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ŁÓDZKIE TOWARZYSTWO NAUKOWE
PL-90-505 Łódź, ul. M. Cuie-Skłodowskiej 11
tel. (42) 665 54 59, fax (42) 665 54 64
sprzedaż wydawnictw: tel. (42) 665 54 48
e-mail: ltn@ltn.lodz.pl

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Skład komputerowy: Zofia Fijarczyk

Druk i oprawa: *AnnGraf* s.c.
92-637 Łódź, ul. Rataja 54
tel. +48 601 24 10 83

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TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

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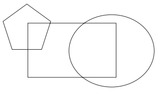

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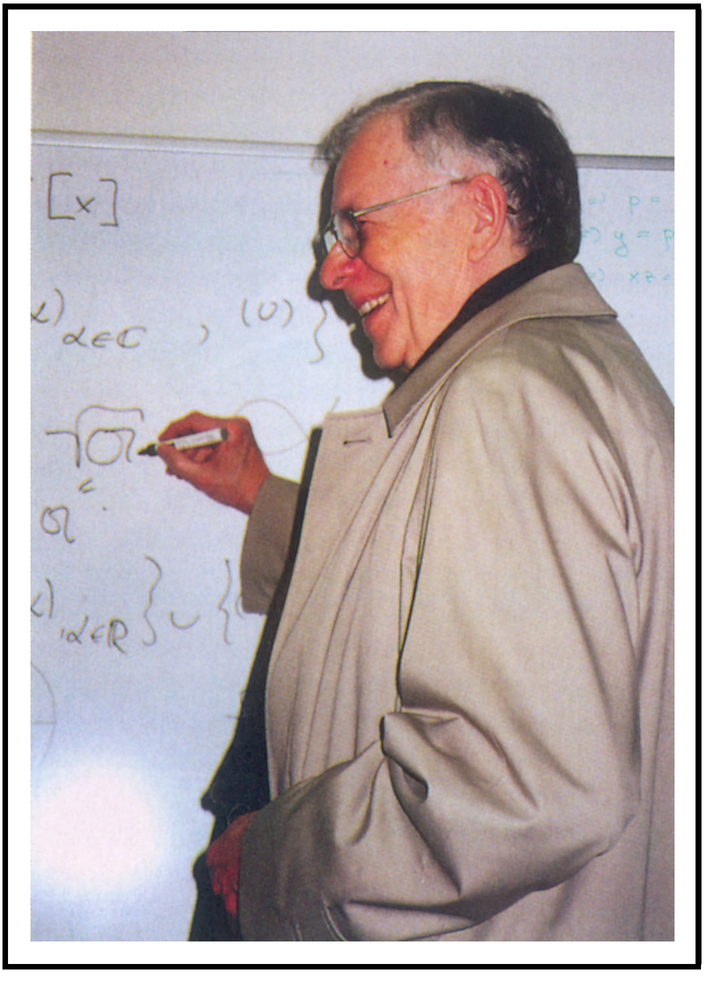
References

[1]

Affiliation/Address

TABLE DES MATIÈRES

Professor
Hans Grauert
*** 8.2.1930 † 4.9.2011**



Professor Hans Grauert (Göttingen), Member of our Editorial Board, will be sadly missed for his extraordinarily inspiring intellect and individuality. He is worldwide known for his works on several complex variables, complex manifolds, applications of sheaf theory, algebraic geometry, complex spaces, and some physical concepts. His passing away is a considerable loss.

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B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

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Recherches sur les déformations no. 2

pp. 7–13

In memory of
Professor Roman Stanisław Ingarden

Ralitza K. Kovacheva

MONTEL'S TYPE RESULTS AND ZERO DISTRIBUTION OF SEQUENCES OF RATIONAL FUNCTIONS

Summary

A new generalization of the classical result by Montel about normal families is provided. As a application, a theorem of Picard's type for rational functions is derived.

Given a domain D in the complex plane $\mathbb{C}$, denote by $\mathcal{A}(D)$ the class of holomorphic (analytic and single valued) functions in the domain D; $\mathcal{A}(D)$ is endowed with the uniform (max-)norm $\|\cdots\|_K$ on compact subsets K.

By the classical *theorem of Montel* (called also the *Second Fundamental Theorem*, [1]), if $\mathcal{F} \subset \mathcal{A}(D)$ is a family of functions with the following characteristics: there are two distinct complex numbers a and b in $\mathbb{C}$ such that each function $f \in \mathcal{F}$ omit in D the value of a and takes the value of b at no more than N points, then the family $\mathcal{F}$ is *normal*, i.e., from each sequence $\subset \mathcal{F}$ one can extract a subsequence which converges locally uniformly inside D to infinity or to a finite function (in the max-norm on compact subsets of D). Hence, if under the conditions of Montel's theorem, a sequence converges uniformly to a function f on some regular subset M of D, then f admits an holomorphic continuation from M into D.

A natural question arises as to what happens if a family of holomorphic functions omits merely one finite value in B. This question appears to make sense for sequences of rational functions, armed with additional approximating properties. To make things clear, we recall a classical result by S. N. Bernstein [2]: *f – a continuous and real valued function on $I := [-1, 1]$ and $\mathcal{E}$ – a Jukowski ellipse with foci at ± 1. Assume that all polynomials P_n with real coefficients of best uniform approximation of f on I are nowhere zero in $\mathcal{E}$. Then the sequence $\{P_n\}$ forms a normal family in $\mathcal{E}$.*

We remark that under the given conditions the function f admits an holomorphic continuation from I into $\mathcal{E}$.

As analogous results of Montel's type, we quote the known result by Baker-Graves-Morris [3] about normality of sequences of Padé approximants, as well as the results by Blatt-Saff-Simkani/ Kovacheva about polynomials/rational functions with fixed number of free poles of best uniform approximation on regular sets (see [4], resp. [5]).

Before presenting a generalization of Montel's theorem results, we introduce the notations $\mathbb{N}_0 := \mathbb{N} \bigcup 0$ and, for a pair (n, m), $n, m \in \mathbb{N}_0$, the class $\mathcal{R}_{n,m} := \{r, r = p/q, q \not\equiv 0\}$, p, q − polynomaisl of degrees n, m respectively ($p \in \Pi_n$, $q \in \Pi_m$). Further, given a function g and a set K, denote by $\nu(g, K)$ the number of zeros of g in K.

Theorem 1. [6]. *Given a domain D and a regular continuum $S \subset D$, suppose that the sequence $\{f_n\}$, $f_n \in \mathcal{R}_{n,n} \bigcap \mathcal{A}(D)$, $n = 1, 2, \dots$ converges uniformly on ∂S to a function f, $f \not\equiv 0$ on S in such a way that*

$$\limsup_{n\to\infty} \|f_n - f\|_{\partial S}^{1/n} < 1. \tag{1}$$

Assume that

$$\nu(f_n, K) = o(n) \text{ as } n \to \infty \tag{2}$$

on compact subsets K of D. Then the sequence $\{f_n\}$ is normal in the domain D; herewith, the function f admits a holomorphic continuation into D.

The advantage of this theorem lies in its applications to the subject of holomorphic continuation. We summarize the main result as follows: given a regular compact set S, a function $f \in C(S)$ and a sequence of rational functions $\{r_n\}$ converging on S geometrically to f, assume that $\{r_n\}$ are holomorphic and fulfill condition (2) in a larger domain D that contains the set S. Under the named conditions, $\{f_n\}$ forms a normal family in D; herewith, the function f is analytically continuable from S into D. Now, in what follows, we listen cases to which such a statement applies:

– best rational uniform approximants $r_{n,n} = r_{n,n}(f, E)$ provided $E^o \not\equiv \emptyset$ and $f \in \mathcal{A}(E) \bigcap C(E)$. (Given a compact set K in $\mathbb{C}$, a function $g \in C(K)$ and a fixed pair (n, m), $n, m \in \mathbb{N}_0$, let $r_{n,m}$ be defined by:

$$\|g - r_{n,m}\|_K := \inf_{r\in\mathcal{R}_{n,m}} \|f - r\|_K.$$

The function $r_{n,m} = r_{n,m}(f, K)$ is called a best uniform approximant of g on K in the class $\mathcal{R}_{n,m}$.) (see [6]);

– best L_p− rational approximants $r_{n,n}(f, \Gamma)$ of $f \in L_p(\Gamma), p > 0$ on a closed analytic curve Γ, $D \supset \Gamma$ (see [7]);

– best rational uniform approximants $r_{n,n}(f, \Delta)$ of a real valued and continuous function on a finite segment $\Delta \subset \mathbb{R}$ (see [7]).

We now pose the question whether an analogue of Theorem 1 is valid with respect to meromorphic functions. To be exact, let us formulate the question: provided the conditions of Theorem 1 are fulfilled with $f_n, n = 1, 2, \cdots$ being meromorphic in D, do the sequence $\{f_n\}$ possess the normality in D (in he spherical metric?)

Before presenting the results, we introduce needed notations and definitions. Given a set A in $\mathbb{C}$, we set $\mathcal{M}(A)$ for the class of the meromorphic in A functions; as usual, poles will be counted with their multiplicities. We mean by $\mathcal{M}_m(A)$, $m \in \mathbb{N}$, functions in $\mathcal{M}(A)$ with no more that m poles in A. Given a function $g \in \mathcal{M}(A)$, we engage the notation $\mu(g, A)$ for the number of poles of g in A. Obviously, $\mu(g, A) := \nu(1/g, A)$.

For our further purposes, we need the term of m_1-*measure* (cf. [8]). Given a set e in $\mathbb{C}$, we introduce

$$m_1(e) := \inf \left\{ \sum_{\nu} |U_\nu| \right\}$$

where the infimum is taken over all coverings $\{U_\nu\}$ of e by disks U_ν and $|U_\nu|$ is the radius of the disk U_ν.

Let D be a domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in D, is said to converge to a function φ m_1-*almost uniformly inside* D if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and the sequence $\{\varphi_n\}$ converges uniformly to φ on K_ε. The sequence $\{\varphi_n\}$ converges m_1- *almost geometrically to the a function* φ *on* K, if for every ε there exists a set $K_\varepsilon \subset K$ such that $m_1(K_\varepsilon) < \varepsilon$ and

$$\limsup \|\varphi_n - \varphi\|^{1/n}_{K \setminus K_\varepsilon} < 1.$$

The next result provides an answer to the posed question.

Theorem 2. [9], [10]. *Given a regular continuum $S \subset \mathbb{C}$, suppose that $\{f_n\}$, $f_n \in \mathcal{R}_n$, $n = 1, 2, \cdots$ is a sequence of rational functions which converges uniformly on ∂S to a function f with $f \not\equiv 0$ on some regular subset of ∂S at a speed of a geometric progression, i.e.,*

$$\limsup_{n\to\infty} \|f_n - f\|^{1/n}_{\partial S} < 1. \tag{1'}$$

Assume that there is a domain $\mathcal{U} \supset \mathcal{S}$ and a number m, $m \in \mathbb{N}_0$ such that each $f_n \in \mathcal{M}_m(\mathcal{U})$ and, in addition,

$$\nu(f_n, K) = o(n) \text{ as } n \to \infty \tag{2'}$$

on compact subsets K of $\mathcal{U}$. Then the sequence $\{f_n\}$ converges m_1 – almost uniformly inside $\mathcal{U}$; herewith f admits a continuation into $\mathcal{U}$ as a m – meromorphic function.

We draw reader's attention to the fact that for the case when $m = 0$, Theorem 2 coincides with Theorem 1. Theorem 2 establishes a m – meromorphic continuation into $\mathcal{U}$ but not as a function having exactly m poles in D. Consider, for instance, the sequence

$$f_n(z) = \frac{z - \frac{1}{2^n}}{z - \frac{1}{3^n}}$$

in the unit disk $\mathcal{D}$. Notice that $\{f_n\} \in \mathcal{M}_1(\mathcal{D})$. Satisfying the conditions of Theorem 2, it converges to m_1- almost geometrically inside $\mathcal{D}$ to $f \equiv 1$. Hence, in contrast to Theorem 1, Theorem 2 does not imply a normality of the sequence $\{f_n\}$, even in the case when each f_n has exactly m poles in $\mathcal{U}$.

Remark. Both theorems, the former and the latter, hold also for "dense enough" sequences. Following the same line of reasoning, as in the proofs of Theorem 2/1, one can show the validity of

Corollary 1. [9]. *Given a regular continuum S, suppose that the sequence $\{f_{n_k}\}$, $f_{n_k} \in \mathcal{R}_{n_k}$, $n_k < n_{k+1}$, $k = 1, 2, \dots$ with*

$$\limsup_{n_k \to \infty} \frac{n_{k+1}}{n_k} < \infty \tag{3}$$

converges uniformly on ∂S as $n_k \to \infty$, to a function f, $f \not\equiv 0$ on S such that an analogue of (1) *holds, i.e.*

$$\limsup_{k \to \infty} \|f_{n_k} - f\|_{\partial S}^{1/n} < 1. \tag{4}$$

Assume further that there is a domain $\mathcal{U} \supset S$ and a fixed number m (a zero or an integer) such that $f_{n_k} \in \mathcal{M}_m(\mathcal{U}), k = 1, 2, \cdots$ and

$$\nu(n_k, K) = o(n_k),\ n_k \to \infty \tag{5}$$

on each compact subset K of $\mathcal{U}$. Then the statements of Theorem 2/*Theorem* 1 *hold with $\{f_n\}$ replaced by $\{f_{n_k}\}$ and $f \in \mathcal{M}_m(\mathcal{U})$ (resp. $f \in \mathcal{A}(\mathcal{U})$).*

The application of Theorem 2 and of the preceding corollary are of importance in establishing theorems of Picard-type for sequences of rational functions. Before, we introduce the term of an α – point. Given a set M, a function $g \in \mathcal{M}(M)$ and a number $\alpha \in \mathbb{C}$, we introduce the notation $\nu_\alpha(g, M)$ as the number of all of $\alpha-$ points of g in M; e.g. $\nu_\alpha(g, M) := \nu(g - \alpha, M)$. For $\alpha = \infty$, we set $\nu_\alpha := \mu(g, M)$.

The main advantage of Theorem 2 is

Theorem 3. [10]. *Let D be a domain in $\mathbb{C}$, and $\{r_{n,n}\}$, $r_{n,n} \in \mathcal{R}_{n,n}$ that converges m_1 – almost geometrically to a function f on compact subsets of D. Let z_0 be a boundary point of D that is not a point of regularity for f. Let a and b be two distinct values in $\overline{\mathbb{C}}$. Then the following distribution result holds for the α-values and the β-values in every neighborhood U of z_0 :*

$$\textit{if} \quad \nu_\alpha(r_n, U) = o(n) \quad \textit{as} \quad n \to \infty, \quad \textit{then} \quad \limsup \nu_\beta(r_n, U) = \infty.$$

From Theorem 3, after involving all arguments of its proof, we obtain

Corollary 2. *Let D be a domain in $\mathbb{C}$, and $\{r_{n,n}\}$, $r_{n,n} \in \mathcal{R}_{n,n}$ that converges m_1 – almost geometrically to a function f on compact subsets of D. Let z_0 be a boundary point of D that is not a point of regularity for f. Let a and b be two distinct values in $\overline{\mathbb{C}}$. Then the following distribution result holds for the α-values and the β-values in every neighborhood U of z_0 :*

$$\text{either} \qquad \limsup \tfrac{\nu_\alpha(r_n,U)}{n} > 0$$

$$\text{or} \qquad \limsup \nu_\beta(r_n, U) = \infty \quad \text{as} \quad n \to \infty.$$

In the context of *normal families*, we see that no nonregular point of ∂D is a normal point for the sequence $\{r_n\}$.

Corollary 3. *Under the conditions of Theorem* 3*, assume that z_0 is a nonregular boundary point of D. Then for any neighborhood U of z_0 and for all $a \in \overline{\mathbb{C}}$, with at most one exception,*

$$\limsup \nu_a(r_{n,m_n}, U) = \infty.$$

Recalling the classical theorem of Picard concerning the behavior of a holomorphic function in a neighborhood of an isolated essential singularity, we can summarize Corollary 2 by saying the the sequence $\{r_{n,m_n}\}$ has an "asymptotic essential singularity" at each nonregular boundary point on ∂D, provided $m_n = o(n)$.

Using now Corollary 1, one get an information about the denseness of the zeros, resp. poles of the approximating rational functions around the nonpolar singularities.

Corollary 4. *Under the conditions of Theorem* 3*, suppose that for the infinite sequence $\Lambda := \{n_k\}$ there holds*

$$\limsup \nu_\beta(r_n, U) < \infty.$$

Then

$$\text{(6)} \qquad \text{either} \quad \limsup_{n_k \to \infty} \frac{\nu_\alpha(r_{n_k}, U)}{n_k} > 0 \quad \text{or} \quad \limsup \frac{n_{k+1}}{n_k} = \infty.$$

This observation is important for the case when $m_n = o(n)$. If $\{r_{n,m_n}\}$ converges geometrically to f m_1- almost uniformly inside D and $z_0 \in \partial D$ is a nonregular point of f, and if

$$\limsup_{n_k \in \Lambda} \nu_a(r_{n_k, m_{n_k}}, U) < \infty$$

for some neighborhood U and some infinite sequence $\Lambda = \{n_k\}$, then Λ is necessarily *rare* in the sense of (6).

Examples. Let

$$f(z) := (z+1)\log(z+1) + \sum_{i=1}^{\infty} \frac{A_i}{z - \alpha_i}, \ \limsup |A_i|^{1/i} = 1/2, |\alpha_i| > 2$$

with $E :=$ an open disk $D_0(r)$, $r < 1$. Let $m_n = o(n)$ as $n \to \infty$; set

$$r_{n,m_n}(z) := z + \sum_{k=2}^{n-m_n} \frac{(-z)^k}{k(k-1)} + \sum_{i=1}^{m_n} \frac{A_i}{z-\alpha_i}.$$

It is easy to verify that the sequence $\{r_{n,m_n}\}$ converges to f uniformly inside the unit disk at a speed of a geometric progression. At the point $z = -1$ lies a nonregular singularity. Applying Theorem 3 and Corollary 4, we see that $z = -1$ attracts "almost all" zeros of r_{n,m_n} as $n \to \infty$. If there is some sequence $\Lambda := \{n_k\}$ which is running away then it should be necessarily rear in the sense of (6).

Theorems 1–3 and Corollaries do not provide an information about an asymptotical behavior of the zeros of the approximating sequences. Theorem 4 below gives an information about for some classes of functions. Before stating it, we introduce the concept of a *radius of meromorphy*. Let E be a regular compact set in $\mathbb{C}$. We denote by $G_E(z,\infty)$ its Green function with (logarithmic) pole at infinity. Given a number $\rho > 1$, we set $E_\rho := \{z, G_E(z,\infty) < \log\rho.\}$ Let $f \in \mathcal{A}(E)$. We define *the radius of meromorphy* $\rho(f)$ as follows:

$$\rho(f) := \sup\{\rho, f \in \mathcal{M}(E_\rho).\}$$

A sequence r_{n,m_n} is called *maximal convergent to* f if it converges m_1- almost geometrically inside $E_{\rho(f)}$ and the speed of convergence on each compact subset K equals $\exp \|G_E(z,\infty)\|_K/\rho(f)$. In [11], a result of Jentzsch-Szegö type was proved:

Theorem 4. *Let E be a regular compact set and $f \in \mathcal{A}(E)$. Assume that*

$$m_n \leq n, m_n \leq m_{n+1} \leq m_n + 1 \quad \textit{and} \quad m_n = 0(n/\log n).$$

Let $\{r_{n,m_n}\}$, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$ be maximal convergent to f inside $E_{\rho(f)}$. If $\rho(f) < \infty$ and if there exists a singularity of multivalued character of f on $\partial E_{\rho(f)}$, then the normalized zero counting measures ν_n of the numerators of r_{n,m_n} converge weakly to the equilibrium distribution of $\overline{E}_{\rho(f)}$, at least for a subsequence $\Lambda \subset \mathbb{N}$ as $n \to \infty$ with $n \in \Lambda$.

Examples to which Theorems 3/4 apply are Pade approximants and best uniform rational approximants of a continuous real valued function on a finite segment on the real axes.

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Institut of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. Bonchev Str. 8
BG-1113-Sofia, Bulgaria

Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 24, 2011

WYNIKI TYPU MONTELA I ROZKŁAD ZER DLA CIĄGÓW FUNKCJI WYMIERNYCH

S t r e s z c z e n i e

Uzyskano nowe uogólnienie klasycznego wyniku Montela dla rodzin normalnych. Jako zastosowanie wyprowadzone jest twierdzenie typu Picarda dla rodzin normalnych.

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B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

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Recherches sur les déformations no. 3

pp. 15–22

In memory of
Professor Roman Stanisław Ingarden

Yuri Zelinskiĭ

SOME QUESTIONS OF INTEGRAL COMPLEX GEOMETRY

Summary

A subject, which is treated in this report, combines in one bundle some questions of complex analysis, geometry and probability theory. Our purpose is to give review of the row of the open problems and known results. First investigations of geometric probabilities were started from well known Buffoons needle problem and related Bertrand paradoxes. The paper introduces original conjectures and results of the present author.

1. Probabilities paradoxes

Let a needle (real line) intersect the ball $B \subset R^2$. What is probability that this needle intersects the ball $B_1 \subset B$? (Fig. 1)

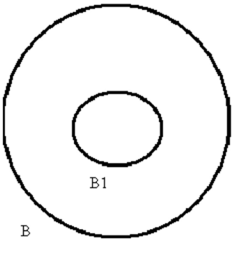

Fig. 1.

Let $R = r_B = 2r_{B1}$. In this case the problem is equivalent to the following. Find the probability, that a chord, chosen at random, be longer than the side of an inscribed equilateral triangle.

Let a needle be considered as a real line, and then the problem reduces to finding some invariant measure of the set relative to movement (L. Santalo, G. Matheron, R. Ambarcumian and other [1–3]). Then the sought probability is found as attitude of the measures. It is well known that for convex sets the invariant measure is the length of perimeter and

$$p = \frac{2\pi r}{2\pi R} = \frac{r}{R}.$$

For any connected set $E \subset \mathbb{R}^2$ the invariant measure is the length of perimeter of the convex hull of E.

Generalizing this construction, the approach relies upon a consideration of a family of linear submanifolds of an Euclidean space, which crosses the given set. In the real case this question is well studied [2]. The case of complex and more general space, as it is noted in [2], has not got the sufficient development yet. In the complex case the following two classes will be a natural generalization of the class of convex sets.

Definition 1. A set $E \subset \mathbb{C}^n$ is called *linearly convex* if for every point $z \in \mathbb{C}^n \setminus E$ there exists a hyperplane l such that $z \in l \subset \mathbb{C}^n \setminus E$.

Example 1. All convex domains and compacts are linearly convex.

Example 2. The Cartesian product $E = E_1 \times E_2 \times \ldots \times E_n$ of arbitrary flat sets $E_i \subset \mathbb{C}$ is a linearly convex set, in particular, torus $T = S^1 \times S^1 \times \ldots \times S^1$.

Definition 2. A set $E \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if for every complex line γ sets $\gamma \cap E$ and $\gamma \setminus \gamma \cap E$ are connected.

For the first the concept of linear convexity in $\mathbb{C}^2$ was introduced in 1935 in the paper of Behnke and Peschl [4] and was used widely by Martino [5] and Aizenberg [6] from the sixties of last century.

Linearly convex sets are very useful in complex analysis and in the questions of the integral geometry and tomography. On the base of these sets in complex analysis there is built linearly convex complex analysis, similar to real convex analysis. More results of linearly convex analysis can be viewed in monographs [7–9] and in the review article [10].

In spite of abundance of results, many unsolved problems remained concerning topological characteristics of these sets, a part of them is possible to find in [7, 10, 11]. One of the problems, put in [11], is solved in the work [12].

It seems interesting for the author to formulate the following open problem of the sphere.

Problem 1 (sphere problem). Is there a linearly convex compact in $\mathbb{C}^2$, for which all cohomology groups coincide with the corresponding cohomology group of the two-dimensional sphere S^2?

Some of the problems of this theme are connected with the known Ulam problem from the Scottish book [13].

