 + a(t-t_0)$. If $\alpha(x,y,t) = \alpha_0 + a(x-x_0)(t-t_0)$ we call such situation to be first temperature gradient. In case of $\alpha(x,y,t) = \alpha_0 + 2a(x-x_0)(t-t_0)$ we name it second temperature gradient. Having given α field in dependence on time and space we can trace the time dependence of superconducting order parameter distribution in the structure. This is depicted in Fig. 6, 7, 8, 9, 10, 11. We can

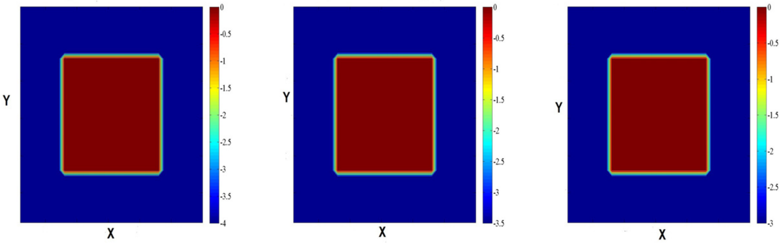

Fig. 3: $\alpha(x, y, t)$ for zero temperature gradient in times $t_0 < t_1 < t_2$

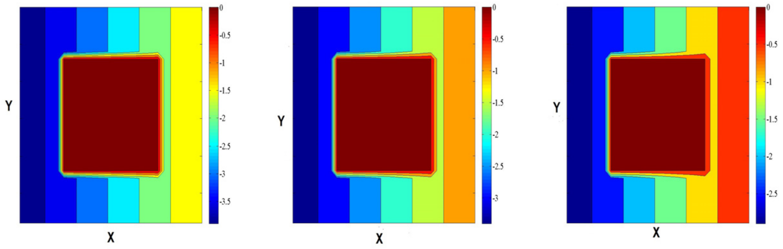

Fig. 4: $\alpha(x, y, t)$ for first temperature gradient with times $t_0 < t_1 < t_2$

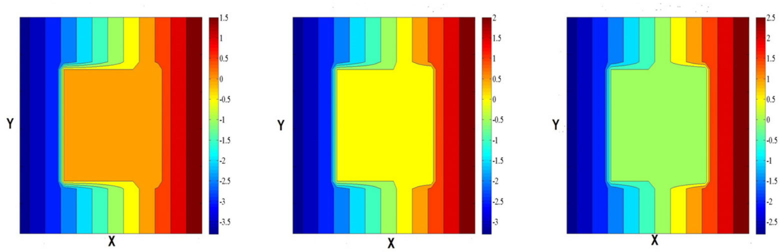

Fig. 5: $\alpha(x, y, t)$ for second temperature gradient with times $t_0 < t_1 < t_2$

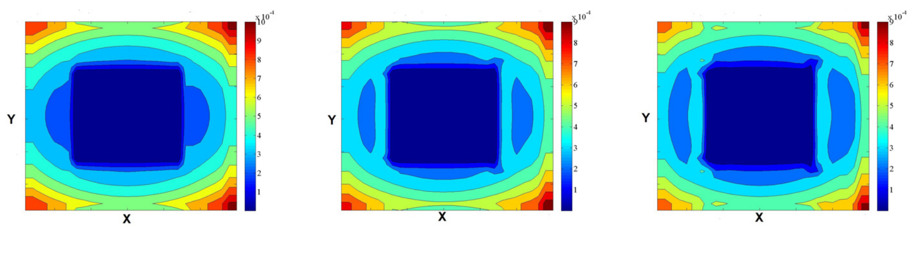

Fig. 6: Order parameter for zero temperature gradient with times $t_0 < t_1 < t_2$, $\gamma = -200$.