2. Ulam problem

Let M_n be an n-dimensional manifold and every section of M^n by hyperplanes L be homeomorphic to the $(n-1)$-dimensional sphere S^{n-1}. Is it true that M^n is n-dimensional sphere?

In the real case this problem is solved by Kosiński in 1962 [15]. The repetition of this result was obtained by Montejano in 1990 [16]. In the complex a case similar result was obtained by Zelinskiĭ in 1993 [7].

Other problems of this group are to find: an estimation of the properties of a set if we know the properties of its intersections with the families of some sets:

1) with the planes of a fixed dimension:
   a) in the real case (Auman, Kosiński, Shchepin [15, 17, 18]);
   b) in the complex case (Zelinskiĭ [7]);

2) with a set of vertices of an arbitrary rectangle (Besicowitch, Danzer, Zamfirescu, Tkachuk [19–22]).

The latter problem is known in literature as Mizel problem.

3. Mizel problem (characterization of a circle)

Let $C \subset R^2$ be a convex Jordan curve with the following property:

For every rectangle *abcd* if any three vertices are on C, will the fourth vertex be also on C? Is it true that C is a circle? This problem is solved by Besicovitch and Danzer independently [19, 20].

In 1989 Zamfirescu [21] proved the similar result for a Jordan curve (not convex à prioriy) and for a rectangle with an infinitesimal relation between sides:

$$\left|\frac{ac}{ab}\right| \leq \varepsilon > 0.$$

In 2006 my PhD student Tkachuk [22] obtained the most general result in this area for compact $C \subset R^2$, where $R^2 \setminus C$ is not connected. Similar open problems in the plane and in n-dimensional case appear in connection with the Mizel problem. Further we shall bring the related known results for linearly convex sets.

Theorem 1. *For convexity of a domain (compact) it is necessary and sufficient that all sections of this domain (compact) k-plane for fixed k, $1 \leq k \leq n-1$, are acyclic.*

Theorem 2. *A $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ is linearly convex.*

Definition. By *conjugate set* to the a $E \subset \mathbb{C}^n$ we call the set

$$E^* = \{w | \langle w, z\rangle \neq 1 \quad \text{for all} \quad z \in E\},$$

where $w = (w_1, w_2, \dots w_n)$, $z = (z_1, z_2, \dots z_n)$ are points in $\mathbb{C}^n$ and $\langle w, z\rangle = w_1 z_1 + w_2 z_2 + \dots + w_n z_n$.

Theorem 3. *Let $E \subset \mathbb{C}^n$ be a linearly convex set such that $\mathbb{C}^n \setminus E$ is not connected. Then E there is a cylinder formed by parallel to each other hyperplanes and base is the set on line γ; moreover the set component $\gamma \setminus Q$ corresponds one-to-one to the set component of $\mathbb{C}^n \setminus E$, but E^* is on $\overset{0}{\gamma} \setminus Q$ on the line passing through the initial coordinates, where $\overset{0}{\gamma} = \gamma \cup (\infty)$.*

Theorem 4. *In $\mathbb{C}^n$ every linearly convex domain with connected smooth boundary is homeomorphe to a ball.*

Theorem 5. *Let $D \subset \mathbb{C}^n$ be a linearly convex domain with smooth not connected boundary. Then D is a cylinder formed by parallel to each other hyperplanes and base is the flat domain Q with a smooth boundary, lying on a complex line l (the additional subspace to form the cylinder). The number of components ∂Q coincides with number the of components ∂D.*

The conjugated compact D^* consists of an union of flat 2-dimensional compacts homeomorphe to circles and resting on line, getting through the initial coordinates. The boundary of this compact is smooth in all point, with the exclusion of, possibly, of that with the initial coordinates, on which can be crossed by some compact.

If we have a set $E \subset \mathbb{C}^n$, $\theta = (0, 0, \dots, 0) \in E$, and a point $z^0 \in \mathbb{C}^n \setminus E$, we denote by $\Gamma(z^0)$ a set of points $w \in \mathbb{C}^n$, such that the hyperplane $\{z | \langle w, z\rangle = 1\}$ passes through z^0 and does not cross E.

4. The conjecture of Aizenberg

A bounded linearly convex domain D, $\theta \in D$, is $\mathbb{C}$-convex iff the sets $\Gamma(z)$ are connected for all $z \in \partial D$.

Theorem 6. *Let $K \subset \mathbb{C}^n$, $\theta \in K$, be such compact that all sections K by tangent hyperplanes are connected. Then each connected component of the set, K^* is a $\mathbb{C}$-convex domain.*

Our next results solve Aizenberg's conjecture.

Theorem 7. [7]. *A bounded domain $D \subset \mathbb{C}^n$, is $\mathbb{C}$-convex iff the sets $\Gamma(z)$ are nonempty and connected for all point $z \in \partial D$.*

Theorem 8. *Let $K \subset \mathbb{C}^n$ be $\mathbb{C}$-convex compact; then its interior* int K *consists of $\mathbb{C}$-convex domains.*

Example 3. Let K be the union of two circles

$$K = \{z | (|z-1| \leq 1) \vee (|z+i| \leq 1)\} \subset \mathbb{C}.$$

Obviously K is $\mathbb{C}$-convex compact. Except for this the interior of compact $\operatorname{int} K$ is not connected.

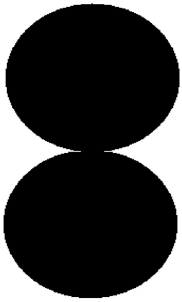

Fig. 2.

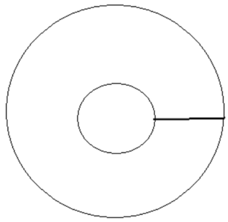

Fig. 3.

Example 4. Let $A = \{z = (z_1, z_2) \| z| \leq 1,\ \operatorname{Im} z_2 \geq 0\}$ be hemiball, but $B = \{z = (z_1, z_2) \| z_2 - i| \leq 1\}$ be an open unlimited cylinder in $\mathbb{C}^2$. We shall consider compact $K = A \setminus B$. Any section of compact K by line, different from $z_2 = \text{const}$, is of the form of intersection of two sets:

1) Halfline $\operatorname{Im} z_2 \geq 0$ with the ball $|z_2 - i| < 1$ thrown away and

2) the ball of the radius not more than 1; moreover if the ball completely lies in halfline $\operatorname{Im} z_2 \geq 0$, and its radius is less then 1.

Hence K is $\mathbb{C}$-convex compact. Obviously $\operatorname{int} K$ consists of two components.

Remark. We shall notice that the equality $\bar{D}^* = \operatorname{int} D^*$ used in the proof of Theorem 5 is true for any bounded (not only $\mathbb{C}$-convex) domain, but for unbounded domain it can be broken.

Example 5. Let $D = D_1 \times \mathbb{C}^{n-1}$, $n > 1$, where D_1 is a flat domain. Then

$$D^* \approx \overset{0}{\mathbb{C}} \setminus D_1 \subset \mathbb{C} \quad \text{but} \quad (\bar{D})^* \approx \overset{0}{\mathbb{C}} \setminus \bar{D}_1 \subset \mathbb{C},$$

and, consequently, $(\bar{D})^* \neq \operatorname{int} D^* = \emptyset$.

Example 6. Let

$$D = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n | 1 < |z_1| < 2,\ z_1 \notin [1, 2]\}.$$

It is easy to check that D is a $\mathbb{C}$-convex domain, but $\bar{D}$ is already not $\mathbb{C}$-convex compact.

Theorem 9. *A domain or a compact, being Cartesian product is $\mathbb{C}$-convex iff it is convex.*

Theorem 10. *Let $E \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex closed set being kept in some real hyperplane. Then either E is in one of the complex hyperplanes or it is a convex set.*

Theorem 11. *Let $K \subset \mathbb{C}^n$ be $\mathbb{C}$-convex compact not lying in a real hyperplane; then for projection K on an arbitrary line* (*with the exclusion of, possibly, one line*) int $\pi(K) \neq \emptyset$, *where $\pi(K)$ is an image of compact at projections π.*

Theorem 12. [23]. *For an acyclic compact $K \subset \mathbb{R}^n$ it be convex it is necessary and sufficient that all its sections by supporting m-planes for fixed m, $1 \leq m \leq n-1$, are acyclic.*

An example illustrating the need (minimality) imposed conditions may be as follows.

Example 7. Hemisphere

$$S^- \left\{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^2 = 1 \; x_3 \leq 0\right\}.$$

The supporting plane $x_3 = 0$ crosses it along the one-dimensional cycle (circle). Intersection with any other supporting plane, such as L, is the only relevant point of a hemisphere.

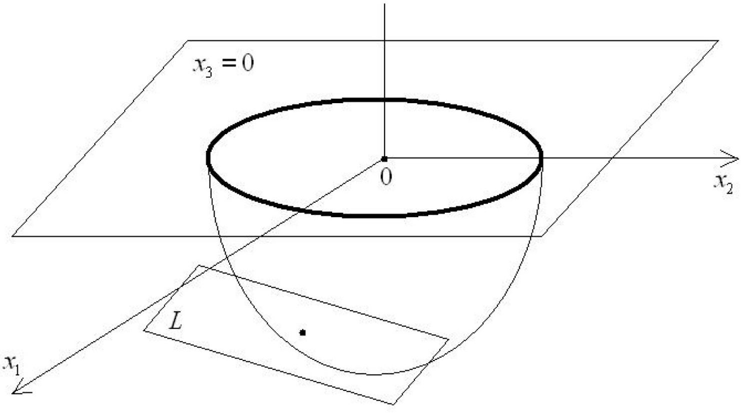

Fig. 4.

Theorem 13. [23]. *For an acyclic compact $K \subset \mathbb{C}^n$ with not empty the interior to be $\mathbb{C}$-convex it is necessary and sufficient that all its sections by supporting complex m-planes for fixed m, $1 < m < n-1$, are acyclic and in the case where $m = n-1$, that they are $\mathbb{C}$-convex.*

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Institute of Mathematics
National Academy of Science of Ukraine
Tereshchenkivs'ka vul. 3, UA-01601 Kyiv
Ukraine
e-mail: zel@imath.kiev.ua; yuzelinski@gmail.com

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O KILKU ZAGADNIENIACH GLOBALNEJ GEOMETRII ZESPOLONEJ

S t r e s z c z e n i e

Temat pracy uwzględnia jednocześnie pewne zagadnienia analizy zespolonej, geometrii i rachunku prawdopodobieństwa. Naszym celem jest zarówno przegląd nierozwiązanych problemów jak i niedawno rozstrzygniętych. Pierwsze badania prawdopodobieństw geometrycznych rozpoczyna znany problem igiełek Buffoona i związane z nim paradoksy Bertranda. Praca uwzględnia oryginalne hipotezy i wyniki obecnego autora.

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In memory of
Professor Roman Stanisław Ingarden

Janusz Garecki

IS TORSION NEEDED IN A THEORY OF GRAVITY? A REAPPRAISAL II

THEORETICAL ARGUMENTS AGAINST TORSION

Summary

It is known that General Relativity (**GR**) uses a Lorentzian Manifold $(M_4; g)$ as a geometrical model of the physical spacetime. The metric g is required to satisfy Einstein's equations. Since the 1960s many authors have tried to generalize this geometrical model of the physical space-time by introducing torsion. In the second part of the paper we discuss theoretical arguments against torsion. Our conclusion is that the general-relativistic model of the physical spacetime is sufficient for the all physical applications and it seems to be the most satisfactory.

5. Theoretical arguments against torsion

We begin this Section with the remark that if one utilizes the so–called "Ockham's razor" then torsion is needn't for him in a theory of gravity because the *wonderful, the most simple and most symmetric* Levi-Civita connection is sufficient for the all physical requirements. By "Ockham's razor" we mean a Philosophical Principle which states: "Entities are not to be multiplied without necessity".

The first our argument against torsion is given in the very important paper by J. Ehlers, F. A. E. Pirani, and A. Schild [73]. These authors have showed that requiring compatibility between conformal geometry **C** defined by rays of light and the projective structure **P** of spacetime determined by trajectories of freely-falling test particles leads to Weyl spacetime with a *symmetric connection* ω. Then, admitting some, very natural axioms [73], we obtain Riemannian geometry.

So, studying the rays of light and freely-falling particles, leads us to Riemannian spacetime.

Now, let us pay our attention to the other, disadvantegeous properties of torsion and metric-compatible spacetimes with torsion:

1. In a spacetime with torsion *do not exist* infinitesimal parallelograms [12,29] because the operation of invariant geometric addition of infinitesimal coordinate segments is noncommutative. So, such spacetimes seem *physically inadmissible* as this result is in direct conflict with the operational and epistemological basis of our difference physics [30]. Besides, such spacetime *cannot be approximated locally* by a flat, Minkowskian spacetime already on classical level.
2. *Torsion is topologically trivial.* This means that the topological invariants of a real manifold M and characteristic classes of vector bundles over M, as defined in [31–33] depend only on curvature and can be fully determined by the curvature ${}_{LC}\Omega^i{}_k$ of the Levi-Civita connection. Roughly speaking, one can continuously deform any metric-compatible connection (or even general linear connection) into Levi-Civita connection *without changing topological invariants and characteristic classes.* So, torsion is not relevant for topological invariants and characteristic classes. Some authors say that torsion which satisfies differential field equations might be topologically non-trivial. But this seems to be incorrect because one can still continuously deform the connection in the case into torsionless Levi-Civita connection *without changing topological invariants and characteristic classes.* The field equations will, of course, change during such deformation. So, it seems to us that one can say only that the torsion which satisfies differential field equations *might be physically* non-trivial. Of course, one cannot exclude that there exist other topological properties of spacetime which can substantially depend on torsion.
3. *Torsion is not relevant from the dynamical point of view either.* Namely, one can reformulate every metric theory of gravitation with a metric- compatible connection $\omega^i{}_k$ as a "Levi-Civita theory". Torsion is then treated as *a matter field.* Such reformulation preserves the all dynamical properties of the theory. An obvious example is given by **ECSK** theory in the so–called "combined formulation" [34]. In this formulation **ECSK** theory is dynamically fully equivalent to the ordinary **GR** [35].
   In general, one can prove [36] that any total Lagrangian of the type
   $$L_t = L_g(\vartheta^i, \omega^i{}_k) + L_m(\Psi, D\Psi) \tag{7}$$
   admits an unique decomposition into a pure geometric part $\tilde{L}_g(\vartheta^i, {}_{LC}\,\omega^i{}_k)$ containing no torsion plus a generalized matter Lagrangian
   $$\tilde{L}_m(\Psi, {}_{LC}\,D\Psi, K^i{}_k, {}_{LC}\,DK^i{}_k)$$
   which collects the pure matter terms and all the terms involving torsion
   $$L_t = L_g + L_m = \tilde{L}_g + \tilde{L}_m. \tag{8}$$

Here ${}_{LC}D$ means the exterior covariant derivative with respect to the Levi-Civita connection ${}_{LC}\omega^i{}_k$.
From the Lagrangian

$$L_t = \tilde{L}_g + \tilde{L}_m \tag{9}$$

there follow the *Levi-Civita equations associated with* L_t.
So, torsion *can always be treated as a matter field.* This point of view is preferred e.g. in [37, 38] and it is supported by transfromational properties of torsion: torsion transforms like a matter field i.e., it transforms as a tensor–valued form.

4. A gravitational theory with torsion *violates* **EEP**, which has so very good experimental evidence. It is because in a spacetime with torsion a tangent space $T_p(M)$ *cannot be identified with Minkowskian spacetime*, i.e., there do not exist holonomic frames such that $g_{ik}(P) = \eta_{ik}$, $\Gamma^i{}_{kl} = 0$, and, in which geometry, in an infinitesimal vicinity of the point P, is Minkowskian. P is here a preselected point. So, a gravitational theory with torsion *is not a covering theory for* **SRT** [54] and violates **EEP** (Strictly speaking, it violates **LLI**). A correct relativistic theory of gravity should be a covering theory for the both theories, **SRT** and Newton's theory of gravity. Of course, **GR** satisfies this condition.
We also lose Fermi coordinates [12, 39, 40, 77] in a Riemann-Cartan spacetime. Fermi coordinates realize in **GR** a local (freely-falling and non-rotating) inertial frame along a curve in which **SRT** is valid.
Some authors [41,42,44] formulate **EEP** in a weaker form than the constructive Will's formulation, which we have adopted in this paper. Namely, in their formulation this Principle reads: there exists (anholonomic for a connection with torsion) *normal frame* $\{\vartheta^i\}$ such that in a preselected point P one has

$$\Gamma^i{}_{kl}(P) = 0, \quad g_{ik}(P) = \eta_{ik}. \tag{10}$$

But this *Equivalence Principle is a tautology* because, as it was showed in past [45], *every linear connection on a metric manifold* satisfies it.
Moreover, if the metric-compatible connection has torsion, then, the so-called *transposed connection* (see, e.g., [4]) $\hat{\omega}^i{}_k(P) := \omega^i{}_k(P) + Q^i{}_{kl}(P)\vartheta^l$, torsion $Q^i{}_{kl}(P)$ and the symmetric part $\Gamma^i{}_{(kl)}(P)$ of the connection $\omega^i{}_k = \Gamma^i{}_{lk}\vartheta^l$ *do not vanish* in P even, if in P, $\omega^i{}_k(P) = \Gamma^i{}_{lk}(P)\vartheta^l = 0$.
In consequence, even in a *normal frame*, the geometry of tangent space $T_p(M)$ *is not Minkowskian* i.e., the constructive Will's formulation of the **EEP** *is violated.* As we have already emphasized, Will's formulation of the **EEP** has very good experimental evidence.
The Equivalence Principle formulated in the form (10) *needs holonomic frames* in order to efectively work. Namely, in the set of the holonomic frames *it chooses* a symmetric, linear connection. Then, adding the most natural *metricity postulate* (or Hamiltonian Principle for trajectories of the test particles)

univocally leads us to (pseudo)-Riemannian geometry i.e., to the Levi-Civita connection.

5. A connection having torsion can be determined neither by its own autoparallels (paths) nor by geodesics [12]. So, one cannot determine unequivocally a connection which has torsion by observation of the test particles (which could move along geodesics or autoparallels).

6. Study of the Einsteinian strength of the field equations of the proposed gravity theories *favorize* the purely metric theories of gravity (obtained with the help of Hilbert variational principle) which use Levi-Civita connection, $_{LC}\omega$, in comparison with competitive *Palatini's* theories of gravity (apart from **ECSK** theory) which use metric-compatible connection admitting torsion (see, e.g., [47]. Namely, the purely metric gravity theories have *much more smaller* strengths (48 in four dimensions) and numbers of dynamical degrees of freedom (16 in four dimenions) than the competitive *Palatini's* **PGT** (120 and 40 in four dimensions respectively).
   Following Einstein, from the two competitive gravity theories this one is better, which has smaller strength and smaller number dynamical degrees of freedom because such theory determines gravitational field *more precisely*. More precisely in the sense: it admits a *smaller number* of arbitrary initial data (putting in "by hand") in the Cauchy problem, i.e., it admits smaller freedom in obtaining a solution to the field equations.

7. Reduction of the principal bundle of the linear frames $L[M_n,\ GL(n;r),\ \pi]$ over M_n to subbundle of the (pseudo)orthonormal frames $O[M_n,\ O(n;k)\ \pi]$ (for $n=4,\ k=1$ one has Lorentz group L) *leads us univocally* to the Levi-Civita connection. Namely, we have the Theorem [76].
   Theorem
   Let $[M_n,\ g]$ be a (pseudo)Riemannian manifold of an arbitrary signature, k,. Then, there exists *one and only one* linear connection ω on $L[M_n,\ GL(n;r),\ \pi]$ with *null torsion* $\Theta = D\theta = 0$ which can be reduced to the group $O(n;k)$, i.e., to the connection ω_R on the principial bundle $[M_n,\ O(n;k),\ \pi]$.
   Interestingly, that ω, and reduced connection ω_R, *are exactly the Levi-Civita connection* $_{LC}\omega$ for the metric g.
   So, the fibre bundle approach suggests choosing of the symmetric and metric Levi-Civita connection for the mathematical model $M_4(g_l,\ \Gamma)$ of the physical spacetime.

Torsion leads to ambiguities:

1. The *Minimal Coupling Principle* (**MCP**) differs from the *Minimal Action Principle* (**MAP**) in a spacetime with torsion [48].
   The **MCP** can be formulated as follows. In **SRT** field equations obtained from the **SRT** Lagrangian density $L = L(\Psi, \partial_i\Psi)$ we replace

$$\partial_i \longrightarrow \nabla_i, \quad \eta_{ik} \longrightarrow g_{ik}$$

and get *covariant field equations* in (M_4, g).
By the **MAP** we mean an application of the Minimal Action Principle (Hamiltonian Principle) to the covariant action integral

$$S = \int_{\Omega} L(\Psi,\ D\Psi) d^4\Omega, \quad \text{where} \quad L(\Psi,\ D\Psi)$$

is a covariant Lagrangian density obtained from the **SRT** Lagrangian density $L(\Psi,\ \partial_i\Psi)$ by **MCP**.
It is natural to expect that the field equations in (M_4, g) obtained by using **MCP** on **SRT** equations should coincide with the Euler-Lagrange equations obtained from $L(\Psi,\ D\Psi)$ by **MAP**. *This holds in GR but not in the framework of the Riemann-Cartan geometry.* So, we have there an ambiguity of the field equations. Axial torsion removes this ambiguity. By (M_4, g) we mean here a general metric manifold; not necessarily Riemannian.

2. In the framework of the **ECSK** theory of gravity we have four energy-momentum tensors for matter: Hilbert, canonical, combined, formal [34]. Which one is more important?

3. Let us consider now normal coordinates **NC(P)** [12, 49–51] which are so very important in **GR** (see, eg., [49–52]). In the framework of the Riemann-Cartan geometry we have two **NC(P)**: normal coordinates for the Levi-Civita part of the Riemann-Cartan connection $\mathbf{NC}(_{\mathbf{LC}}\omega,\ \mathbf{P})$ and normal coordinates for the symmetric part of the full connection $\mathbf{NC}(_{\mathbf{s}}\omega,\ \mathbf{P})$ [53]. Which one has a greater physical meaning?
The above ambiguity of the normal coordinates leads us to ambiguities in superenergy and supermomentum tensors [53]. Axial torsion removes this ambiguity. Moreover, the obtained expressions are too complicated for practical use. In fact, we lose here a possibility of effective use of the normal coordinates which give a very powerful tool in **GR** to extract physical content hidden in various non-covariant expressions.
Perhaps by use *normal frames* defined in [45, 78] instead of normal coordinates one could avoid these ambiguities and connected problems. This conjecture will be studied in future.

4. In the framework of Riemann-Cartan geometry [12] there holds

(11) $$R_{(ik)lm} = R_{ik(lm)} = 0,$$

but

(12) $$R_{iklm} \neq R_{lmik}.$$

The last asymmetry leads to an ambiguity in construction of the so–called "Maxwellian superenergy tensor" for the field R_{iklm} [53]. This tensor is uniquely constructed in **GR** owing to the symmetry $R_{iklm} = R_{lmik}$ and it is proportional to the Bel-Robinson tensor [53]. In the framework of the Riemann-Cartan geometry the obtained result depends on which antisymmetric pair of the R_{iklm}, the first or second, is used in the construction.

5. In a Riemann-Cartan spacetime we have geodesics and autoparallells (paths). Hamiltonian Principle demands geodesics as trajectories for the test particles [54]. Then, what about the physical meaning of the autoparallells? Axial torsion removes this problem. One can also easily prove in the framework of the **ECSK** theory that spinless test particles move along geodesics.

6. In a spacetime with torsion we have in fact three kinds of parallel displacement defined by

$$dv^k = (-)\Gamma^k_{ij} v^j dx^i, \tag{13}$$

$$dv^k = (-)\Gamma^k_{ij} v^i dx^j, \tag{14}$$

and

$$dv^k = (-)\Gamma^k{}_{(ij)} v^i dx^j, \tag{15}$$

and three different curvatures. These results follow from that three kinds of covariant (and absolute) differentials

$$\nabla^{(L)}_i v^k = \partial_i v^k + \Gamma^k_{il} v^l, \tag{16}$$

$$\nabla^{(R)}_i v^k = \partial_i v^k + \Gamma^k_{li} v^l, \tag{17}$$

$$\nabla^{(s)}_i v^k = \partial_i v^k + \Gamma^k{}_{(li)} v^l. \tag{18}$$

Authors usually use only one of the two first possibilities. What about the others?