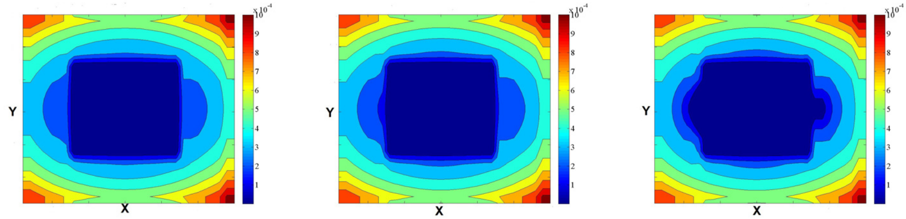

Fig. 7: Order parameter for zero temperature gradient with times $t_0 < t_1 < t_2$, $\gamma = 500$.

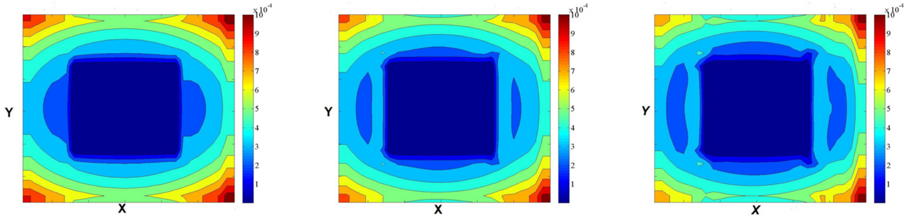

Fig. 8: Order parameter for first temperature gradient with times $t_0 < t_1 < t_2$, $\gamma = -200$.

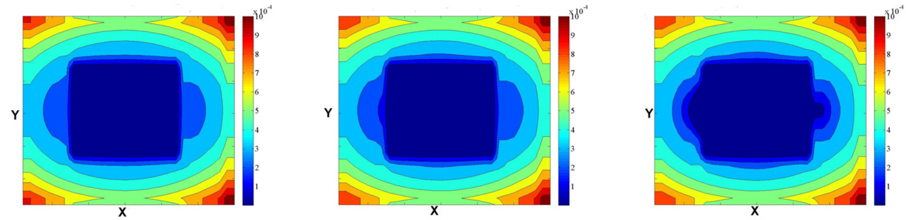

Fig. 9: Order parameter for first temperature gradient with times $t_0 < t_1 < t_2$,$\gamma = 500$

trace the effect of γ coefficient in TDGL equation on superconducting distribution in different time steps.

3.1. Toroidal uJJ SQUID in time dependent temperature gradient

Let us consider the system as depicted in Fig. 12, which is the unconventional torus SQUID made by putting on the top of s-wave torus superconductor parametrized by (r, R) the part of non-superconducting torus (toroidal strip) with hole inside so the system can create 1 unconventional Josephson junction. In analogy to the previous case of unconventional Josephson junction or unconventional Josephson junction SQUID, the presence of nonsuperconducting strip will cause Cooper pairs to diffuse from superconductor into non-superconductor. Also the electrons from the normal toroidal strip will diffuse into superconductor. As the result the super-

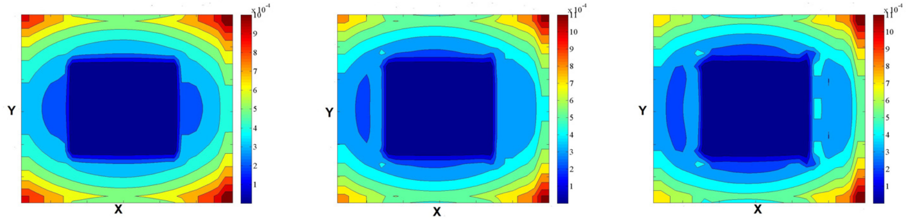

Fig. 10: Order parameter for second temperature gradient with times $t_0 < t_1 < t_2$,$\gamma = -200$.