In a torsionless spacetime the above three possibilities coincide.

The ambiguities (13, 14)–(16, 17) arise from the two possibilities expanding of the local connection forms $\tilde{\omega}^i{}_k$ on the base space M_n in coordinate frames:

$$\tilde{\omega}^i{}_k = \Gamma^i{}_{kl} dx^l, \quad \tilde{\omega}^i{}_k = \Gamma^i{}_{lk} dx^l. \tag{19}$$

In practice, one must consequently use one of the two above possibilities (or conventions) in order to avoid mistakes.

5.1. Symmetry of the energy–momentum tensor of matter

In Special Relativity (**SRT**) the correct energy–momentum tensor for matter (electromagnetic field,continuous medium, dust, elastic body, solids) *must be symmetric* [39, 55].

One can always get such a tensor starting from the *canonical pair* ${}_cT^{ik}, {}_cS^{ikl} = (-){}_cS^{kil}$, where ${}_cT^{ik} \neq {}_cT^{ki}$ is the canonical energy-momentum tensor and ${}_cS^{ikl}$ — the canonical spintensor. These two canonical tensors are connected by the equations

$$\partial_k {}_cT^{ik} = 0, \quad {}_cT^{ik} - {}_cT^{ki} = \partial_l S^{ikl}. \tag{20}$$

By use of the *Belinfante symmetrization procedure* [34, 48, 56, 57] one can get the most simple new pair

(21) $$ {}_sT^{ik} = {}_cT^{ik} - \frac{1}{2}\partial_j\big({}_cS^{ikj} - {}_cS^{ijk} + {}_cS^{jki}\big), $$

(22) $$ S^{ijk} = {}_cS^{ijk} - A^{jki} + A^{ikj} = 0. $$

Here

(23) $$ A^{ikj} = \frac{1}{2}\big({}_cS^{ikj} - {}_cS^{ijk} + {}_cS^{jki}\big). $$

The obtained new "pair" $({}_sT^{ik},\ 0)$ is *the most simple and the most symmetric*. Note that the symmetric tensor ${}_sT^{ik} = {}_sT^{ki}$ gives complete description of matter because the spin density tensor ${}_cS^{ijk}$ is *entirely absorbed* into ${}_sT^{ik}$ by the symmetrization procedure.

Note also that the symmetric tensor ${}_sT^{ik}$ has 10 independent components and this number is exactly the same as the number of integral conserved quantities in an asymptotically flat closed system.

It is interesting that one can easily generalize the above symmetrization procedure onto a general metric manifold (M_4, g) [14, 34] by using the Levi-Civita connection associated with the metric g. The generalized symmetrization procedure has the same form as above with the replacement $\eta_{ik} \longrightarrow g_{ik},\ \ \partial_i \longrightarrow_{LC} \nabla_i$.

So, one can always get on a metric manifold (M_4, g) a symmetric energy-momentum tensor ${}_sT^{ik} = {}_sT^{ki}$ for matter (then, of course, corresponding $S^{ikj} = 0$). Observe that the symmetric tensor ${}_sT^{ik}$, like as in **SRT**, consists of the canonical tensors ${}_cT^{ik}$ and ${}_cS^{ikl}$.

The symmetric energy–momentum tensor for matter *is unique*, i.e., it is uniquely determined by the matter equations of motion and reasonable boundary conditions [58]. This fact is essential for the uniqueness of the gravitational field equations. Moreover, the symetric energy–momentum tensor is covariantly conserved (a canonical energy-momentum tensor is not conserved).

L. Rosenfeld has proved [59] that

(24) $$ {}_sT^{ik} = \frac{\delta L_m}{\delta g_{ik}}, $$

where

$$ L_m = L_m(\Psi,\ {}_{LC}D\Psi) $$

is a covariant Lagrangian density for matter. The tensor ${}_sT^{ik}$ given by (24) is the source in the Einstein equations

(25) $$ G_{ik} = \chi {}_sT_{ik}, $$

where

$$ \chi = \frac{8\pi G}{c^4}. $$

Note that these equations *geometrize both the canonical quantities* ${}_cT^{ik}$ and ${}_cS^{ikl} = (-){}_cS^{kil}$ *in some equivalent way* because the tensor ${}_sT^{ik}$ is built from these two canonical tensors.

So, it is the most natural and most simple to postulate that, in general, the correct energy–momentum tensor for matter is the symmetric tensor ${}_sT^{ik}$. This leads us to a purely metric torsion-free theory of gravity with the field equations

$$\frac{\delta L_g}{\delta g_{ik}} = \frac{\delta L_m}{\delta g_{ik}}. \tag{26}$$

Then, if we take into account the *dynamical universality* of the Einstein equations [38, 60, 61], we will end up with General Relativity (possibly with $\Lambda \neq 0$) which will have a sophisticated, symmetric energy-momentum tensor as a source.

5.2. Some remarks on the "teleparallel equivalent of general relativity"

After presenting the preliminary draft of the old our lectures in arXiv [80], we have got critical remarks from some persons which are working on the so-called *teleparallel equivalent of general relativity* (**TEGR**) in the framework of the *Weitzenböck or teleparallel geometry* [12, 29, 62]. Our reply was the following. This reply was considerably extended and updated in the paper [81]. The *Weitzenböck or teleparallel connection and geometric structure* on spacetime is determined by a tetrad (or other anholonomic frame) field $h^{(a)}{}_b(x)$ and *can always* be introduced *independently* of the geometric structure of the spacetime. Here $(a), (b), \ldots$ are tetrad (= anholonomic) indices and $a, b, c, \ldots$ mean holonomic (= world) indices.

The fundamental formulas of the teleparallel geometry read

$$g_{ik} := \eta_{(a)(b)} h^{(a)}{}_i h^{(b)}{}_k, \tag{27}$$

$$\Gamma^i{}_{kl} := h_{(a)}{}^i \partial_k h^{(a)}{}_l, \tag{28}$$

$$\nabla_i h^{(a)}{}_k = 0, \tag{29}$$

$$\Gamma^i{}_{kl} =_{LC} \Gamma^i{}_{kl} + K^i{}_{kl}, \tag{30}$$

$$K^i{}_{kl} := 1/2\big(T_k{}^i{}_l + T_l{}^i{}_k - T^i{}_{kl}\big), \tag{31}$$

$$T^i{}_{kl} := \Gamma^i{}_{lk} - \Gamma^i{}_{kl}, \tag{32}$$

and

$$R^i{}_{klm} =_{LC} R^i{}_{klm} + Q^i{}_{klm} \equiv 0, \tag{33}$$

where $Q^i{}_{klm}$ is a tensor written in terms of the *contortion* $K^i{}_{kl}$ and its covariant derivatives with respect to the Levi-Civita connection ${}_{LC}\Gamma^i{}_{kl}$ of the metric g_{ik}.

Here $\eta_{(a)(b)}$ means the *interior metric* (usually Minkowskian) of a tangent space and the duals $h_{(a)}{}^i$ are defined by

$$h_{(a)}{}^i h^{(a)}{}_k = \delta^i_k. \tag{34}$$

Those authors which work on **TEGR**, by use the formulas (27), (30), and (33) of the teleparallel geometry *rephrase*, step-by-step, the all formalism of **GR** in terms of the Weitzenböck connection $\Gamma^i{}_{kl}$ and its *torsion* $T^i{}_{kl}$. Then, they call this formal reformulation of **GR** in terms of the Weitzenböck geometry *the teleparallel equivalent of general relativity* (**TEGR**) (What kind of "equivalence"?).

One can read in the papers [62] the following conclusion: "Gravitational interaction, thus, can be described alternatively in terms of curvature, as is usually done in **GR**, or in terms of torsion, in which case we have the so–called *teleparallel gravity.* Whether gravitation requires a curved or torsional spacetime, therefore, turns out to be a matter of convention".

From the point of view of the **TEGR**, therefore, teleparallel torsion has fundamental physical meaning and it has been already detected.

We cannot agree with such statements. In our opinion, the "teleparallel equivalent of **GR**" is only *formal* and geometrically *trivial* rephrase of **GR** in terms of the Weitzenböck geometry. Such rephrase is, of course, *always possible* not only with **GR** but also with any other purely metric theory of gravity (see eg. [63]) but it *has no profound physical motivation.* It is because, as one can easily show, the teleparallel torsion is entirely expressed in terms of the Van Danzig and Schouten aholonomity object $\Omega^{(a)}{}_{(b)(c)}$ (see eg. [12, 29]). So, the torsion $T^i{}_{kl}$ of a teleparallel connection *describes only anholonomity* of the used field of aholonomic frames $h^{(a)}{}_i(x)$; *not real geometry of the spacetime.* Unless one can physically distinguish a tetrad field (or other anholonomic field of frames) and give it a fundamental geometrical and physical meaning. But we think that this could introduce a cristal–like structure on spacetime and, therefore, it would contradict local Lorentz invariance. Contrary, Levi-Civita part of a Weitzenböck connection can have (and has) geometrical (and physical) meaning.

Resuming, it seems to us that **TEGR** is rather a *mathematical curiosity* which gives, by no means, *anything better* than ordinary **GR** gives and one can doubt into its physical meaning. Precise experimental confirmation of the **EEP** proved non-zero curvature of physical spacetime [39, 66] and supported ordinary **GR**. We think that this fact excludes a physical motivation for *rephrasing* **GR** into **TEGR**. One remark more is in order concerning **TEGR**: **TEGR** resulted in $f(T)$ theories where T means the Lagrangian density [83] of the **TEGR**. In analogy to $f(R)$ extension of the Hilbert action of **GR**, the $f(T)$ theories are generalization of the action of **TEGR**. It seems that the only one positive property of these theories is the fact that they have 2-nd order field equations.

6. Concluding remarks

The **GR** model of the space-time has very good experimental confirmation in a weak-field approximation (Solar System) and in the strong fields (binary pulsars). On the other hand, torsion has no experimental evidence (at least in vacuum) and it is not needed in a theory of gravity. Moreover, the introduction of torsion into the geometric structure of space-time leads to many problems (apart from calculational, of course).

Most of these problems are removed if only axial torsion $A_i = \frac{1}{6}\eta_{iabc}Q^{abc}$, $Q^{[abc]} = Q^{abc}$ exists. So, it would be reasonable to confine themselves to the axial torsion only (If one still want to keep on torsion). This is also supported by the important fact that the *matter fields* (= Dirac's particles) are coupled only to the axial part of torsion in the Riemann-Cartan space-time.

However, if we confine to the axial torsion, then (if we remember the dynamical triviality of torsion and the dynamical universality of the Einstein equations) we effectively will end up with **GR** + additional matter fields. In the most important case of the **ECSK** theory we will end up with **GR** + an aditional pseudovector field A_i (or with an additional pseudoscalar field φ if the field A_i is potential, i.e., if $A_i = \partial_i\varphi$) [48]. But **GR** with an additional dynamical pseudovector field A_i yields local gravitational physics which may have both location and velocity-dependent effects [19] unobserved up to now. Besides, **GR** with an additional pseudoscalar field has a defect because there exist two distinguished frames, *the Einstein frame* and *the Jordan frame*, which are not equivalent physically [67].

Additionally, we would like to emphasize that there exist very strong experimental constraints on the components of the axial torsion: $< 10^{(-15)}m^{(-)1}$ [75].

So, we will finish with the conclusion that the geometric model of the space-time given by ordinary **GR** and "wonderful" Levi-Civita connection *seems to be the most satisfactory.*

Interestingly that this model has a very strong support from the field-theoretic approach to gravity (see e.g., [68]).

It seems to us that the torsion was introduced into a theory of gravity in order to get some link between theory of gravity and quantum fundamental particles theory (It is commonly known that the role of the curvature in an atomic and smaller scale is neglegible). But these trials *were not successful* (see, e.g., [75]). It also seems that what we really need nowadays is a *quantum model* of the Riemannian geometry and a *quantum gravity* which is based on this model. The recent papers given by Ashtekhar [16–18, 74] and co–workers on this problem seems to be very promissing.

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Institute of Mathematics
University of Szczecin
Wielkopolska 15, PL-70–451 Szczecin
Poland
e-mail: garecki@wmf.univ.szczecin.pl

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CZY TORSJA JEST POTRZEBNA W TEORII GRAWITACJI? NOWE SPOJRZENIE II

ARGUMENTY TEORETYCZNE PRZECIWKO TORSJI

S t r e s z c z e n i e

W pracy pokazano, że wprowadzenie skręcenia do modelu matematycznego fizycznej czasoprzestrzeni nie jest *ani konieczne, ani wskazane.*

W drugiej części pracy przedyskutowano argumenty teoretyczne przeciwko torsji. Model matematyczny, który daje ogólna teoria względności *jest wystarczający* dla wszelkich potrzeb fizyki i, jak dotąd, jest *bardzo dobrze* potwierdzony przez eksperymenty.

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B U L L E T I N

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pp. 39–53

In memory of
Professor Roman Stanisław Ingarden

Stanisław Bednarek and Tomasz Bednarek

MODERN HOMOPOLAR MOTORS

Summary

This paper aims to present a construction and operating principles of some types of homopolar motors. These motors are characterized by the fact that the spinning of the rotor occurs in the surrounding of only one of the poles of a permanent magnet. An early demonstration of a homopolar motor is Barlow's wheel. A few versions of such motors were built using easily available and inexpensive materials, such as a neodymium magnet, piece of copper wire and battery in a metal shell. These are motors with a rotating magnet, a battery or a frame of wire. A detailed explanation of principles of operation of such motors has also been provided.

1. Introduction

One of the earlest and simplest homopolar electrical motors to be built is Barlow's wheel. Known from literature, a classic model of the wheel comprises a conductive disk of nonferromagnetic material [1, 2]. The disk is mounted on a conductive axle and can rotate almost without friction. A constant magnetic field is applied perpendicularly to the surface of the disk. Two resilient contact points which are connected to a DC power source make contact with both the rim of the disk and its axle. As a result an electric current passes over the disk in radial direction. Since the passing current is perpendicular to the magnetic field, on the disk there act electrodynamic forces which are directed tangentially to the circumference of the wheel. The momentum of these forces causes the disk to rotate.

A change in direction of the current flow, or a reversal of induction vector of the applied magnetic field cause a reversal of the direction of the disk rotation. An increase in amperage or induction value leads, in turn, to an increase in rotational speed of the disk. In some technical solutions the rim of the disk is dipped into mercury contained in a small trough to which one pole of the electric source is attached. This ensures good electrical contact at a fairly high amperage necessary to cause the rotation of the disk.

At present, Barlow's wheel is used in some special driving systems, or in electromagnetic generators/dynamos that have to operate at low amperage and high current magnitude. In recent years among several household goods there appeared on the market round batteries or battery cells in metal shells, as well as strong conductive nickel coated neodymium magnets, and aluminum cans used as packaging for beverages and deodorants. These articles allow us to easily build some simple but interesting models of unipolar electric motors which constitute modern versions of the earlier types. More examples of such motors will be demonstrated further on in this paper. One good point about them, except their simplicity, is that they make an excellent teaching tool.

2. Motor with a rotating magnet

A simple homopolar motor is very easy to build with the following components: a nickel coated neodymium magnet in the shape of cylinder $1-2$ cm in diameter and 1 cm thick, R6 battery cell in a metal shell, a piece of conductive wire with bare ends 10 to 15 cm long, a two-inch steel nail or a drywall screw. If the copper wire is isolated you need to remove the insulation. The flattened boss on the end of the nail should be centered over the the flat surface of the cylindrical magnet. As a result of magnetic attraction the nail will adhere firmly to the magnet, see Fig. 1. The point of the nail, in turn, should be aplied to one of the terminals of the battery. Due to its metal shell, the nail will also be attracted to the battery. The other end of the battery should be held with the fingers of one hand and the bare end of the wire should be pressed into it. The magnet and the nail become suspended vertically held by the forces of magnetic attraction. The remaining bare end of the wire is to be held with the fingers of the the other hand and applied to the side surface of the magnet. It turns out then that the magnet and the nail begin to spin around their vertical axis.

The observed rotation of the magnet can be explained as follows, see Fig. 2. From the battery terminal that contacts the nail there flows an electric current along the nail towards the centre of the magnet, and then in a radial direction through the magnet to the wire end that makes contact with the side surface of the magnet. The current contues to flow on to the other terminal of the battery. In this way, the circuit is completed through the nickel coating of a neodymium magnet. The current that is flowing radially through the magnet is in the magnetic field that is induced by the same magnet. The direction of induction vector of the magnetic field is vertical, or perpendicular to the direction of a current. In such a case, an electrodynamic force acts on the magnet which is directed horizontally and

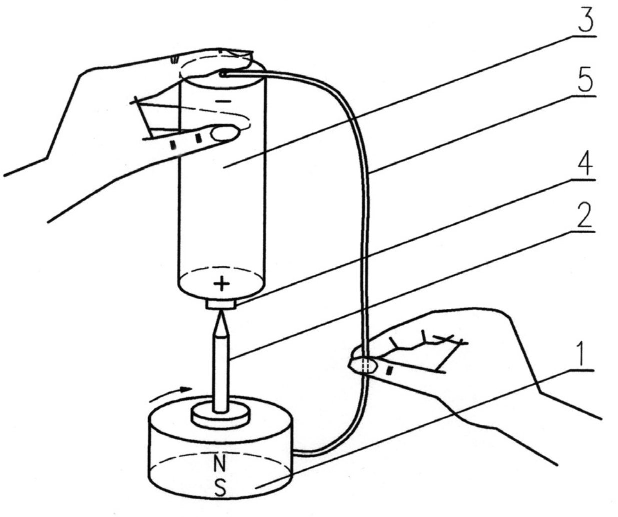

Fig. 1: Motor with a rotating magnet construction; 1 – a neodymium magnet, 2 – a steel nail, 3 – R6 battery cell, 4 – the positive battery terminal, 5 – a thin resilient wire.

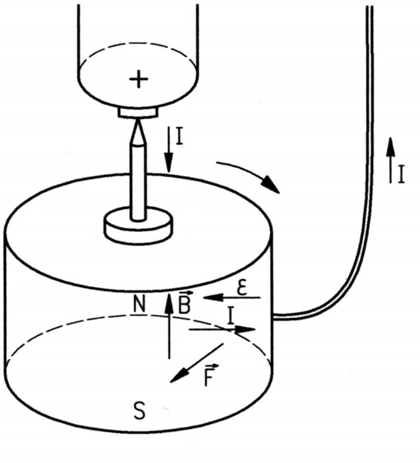

Fig. 2: Basic explanation of the working principle of operation of the motor with a rotating magnet; I – the electric current intensity, **B** – magnetic field or magnetic induction, **F** – electrodynamics force.

tangentially to the circumference of the magnet. The momentum of the force sets the magnet in rotation.

By turning the magnet over so that its other pole makes contact with the nail head causes a change of direction of the magnet's rotation to the opposite. The same effect is obtained by changing the battery terminal in contact with the point of the nail. Moreover, the speed of magnetic rotation is dependent on the place where the end of the wire contacts the side surface of the magnet and is the highest where the contact with the surface is at mid-height of the magnet. A drywall screw with a pointed end can be used in place of a nail. In such a case the rotation will be more observable. A bigger sized batteries, such as R14 or R20, can also be used. Batteries of the type have lower internal resistance and can supply more current, which results in a higher speed of magnet rotation, see Photo 1. The experiment comes off also with smaller batteries of R3 type, see Photo 2.

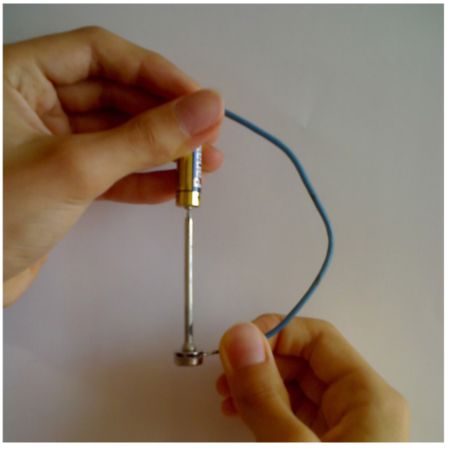

Photo 1: The homopolar motor made with the use of the smallest round R3 battery cell, the so called pencil battery.

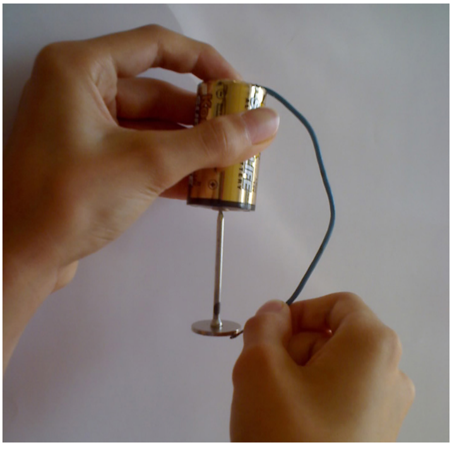

Photo 2: The homopolar motor in which R20 battery cell was used.

A ferrite cylinder magnet can be used in place of a neodymium magnet, but then it should be wrapped carefully in an aluminum foil, to allow a current to flow over the surface of the magnet. The nail or screw cannot be too long and their cross-sectional area should not be too small, otherwise it will not be attracted firmly enough to the outer metal shell of the battery and will not be able to hold the weight of the magnet. When rotating, the magnet is prone to sway slightly to the sides and the endpoint of the wire may fail to contact the side surface of the magnet. Despite that, due to the inertia effect the magnet continues to rotate. During rotation a pondermotoric force is induced in the magnet which is oriented in the opposite direction to the electromotive force of the battery. As a result, the intensity of the current passing through the magnet becomes dicreased and a state of balance is achieved, which prevents the motor from reaching the warming-up phase. Despite that, the battery works close to the level of short circuit and a significant ammout of current is flowing through it at the intensity level of up to a few amperes. This causes noticeable heating of the motor components and excessive discharge of the battery.

To prevent the point of the nail from deviating off the centre of battery terminals, which occurs during fast rotations of the motor, it is useful to make small dents at the tops of the terminals with a nail or a slightly blunted point of a point chisel. Attention should be paid not to go through the battery shell, otherwise in the case of alkaline batteries this may cause a leak of electrolyte and the irrepareable damage to the battery. Making such dents in the top of the battery also does the trick in the case of motors to be described in the further parts of our paper. The point of the nail should be applied to the opposite surface of the battery. By holding the nail head and lifting it we check wheather the battery and magnet do not come off the nail.

3. Motor with rotating battery and magnet

The motor is built from the same components as the one described earlier, but its elements are arranged differently in relation to one another, see Fig. 3. Due to that, the battery in the motor rotates together with the magnet while the nail remains immobile, see Photo 3. Such a configuration of elements, unknown in literature, has been proposed by one of the co-authors of this paper. To build the motor, the flat surface of the neodymnium magnet should be applied to one of the surfaces of the battery in a metal shell.

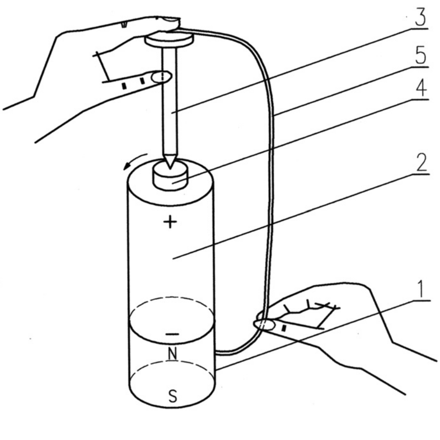

Fig 3: The motor with a rotating battery and magnet; 1 – a neodymium magnet, 2 – R6 battery cell, 3 – a steel nail, 4 – the positive battery terminal, 5 – a thin resilient wire.

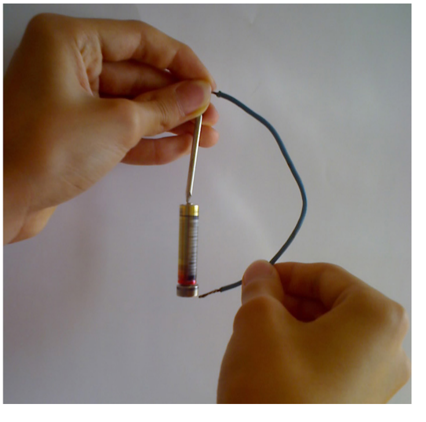

Photo 3: The homopolar motor where R3 battery rotates together with the magnet.

If it does, a stronger magnet, or a shorter nail with a bigger cross section area is needed. It should be remembered that the battery and magnet hold on better when its wider end is in good contact with the flattened nail head, which constitutes the negative terminal. The bare end of a copper wire is applied to the nail head and pressed with a finger. The other bare end of the wire is to be held with the fingers of the other hand and applied to the side surface of the neodymium magnet. A spin of a magnet and a battery can be observed. The operation principle of the motor is similar to the one described earlier. The same remarks apply also to the change in the direction of the rotation of the motor and to using other battery options.

Let's have a closer look at the dynamics of the motor. On the magnet and battery, or the rotor there acts the momentum of electromagnetic force, getting ready to set it in motion. In addition to that, on the magnet there acts the momentum of friction of the end of the wire against its side surface, and the slight momentum of friction of the magnet against the point of the nail. The momenta of friction counteract the movement of the rotar. Assuming that the values of these momenta are constant and the momentum of electrodynamic force is greater than the momenta of friction, then the angular velocity of the rotor would be constant and its velocity could increase indefinitely with the passing of time. As a result, it could lead to excessive velocity, or the so-called warming up of the motor, resulting in its damage. However, this has not been observed. Why is it so?

The momentum of friction of the air acts also on the rotor, which, in accordance with Stokes' theorem, is directly proportional to its velocity. In this situation, however, the momentum is rather insignificant. A really essential factor that curbs the speed of the rotor is the aforementioned pondermotive force. The force increases proportionally to the speed of rotation and decreases the intensity of a current, on which there depends the momentum of electromagnetic force. Due to that, with the passing of time, the resultant forces that act on the rotor and its acceleration decrease and the angular velocity of the rotor approaches exponentially the established critical value, Fig. 4. The critical value is dependent only on the parameters that characterize the elements of the motor, for example, the mass and the size of a magnet,or the electromotive force of a battery. The curve shown in Fig. 2 describes an increase in the angular velocity of the rotor for an ideal motor in which the end of the wire is in constant contact with the side surface of a magnet.

During experiments, it is easy to observe that the magnet deviates from a straight line and contact between the wire ends and the magnet is lost. Severance of contact leads to the disappearance of the momentum of the electrodynamics force that propells the rotor. Friction forces are still at work causing the angular velocity of the rotor to decrease. Repeated engagement of the wire end with the surface of the magnet causes an increase in angular velocity. Such situations repeat, and a dependence graph for angular velocity of the rotor on the time assumes the shape as shown in Fig. 5. On breaking off the connection between the wire end and the magnet, little sparks can be observed signifying a high intensity of the current flowing through the motor. Similar experiments can be carried out with the motor as with the motor described in the previous paper, and so to reverse the battery terminals or the poles on the magnet, to test the batteries and magnets of various sizes, or to apply the wire end at different points over the surface of the magnet.

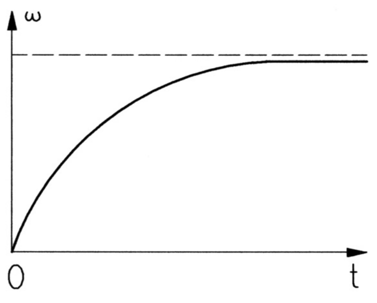

Fig 4: The dependence of an angular velocity of the rotor ω on the time t for an ideal motor.

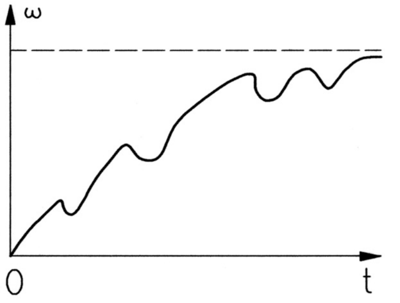

Fig. 5: The dependence of an angular velocity of the rotor ω on the time t for a real motor.

4. Homopolar motor with a rotating battery

If you have a round R20 non-alkaline battery and a big enough neodymium magnet, you can build a homopolar motor in which only a battery itself will rotate. In order to do that the outer steel shell has to be removed. The metal sheet is bent back by means of a screwdriver or a knife along the edge and pulled off with a pair of pliers. A zinc case is the negative terminal, whereas the carbon rod with a metal cap is positive. An alkaline battery is not fit for the experiment of this kind, as it has no zinc case and the unbending of its shell may cause damage to the battery and the spilling of electrolyte.

The next step will be putting a brass cap (recovered from the used battery) on top of the carbon rod sticking out of R20 battery cell. In the centre of the metal cap we make a slight dent with the point of a nail. If the R20 battery rod has too large a diameter, to fit the metal cap we make it smaller by scraping it. The small cap will serve as a resistant bearing for the motor and ensure the flow of a current. Close to the end of a piece of non-ferromagnetic sheet we cut out a round hole with a pair of scissors and finish off the edges with a file – of a diameter that is larger than the outer diameter of the zinc case of R20 battery. Then we bend the strip twice at right angle to form a bracket-like structure in the shape of letter C, as shown in Fig. 4. The perpendicular arm of the bracket must be short – around 1 cm in length. At the lower arm of the bracket a small hole is pierced through with the point through which a thumb-tack will be pressed from the outside. The hole should be made under the hole in the upper arm of the bracket.

When the above components are prepared, we can go about carrying out the experiment with the motor, see Fig. 6. The bracket is adjusted on top of a cylindrical battery, right at its centre, and R20 round battery is stripped off the metal outer shell and inserted into its opening from the top. Thanks to the ferromagnetic thumb-tack, the bracket will be firmly drawn onto the magnet. One end of the battery with the metal cap should be turned downwards so that it can rest on the point of a thumb-tack put in the dent in the cap. A slow battery spinning can be observed, see Photo 4. An electric current in the motor passes from the metal cap on the battery positive terminal, then it flows on through the thumb-tack and the lower arm of the bracket up to its vertical and upper arm and then over the surface of the battery zinc case. The current inside the battery flows radially through the electrolyte to the zinc case and then to the carbon rod. An electrolyte and an electric current are in the strong vertical magnetic field that is directed perpendicularly. In this situation, an electrodynamics force acts on the battery, which is directed horizontally and tangentially to the battery. The momentum of the force gets the battery to spin around its axis. In this type of a motor it is easy to change the direction of rotations by reversing the battery or magnet polarity.

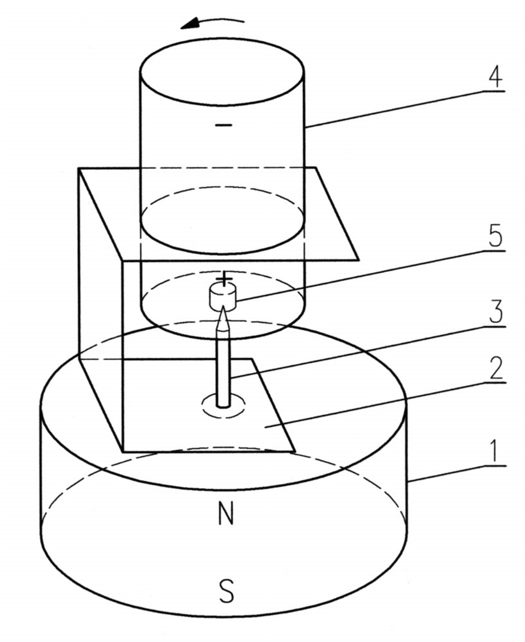

Fig 6: The construction of the motor with a rotating battery cell only; 1 – a neodymium magnet, 2 – a non-ferromagnetic bracket, 3 – a thumb tack, 4 – a round battery, 5 – a brass cap.

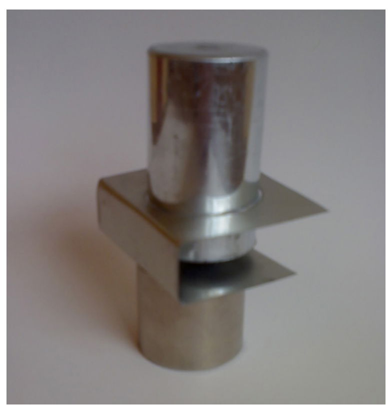

Photo 4: The homopolar motor in which only R20 battery rotates slowly.

5. Motors with rotating frames

The source of a magnetic field in the motors is a cylindrical neodymium magnet 1 coated with a protective layer of nickel, Fig. 7. The diameter of the magnet equals or is greater than the diameter of the battery cell 2 to be used here. Any of the aforementioned round battery cells of R6, R14 or R20 type can be effectively used. The magnet should be 1 cm or more in height. A relevant battery in a metal shell is placed co-axially on the magnet, and therefore it is strongly attracted by the magnet. The battery can be placed either with the positive terminal 3 facing upwards, or the other way round. A movable element in the motor is the frame made of a non-ferromagnetic wire 1 mm in diameter. It can be a coper wire, brass or just a wire coated with a thin layer of silver, the so called silver plating for applications in electronics. Wires that are made from such materials are easy to bend and solder.

Some specific elements can be singled out in the frame. The bottom of the frame has a ring 4, its inner diameter being slightly wider than that of the magnet. The ring is obtained by bending a wire on the magnet which is wrapped up with a few layers of paper to enlarge the diameter of the ring. On the opposite sides from the ring along its diameter there diverge two horizontal segments of wire 5 which at some distance from the battery are bent at right angle upwards forming vertical segments 6 that jut out over the battery. The segments are bent again over the battery and pass over into the horizontal section 7. At mid-length the last segment has a vertical fragment 8 which is bent downwards, is pointed at the end and rests on the terminal directed upwards. In sum, the frame comprises a horizontal ring that is in contact with the side surface of the magnet, and a vertical rectangle that encloses the battery. It takes no more than a few minutes to bend a frame shape from a single length of wire and to join its segments together by soldering.

As soon as the frame is put over the battery, it begins to spin, Photo 5. The direction of rotation of the frame can be reversed by reorienting the magnet and battery polarities. Batteries of bigger sizes give more amperage and higher rotational rate of the frame. The cause of rotation of the frame is the resultant momentum of electromagnetic forces that act on the separate sides of its rectangular part, Fig. 8. When the frame is put over the battery, the electric current flows through the vertical bent section 8, and then it branches off into separate sides of the rectangular part of the frame and runs into the ring contacting with the side surface of the magnet. The current enters the other terminal of the battery through the protective, conductive layer of nickel.

All of the segments of the rectangular part of the frame which exhibit conductivity are in the magnetic field whose induction vectors have, in general, a diagonal orientation to that section. Due to that, each of the component vectors is perpendicular to each other. Since an electric current passes through these segments, the electromagnetic forces act on them in the direction of the circuit. Applied to the opposite sides of the frame the forces have opposite directions and the same values,

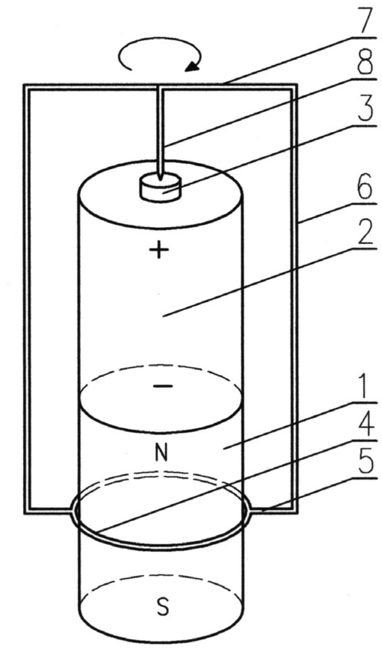

Fig 7: The motor with a rotating frame construction; 1 – a neodymium magnet, 2 – R6 battery cell, 3 – the positive battery terminal, 4 – a frame ring, 5 – the lower section of the frame, 6 – the vertical section of the frame, 7 – the upper section of the frame, 8 – the pointed end of the frame.

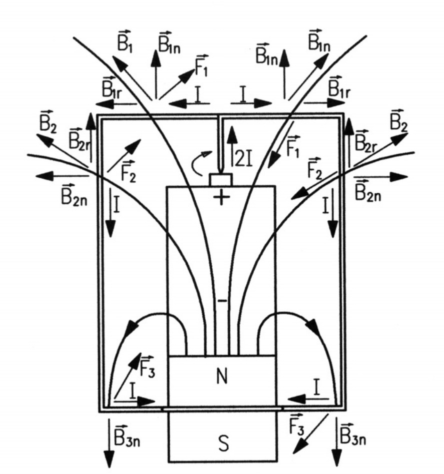

Fig 8: The explanation of the effects of the frame in a magnetic field; I – the electric current intensity, B_1, B_2, B_3 – the magnetic induction vectors on separate sides of the frame, B_{1n}, B_{2n}, B_{3n} – the components of the magnetic induction vectors parallel to the sides of the frame, F_1, F_2, F_3 – the electrodynamics forces that act on the sides of the frame.

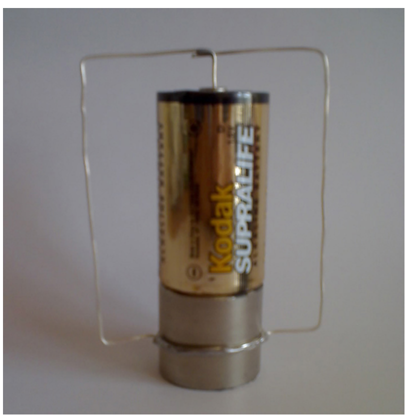

Photo 5: An example of a homopolar motor with a rotating non-ferromagnetic frame.

or constitute pairs of forces. This leads to creating the momentum of forces that rotate the frame. The frame undergoes a swinging motion and its ring need not make contact with the side surface of the magnet all the time because whenever there is no contact, the frame that has been set in motion will keep on rotating due to its inertia. Similarly as in the case of the models described earlier, the battery operates in the circuit mode and the amount of current of high intensity passing through it, causes overheating of the motor elements and a fast battery wear. The vertical section of the frame need not have any lower horizontal segments 5, Fig. 9a. This part may be of a different shape than rectangular, for example, trapezium, or it may take the form of a complex polygon, Fig. 9b.

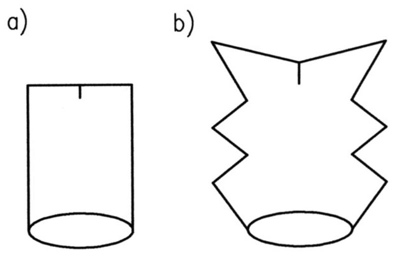

Fig. 9: Examples of various shapes of frames; a) a rectangular frame, b) a zigzag frame.

6. Motors with rotating cans

The use of appropriately prepared cans in place of wire frames in the motors that have been presented so far was an idea of one of the co-writes of this article, not found in available literature, Fig. 10. In such motors, a neodymium magnet 1 with a battery placed on it 2, its positive terminal 3 directed upwards, are the same as in the models described earlier. A used aluminum deodorant can with a sawn-off bottom 4 in an upside down position is placed over the battery. Both the diameter and the length of the cut-off fragment of the can are matched in such a way that the inner surface of the can makes a contact with the surface of the magnet, thus ensuring a good electrical contact. To reduce friction and keep the can on the axis of the system, a thumb-tack 5 or a short nail is driven through right at the centre of the bottom.

When the can is put over the magnet and the battery, it begins to rotate quickly, Photo 6. The principle of operation of this version of a motor is similar to that of motors with frames. Here, an electric current flows from the battery cell terminal through the thumb-tack, and then is dispersed radially over the bottom of the can towards its side surface. Over this surface the current flows perpendicularly into a lower edge of the can, through which it reaches the side surface of the magnet and then further on into the other battery terminal. Within the area of the bottom of the can and its side surface there is a perpendicular component of magnetic induction

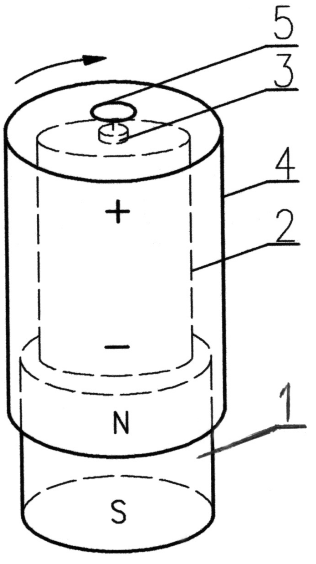

Fig 10: The external view of the motor with a rotating can; 1 – a neodymium magnet, 2 – a round battery cell, 3 – the positive terminal of the battery, 4 – an aluminum can, 5 – a thumb-tack.

Photo 6: A view of the homopolar motor in which a deodorant can is rotating. The image of a can is blurred due to a high rotation rate.

Photo 7: The homopolar motor constructed with usage of the most popular aluminum can.

Photo 8: The homopolar motor with rotating aluminum energy drink can.

Photo 9: The homopolar motor with rotating aluminum beer can.

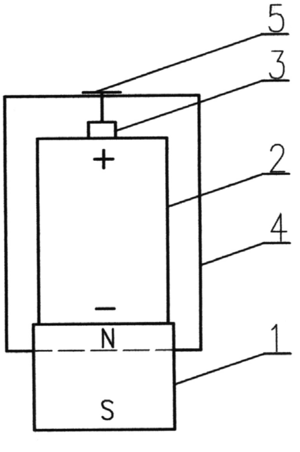

Fig. 11: A cross-section of the motor with a rotating soft drink can – the denotations of the elements are the same as in Fig. 10.

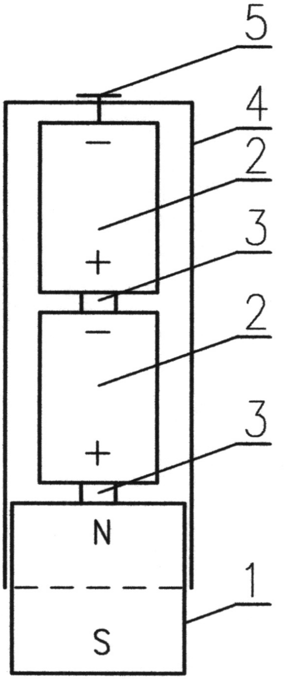

Fig. 12: A cross-section of the assembled motor with a rotating beer can – the denotations of the elements are the same as in Fig. 10.

vector of the magnetic field generated by the magnet. As a result, electromagnetic forces act on the can, which are oriented in the direction of the circuit, their momentum causing the rotation of the can.

Apart from the used deodorant cans of a diameter slightly wider than that of the magnet, aluminum cans for soft and energy drinks and beer with a volume of 330 ml, 200 ml and 500 ml, Photos 7, 8, 9 were successfully used for creating homopolar motors. In the case of these cans, co-centered holes were cut out in their bottoms of diameter slightly wider than that of a neodymium magnet. For that purpose, a pair of curved scissors for nails was used, any rough edges being smoothed with a file. A thumb tack was stuck in the centre of the can top. And the can thus prepared was put over the battery which, in turn, was placed on the neodymium magnet, Fig. 11. The edge of the hole in a can bottom was then in contact with the side surface of the magnet. As for the beer and energy drinks tall cans, two battery cells were placed on the magnet, one on top of the other. The battery terminals were oriented in such a way that they were connected in series, Fig. 12, which ensured a proper adjustment of the series-connected batteries to the height of a can and caused a higher intensity of current required to set a heavier beer can in rotation.

7. Summary

Some interesting educational demonstration experiments described in the books several years ago seem to have been forgotten. Very often some forbidden or hazardous and inaccessible substances were used in the experiments, such as mercury in Barlow's wheel. It turns out, however, that the appearance and availability of new materials, a wide range of packaging options for a variety of products or gadgets that make our life easier allow us to conduct these already forgotten and once difficult experiments in an easy and attractive way. The homopolar motors described here are a good example of that. Their presentation in a new format was made possible thanks to, among other things, the availability of strong neodymium magnets covered with anti-corrosion coating of nickel, battery cell enclosed in a metal shell, which protects electrical appliances from damage caused by a battery electrolyte leak, as well as the appearance of new types of packaging materials, such as steel and aluminum beverage cans. Obviously, for all that we also require some imaginative contrivance or inventive skill, as well as coming to the realization that the laws of physics universal in character as they are, they are also in operation with regard to these objects.

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Chair of Modelling the Teaching and Learning Processes
University of Łódź
Pomorska 149/153, PL-90-236 Łódź
Poland
e-mail: bedastan@uni.lodz.pl

Low Secondary School No 1
Sterlinga 24, PL-90-212 Łódź
Poland

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WSPÓŁCZESNE SILNIKI HOMOPOLARNE

S t r e s z c z e n i e

W początkowej części artykułu opisano krótko znane z literatury koło Barlowa, jako przykład silnika elektrycznego homopolarnego, czyli jednobiegunowego. W następnych częściach przedstawione zostały przykłady budowy takiego silnika przy użyciu współcześnie dostępnych przedmiotów i materiałów codziennego użytku, takich jak: okrągłe baterie, magnes neodymowy, kawałek miedzianego drutu, gwóźdź stalowy oraz aluminiowe puszki od napojów i dezodorantów. Opisane zostały silniki wykonane z wymienionych przedmiotów, w których elementami wirującymi są: magnes neodymowy, bateria, druciana ramka oraz aluminiowe puszki. Podano wskazówki techniczne oraz wyjaśnienie zasady działania zbudowanych silników. Przykłady te są bardzo proste w realizacji i dają widowiskowy efekt. Ponadto dobrze nadają się do stworzenia sytuacji problemowej, zachęcającej osoby oglądające te silniki do lepszego zrozumienia praw fizyki.

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In memory of
Professor Roman Stanisław Ingarden

Marek Stojecki

THE PROBLEM OF REGRESSION FOR THE HILBERT SPACE-VALUED FUNCTIONS

Summary

The regressive polynomials play an important role in analysis of empiric data represented by the pair of finite sequences x and y. The linear dependence most common in practice, expressed for example in physical and chemical laws, brings too much simplification in searched dependence between the data. The generalized regression problem considered in this paper leads to solution of a certain extremal problem, defined in a finite-dimensional Hilbert space.

1. Formulation of the regression problem

In order to solve mentioned above extremal problem, we ought to recall first the regression structure, cf. [3]. By the *regression structure* we maen a structure $\mathfrak{R} := (A, B, \delta, x, y)$ where

I.1 A, B are nonempty sets;

I.2 $x\colon \Omega_1 \to A$, $y\colon \Omega_2 \to B$ for some nonempty sets Ω_1 and Ω_2;

I.3 $\delta\colon (\Omega_1 \to B) \times (\Omega_2 \to B) \to \bar{\mathbb{R}}$.

Having desposed a regression structure $\mathfrak{R}$ we may consider the *functional model* of $\mathfrak{R}$, i.e. a nonempty subclass $\mathcal{F}$ of the class $A \to B$.
We will seek optimal theoretic functions $f_0 \in \mathcal{F}$ which are the best fitted to empirical data functions x and y with respect to the criterion δ, i.e. all functions $f_0 \in \mathcal{F}$ satisfying inequality

$$F(f_0) \leq F(f), \qquad f \in \mathcal{F}, \tag{1}$$

where F is the functional defined as follows

$$\mathcal{F} \ni f \to F(f) := \delta(f \circ x, y) \in \bar{\mathbb{R}}. \tag{2}$$

The set of all functions satisfying (1) we will denote by $\mathrm{Reg}(\mathcal{F}, \mathfrak{R})$.

From now we shall consider the family of regression structures $\mathfrak{R}$ in the case, that

II.1 B is the support of a complex (resp.real) Hilbert Space, which means that $\mathbf{B} := (B, +, \cdot; \langle\cdot|\cdot\rangle_B)$ is a complex (resp. real) Hilbert Space.