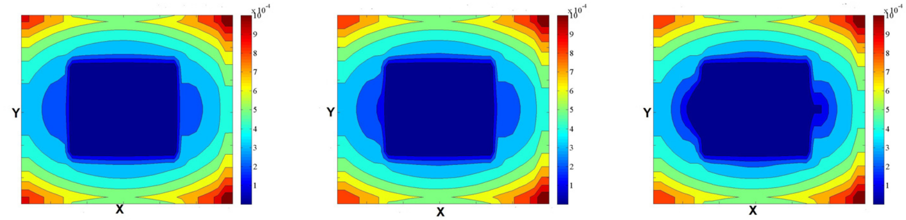

Fig. 11: Order parameter for second temperature gradient with times $t_0 < t_1 < t_2$,$\gamma = 500$.

conducting order parameter under toroidal non-superconducting strip will be decreased. This is described by the solutions of GL equation for the limiting case when $R >> r$. Then the whole system can be treated as superconducting cylinder with non-superconducting cylinder with hole placed on the superconducting cylinder with certain periodic boundary conditions. Superconducting order parameter in cylindrical unconventional SQUID is given by solution of GL with time dependent or space uniform time dependent linear temperature gradient as depicted in Fig. 14. The presence of the time-dependent temperature gradient allows to introduce additional barrier in the superconductor order parameter. The dependence of temperature with zero temperature gradient on superconducting order parameter distribution is depicted in Fig. 13. The validity of solutions of GL equations for d-wave superconductor was check basing on Fig. 15.

$$\gamma\frac{d}{dt}\psi(r,\phi,\theta,t) = \alpha(r,\phi,\theta,t)\psi(r,\phi,\theta,t) + \beta\psi(r,\phi,\theta,t)|\psi(r,\phi,\theta,t)|^2 - \frac{\hbar^2}{2m}(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})\psi(r,\phi,\theta,t). \tag{13}$$

Time dependent uniform gradient of temperature occurs in ϕ direction and is linearly time and temperature dependent so $\alpha(r,\psi,\theta,t) = (\alpha_0 + a_1(t-t_0)(\psi-\psi_0))$ is non-zero inside superconductor and 0 outside. The physical situation becomes even more interesting if the non-superconducting strip is magnetized or when it is ferroelectric material. This situation will be the subject of investigations in future research.

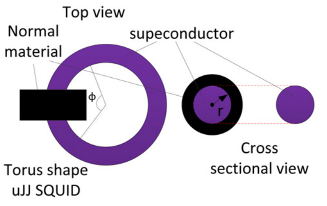

Fig. 12: Torus shape s-wave SQUID in unconventional Josephson junction architecture.

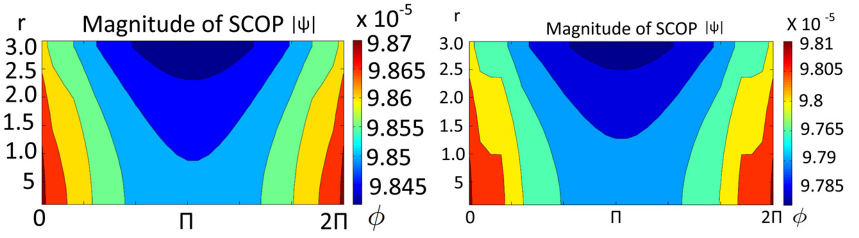

Fig. 13: Distribution of s-wave superconducting order parameter inside torus uJJ SQUID for different temperatures T_1(left) $> T_2$(right). No temperature gradient is present. α is linearly time dependent.

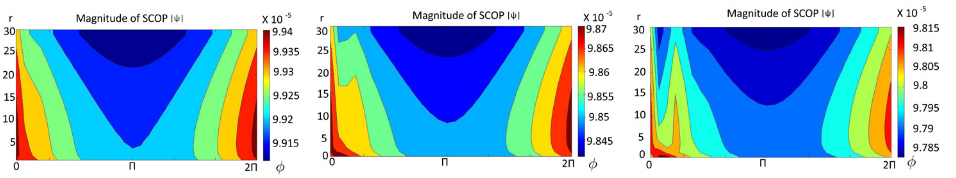

Fig. 14: Distribution of superconducting order parameter in superconductor region parameterized by angle ϕ and radius r are obtained by use of the relaxation algorithm. Gradient of temperatures occurs in the direction of ϕ.

Because of various symmetries this system can be quite easy studied with more advanced formalism as Bogoliubov-de Gennes or equilibrium and non-equilibrium Usadel formalism as it is used in [9–11].

3.2. Case of d-wave Ginzburg-Landau equation

One of the study the d-wave GL equation for superconducting vortices was performed by J. Alvarez [13]. Vortex in superconductor is the example of defect in the superconducting order parameter. Another type of defects in superconductor is the Josephson junction that can be induced by proximity effect and will be the subject of our interest as in the SQUID configuration depicted in the Fig. 15, 16. It should be noticed that the d-wave order parameter in the neighborhood of superconductivity

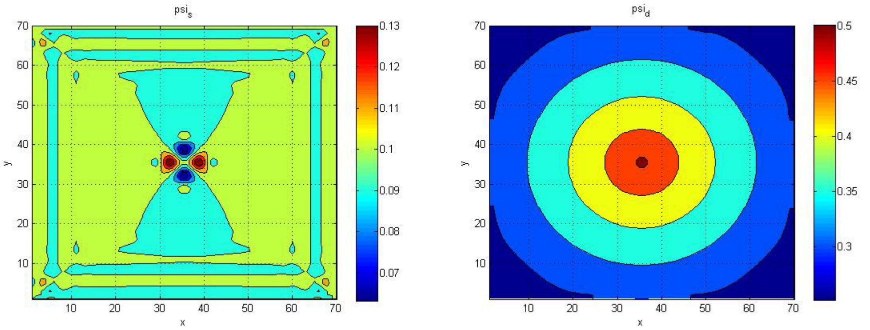

Fig. 15: Distribution of s and d-wave superconducting order parameter components in d-wave superconducting square in ab-plane.

disturbing factor diminish while s-wave order parameter is initially enhanced and later lowered.

3.2.1. Testing relaxation algorithm

Highly correlated materials as d-wave superconductors are strongly anisotropic what is reflected for example in the electron effective massive that is different in ab-plane than in c axe by factor of 100. Properties of superconductor state in ab-plane can be described by Ginzburg-Landau equations for d-wave superconductor also known as GL $x^2 - y^2$ equations and are the set of 2 coupled non-linear partial differential equation given as

$$(-\gamma_d \nabla^2 + \alpha_d)\psi_d + \gamma_v(\nabla_x^2 - \nabla_y^2)\psi_s + 2\beta_2|\psi_d|^2\psi_d + \beta_3|\psi_s|^2\psi_d + 2\beta_4\psi_s^2\psi_d^* = 0 \tag{14}$$

$$(-\gamma_s \nabla^2 + \alpha_s)\psi_s + \gamma_v(\nabla_x^2 - \nabla_y^2)\psi_d + 2\beta_1|\psi_s|^2\psi_s + \beta_3|\psi_d|^2\psi_s + 2\beta_4\psi_d^2\psi_s^* = 0 \tag{15}$$

where $\gamma_\rho \equiv \hbar^2/2m_\rho$, and $\rho = d, s, v$. The parameters $\alpha_d, \alpha_s, \gamma_v, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_s$, γ_d can be derived from the extended Hubbard model. The electric current density is given by

$$\begin{aligned}\mathbf{J} = &\frac{e\hbar}{im_d}\{\psi_d^*\nabla\psi_d - \text{c.c.}\} + \frac{e\hbar}{im_s}\{\psi_s^*\nabla\psi_s - \text{c.c.}\}\\ &- \hat{x}\frac{e\hbar}{im_v}\{\psi_s^*\nabla_x\psi_d - \psi_d\nabla_x\psi_s^* - \text{c.c.}\} + \hat{y}\frac{e\hbar}{im_v}\{\psi_s^*\nabla_y\psi_d - \psi_d\nabla_y\psi_s^* - \text{c.c.}\}.\end{aligned}$$