In order to make the mentioned above regression problem well defined on the ground of Hilbert Spaces we have to make additional assumptions:

II.2 There exist a σ-field $\mathcal{B}$ of subsets of the cartesian product $\Omega_1 \times \Omega_2$ and a measure $\mu\colon \mathcal{B} \to [0, +\infty]$ such that the function δ satisfies the following equality for every $u\colon \Omega_1 \to B$, $v\colon \Omega_2 \to B$

$$\delta(u, v) := \int\limits_{\Omega_1 \times \Omega_2} \| u(t_1) - v(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2), \tag{3}$$

provided the function $\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to \| u(t_1) - v(t_2) \|_B$ is $\mathcal{B}$-measureable and $\delta(u, v) = +\infty$ otherwise.

II.3 The function $\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to y(t_2)$ is $\mathcal{B}$-measureable.

From now we confine ourselves to the case that B is a finite-dimensional Hilbert Space. Lets us consider now the set $\mathcal{L}_1(\mathfrak{R})$ of all functions $f\colon A \to B$ such that $\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to \| f \circ x(t_1) \|_B$ is $\mathcal{B}$-measureable and

$$\int\limits_{\Omega_1 \times \Omega_2} \| f \circ x(t_1) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) < +\infty \tag{4}$$

and the set $\mathcal{L}_2(\mathfrak{R})$ of all functions $g\colon B \to B$ such that $\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to \| g \circ y(t_2) \|_B$ is $\mathcal{B}$-measureable and

$$\int\limits_{\Omega_1 \times \Omega_2} \| g \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) < +\infty. \tag{5}$$

From the Schwarz inequality and the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, $a, b \in \mathbb{R}$ we obtain

$$|\langle z|w\rangle_B| \leq \frac{1}{2}(\| z \|_B^2 + \| w \|_B^2), \qquad z, w \in B. \tag{6}$$

Since the sum of two $\mathcal{B}$-measureable functions is $\mathcal{B}$-measureable we conclude from (6) that the functional

$$\mathcal{L}_1(\mathfrak{R}) \times \mathcal{L}_1(\mathfrak{R}) \ni (u, v) \mapsto \langle u|v\rangle_\star := \int\limits_{\Omega_1 \times \Omega_2} \langle u \circ x(t_1)|v \circ x(t_1)\rangle_B \mathrm{d}\mu(t_1, t_2) \tag{7}$$

is well defined.

Hence $\langle u|u\rangle_\star \geq 0$ for every $u \in \mathcal{L}_1(\mathfrak{R})$ and in consequence functional

$$\mathcal{L}_1(\mathfrak{R}) \ni u \mapsto \| u \|_\star := \sqrt{\langle u|u\rangle_\star} = \left(\int_{\Omega_1 \times \Omega_2} \| u \circ x(t_1) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \right)^{\frac{1}{2}} \tag{8}$$

is well defined.

Combining the inequality (6) with (4) and (5) we see that for every $g \in \mathcal{L}_2(\mathfrak{R})$ functional

$$\mathcal{L}_1(\mathfrak{R}) \ni u \mapsto g^\star(u) := \int_{\Omega_1 \times \Omega_2} \langle u \circ x(t_1) | g \circ y(t_2) \rangle_B \mathrm{d}\mu(t_1, t_2) \tag{9}$$

is also well defined.

Lemma 1.1. *The structure $\mathcal{H}(\mathfrak{R}) := (\mathcal{L}_1(\mathfrak{R}), +, \cdot; \langle\cdot|\cdot\rangle_\star)$ is a complex (resp. real) p-Hilbert Space, i.e. $(\mathcal{L}_1(\mathfrak{R}), +, \cdot)$ is a linear space and the following properties hold for $u, v, w \in \mathcal{L}_1(\mathfrak{R})$ and $\alpha, \beta \in \mathbb{C}$ (resp. $\alpha, \beta \in \mathbb{R}$).*

$$\begin{aligned} \langle \alpha u + \beta v | w \rangle_\star &= \alpha \langle u|w\rangle_\star + \beta \langle v|w\rangle_\star \\ \langle u|v\rangle_\star &= \overline{\langle v|u\rangle_\star} \\ \langle u|u\rangle_\star &\geq 0 \end{aligned} \tag{10}$$

Moreover, every Cauchy sequence in $\mathcal{L}_1(\mathfrak{R})$ is convergent to a certain function in $\mathcal{L}_1(\mathfrak{R})$ with respect to the norm $\| \cdot \|_\star$.

Proof. Without losing the generality we may confine ourselves to the complex case only.
By the equality

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2), \qquad x, y \in X$$

which holds for every Hilbert Space X, we get

$$\| z + w \|_B^2 \leq 2(\| z \|_B^2 + \| w \|_B^2), \qquad z, w \in B. \tag{11}$$

By (11) and (4) we see that for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u, v \in \mathcal{L}_1(\mathfrak{R})$

$$\begin{aligned}
&\int\limits_{\Omega_1\times\Omega_2} \| (\lambda_1 u + \lambda_2 v)\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) \\
&= \int\limits_{\Omega_1\times\Omega_2} \| \lambda_1 u\circ x(t_1) + \lambda_2 v\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) \\
&\overset{(11)}{\leq} 2 \int\limits_{\Omega_1\times\Omega_2} \| \lambda_1 u\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) + 2 \int\limits_{\Omega_1\times\Omega_2} \| \lambda_2 v\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) \\
&= 2|\lambda_1|^2 \int\limits_{\Omega_1\times\Omega_2} \| u\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) \\
&\quad +2|\lambda_2|^2 \int\limits_{\Omega_1\times\Omega_2} \| v\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) < +\infty.
\end{aligned}$$

Thus $\lambda_1 u + \lambda_2 v \in \mathcal{L}_1(\mathfrak{R})$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u, v \in \mathcal{L}_1(\mathfrak{R})$. Therefore $\mathcal{L}_1(\mathfrak{R})$ is a linear set.

From the properties of the inner product $\langle\cdot|\cdot\rangle_B$ and the formula (7) we obtain the properties (10). Now we shall prove the completeness of $\mathcal{H}(\mathfrak{R})$. The mapping $x\colon \Omega_1 \to A$ induces the σ-field $\mathcal{B}_x := \{V \in 2^A\colon x^{-1}(V)\times\Omega_2 \in \mathcal{B}\}$ and the measure $\mathcal{B}_x \ni V \to \mu_x(V) := \mu(x^{-1}(V)\times\Omega_2)$. Fix $u \in \mathcal{L}_1(\mathfrak{R})$. Since the function $\Omega_1\times\Omega_2 \ni (t_1,t_2) \to u\circ x(t_1)$ is $\mathcal{B}$-measureable we see that for every Borel set $U \subset B$:

$$x^{-1}\circ(u^{-1}(U))\times\Omega_2 = (u\circ x)^{-1}(U)\times\Omega_2 \in \mathcal{B}.$$

Hence $u^{-1}(U) \in \mathcal{B}_x$. Thus u is $\mathcal{B}_x$-measureable as well. Moreover

$$\begin{aligned}
\| u \|_\star^2 &= \int\limits_{\Omega_1\times\Omega_2} \| u\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2) \\
&= \int\limits_A \| u(t) \|_B^2 \,\mathrm{d}\mu_x(t) = \| u \|_2^2,
\end{aligned} \tag{12}$$

Since the space $(B,+,\cdot)$ is finite dimensional space, there exist $n \in \mathbb{N}$ and the orthornormal basis $\{e_1, e_2, \ldots, e_n\} \subset B$. Hence

$$u(t) = \sum_{k=1}^n \langle u(t)|e_k\rangle_B e_k$$

for $t \in A$, which together with (12) gives

$$\| u \|_\star^2 = \int_A \| u(t) \|_B^2 \, \mathrm{d}\mu_x(t) \tag{13}$$
$$= \int_A \| \sum_{k=1}^n \langle u(t)|e_k\rangle_B e_k \|_B^2 \, \mathrm{d}\mu_x(t)$$
$$= \sum_{k=1}^n \int_A |\langle u(t)|e_k\rangle_B|^2 \, \mathrm{d}\mu_x(t).$$

Let $\mathbb{N} \ni n \to u_n \in \mathcal{L}_1(\mathfrak{R})$ be a Cauchy sequence in the space $\mathcal{H}(\mathfrak{R})$. From (13) we have for any $k \in \mathbb{Z}_{1,n}$

$$\int_A |\langle u_n(t)|e_l\rangle_B - \langle u_m(t)|e_l\rangle_B|^2 \mathrm{d}\mu_x(t)$$
$$= \int_A |\langle u_n(t) - u_m(t)|e_l\rangle_B|^2 \mathrm{d}\mu_x(t)$$
$$\leq \sum_{k=1}^n \int_A |\langle u_n(t) - u_m(t)|e_k\rangle_B|^2 \, \mathrm{d}\mu_x(t)$$
$$\overset{(13)}{=} \| u_n - u_m \|_\star^2 \to 0 \quad \text{as} \quad n, m \to \infty.$$

By the completeness of $L^2(A, \mathcal{B}_x, \mu_x)$ we deduce, that there exist functions $\tilde{u}_l \in L^2(A, \mathcal{B}_x, \mu_x)$, $l \in \mathbb{Z}_{1,n}$, such that

$$\int_A |\langle u_n(t)|e_l\rangle_B - \tilde{u}_l(t)|^2 \mathrm{d}\mu_x(t) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad l \in \mathbb{Z}_{1,n}.$$

Puting $u(t) := \sum_{k=1}^n \tilde{u}_k(t) e_k$ we see that $u \in \mathcal{L}_1(\mathfrak{R})$ and

$$\| u_n(t) - u(t) \|_\star^2 = \| \sum_{k=1}^n \langle u_n(t)|e_k\rangle_B e_k - \sum_{k=1}^n \langle u(t)|e_k\rangle_B e_k \|_\star^2 \tag{14}$$
$$= \| \sum_{k=1}^n \langle u_n(t) - u(t)|e_k\rangle_B e_k \|_\star^2$$
$$= \int_A \sum_{k=1}^n |\langle u_n(t) - u(t)|e_k\rangle_B|^2 \mathrm{d}\mu_x(t)$$
$$= \sum_{k=1}^n \int_A |\langle u_n(t)|e_k\rangle_B - \tilde{u}_k(t)|^2 \mathrm{d}\mu_x(t) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence the completeness is proved.

Lemma 1.2. *The structure* $(\mathcal{L}_2(\mathfrak{R}), +, \cdot)$ *is a complex (resp. real) linear space. Moreover, for each* $g \in \mathcal{L}_2(\mathfrak{R})$ *the functional* $g^\star$ *defined in* (9) *is bounded on* $\mathcal{H}(\mathfrak{R})$ *and the supremum norm of* $g^\star$ *satisfies the following inequality:*

$$\sup\{|g^\star(f)|\colon f \in \mathcal{L}_1(\mathfrak{R}) \tag{15}$$

and

$$\| f \|_\star \leq 1\} \leq \left(\int\limits_{\Omega_1 \times \Omega_2} \| g \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \right)^{\frac{1}{2}}$$

Proof. From the inequality (11) and by (5) we have that for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $g, h \in \mathcal{L}_2(\mathfrak{R})$

$$\begin{aligned}
&\int\limits_{\Omega_1 \times \Omega_2} \| (\lambda_1 g + \lambda_2 h) \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \int\limits_{\Omega_1 \times \Omega_2} \| \lambda_1 g \circ y(t_2) + \lambda_2 h \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&\overset{(11)}{\leq} 2 \int\limits_{\Omega_1 \times \Omega_2} \| \lambda_1 g \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&\qquad + 2 \int\limits_{\Omega_1 \times \Omega_2} \| \lambda_2 h \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= 2|\lambda_1|^2 \int\limits_{\Omega_1 \times \Omega_2} \| g \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&\qquad + 2|\lambda_2|^2 \int\limits_{\Omega_1 \times \Omega_2} \| h \circ y(t_2) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) < \infty.
\end{aligned}$$

Thus

$$\lambda_1 g + \lambda_2 h \in \mathcal{L}_2(\mathfrak{R}) \tag{16}$$

for all

$$\lambda_1, \lambda_2 \in \mathbb{C}, \quad g, h \in \mathcal{L}_2(\mathfrak{R})$$

and so $\mathcal{L}_2(\mathfrak{R})$ is a linear set. Then the structure $(\mathcal{L}_2(\mathfrak{R}), +, \cdot)$ is a linear space. From the algebraic properties of Lebesgue integral for all $u, v \in \mathcal{L}_1(\mathfrak{R})$, $g \in \mathcal{L}_2(\mathfrak{R})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ we get

$$\begin{aligned}
g^\star(\lambda_1 u + \lambda_2 v) &= \int\limits_{\Omega_1 \times \Omega_2} \langle (\lambda_1 u + \lambda_2 v) \circ x(t_1) | g \circ y(t_2) \rangle_B \mathrm{d}\mu(t_1, t_2) \\
&= \int\limits_{\Omega_1 \times \Omega_2} \langle \lambda_1 u \circ x(t_1) + \lambda_2 v \circ x(t_1) | g \circ y(t_2) \rangle_B \mathrm{d}\mu(t_1, t_2)
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\Omega_1\times\Omega_2} \langle \lambda_1 u\circ x(t_1)|g\circ y(t_2)\rangle_B + \langle \lambda_2 v\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\\
&= \int\limits_{\Omega_1\times\Omega_2} \langle \lambda_1 u\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\\
&\quad + \int\limits_{\Omega_1\times\Omega_2} \langle \lambda_2 v\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\\
&= \lambda_1 \int\limits_{\Omega_1\times\Omega_2} \langle u\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\\
&\quad + \lambda_2 \int\limits_{\Omega_1\times\Omega_2} \langle v\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\\
&= \lambda_1 g^\star(u) + \lambda_2 g^\star(v)
\end{aligned}
$$

so the functional $g^\star$ is linear.

$$
g^\star(\lambda_1 u + \lambda_2 v) = \lambda_1 g^\star(u) + \lambda_2 g^\star(v) \quad \text{for} \quad u,v\in\mathcal{L}_1(\mathfrak{R}) \quad \text{and} \quad g\in\mathcal{L}_2(\mathfrak{R}). \tag{17}
$$

Now using twice Schwarz inequality we shall evaluate the quantity $|g^\star(f)|$ for all $f\in\mathcal{L}_1(\mathfrak{R})$ and $g\in\mathcal{L}_2(\mathfrak{R})$. We have

$$
\begin{aligned}
|g^\star(f)| &= \left| \int\limits_{\Omega_1\times\Omega_2} \langle f\circ x(t_1)|g\circ y(t_2)\rangle_B \mathrm{d}\mu(t_1,t_2)\right|\\
&\le \int\limits_{\Omega_1\times\Omega_2} |\langle f\circ x(t_1)|g\circ y(t_2)\rangle_B|\, \mathrm{d}\mu(t_1,t_2)\\
&\le \int\limits_{\Omega_1\times\Omega_2} \| f\circ x(t_1) \|_B \cdot \| g\circ y(t_2)\|_B \,\mathrm{d}\mu(t_1,t_2)\\
&\le \left(\int\limits_{\Omega_1\times\Omega_2} \| f\circ x(t_1) \|_B^2 \,\mathrm{d}\mu(t_1,t_2)\right)^{\frac{1}{2}}\\
&\quad \cdot \left(\int\limits_{\Omega_1\times\Omega_2} \| g\circ y(t_2) \|_B^2 \,\mathrm{d}\mu(t_1,t_2)\right)^{\frac{1}{2}}\\
&= \| f \|_\star \cdot \left(\int\limits_{\Omega_1\times\Omega_2} \| g\circ y(t_2) \|_B^2 \,\mathrm{d}\mu(t_1,t_2)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence

$$\sup\{|g^{\star}(f)|\colon f \in \mathcal{L}_1(\mathfrak{R}) \text{and} \parallel f \parallel_{\star} \leq 1\} \leq \left(\int\limits_{\Omega_1 \times \Omega_2} \parallel g \circ y(t_2) \parallel_B^2 \mathrm{d}\mu(t_1, t_2) \right)^{\frac{1}{2}}$$

and the proof is complete.

Remark 1.4. Given a regression structure $\mathfrak{R} := (A, B, \delta; x, y)$ satisfying the properties I.1 - I.3 we see that for each function $g\colon B \to B$, $\mathfrak{R}_g := (A, B, \delta; x, g \circ y)$ is a regression structure too.

Let the sequence $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ be the orthogonal basis in $\mathbf{B}$. In later applications, the following definitions will be required. For every $f \in \mathcal{F}, g \in \mathcal{L}_1(\mathfrak{R})$ and $k \in \mathbb{Z}_{1,n}$ we define

$$A \ni t \to f_k(t) := \frac{\langle f(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2}, \tag{18}$$

$$\mathcal{F}_k := \{A \ni t \to \frac{\langle f(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2} : f \in \mathcal{F}\}, \tag{19}$$

$$B \ni t \to g_k(t) := \frac{\langle g(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2}, \tag{20}$$

$$\mathfrak{R}_g^k := (A, \mathbb{C}, \delta^{\star}; x, g_k \circ y), \tag{21}$$

where

$$\delta^{\star}(u, v) := \int\limits_{\Omega_1 \times \Omega_2} |u(t_1) - v(t_2)|^2 \mathrm{d}\mu(t_1, t_2) \text{ for every } \ u\colon \Omega_1 \to \mathbb{C}, v\colon \Omega_2 \to \mathbb{C}.$$

The following lemmas hold:

Lemma 1.5. *If $\mathcal{F}$ ($\mathcal{F} \neq \emptyset$) is a linear set in $\mathcal{H}(\mathfrak{R})$ and the sequence $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is the orthogonal basis in $\mathbf{B}$ then the set $\mathcal{F}_k$ is a linear set in the linear space $(A \to \mathbb{C}, +, \cdot)$ for every $k \in \mathbb{Z}_{1,n}$.*

Proof. Fix $k \in \mathbb{Z}_{1,n}$. For every $h_1, h_2 \in \mathcal{F}_k$ exist $\tilde{h_1}, \tilde{h_2} \in \mathcal{F}$ such that for $t \in A$

$$h_1(t) = \frac{\langle \tilde{h_1}(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2}, \qquad h_2(t) = \frac{\langle \tilde{h_2}(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2}.$$

For every $\lambda_1, \lambda_2 \in \mathbb{C}$ we get

$$\begin{aligned}(\lambda_1 h_1 + \lambda_2 h_2)(t) &= \lambda_1 h_1(t) + \lambda_2 h_2(t) \\ &= \frac{\langle \lambda_1 \tilde{h_1}(t) + \lambda_2 \tilde{h_2}(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2} = \frac{\langle (\lambda_1 \tilde{h_1} + \lambda_2 \tilde{h_2})(t)|e_k\rangle_B}{\parallel e_k \parallel_B^2}\end{aligned}$$

By the linearity of set $\mathcal{F}$ we have $\lambda_1 \tilde{h_1} + \lambda_2 \tilde{h_2} \in \mathcal{F}$. Then, by the (19) we get $\lambda_1 h_1 + \lambda_2 h_2 \in \mathcal{F}_k$, so $\mathcal{F}_k$ is a linear set in the space $(A \to \mathbb{C}, +, \cdot)$.

Lemma 1.6. *If $\mathcal{F}$ ($\mathcal{F} \neq \emptyset$) is a linear set in $\mathcal{H}(\mathfrak{R})$ and the sequence $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is the orthogonal basis in $\mathbf{B}$ then the set $\mathcal{F}_k \subset \mathcal{L}_1(\mathfrak{R}_g^k)$ for every $k \in \mathbb{Z}_{1,n}$.*

Proof. By (19) for every $f \in \mathcal{F}_k$ there exist $\tilde{f} \in \mathcal{F}$ such that

$$f(t) = \frac{\langle \tilde{f}(t) | e_k \rangle_B}{\| e_k \|_B^2} \quad \text{for} \quad t \in A.$$

Since $\mathcal{F} \subset \mathcal{L}_1(\mathfrak{R})$ the function $\Omega_1 \times \Omega_2 \ni (t_1, t_2) \to \tilde{f} \circ x(t_1)$ is $\mathcal{B}$-measureable and

$$\int_{\Omega_1 \times \Omega_2} \| \tilde{f} \circ x(t_1) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) < \infty. \tag{22}$$

From the continuity of the inner product $\langle \cdot | \cdot \rangle_B$ we conclude that the function f is $\mathcal{B}$-measureable and by (22)

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} |f \circ x(t_1)|^2 \mathrm{d}\mu(t_1, t_2) &= \int_{\Omega_1 \times \Omega_2} \left| \frac{\langle \tilde{f} \circ x(t_1) | e_k \rangle_B}{\| e_k \|_B^2} \right|^2 \mathrm{d}\mu(t_1, t_2) \\ &\leq \int_{\Omega_1 \times \Omega_2} \| \tilde{f} \circ x(t_1) \|_B^2 \, \mathrm{d}\mu(t_1, t_2) < \infty. \end{aligned}$$

Lemma 1.7. *If a function $f \colon \Omega_1 \times \Omega_2 \to B$ is $\mathcal{B}$-measureable, then the function $\Omega_1 \times \Omega_2 \ni t \to \langle f(t) | e \rangle_B$ is also $\mathcal{B}$-measureable for every $e \in B$.*

Proof. The functional $B \ni x \to F(x) := \langle x | e \rangle_B$ is continuous. Let U be an open set in $\mathbb{C}$. Then the set $F^{-1}(U)$ is open in B. Since $f \colon \Omega_1 \times \Omega_2 \to B$ is $\mathcal{B}$-measureable, then the set $f^{-1}(F^{-1}(U)) = (F \circ f)^{-1}(U)$ is $\mathcal{B}$-measureable. Hence the function $\Omega_1 \times \Omega_2 \ni t \to \langle f(t) | e \rangle_B$ is $\mathcal{B}$-measureable.

2. Solution of the regression problem

The next lemmas enable us to reduce our regression problem to the simplest case $B = \mathbb{C}$ (resp. $B = \mathbb{R}$).

Lemma 2.8. *If $\mathcal{F}$ ($\mathcal{F} \neq \emptyset$) is a linear set in $\mathcal{H}(\mathfrak{R})$, $g \in \mathcal{L}_2(\mathfrak{R})$ and the sequence $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is the orthonormal basis in $\mathbf{B}$, then for every $f \in \mathcal{F}$ and $k \in \mathbb{Z}_{1,n}$ holds*

$$f_k \in \operatorname{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k) \Longrightarrow f \in \operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g) \tag{23}$$

Proof. Fix $h \in \mathcal{F}$. We have

$$
\begin{aligned}
\delta(h \circ x, g \circ y) &= \int\limits_{\Omega_1 \times \Omega_2} \| \, h \circ x(t_1) - g \circ y(t_2) \, \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \int\limits_{\Omega_1 \times \Omega_2} \| \sum_{k=1}^{n} h_k \circ x(t_1) e_k - \sum_{k=1}^{n} g_k \circ y(t_2) e_k \, \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \int\limits_{\Omega_1 \times \Omega_2} \| \sum_{k=1}^{n} (h_k \circ x(t_1) - g_k \circ y(t_2)) e_k \, \|_B^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \int\limits_{\Omega_1 \times \Omega_2} \sum_{k=1}^{n} |(h_k \circ x(t_1) - g_k \circ y(t_2))|^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \sum_{k=1}^{n} \int\limits_{\Omega_1 \times \Omega_2} |(h_k \circ x(t_1) - g_k \circ y(t_2))|^2 \, \mathrm{d}\mu(t_1, t_2) \\
&= \sum_{k=1}^{n} \delta^\star(h_k \circ x, g_k \circ y)
\end{aligned} \tag{24}
$$

Hence we get the following equivalence:

$$
\begin{aligned}
&\left[\sum_{k=1}^{n} \delta^\star(f_k \circ x, g_k \circ y) \leq \sum_{k=1}^{n} \delta^\star(h_k \circ x, g_k \circ y) \right] \\
&\qquad \Longleftrightarrow \Big[\delta(f \circ x, g \circ y) \leq \delta(h \circ x, g \circ y) \Big] \qquad \text{for} \quad f, h \in \mathcal{F}
\end{aligned} \tag{25}
$$

Let's assume now, that $f_k \in \mathrm{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k)$ for $k \in \mathbb{Z}_{1,n}$. Then

$$
\delta^\star(f_k \circ x, g_k \circ y) \leq \delta^\star(h_k \circ x, g_k \circ y) \qquad \text{for} \quad h \in \mathcal{F}, k \in \mathbb{Z}_{1,n}. \tag{26}
$$

Hence

$$
\sum_{k=1}^{n} \delta^\star(f_k \circ x, g_k \circ y) \leq \sum_{k=1}^{n} \delta^\star(h_k \circ x, g_k \circ y) \qquad \text{for} \quad h \in \mathcal{F}.
$$

From (24) we obtain

$$
\delta(f \circ x, g \circ y) \leq \delta(h \circ x, g \circ y) \qquad \text{for} \quad h \in \mathcal{F}, \tag{27}
$$

which means, that $f \in \mathrm{Reg}(\mathcal{F}, \mathfrak{R}_g)$.

The converse implication is not true in general. If we wish to get equivalence in (23) we should make additional assumptions. First we shall define a new notion.

Definition 2.2. A linear set $\mathcal{G} \subset (A \to B)$ is said to be linearly closed in the direction of a vector $e \in B$ if the condition holds:

$$
f + h \bullet e \in \mathcal{G} \quad \text{for } f \in \mathcal{G} \quad \text{and} \quad h \in P_e(\mathcal{G}), \tag{28}
$$

where

$$A \ni t \to P_e(\phi)(t) := \frac{\langle \phi(t)|e\rangle_B}{\| e \|_B^2} \tag{29}$$

and for all $\psi : A \to \mathbb{C}$

$$A \ni t \to \psi \bullet e(t) := \psi(t)e \tag{30}$$

We have the following lemma:

Lemma 2.3. *If $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is an orthogonal basis in $\mathbf{B}$ and $\mathcal{F}$ ($\mathcal{F} \neq \emptyset$) is the linearly closed in each direction $e_k \in B, k \in \mathbb{Z}_{1,n}$, then for all $f \in \mathcal{F}$ and $k \in \mathbb{Z}_{1,n}$ holds*

$$f \in \operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g) \Longrightarrow f_k \in \operatorname{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k) \tag{31}$$

Proof. Let $f \in \operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g)$. By the equivalence (24) we obtain the condition (27). Fix $l \in \mathbb{Z}_{1,n}$ and $h^\star \in \mathcal{F}_l$. Let's consider the function $h := f + h^\star \bullet e_l$. Since $P_{e_l}(\mathcal{F}) = \mathcal{F}_l$ we conclude from the fact that $\mathcal{F}$ is linearly closed in each direction e_k, $k \in \mathbb{Z}_{1,n}$, that $h \in \mathcal{F}$. From this observation we have

$$h = \sum_{k=1}^{n} f_k \bullet e_k + h^\star \bullet e_l = (f_l + h^\star) \bullet e_l + \sum_{l \neq k=1}^{n} f_k \bullet e_k.$$

Hence $h_l = f_l + h^\star$ and $h_k = f_k$ for $k \in \mathbb{Z}_{1,n} \backslash \{l\}$. By (24) we have

$$\sum_{k=1}^{n} \delta^\star(f_k \circ x, g_k \circ y) \leq \sum_{k=1}^{n} \delta^\star(h_k \circ x, g_k \circ y) \qquad \text{for} \quad h \in \mathcal{F}.$$

and so

$$\delta^\star(f_l \circ x, g_l \circ y) \leq \delta^\star((f_l + h^\star) \circ x, g_l \circ y) \qquad \text{for} \quad h^\star \in \mathcal{F}_l.$$

This yields $f_l \in \operatorname{Reg}(\mathcal{F}_l, \mathfrak{R}_g^l)$, which completes the proof.

Lemma (2.3) together with lemma 2.8 gives the following theorem:

Theorem 1. *Suppose that $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is an orthogonal basis in $\mathbf{B}$ and $\mathcal{F}$ ($\mathcal{F} \neq \emptyset$) is a linear set in $\mathcal{H}(\mathfrak{R})$, which is linearly closed in each direction e_k. Then for every $f \in \operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g)$ there exist a sequence*

$$\mathbb{Z}_{1,n} \ni k \mapsto f_k \in \operatorname{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k)$$

such that

$$f = \sum_{k=1}^{n} f_k \bullet e_k. \tag{32}$$

Conversely, for every sequence $\mathbb{Z}_{1,n} \ni k \in \mathrm{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k)$

$$\sum_{k=1}^{n} f_k \bullet e_k \in \mathrm{Reg}(\mathcal{F}, \mathfrak{R}_g) \tag{33}$$

Remark 2.5 In the other words Theorem 1 states that

$$\mathrm{Reg}(\mathcal{F}, \mathfrak{R}_g) = \sum_{k=1}^{n} \{f_k \bullet e_k \colon f_k \in \mathrm{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k)\}.$$

Theorem 6. *Suppose that* $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ *is an orthogonal basis in* $\mathbf{B}$ *and* $\mathcal{F}$ $(\mathcal{F} \neq \emptyset)$ *is a linear set in* $\mathcal{H}(\mathfrak{R})$*, linearly closed in each direction* e_k*.. Then* $\mathcal{F} := \sum_{k=1}^{n} \mathcal{F}_k \bullet e_k$*, where* $\mathbb{Z}_{1,n} \ni k \to \mathcal{F}_k \subset (A \to \mathbb{C})$ *is a sequence such that* $\mathcal{F}_k$ *is a linear set in* $(A \to \mathbb{C}, +, \cdot)$*.*

Proof. Fix $f, h \in \mathcal{F}$. By the definition of $\mathcal{F}$ there exists a sequence

$$\mathbb{Z}_{1,n} \ni k \to f_k \in \mathcal{F}_k \quad \text{and} \quad \mathbb{Z}_{1,n} \ni k \to h_k \in \mathcal{F}_k,$$

such that

$$f = \sum_{k=1}^{n} f_k \bullet e_k, \qquad h = \sum_{k=1}^{n} h_k \bullet e_k. \tag{34}$$

Then for each $l \in \mathbb{Z}_{1,n}$ and $t \in A$

$$\begin{aligned} f(t) + \frac{\langle h(t)|e_l\rangle_B}{\| e_l \|_B^2} \cdot e_l &= \sum_{k=1}^{n} f_k(t) \cdot e_k + \frac{\langle \sum_{k=1}^{n} h_k(t) \cdot e_k | e_l\rangle_B}{\| e_l \|_B^2} \cdot e_l \\ &= \sum_{k=1}^{n} f_k(t) \cdot e_k + \frac{\sum_{k=1}^{n} \langle h_k(t) \cdot e_k | e_l\rangle_B}{\| e_l \|_B^2} \cdot e_l \\ &= \sum_{k=1}^{n} f_k(t) \cdot e_k + \sum_{k=1}^{n} h_k(t) \frac{\langle e_k | e_l\rangle_B}{\| e_l \|_B^2} \cdot e_l \end{aligned}$$

$$= \sum_{k=1}^{n} f_k(t) \cdot e_k + h_l(t) \cdot e_l = \sum_{k=1 \neq l}^{n} f_k(t) \cdot e_k + (f_l + h_l)(t) \cdot e_l.$$

Hence $f + P_{e_l}(h) = \sum_{k=1\neq l}^{n} f_k \bullet e_k + (f_l + h_l) \bullet e_l \in \mathcal{F}$ because $f_l + h_l \in \mathcal{F}_l$. Therefore $\mathcal{F}$ is linearly closed in each direction e_l, as $l \in \mathbb{Z}_{1.n}$.
Conversely, suppose now, that $\mathcal{F} \subset (A \subset B)$ is a linear set linearly closed in each direction e_l, as $l \in \mathbb{Z}_{1.n}$. For each $k \in \mathbb{Z}_{1,n}$ we define

$$\mathcal{F}_k := \{P_{e_k}(h) \colon h \in \mathcal{F}\}.$$

Fix $f \in \mathcal{F}$. Since

$$f(t) = \sum_{k=1}^{n} \frac{\langle f(t)|e_k\rangle_B}{\| e_k \|_B^2} \cdot e_k, \qquad t \in A,$$

we have

$$f = \sum_{k=1}^{n} P_{e_k}(f) \bullet e_k \in \sum_{k=1}^{n} \mathcal{F}_k \bullet e_k.$$

This implies the following inclusion

$$\mathcal{F} \subset \sum_{k=1}^{n} \mathcal{F}_k \bullet e_k. \tag{35}$$

Given $k \in \mathbb{Z}_{1,n}$ fix $f_k \in \mathcal{F}_k$. Then $f_k \in P_{e_k}(f)$ for certain $f \in \mathcal{F}$. Since $\mathcal{F}$ is linearly closed in the direction e_k and $\Theta, f \in \mathcal{F}$, we see that

$$f_k \bullet e_k = \Theta + f_k \bullet e_k = \Theta + P_{e_k}(f) \bullet e_k \in \mathcal{F}.$$

Thus $\mathcal{F}_k \bullet e_k \subset \mathcal{F}$ and consequently $\sum_{k=1}^{n} \mathcal{F}_k \bullet e_k \subset \mathcal{F}$, because $\mathcal{F}$ is a linear set. This inclusion together with the inverse one (35) gives the equality

$$\mathcal{F} = \sum_{k=1}^{n} \mathcal{F}_k \bullet e_k \tag{36}$$

We can now apply the theory elaborated by D. Partyka and J. Zając.

Theorem 7. [Partyka, Zając, 2010] *Given $p \in \mathbb{N}$. Let $\mathbb{Z}_{1,p} \ni k \to h_k \in \mathcal{F} \backslash \Theta$ be a sequence satisfying the following two conditions:*

$$\operatorname{lin}(\{h_k \colon k \in \mathbb{Z}_{1,p}\}) = \mathcal{F} \tag{37}$$

as well as

$$h_k \perp h_l, \qquad k, l \in \mathbb{Z}_{1,p}, k \neq l. \tag{38}$$

If $y \in \mathcal{L}_2(\mathfrak{R})$ then

$$\operatorname{Reg}(\mathcal{F}, \mathfrak{R}) = (\Theta \cap \mathcal{F}) + \sum_{k=1}^{p} \frac{\overline{y^\star(h_k)}}{\| h_k \|^2} h_k, \tag{39}$$

where $\Theta := \{h \in \mathcal{L}_1(\mathfrak{R}) \colon \ \| h \| = 0\}$.

Corollary 1. *Suppose that $\mathbb{Z}_{1,n} \ni k \to e_k \in B$ is an orthogonal basis in* $\mathbf{B}$. *Given a sequence $\mathbb{Z}_{1,n} \ni k \to p_k \in \mathbb{N}$. Let for each $k \in \mathbb{Z}_{1,n}$, $\mathbb{Z}_{1,p_k} \ni l \to h_{l,k} \in \mathcal{L}_1(\mathfrak{R}^k) \setminus \Theta_k$ be an orthogonal sequence in $\mathcal{H}(\mathfrak{R}^k)$, i.e.*

$$\langle h_{l,k} | h_{j,k} \rangle = 0 \qquad \text{as} \quad l \neq j \tag{40}$$

Then for every sequence $\mathbb{Z}_{1,n} \ni k \to g_k \in \mathcal{L}_2(\mathfrak{R}^k)$

$$\operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g) = (\Theta \cap \mathcal{F}) + \sum_{k=1}^{n} \sum_{l=1}^{p_k} \frac{\overline{g_k^*(h_{k,l})}}{\| h_{l,k} \|^2} h_{l,k} \bullet e_k \tag{41}$$

where

$$\mathcal{F}_k := \operatorname{lin}(\{h_{k,l} \colon l \in \mathbb{Z}_{1,p_k}), \qquad k \in \mathbb{Z}_{1,n} \tag{42}$$

$$\mathcal{F} := \sum_{k=1}^{n} \mathcal{F}_k \bullet e_k \tag{43}$$

$$g := \sum_{k=1}^{n} g_k \bullet e_k \tag{44}$$

Proof. By Theorem 6 the set $\mathcal{F}$, given by the formula (43), is linearly closed in each direction e_k, $k \in \mathbb{Z}_{1,n}$. Then Theorem 1 shows that

$$\mathrm{Reg}(\mathcal{F}, \mathfrak{R}_g) = \sum_{k=1}^{n} \mathrm{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k) \bullet e_k \tag{45}$$

Applying now Theorem 7 we conclude from the assumption (40) that

$$\mathrm{Reg}(\mathcal{F}_k, \mathfrak{R}_g^k) = (\Theta_k \cap \mathcal{F}_k) + \sum_{l=1}^{p_k} \frac{\overline{g_k^*(h_{k,l})}}{\| h_{l,k} \|^2} h_{l,k} \quad \text{for} \quad k \in \mathbb{Z}_{1,n} \tag{46}$$

Combining this with (45) we have

$$\begin{aligned} \mathrm{Reg}(\mathcal{F}, \mathfrak{R}_g) &= \sum_{k=1}^{n} \left[(\Theta_k \cap \mathcal{F}_k) + \sum_{l=1}^{p_k} \frac{\overline{g_k^*(h_{k,l})}}{\| h_{l,k} \|^2} h_{l,k} \right] \bullet e_k \\ &= \sum_{k=1}^{n} (\Theta_k \cap \mathcal{F}_k) \bullet e_k + \sum_{k=1}^{n} \sum_{l=1}^{p_k} \frac{\overline{g_k^*(h_{k,l})}}{\| h_{l,k} \|^2} h_{l,k} \bullet e_k. \end{aligned} \tag{47}$$

Fix

$$f = \sum_{k=1}^{n} (\Theta_k \cap \mathcal{F}_k) \bullet e_k.$$

Then

$$f = \sum_{k=1}^{n} f_k \bullet e_k \quad \text{for} \quad \mathbb{Z}_{1,n} \ni k \to f_k \in \Theta_k \cap \mathcal{F}_k.$$

Hence $f_k \in \mathcal{F}_k$ and $\| f_k \| = 0$ as $k \in \mathbb{Z}_{1,n}$. Moreover

$$\| f \|_\star^2 = \sum_{k=1}^{n} \| f_k \|^2 \| e_k \|_B^2 = 0 \quad \text{i.e.} \quad f \in \Theta$$

Finally $f \in \Theta \cap \mathcal{F}$ which gives the following inclusion

$$\sum_{k=1}^{n} (\Theta_k \cap \mathcal{F}_k) \bullet e_k \subset \Theta \cap \mathcal{F} \tag{48}$$

Conversely, fix $f \in \Theta \cap \mathcal{F}$. By (43)

$$f = \sum_{k=1}^{n} f_k \bullet e_k$$

for a sequence $\mathbb{Z}_{1,n} \ni k \to f_k \in \mathcal{F}_k$. Since $f \in \Theta$ we have

$$0 = \| f \|_\star^2 = \sum_{k=1}^{n} \| f_k \|^2 \| e_k \|_B^2,$$

and so $f_k \in \Theta_k$ as $k \in \mathbb{Z}_{1,n}$. Hence $f_k \in \Theta_k \cap \mathcal{F}_k$ as $k \in \mathbb{Z}_{1,n}$. Therefore $\Theta \cap \mathcal{F} \subset \sum_{k=1}^{n} (\Theta_k \cap \mathcal{F}_k) \bullet e_k$. This inclusion together with the inverse one (48) yields the equality

$$\Theta \cap \mathcal{F} = \sum_{k=1}^{n} (\Theta_k \cap \mathcal{F}_k) \bullet e_k. \tag{49}$$

Combining (47) with (49) we obtain the equality (41), which completes the proof.

Remark 2.9. The equality (41) holds under the ortogonality assumption. Otherwise we imply the orthogonalization procedure.

Setting

$$h_{l,k}^* := h_{l,k} \bullet e_k \quad \text{for} \quad k \in \mathbb{Z}_{1,n}, l \in \mathbb{Z}_{1,p_k} \tag{50}$$

we can rephrase Corollary 1 in the following form.

Corollary 10. *Under assumption of Corollary 1 the equality holds*

$$\operatorname{Reg}(\mathcal{F}, \mathfrak{R}_g) = (\Theta \cap \mathcal{F}) + \sum_{k=1}^{n} \sum_{l=1}^{p_k} \frac{\overline{g^*(h_{l,k}^*)}}{\| h_{l,k}^* \|_*^2} h_{l,k}^*. \tag{51}$$

Proof. By (7) and (44) we see that

$$g^*(h_{l,k}^*) = g_k^*(h_{l,k}) \| e_k \|_B^2, \quad k \in \mathbb{Z}_{1,n}, l \in \mathbb{Z}_{1,p_k}. \tag{52}$$

By (6) and (50) we get

$$\| h_{l,k}^* \|_\star^2 = \| h_{l,k} \bullet e_k \|_*^2 = \| h_{l,k} \|^2 \| e_k \|_B^2 \quad k \in \mathbb{Z}_{1,n}, l \in \mathbb{Z}_{1,p_k} \tag{53}$$

Combining (50), (52) and(53) we deduce from (41) the equality (51), which is desired conclusion.

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State University of Applied Science in Chełm
Pocztowa 54, PL-22-100 Chełm
Poland

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PROBLEM REGRESJI DLA FUNKCJI O WARTOŚCIACH W PRZESTRZENI HILBERTA

S t r e s z c z e n i e

Wielomiany regresyjne są istotne w analizie danych doświadczalnych reprezentowanych przez parę ciągów x i y. Najczęstsza w praktyce zależność liniowa, wyrażona np. przez prawa fizyczne i chemiczne, prowadzi do zbyt znacznego uproszczenia w poszukiwanej zależności między danymi. Uogólniony problem regresji rozważany w tej pracy, prowadzi do rozwiązania pewnego zagadnienia ekstremalnego, określonego w skończenie wymiarowej przestrzeni Hilberta.

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In memory of
Professor Roman Stanisław Ingarden

Andrzej Polka

MULTIPRODUCTS OF VECTORS IN DESCRIPTION OF SPHERICAL MOTION I

VELOCITY, ACCELERATION, AND MASS MOMENTS OF INERTIA

Summary

In the paper a possibility of the application of vector and versor multiproducts for the description of motion of a rigid spherical body has been presented. Both the classical notation of vectors and the corresponding matrix notation, with the use of an outer product of vectors, i.e. a dyad of a scalar product and a dyad of a vector product of two vectors were employed. In the description of spherical motion a reference system, related to the instantaneous axis of rotation, called an umbrella in the present work, has been used. In the first part of the paper formulae for the velocity and acceleration of any point of a body, and mass moments of inertia of a body for the umbrella system have been derived.

1. Introduction

In the present work two methods of notation of vectors and multiptoducts of vectors have been applied. The classical notation, in which the vector of projection $\vec{\boldsymbol{b}}_l$ of the vector $\vec{\boldsymbol{b}}$ onto any axis of a versor $\vec{\boldsymbol{e}}_l$ is determined by its coordinate on this axis (i.e. the scalar product of the vector and versor of the axis) multiplied by the versor of this axis; and the matrix notation, in which the coordinates of the vector of projection $\vec{\boldsymbol{b}}_l$ are entries of the column matrix $\boldsymbol{b}_l$ (or of the row matrix $\boldsymbol{b}_l^T$) in the orthogonal coordinate system adopted. In the matrix notation, dyads – i.e. matrices of the outer product of two vectors or versors – were used, described in more detail in work [3].

Dyad, i.e. an outer product of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ in space K^3 is a square matrix $\boldsymbol{P}_{ab}$ of the entries $p_{ij} = a_i b_j$ $(i, j = 1, 2, 3)$. The dyad is sensitive to the order of its elements. A change in the order of the dyad vectors, i.e. when $\boldsymbol{P}_{ba} = [b_i a_j]$, yields a transposed matrix $\boldsymbol{P}_{ba} = \boldsymbol{P}^t_{ab}$.

Thus, the dyad of vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ and, by analogy, the dyad of versors $\vec{\boldsymbol{e}}_a$ and $\vec{\boldsymbol{e}}_b$ have the following forms

$$(1)\qquad \boldsymbol{P}_{ab} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{P}_{e_a e_b} = \begin{bmatrix} e_{a1}e_{b1} & e_{a1}e_{b2} & e_{a1}e_{b3} \\ e_{a2}e_{b1} & e_{a2}e_{b2} & e_{a2}e_{b3} \\ e_{a3}e_{b1} & e_{a3}e_{b2} & e_{a3}e_{b3} \end{bmatrix}$$

For the multiproduct of vectors $(\vec{\boldsymbol{v}}\underline{\vec{\boldsymbol{a}})\vec{\boldsymbol{b}}}$ defined in the matrix notation $(\boldsymbol{v}^T\underline{\boldsymbol{a})\boldsymbol{b}^T}$ or $\underline{\boldsymbol{b}(\boldsymbol{a}^T}\boldsymbol{v})$, a dyad replaces the non-multipliable product of the two matrices underlined, shown below. Therefore, identifiably speaking, $(\boldsymbol{v}^T\underline{\boldsymbol{a})\boldsymbol{b}^T} = \boldsymbol{v}^T\boldsymbol{P}_{ab}$ or $\underline{\boldsymbol{b}(\boldsymbol{a}^T}\boldsymbol{v}) = \boldsymbol{P}_{ba}\boldsymbol{v}$, where both the identities are reciprocal transpositions.

Thus, the projection $\vec{\boldsymbol{b}}_l$ of the vector $\vec{\boldsymbol{b}}$ onto the direction of the versor $\vec{\boldsymbol{b}}_l$ can be expressed in the following manner – in the classical notation and as row and column matrices of coordinates:

$$(2)\qquad \vec{\boldsymbol{b}}_l = (\vec{\boldsymbol{b}}\underline{\vec{\boldsymbol{e}}_l)\vec{\boldsymbol{e}}_l} = b(\vec{\boldsymbol{e}}_b\underline{\vec{\boldsymbol{e}}_l)\vec{\boldsymbol{e}}_l}$$

that

$$\boldsymbol{b}_l^T = \boldsymbol{b}^T\boldsymbol{P}_{e_l e_l} = b\boldsymbol{e}_b^T\boldsymbol{P}_{e_l e_l} \qquad \text{or} \qquad \boldsymbol{b}_l = \boldsymbol{P}_{e_l e_l}\boldsymbol{b} = b\boldsymbol{P}_{e_l e_l}\boldsymbol{e}_b.$$

The system of three elements shown in Fig. 1, a pole S – a straight line l – a plane π; (where the straight line l and the plane π are reciprocally perpendicular) has been treated as a system of reference and named an umbrella [2, 3]. The umbrella system thus defined will be the basis for the description of spherical motion of a rigid body proposed in the present paper.

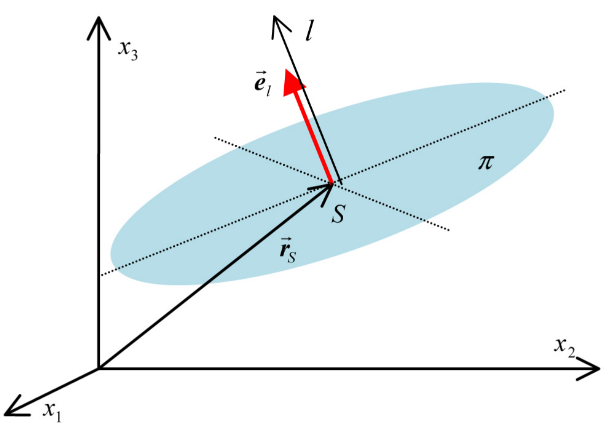

Fig. 1: An umbrella in a space K^3.

The position of an umbrella in the space K^3 is determined by six coordinates:
– three coordinates of the vector-radius of the point S, $\vec{\boldsymbol{r}}_S$; $\boldsymbol{r}_S = [x_{S1} x_{S2} x_{S3}]^T$,
– three coordinates of the versor $\vec{\boldsymbol{e}}_l$, $\boldsymbol{e}_l = [e_{l1} e_{l2} e_{l3}]^T$.