In d-wave superconductors the superconducting order parameter $\Delta(x, y, z)$ is given as

$$\Delta(x, y, z) = \psi_s(x, y, z) + cos(2\phi)\psi_d(x, y, z) \tag{16}$$

and boundary conditiones are expressed as

$$(17)\quad \frac{i}{\kappa}\underline{n}(\Pi\psi_s + \frac{1}{2}(\Pi_x - \Pi_y)\psi_d) = -V_s(\psi_s), \frac{i}{\kappa}\underline{n}(\Pi\psi_d + (\Pi_x - \Pi_y)\psi_s) = -V_d(\psi_d).$$

Here V_s and V_d depends on the material constants that can be determined from the extended Hubbard model as from [13] and $\underline{n}$ is the unit vector normal to the surface of superconductor-non-superconductor or superconductor-vacuum interface. The Δ is the global superconducting order parameter, which is the complex scalar field and is obtained from as the sum of 2 complex scalar superconducting fields ψ_s and ψ_d. Here $\kappa = \lambda/\xi$ is the ratio of magnetic field penetration field λ to superconducting coherence length ξ. We also define canonical momentum operator in various directions as

$$(18)\qquad \Pi = \frac{\hbar}{i}(\Pi_{ab} + \eta\Pi_c) - \frac{2e}{\hbar c}(A_{ab} + \eta A_c),$$

where canonical momentum in ab-plane is

$$(19)\qquad \Pi_{ab} = \underline{i}\nabla_x + \underline{j}\nabla_y,$$

vector potential in ab-plane $A_{ab} = \underline{i}A_x + \underline{j}A_y$. In analogy we can define $\Pi_c = \nabla_z$, η-parameter accounting electron effective mass anisotropy,

$$(20)\qquad \Pi_x = \frac{\hbar}{i}\nabla_x - \frac{2e}{\hbar c}A_x, \Pi_y = \frac{\hbar}{i}\nabla_y - \frac{2e}{\hbar c}A_y$$

Using only GL $(x^2 - y^2)$ equations we confirm the results of [16] in static case as presented in Fig. 14. The coeffcients describing GL functional was taken from [15]. After solving d-wave GL equations in static case, the time-dependent GL $(x^2 - y^2)$ was obtained in the case of superconducting 2 dimensional rectangular body in ab plane, which is in the linearly dependent temperature gradient as it is depicted for different time steps in Fig. 17. In comparison with Fig. 15. we note that s component of superconducting order parameter has changed its distribution and become asymmetric along temperature gradient. Therefore it will have non-trivial impact on transport properties of such structure.

4. Conclusions and future work

We have conducted the computations of superconducting order parameter with use of relaxation method. We have applied it to solve Time Dependent Ginzburg Landau equation in s-wave superconductor or equations in d-wave superconductor. This algorithm, which was originally presented by [7, 17] and others, turns out to be simple in implementation, fast and numerically stable even in the case of occurrence of more than 2 coupling scalar fields. The conducted computations provide preliminary knowledge necessary in determination of the transport properties of the superconducting structures especially in the case of zero electric current limit. In the conducted work the particular interest should be paid to the unconventional Josephson junction (uJJ) made by putting non-superconducting strip on the top of superconductor strip. It is the continuation of previous work given by [5] and [14].

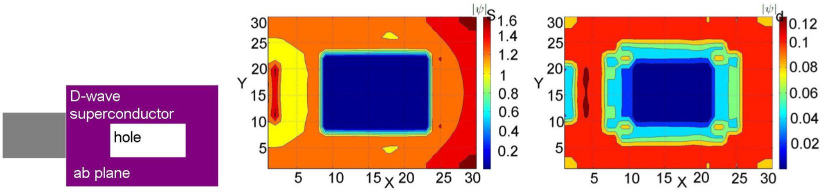

Fig. 16: Rectangular d-wave superconductor SQUID scheme (left) made in unconventional Josephson junction architecture. S-wave (center) and d-wave (right side) superconducting orders parameter distribution inside superconductor region are obtained by use of the relaxation algorithm. No magnetic field is present in the system.