The versor $\vec{\boldsymbol{e}}_l$ determines a positive sense of the axis Sl, the direction of the straight line l and the position of the plane π in a space, thereby, it is simultaneously the versor $\vec{\boldsymbol{e}}_l$ of the straight line and the versor $\vec{\boldsymbol{e}}_\pi$ of the plane, hence $\vec{\boldsymbol{e}}_l = \vec{\boldsymbol{e}}_\pi$. An umbrella forms in the space K^3 a specific reference system in which – in a relatively simple manner – one can describe any vector $\vec{\boldsymbol{b}}$ and the vector product of two vectors $\vec{\boldsymbol{w}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$. The projections of the vector $\vec{\boldsymbol{b}}$ onto the elements of the umbrella in the space K^3 are shown in Fig. 2.

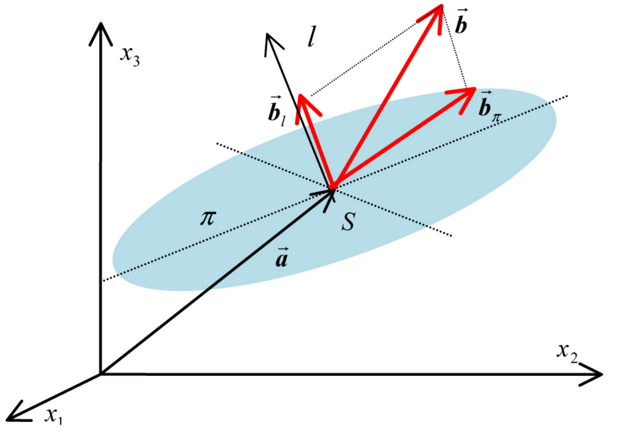

Fig. 2: Projections of the vector in the umbrella system in the space K^3.

Let vectors $\vec{\boldsymbol{a}} = a\vec{\boldsymbol{e}}_a$ and $\vec{\boldsymbol{b}} = b\vec{\boldsymbol{e}}_b$ be described by the matrices of coordinates

$$\boldsymbol{a}^T = |a_1 a_2 a_3|^T = a|c_{a1} c_{a2} c_{a3}|^T, \qquad \boldsymbol{b}^T = |b_1 b_2 b_3|^T = b|c_{b1} c_{b2} c_{b3}|^T.$$

The vector $\vec{\boldsymbol{b}}$ projected onto the axis of the umbrella $\vec{\boldsymbol{l}} = l\vec{\boldsymbol{e}}_l$ is a vector $\vec{\boldsymbol{b}}_l$ and can be written in the matrix form $\boldsymbol{b}_l$ by means of a dyad $\boldsymbol{P}_{e_l e_l}$ as

$$(3) \qquad \vec{\boldsymbol{b}}_l = \left(\vec{\boldsymbol{b}}\vec{\boldsymbol{e}}_l\right)\vec{\boldsymbol{e}}_l = b(\vec{\boldsymbol{e}}_b\vec{\boldsymbol{e}}_l)\vec{\boldsymbol{e}}_l, \qquad \boldsymbol{b}_l = b\boldsymbol{P}_{e_l e_l}\boldsymbol{e}_b = \boldsymbol{P}_{e_l e_l}\boldsymbol{b}.$$

The projection $\vec{\boldsymbol{b}}$ onto the plane π of the umbrella is a vector $\vec{\boldsymbol{b}}_\pi$. Since $\vec{\boldsymbol{b}} = \vec{\boldsymbol{b}}_l + \vec{\boldsymbol{b}}_\pi$, then $\vec{\boldsymbol{b}}_\pi = \vec{\boldsymbol{b}} - \vec{\boldsymbol{b}}_l = \vec{\boldsymbol{b}} - (\vec{\boldsymbol{b}}\vec{\boldsymbol{e}}_l)\vec{\boldsymbol{e}}_l$. The matrix $\boldsymbol{b}_\pi$ of the coordinates of projection of the vector $\vec{\boldsymbol{b}}$ onto the plane π has the form

$$(4) \qquad \boldsymbol{b}_\pi = \boldsymbol{b} - \boldsymbol{P}_{e_l e_l}\boldsymbol{b} = \boldsymbol{I}_3\boldsymbol{b} - \boldsymbol{P}_{e_l e_l}\boldsymbol{b} = (\boldsymbol{I}_3 - \boldsymbol{P}_{e_l e_l})\boldsymbol{b},$$

where $\boldsymbol{I}_3$ is a diagonal unit matrix of the third order.

The matrices $\boldsymbol{b}_l$ (3) and $\boldsymbol{b}_\pi$ (4) of the coordinates of projections of the vector $\vec{\boldsymbol{b}}$ onto the elements of the umbrella have, in the space K^3, has the following form

$$\boldsymbol{b}_l = \begin{vmatrix} c_{l1}^2 & c_{l1}c_{l2} & c_{l1}c_{l3} \\ c_{l2}c_{l1} & c_{l2}^2 & c_{l2}c_{l3} \\ c_{l3}c_{l1} & c_{l3}c_{l2} & c_{l3}^2 \end{vmatrix} \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}, \qquad \boldsymbol{b}_\pi = \begin{vmatrix} 1-c_{l1}^2 & c_{l1}c_{l2} & c_{l1}c_{l3} \\ c_{l2}c_{l1} & 1-c_{l2}^2 & c_{l2}c_{l3} \\ c_{lz}c_{lx} & c_{l3}c_{l2} & 1-c_{l3}^2 \end{vmatrix} \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}. \tag{5}$$

The vector of the vector product $\vec{\boldsymbol{w}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ in the space K^3 can be written in the form of the following matrices, row or column ones. If

$$\vec{\boldsymbol{w}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} \quad \text{then} \quad \boldsymbol{w}^T = \boldsymbol{a}^T \boldsymbol{P}_{eb}^* \quad \text{or} \quad \boldsymbol{w} = \boldsymbol{P}_{ae}^* \boldsymbol{b}. \tag{6}$$

The dyads $\boldsymbol{P}_{eb}^*$ and $\boldsymbol{P}_{ae}^*$ of the vector product $\vec{\boldsymbol{w}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ are matrices of the entries

$$\begin{aligned} \boldsymbol{P}_{eb}^* &= \left[p_{ij} = \sum_{k=1}^{3} \{\operatorname{sgn}[(i-j)(k-i)(k-j)]\} \, b_k \right] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \\ \boldsymbol{P}_{ae}^* &= \left[p_{ij} = \sum_{k=1}^{3} \{\operatorname{sgn}[(i-j)(k-i)(k-j)]\} \, a_k \right] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

The projection of the vector $\vec{\boldsymbol{w}}$ onto the axis $\vec{\boldsymbol{l}}$ of the umbrella is the vector $\vec{\boldsymbol{w}}_l = (\vec{\boldsymbol{w}}\vec{\boldsymbol{e}}_l)\vec{\boldsymbol{e}}_l$ and its coordinates are contained in one of the two matrix forms, the row or the column one

$$\boldsymbol{w}_l^T = \boldsymbol{a}^T \boldsymbol{P}_{eb}^* \boldsymbol{P}_{e_l e_l} \quad \text{or} \quad \boldsymbol{w}_l = \boldsymbol{P}_{e_l e_l} \boldsymbol{P}_{ae}^* \boldsymbol{b}, \tag{8}$$

whereas the vector of projection $\vec{\boldsymbol{w}}_\pi$ of the vector $\vec{\boldsymbol{w}}$ onto the plane π of the umbrella and the matrix of coordinates of this projection have the form

$$\begin{aligned} &\vec{\boldsymbol{w}}_\pi = \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}_l = \vec{\boldsymbol{w}} - (\vec{\boldsymbol{w}}\vec{\boldsymbol{w}}_l)\vec{\boldsymbol{e}}_l, \\ &\boldsymbol{w}_\pi = \boldsymbol{w} - \boldsymbol{w}_l = \boldsymbol{P}_{ae}^* \boldsymbol{b} - \boldsymbol{P}_{e_l e_l} \boldsymbol{P}_{ae}^* \boldsymbol{b} = (\boldsymbol{I}_3 - \boldsymbol{P}_{e_l e_l}) \boldsymbol{P}_{ae}^* \boldsymbol{b}. \end{aligned} \tag{9}$$

The dyad of the versors $\boldsymbol{P}_{e_l e_l}$ and the dyads of the vector product $\boldsymbol{P}_{ae}^*$ and $\boldsymbol{P}_{eb}^*$ occurring here have been described above.

2. The kinematics of spherical motion of a rigid body

Spherical motion is rotary motion of a rigid body whose one point S, the centre of spherical motion, is permanently stationary, which means that the vectors of velocity $\vec{\boldsymbol{v}}_S$ and acceleration $\vec{\boldsymbol{p}}_S$ of this point of the body are constantly zero vectors. It has been assumed that the centre of spherical motion, the point S, coincides with the centre of the orthogonal coordinate system $Ox_1x_2x_3$. The spherical motion of a body can be considered as a rotation around the axis of a momentary rotation, always passing through the centre of motion, the point S, repeatedly occupying a different position in a space. The vector $\vec{\omega}$ of the angular velocity of spherical motion lies on this axis. The point S is the origin of coordinate system of umbrella system (Fig. 3), whose axis coincides with the axis of momentary rotation and the vector $\vec{\omega}$, which means that the position of the umbrella in the space is determined by the coordinates of the versor $\vec{\boldsymbol{e}}_\omega$ of the angular velocity $\vec{\omega} = \omega\vec{\boldsymbol{e}}_\omega$ of the matrix $\boldsymbol{e}_\omega = [e_{\omega 1} e_{\omega 2} e_{\omega 3}]^T$.

The velocity of any point of a body whose position is determined by the radius-vector $\vec{\boldsymbol{r}}$, $\boldsymbol{r}[x_1 x_2 x_3]^T$, is a vector product $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and the matrices $\boldsymbol{v}$ of its coordinates have the form

$$\boldsymbol{v}^T = \boldsymbol{\omega}^T \boldsymbol{P}^*_{er} \quad \text{or} \quad \boldsymbol{v} = \boldsymbol{P}^*_{\omega e}\boldsymbol{r}. \tag{10}$$

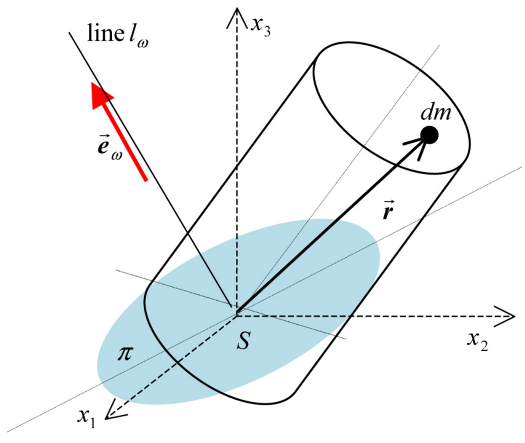

Fig. 3: An umbrella in spherical motion of a rigid body.

The vector of angular acceleration is a vector derivative of the vector $\vec{\boldsymbol{\omega}}$

$$\vec{\boldsymbol{\varepsilon}} = \frac{d\vec{\boldsymbol{\omega}}}{dt} = \frac{d}{dt}(\omega\vec{\boldsymbol{e}}_\omega) = \frac{d\omega}{dt}\vec{\boldsymbol{e}}_\omega + \omega\frac{d\vec{\boldsymbol{e}}_\omega}{dt}. \tag{11}$$

Assuming $\frac{d(\;)}{dt} = (\dot{\;})$ and after taking into consideration

$$\frac{d\vec{\boldsymbol{e}}_\omega}{dt} = \vec{\boldsymbol{\omega}}_u \times \vec{\boldsymbol{e}}_\omega,$$

where $\vec{\boldsymbol{\omega}}_u$ is – lying in the plane π of the umbrella – the vector of angular velocity of the umbrella, we obtained

$$\vec{\boldsymbol{\varepsilon}} = \dot{\omega}\vec{\boldsymbol{e}}_\omega + \omega\dot{\vec{\boldsymbol{e}}}_\omega = \dot{\omega}\vec{\boldsymbol{e}}_\omega + \omega\vec{\boldsymbol{\omega}}_u \times \vec{\boldsymbol{e}}_\omega = \vec{\boldsymbol{\varepsilon}}_\omega + \vec{\boldsymbol{\varepsilon}}_\pi. \tag{12}$$

Thus, in the umbrella system, the vector $\vec{\boldsymbol{\varepsilon}}$ is projected onto two reciprocally perpendicular directions: onto the umbrella axis as a vector $\vec{\boldsymbol{\varepsilon}}_\omega = \dot{\omega}\vec{\boldsymbol{e}}_\omega$ and onto the plane π of the umbrella as a vector $\vec{\boldsymbol{\varepsilon}}_\pi = \omega\vec{\boldsymbol{\omega}}_u \times \vec{\boldsymbol{e}}_\omega$, such that $\vec{\boldsymbol{\varepsilon}}_\pi = \vec{\boldsymbol{\varepsilon}} - \vec{\boldsymbol{\varepsilon}}_\omega$. It can be proved that – since $\vec{\boldsymbol{\omega}}_u$ and $\vec{\boldsymbol{e}}_\omega$ are orthogonal – the vector of angular velocity of the umbrella $\vec{\boldsymbol{\omega}}_u = \omega^{-1}\vec{\boldsymbol{e}}_\omega \times (\vec{\boldsymbol{\varepsilon}} - \vec{\boldsymbol{\varepsilon}}_\omega) = \omega^{-1}\vec{\boldsymbol{\varepsilon}}_\omega \times \vec{\boldsymbol{\varepsilon}}$.

The acceleration $\vec{\boldsymbol{p}}$ of any point of a body, the point whose position is determined by the radius-vector $\vec{\boldsymbol{r}}$ is a vector derivative of the vector of velocity $\vec{\boldsymbol{v}}$ of this point

$$\vec{\boldsymbol{p}} = \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) = \frac{d\vec{\boldsymbol{\omega}}}{dt} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{\varepsilon}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}. \tag{13}$$

Following the use of (12) and transformation, the following vector equation was obtained

$$\vec{\boldsymbol{p}} = \dot{\omega}\vec{\boldsymbol{e}}_\omega \times \vec{\boldsymbol{r}} + (\vec{\boldsymbol{\omega}}_u \times \vec{\boldsymbol{\omega}}) \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}), \tag{14}$$

which, making use of the following kinematic and vector identities

$$\dot{\omega}\vec{\boldsymbol{e}}_\omega \times \vec{\boldsymbol{r}} = \frac{\dot{\omega}}{\omega}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} = \frac{\dot{\omega}}{\omega}\vec{\boldsymbol{v}};$$

$$\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) = (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} - (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{\omega}})\vec{\boldsymbol{r}} = (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} - \omega^2\vec{\boldsymbol{r}};$$

$$(\vec{\boldsymbol{\omega}}_u \times \vec{\boldsymbol{\omega}}) \times \vec{\boldsymbol{r}} = (\vec{\boldsymbol{\omega}}_u\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} - (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}}_u$$

presented as a sum of five component vectors of acceleration of the point

$$\vec{\boldsymbol{p}} = \frac{\dot{\omega}}{\omega}\vec{\boldsymbol{v}} + (-\omega^2)\vec{\boldsymbol{r}} + (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} + (\vec{\boldsymbol{\omega}}_u\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} + (-\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}}_u, \tag{15}$$

denoted successively as

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_t + \vec{\boldsymbol{p}}_r + \vec{\boldsymbol{p}}_{\pi\omega} + \vec{\boldsymbol{p}}_{u\omega} + \vec{\boldsymbol{p}}_{\omega u}.$$

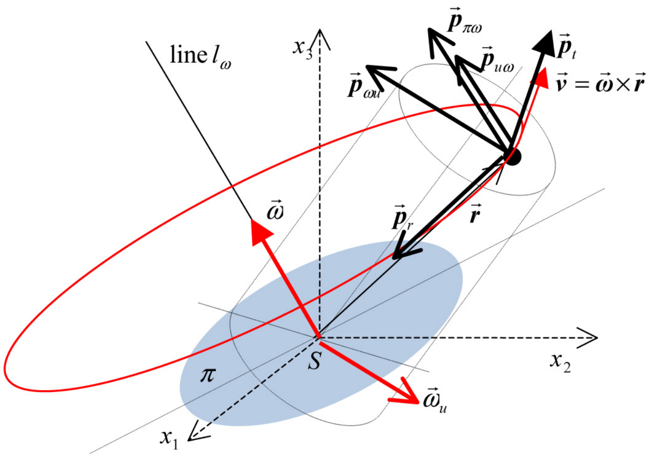

Fig. 4: Accelerations of a point of a rigid body.

The successive component vectors of acceleration in matrix notation have the form:

– the tangent acceleration $\vec{\boldsymbol{p}}_t$ lies on the direction of the velocity vector of the point under consideration

$$\vec{\boldsymbol{p}}_t = \frac{\dot{\omega}}{\omega}\vec{\boldsymbol{v}} = \frac{\dot{\omega}}{\omega}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}; \qquad \boldsymbol{p}_t = \frac{\dot{\omega}}{\omega}\boldsymbol{v} = \frac{\dot{\omega}}{\omega}\boldsymbol{P}^*_{\omega e}\boldsymbol{r},$$

– the centripetal acceleration $\vec{\boldsymbol{p}}_r$ lies on the direction of the radius-vector of the point and its vector is turned toward the centre of spherical motion;

$$\vec{\boldsymbol{p}}_r = -\omega^2 \vec{\boldsymbol{r}}; \quad \boldsymbol{p}_r = -\omega^2 \boldsymbol{r},$$

– the acceleration $\vec{\boldsymbol{p}}_{\pi\omega}$ is a vector lying in parallel to the vector of angular velocity $\vec{\boldsymbol{\omega}}$ of spherical motion, i.e. it is perpendicular to the umbrella plane π and its sense is the same as that of $\vec{\boldsymbol{\omega}}$;

$$\vec{\boldsymbol{p}}_{\pi\omega} = (\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}}; \quad \boldsymbol{p}_{\pi\omega} = \boldsymbol{P}_{\omega\omega}\boldsymbol{r},$$

– the acceleration $\vec{\boldsymbol{p}}_{u\omega}$ is also a vector parallel to the vector of angular velocity $\vec{\omega}$ of the same sense as that of $\vec{\omega}$;

$$\vec{\boldsymbol{p}}_{u\omega} = (\vec{\boldsymbol{\omega}}_u\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}}; \quad \boldsymbol{p}_{u\omega} = \boldsymbol{P}_{\omega\omega_u}\boldsymbol{r},$$

– the acceleration $\vec{\boldsymbol{p}}_{\omega u}$ is a vector parallel to the vector $\vec{\boldsymbol{\omega}}_u$ of the angular velocity of the umbrella plane π of the sense opposite to the vector $\vec{\boldsymbol{\omega}}_u$. Thus, it lies in the plane parallel to the plane π;

$$\vec{\boldsymbol{p}}_{\omega u} = -(\vec{\boldsymbol{\omega}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}}_u; \quad \boldsymbol{p}_{\omega u} = -\boldsymbol{P}_{\omega_u\omega}\boldsymbol{r}.$$

Having considered the above notations, the matrix $\boldsymbol{p}$ of the coordinates of the acceleration vector $\vec{\boldsymbol{p}}$ (15) has the form of the equation

$$\boldsymbol{p} = \left[\frac{\dot{\omega}}{\omega}\boldsymbol{P}^*_{\omega e} - \omega^2\boldsymbol{I}_3 + \boldsymbol{P}_{\omega\omega} + \boldsymbol{P}_{\omega\omega_u} - \boldsymbol{P}_{\omega_u\omega}\right]\boldsymbol{r}. \tag{16}$$

The successive components of the vector $\vec{\boldsymbol{p}}$ of acceleration of a point are shown in Fig. 4.

3. Moments of inertia of a body in an umbrella system

The author used the definition of moments of inertia of a rigid body of a mass m relative to the centre of spherical motion, the point $S(J_S)$; relative to the umbrella plane π, (J_π); and relative to the axis of rotation ω, (J_ω); expressed by multiproducts of the radius-vector $\vec{\boldsymbol{r}}$ determining the position of the elementary mass dm of a body and the versor $\vec{\boldsymbol{e}}_\omega$, determining the position of the axis of rotation ω and, at the same time, the umbrella plane π in the coordinate system adopted, shown in Fig. 3. Moments of inertia of a body are sums of moments of inertia of all the elementary masses dm of this body.

The moment of inertia relative to the point S,

$$J_S = \int\limits_m \vec{\boldsymbol{r}}^2 dm. \tag{17}$$

The moment of inertia relative to the plane π,

$$J_\pi = \int\limits_m (\vec{\boldsymbol{e}}_\omega\vec{\boldsymbol{r}})^2\, dm. \tag{18}$$

The moment of inertia relative to the axis of rotation ω,

$$J_\omega = \int\limits_m (\vec{\boldsymbol{e}}_\omega \times \vec{\boldsymbol{r}})^2 \, dm. \tag{19}$$

From Lagrange's vector identity it results that sub-integral functions (19) satisfy the condition

$$(\vec{\boldsymbol{e}}_\omega \times \vec{\boldsymbol{r}})^2 = \vec{\boldsymbol{e}}_\omega^2 \vec{\boldsymbol{r}}^2 - (\vec{\boldsymbol{e}}_\omega \vec{\boldsymbol{r}})^2 = \vec{\boldsymbol{r}}^2 - (\vec{\boldsymbol{e}}_\omega \vec{\boldsymbol{r}})^2,$$

which means that for any umbrella, moments of inertia of a body relative to the three elements of an umbrella: the pole, plane and the straight line are bound by the equation $J_\omega = J_S - J_\pi$, hence

$$J_S = J_\omega + J_\pi. \tag{20}$$

The moment of inertia of a rigid body relative to any pole is equal to the sum of moments of inertia relative to the reciprocally perpendicular elements, the straight line and the plane passing through this pole.

In particular, this condition is satisfied for the centre of spherical motion and the axis of momentary rotation and the plane π of the umbrella, shown in Fig. 3.

In the matrix notation, the moments of inertia (17–19) have the form:

– the moment of inertia relative to the point S, J_S (17) has its corresponding matrix

$$\boldsymbol{I}_S = J_S \boldsymbol{I}_3, \tag{21}$$

where $\boldsymbol{I}_3$ is a diagonal unit matrix;

– the moment of inertia relative to the plane π, J_π (18), after the identity

$$J_\pi = \int\limits_m (\boldsymbol{e}_\omega \boldsymbol{r})^2 dm = \int\limits_m (\boldsymbol{e}_\omega \underline{\boldsymbol{r})(\boldsymbol{r}} \boldsymbol{e}_\omega) dm = \boldsymbol{e}_\omega^T \left[\int\limits_m \boldsymbol{P}_{rr} dm \right] \boldsymbol{e}_\omega = \boldsymbol{e}_\omega^T \boldsymbol{I}_{rr} \boldsymbol{e}_\omega$$

has been taken into consideration, has its corresponding matrix notation

$$J_\pi = \boldsymbol{e}_\omega^T \boldsymbol{I}_{rr} \boldsymbol{e}_\omega, \tag{22}$$

where $\boldsymbol{I}_{rr} = \int_m \boldsymbol{P}_{rr} dm$ is a *matrix of plane moments of inertia of a body*, and integrals of the elements of the dyad $\boldsymbol{P}_{rr}$ are the entries of the matrix.

The matrix $\boldsymbol{I}_{rr}$ contains – along the main diagonal – moments of inertia relative to the planes of the orthogonal coordinate system $Sx_1x_2x_3$,

$$J_{11} = \int\limits_m x_1^2 dm \quad \text{relative to the plane} \quad Sx_2x_3,$$

$$J_{22} = \int\limits_m x_2^2 dm \quad \text{and} \quad J_{33} + \int\limits_m x_3^2 dm \quad \text{relative to} \quad Sx_1x_3 \quad \text{and} \quad Sx_1x_2,$$

respectively. The entries of the matrix lying outside the main diagonal contain deviation moments of inertia of a body,

$$J_{12} = \int\limits_m x_1 x_2 dm, \quad J_{13} = \int\limits_m x_1 x_3 dm \quad \text{and} \quad J_{23} = \int\limits_m x_2 x_3 dm,$$

respectively.