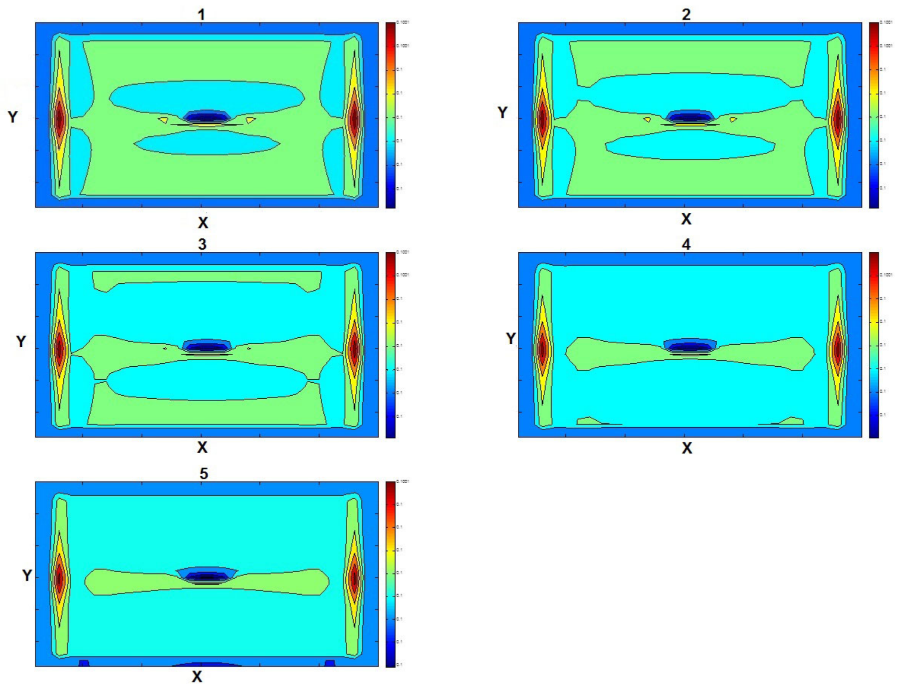

Fig. 17: The s-wave superconducting order parameter distribution in the rectangular shape d-wave superconductor placed in vacuum (2-dimensional GL $x^2 - y^2$) subjected to the linearly changing in time and space temperature gradient for different times $t = t, t+\Delta t, t+2\Delta t, t+3\Delta t, t+4\Delta t$ (from top to bottom). The vertical and horizontal axes correspond to y and x coordinates.The linear time-dependent temperature gradient occurs in y direction.

Such structure is easy in fabrication process and hence can be used in the superconducting circuits of high integration. Studying the properties of uJJ is the subject of experiments described by [2–4, 6]. It is important to underline that Ginzburg Landau formalism is the most phenomenological tool to study the superconducting structure properties. Its application to various structures is the preliminary step before the application of more complex formalism as the Bogoliubov-de Gennes, Usadel or Keldysh techniques. It is not big suprise that the temperature gradient diminish superconducting order parameter. However in certain cases as depicted in the left side of right plot among Fig. 14 that can generate "superconductor barrier" for quasiparticles and partly block their transport via Josephson junction. Also determination of γ coefficients occurring in TDGL from microscopic theory is not easy task since it deals with non-equilibrium processes, but is the future research target.

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NUMERYCZNE ROZWIĄZANIA RÓWNAŃ GINZBURGA-LANDAUA ZALEŻNYCH OD CZASU DLA RÓŻNYCH STRUKTUR NADPRZEWODZACYCH

S t r e s z c z e n i e

W niniejszej pracy prezentujemy algorytm relaksacyjny rozwiązywania równań Ginzburga-Landaua zależnych od czasu dla różnych geometrii nadprzewodzacych struktur w zależnych od czasu polach temperaturowych oraz zależnym od czasu prądzie elektrycznym przepływającym przez nadprzewodnik.

Rapporteurs – Referees

CONTENU DU VOLUME LXI, no. 3