The moment of inertia relative to the axis of momentary rotation ω, J_ω (19) has its corresponding matrix notation

$$J_\omega = \boldsymbol{e}_\omega^T \boldsymbol{I} \boldsymbol{e}_\omega, \tag{23}$$

in which the matrix $\boldsymbol{I}$ is a *matrix of moments of inertia of a body* of the entries resulting from formula (19) and identity (20). The matrix equivalent of identity (20) is the identity

$$J_S \boldsymbol{I}_3 = \boldsymbol{I}_{rr} + \boldsymbol{I}. \tag{24}$$

In the matrix of moments of inertia of a body $\boldsymbol{I}$, the entries lying along the main diagonal are moments of inertia relative to the successive axes of the system $Sx_1x_2x_3$;

$$J_1 = \int\limits_m (x_2^2 + x_3^2) dm \quad \text{relative to the axis} \quad Sx_1,$$

$$J_2 = \int\limits_m (x_1^2 + x_3^2) dm \quad \text{and} \quad J_3 = \int\limits_m (x_1^2 + x_2^2) dm \quad \text{relative to} \quad Sx_2 \quad \text{and} \quad Sx_3,$$

respectively.

Therefore, the matrices of inertia of a solid body have the form

$$\boldsymbol{I}_S = J_S \boldsymbol{I}_3; \quad \boldsymbol{I}_{rr} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}; \quad \boldsymbol{I} = \begin{bmatrix} J_1 & -J_{12} & -J_{13} \\ -J_{21} & J_2 & -J_{23} \\ -J_{31} & -J_{32} & J_3 \end{bmatrix} \tag{25}$$

and identity (24) holds for them. The matrix $\boldsymbol{I}_3$ is a diagonal unit matrix.

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Division of Dynamics
Technical University of Łódź
Stefanowskiego 1/15 , PL 90-924 Łódź
Poland
e-mail: eia@thepolkadots.net

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MULTIILOCZYNY WEKTORÓW W OPISIE RUCHU SFERYCZNEGO I

PRĘDKOŚĆ, PRZYSPIESZENIE I MASOWE MOMENTY BEZWŁADNOŚCI

S t r e s z c z e n i e
W pracy pokazano możliwość zastosowania multiiloczynów wektorów oraz wersorów dla opisu ruchu sferycznego ciała sztywnego. Wykorzystano przy tym zarówno klasyczny zapis wektorowy, jak i odpowiadający mu zapis macierzowy, z użyciem iloczynu zewnętrznego wektorów to jest diady iloczynu skalarnego oraz diady iloczynu wektorowego dwóch wektorów. W opisie ruchu sferycznego użyto układu odniesienia, nazwanego tu parasolem, związanego z osią chwilowego obrotu. W pierwszej części pracy dla układu parasola wyprowadzono wzory na prędkość i przyspieszenie dowolnego punktu ciała oraz masowe momenty bezwładności ciała ruchu sferycznego.

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DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2011 Vol. LXI

Recherches sur les déformations no. 3

pp. 81–90

In memory of
Professor Roman Stanisław Ingarden

Andrzej Polka

MULTIPRODUCTS OF VECTORS IN DESCRIPTION OF SPHERICAL MOTION II

DYNAMIC EQUATION FOR SPHERICAL MOTION

Summary

In the paper a possibility of the application of vector and versor multiproducts for the description of motion of a rigid spherical body has been presented. Both the classical notation of vectors and the corresponding matrix notation, with the use of an outer product of vectors, i.e. a dyad of a scalar product and a dyad of a vector product of two vectors were employed. In the description of spherical motion a reference system, related to the instantaneous axis of rotation, called an umbrella in the present work, has been used. In the second part of the paper a dynamic equation for spherical motion for the umbrella system has been derived.

For the first part of this paper, see [6].

4. Dynamics of spherical motion

4.1. Angular momentum and the angular momentum plane

An angular momentum of a rigid body – for a body moving with spherical motion at an angular velocity $\vec{\boldsymbol{\omega}}$, lying on the axis of momentary rotation – calculated relative to the constant pole S, the centre of spherical motion, is a vector

$$\vec{\boldsymbol{k}}_S = \int_m \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}} dm = \int_m \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) dm. \tag{26}$$

The matrix $\boldsymbol{k}_S$ of coordinates of the angular momentum vector $\vec{\boldsymbol{k}}_S$ has the well-known form

$$\boldsymbol{k}_S = \boldsymbol{I}\boldsymbol{\omega}, \tag{27}$$

where $\boldsymbol{I}$ is matrix (25) of inertia of a body, while $\boldsymbol{\omega}$ is a matrix of coordinates of the vector of the angular velocity of spherical motion, $\boldsymbol{\omega} = |\omega_1\omega_2\omega_3|^T$.

After applying the identity

$$\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}) = (\vec{\boldsymbol{a}}\vec{\boldsymbol{a}})\vec{\boldsymbol{b}} - (\vec{\boldsymbol{a}}\vec{\boldsymbol{b}})\vec{\boldsymbol{a}}$$

the sub-integral expression of equation (26) can be written in the form

$$\vec{\boldsymbol{k}}_S = \int\limits_m [(\vec{\boldsymbol{r}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} - (\vec{\boldsymbol{r}}\vec{\boldsymbol{\omega}})\vec{\boldsymbol{r}}]\, dm. \tag{28}$$

The first of integrals (28) is a *resultant* (*total*) *vector of the angular momentum* of a body in spherical motion $\vec{\boldsymbol{k}}$ for an umbrella whose axis coincides with the momentary axis of rotation, suspended in the pole S,

$$\int\limits_m (\vec{\boldsymbol{r}}\vec{\boldsymbol{r}})\vec{\boldsymbol{\omega}} dm = \vec{\boldsymbol{\omega}} \int\limits_m r^2 dm = J_S\vec{\boldsymbol{\omega}}, \quad \vec{\boldsymbol{k}} = J_S\vec{\boldsymbol{\omega}} \tag{29}$$

it results from the formula that the vector of angular momentum of spherical motion $\vec{\boldsymbol{k}}$ lies on the momentary axis of rotation, has the same sense as that of the angular velocity of motion and the value equal to the product of moment of inertia J_S of a body relative to the centre of spherical motion and the vector of the angular velocity of spherical motion $\vec{\boldsymbol{\omega}}$. In the matrix notation the matrix $\boldsymbol{k}$ of the vector of angular momentum $\vec{\boldsymbol{k}}$ has the form

$$\boldsymbol{k} = J_S\boldsymbol{I}_3\boldsymbol{\omega}. \tag{30}$$

The second integral (28) can be written as

$$\vec{\boldsymbol{k}}_P = \int\limits_m (\vec{\boldsymbol{r}}\vec{\boldsymbol{\omega}})\vec{\boldsymbol{r}} dm \tag{31}$$

and is a vector of angular momentum which can be called a *plane angular momentum* $\vec{\boldsymbol{k}}_P$, due to the fact that its value depends on the values of plane moments of inertia of a body for the same umbrella system. In the matrix notation, after using identity (22) and introducing matrices of plane moments of inertia $\boldsymbol{I}_{rr}$, defined by equation (22), the matrix of coordinates of the plane angular momentum $\boldsymbol{k}_p$ has the form

$$\boldsymbol{k}_P = \int\limits_m \boldsymbol{P}_{rr}\boldsymbol{\omega} dm = \boldsymbol{I}_\pi\boldsymbol{\omega}. \tag{32}$$

Thus, vector equation (28) can first be written as $\vec{\boldsymbol{k}}_S = \vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_P$ and then, after transformation, as

$$\vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_S + \vec{\boldsymbol{k}}_p. \tag{33}$$

This is an equation of the angular momentum of a rigid body in spherical motion about the centre of spherical motion, the stationary pole S, at an angular velocity $\vec{\boldsymbol{\omega}}$, whose vector lies on the momentary axis of rotation, the straight line Sl_ω.

A geometrical illustration of an equation of angular momentum is shown in Fig. 5. Three vectors of angular momentum of equation (33) lie in the common plane of the angular momentum containing the momentary axis of rotation (Fig. 5) and their distribution is always such that the resultant angular momentum $\vec{\boldsymbol{k}}$ lies on the momentary axis of rotation, while the resultant angular momentum (relative to the pole S, $\vec{\boldsymbol{k}}_S$ and the plane angular momentum, $\vec{\boldsymbol{k}}_P$) tilt from the axis of rotation so that their projections onto the umbrella plane π will be reciprocally balanced, whereas the projections onto the axis of the umbrella Sl_ω will be summed and, consequently, yield a vector of the resultant angular momentum $\vec{\boldsymbol{k}}$. Hence

$$\vec{\boldsymbol{k}}_{S\pi} + \vec{\boldsymbol{k}}_{p\pi} = 0, \quad \vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_S + \vec{\boldsymbol{k}}_P = \vec{\boldsymbol{k}}_{S\omega} + \underline{\vec{\boldsymbol{k}}_{S\pi}} + \vec{\boldsymbol{k}}_{p\omega} + \underline{\vec{\boldsymbol{k}}_{p\pi}} = \vec{\boldsymbol{k}}_{S\omega} + \vec{\boldsymbol{k}}_{p\omega}. \tag{34}$$

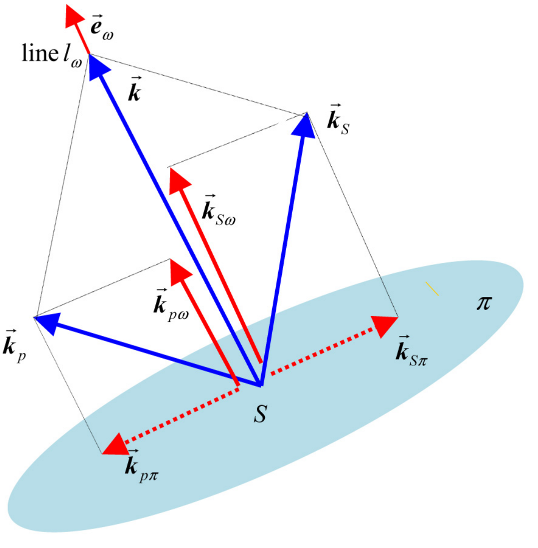

Fig. 5: Vectors of the angular momentum and the angular momentum plane.

Since the projection of the resultant angular momentum vector onto the umbrella plane is equal to zero, the underlined vectors of projections onto the plane π reciprocally balance one another. Three of the vectors of equation (34) lie on the common direction (Fig. 6), thus the vector equation

$$\vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_{S\omega} + \vec{\boldsymbol{k}}_{p\omega}$$

can be replaced with the scalar equation

$$k = k_{S\omega} + k_{p\omega}. \tag{35}$$

The vectors $\vec{\boldsymbol{k}}_{S\omega}$ and $\vec{\boldsymbol{k}}_{p\omega}$ of projections of the angular momentum $\vec{\boldsymbol{k}}_S$ and $\vec{\boldsymbol{k}}_p$ onto the direction of the umbrella axis, i.e. the direction of the versor $\vec{\boldsymbol{e}}_\omega$, are described by the vector and matrix equations

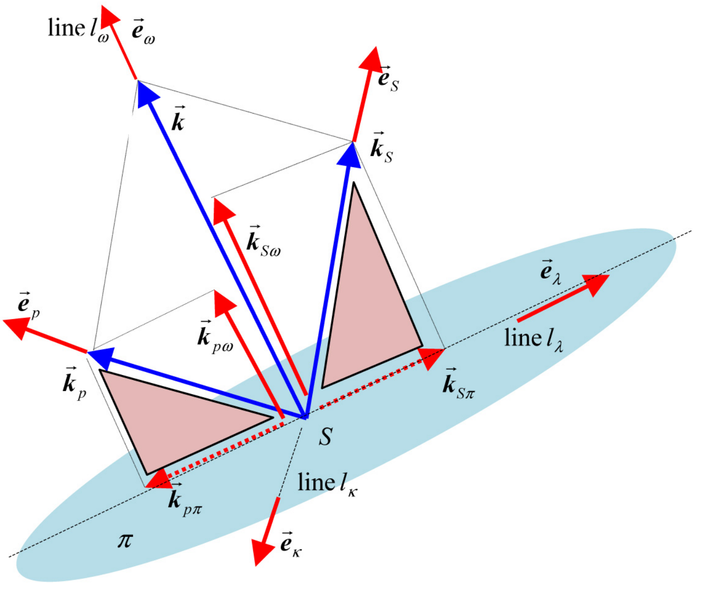

Fig. 6: Versors of angular momenta, planes and edges of intersection.

(36) $$\vec{\boldsymbol{k}}_{S\omega} = (\vec{\boldsymbol{k}}_S \vec{\boldsymbol{e}}_\omega)\vec{\boldsymbol{e}}_\omega, \quad \boldsymbol{k}_{S\omega} = \boldsymbol{P}_{e_\omega e_\omega} \boldsymbol{k}_S;$$

(37) $$\vec{\boldsymbol{k}}_{p\omega} = (\vec{\boldsymbol{k}}_p \vec{\boldsymbol{e}}_\omega)\vec{\boldsymbol{e}}_\omega, \quad \boldsymbol{k}_{p\omega} = \boldsymbol{P}_{e_\omega e_\omega} \boldsymbol{k}_p.$$

After substituting equation (31) $\vec{\boldsymbol{k}}_p = \int_m (\vec{\boldsymbol{r}}\vec{\boldsymbol{\omega}})\vec{\boldsymbol{r}} dm$ in $\vec{\boldsymbol{k}}_{p\omega}$ and then substituting an expression for the moment of inertia J_π of a body relative to the umbrella plane π,

$$J_\pi = \int_m (\vec{\boldsymbol{e}}_\omega \vec{\boldsymbol{r}})^2 dm = \boldsymbol{e}^T \boldsymbol{I}_{rr} \boldsymbol{e}_\omega,$$

a vector of projection of the plane angular momentum onto the umbrella axis were obtained

(38) $$\vec{\boldsymbol{k}}_{p\omega} = (\vec{\boldsymbol{k}}_p \vec{\boldsymbol{e}}_\omega)\vec{\boldsymbol{e}}_\omega = \int_m (\vec{\boldsymbol{e}}_\omega \vec{\boldsymbol{r}})^2 dm \vec{\boldsymbol{\omega}} = J_\pi \vec{\boldsymbol{\omega}} \quad \text{and} \quad k_{p\omega} = J_\pi \omega.$$

In such a case, since $\vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_{S\omega} + \vec{\boldsymbol{k}}_{p\omega}$, the vector of projection of the angular momentum relative to the pole S onto the umbrella axis has the value

(39) $$\vec{\boldsymbol{k}}_{S\omega} = \vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}_{p\omega} = J_S \vec{\boldsymbol{\omega}} - J_\pi \vec{\boldsymbol{\omega}} = J_\omega \vec{\boldsymbol{\omega}} \quad \text{and} \quad k_{S\omega} = J_\omega \omega$$

because, according to (20), $J_\omega + J_\pi = J_S$.

Now scalar equation (35) $k = k_{S\omega} + k_{p\omega}$ can be written in the form

$$J_S\omega = J_\omega\omega + J_\pi\omega \quad \text{and} \quad k = J_S\omega, \tag{40}$$

while vector equation (34) $\vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_{S\omega} + \vec{\boldsymbol{k}}_{p\omega}$ can be presented in the classical notation or the corresponding matrix notation as

$$J_S\vec{\boldsymbol{\omega}} = J_\omega\vec{\boldsymbol{\omega}} + J_\pi\vec{\boldsymbol{\omega}} \quad \text{or} \quad J_S\boldsymbol{\omega} = J_\omega\boldsymbol{\omega} + J_\pi\boldsymbol{\omega}. \tag{41}$$

The vectors of the angular momentum lie in the angular momentum plane, perpendicular to the umbrella axis and containing the umbrella axis. In Fig. 6 both planes and their vectors perpendicular to them are shown: $\vec{\boldsymbol{e}}_\omega$ (for the umbrella plane) and $\vec{\boldsymbol{e}}_\kappa$ (for the angular momentum plane).

Both planes intersect along the common edge, the straight line Sl_λ, passing through the pole S and determined by the versor $\vec{\boldsymbol{e}}_\lambda$. Three versors, $\vec{\boldsymbol{e}}_\kappa\vec{\boldsymbol{e}}_\lambda\vec{\boldsymbol{e}}_\omega$ define the orientation of the space $S\kappa\lambda\omega$ and form an orthogonal, right-handed reference system, which means that $\vec{\boldsymbol{e}}_\kappa \times \vec{\boldsymbol{e}}_\lambda = \vec{\boldsymbol{e}}_\omega$.

In addition, the versors of the angular momentum vector are shown $\vec{\boldsymbol{e}}_\omega$, $\vec{\boldsymbol{e}}_S$, $\vec{\boldsymbol{e}}_p$ by means of which relationships between the lengths of these vectors and their geometrical position in the space have been expressed

$$\vec{\boldsymbol{k}} = k\vec{\boldsymbol{e}}_\omega, \quad \vec{\boldsymbol{k}}_S = k_S\vec{\boldsymbol{e}}_S, \quad \vec{\boldsymbol{k}}_p = k\vec{\boldsymbol{e}}_p. \tag{42}$$

Depending on the position of the versors $\vec{\boldsymbol{e}}_S$ and $\vec{\boldsymbol{e}}_\omega$ in the space one can determine the position of the angular momentum plane (i.e. the versor $\vec{\boldsymbol{e}}_\kappa$) and the edge of intersection of the planes, the straight line Sl_λ (i.e. the versor $\vec{\boldsymbol{e}}_\lambda$) in this space. These versors are determined by the vector products

$$\vec{\boldsymbol{e}}_\kappa = \frac{\vec{\boldsymbol{e}}_S \times \vec{\boldsymbol{e}}_\omega}{|\vec{\boldsymbol{e}}_S \times \vec{\boldsymbol{e}}_\omega|} = \frac{\vec{\boldsymbol{e}}_S \times \vec{\boldsymbol{e}}_\omega}{\sqrt{1-(\vec{\boldsymbol{e}}_S\vec{\boldsymbol{e}}_\omega)^2}} \tag{43}$$

and

$$\vec{\boldsymbol{e}}_\lambda = \vec{\boldsymbol{e}}_\omega \times \vec{\boldsymbol{e}}_\kappa = \frac{\vec{\boldsymbol{e}}_\omega \times (\vec{\boldsymbol{e}}_S \times \vec{\boldsymbol{e}}_\omega)}{|\vec{\boldsymbol{e}}_S \times \vec{\boldsymbol{e}}_\omega|} = \frac{(\vec{\boldsymbol{e}}_\omega\vec{\boldsymbol{e}}_\omega)\vec{\boldsymbol{e}}_S - (\vec{\boldsymbol{e}}_\omega\vec{\boldsymbol{e}}_S)\vec{\boldsymbol{e}}_\omega}{\sqrt{1-(\vec{\boldsymbol{e}}_S\vec{\boldsymbol{e}}_\omega)^2}}. \tag{44}$$

If the matrices of the coordinates $\boldsymbol{e}_S$ and $\boldsymbol{e}_\omega$ of the versor $\vec{\boldsymbol{e}}_S$ and $\vec{\boldsymbol{e}}_\omega$ have the form

$$\boldsymbol{e}_S = |e_{S1}e_{S2}e_{S3}|^T \quad \text{and} \quad \boldsymbol{e}_\omega = |e_{\omega1}e_{\omega2}e_{\omega3}|^T,$$

then the matrix $\boldsymbol{e}_\kappa$ of the versor $\vec{\boldsymbol{e}}_\kappa$ is the matrix

$$\boldsymbol{e}_\kappa = \frac{1}{\sqrt{1-(\boldsymbol{e}_S^T\boldsymbol{e}_\omega)^2}}\boldsymbol{P}^*_{e_S e}\boldsymbol{e}_\omega, \tag{45}$$

while the matrix of the versor of the straight line Sl_λ is

$$\boldsymbol{e}_\lambda = \frac{1}{\sqrt{1-(\boldsymbol{e}_S^T\boldsymbol{e}_\omega)^2}}\left(\boldsymbol{P}_{e_S e_\omega} - \boldsymbol{P}^T_{e_S e_\omega}\right)\boldsymbol{e}_\omega. \tag{46}$$

The projections of the component vectors of the angular momentum onto the umbrella plane are determined by multiproducts in the classical and matrix form

$$\vec{\boldsymbol{k}}_{S\pi} = (\vec{\boldsymbol{k}}_S \vec{e}_\lambda)\vec{\boldsymbol{e}}_\lambda, \quad \boldsymbol{k}_{S\pi} = \boldsymbol{P}_{e_\lambda e_\lambda}\boldsymbol{k}_S,$$

(47)

$$\vec{\boldsymbol{k}}_{p\pi} = (\vec{\boldsymbol{k}}_p \vec{e}_\lambda)\vec{\boldsymbol{e}}_\lambda, \quad \boldsymbol{k}_{p\pi} = \boldsymbol{P}_{e_\lambda e_\lambda}\boldsymbol{k}_p.$$

The versors $\boldsymbol{e}_\kappa$ and $\boldsymbol{e}_\lambda$ $\left[\boldsymbol{e}_\kappa = |e_{\kappa 1} e_{\kappa 2} e_{\kappa 3}|^T,\ \boldsymbol{e}_\lambda = |e_{\lambda 1} e_{\lambda 2} e_{\lambda 3}|^T\right]$ thus determined allow one to determine projections of the vectors of angular velocity, the vectors of angular momentum and the vectors of moments of outer forces onto the edges l_κ and l_λ.

4.2. A dynamic equation of spherical motion

During motion the umbrella system rotates at an angular velocity $\vec{\boldsymbol{\omega}}$ about the umbrella axis and at an angular velocity $\vec{\boldsymbol{\omega}}_u$ about the axis lying in the umbrella plane. The angular momentum plane (Fig. 7) rotates about the momentary axis of rotation, the straight line Ol_ω, at an angular velocity of spherical motion $\vec{\boldsymbol{\omega}}$ and participates in the rotational motion of the umbrella at an angular velocity of the umbrella $\vec{\boldsymbol{\omega}}_u$, whose vector lies in the plane π_ω of the umbrella. In Fig. 7 the vectors of angular velocities of the angular momentum plane are shown. The resultant velocity of the angular momentum plane is a vector

$$\vec{\boldsymbol{\omega}}_k = \vec{\boldsymbol{\omega}} + \vec{\boldsymbol{\omega}}_u. \tag{48}$$

The matrix of coordinates of the vector $\vec{\boldsymbol{\omega}}_k$ in a non stationary system of axes $S\kappa\lambda\omega$, related to the angular momentum plane, has the form

$$\boldsymbol{\omega}_k = |\omega_{u\kappa}\omega_{u\lambda}\omega|^T. \tag{49}$$

The first two coordinates of the matrix (49) are lengths of the projections $\vec{\boldsymbol{\omega}}_{u\kappa}$ and $\vec{\boldsymbol{\omega}}_{u\lambda}$ of the vector of angular velocity of the umbrella $\vec{\boldsymbol{\omega}}_u$ from the umbrella plane π onto the axes $S\kappa$ and $S\lambda$,

$$\begin{array}{llcll} \omega_{u\kappa} = |\vec{\boldsymbol{\omega}}_{u\kappa}|, & \vec{\boldsymbol{\omega}}_{u\kappa} = (\vec{\boldsymbol{\omega}}_u \vec{\boldsymbol{e}}_\kappa)\vec{\boldsymbol{e}}_\kappa & \text{and} & \omega_{u\lambda} = |\vec{\boldsymbol{\omega}}_{u\lambda}|, & \vec{\boldsymbol{\omega}}_{u\lambda} = (\vec{\boldsymbol{\omega}}_u \vec{\boldsymbol{e}}_\lambda)\vec{\boldsymbol{e}}_\lambda; \\ & \boldsymbol{\omega}_{u\kappa} = \boldsymbol{P}_{e_\kappa e_\kappa}\boldsymbol{\omega}_u & \text{and} & & \boldsymbol{\omega}_{u\lambda} = \boldsymbol{P}_{e_\lambda e_\lambda}\boldsymbol{\omega}_u, \end{array} \tag{50}$$

whereas the third coordinate is a length (value ω) of the momentary velocity of spherical motion.

For a rigid body moving with spherical motion an equation of angular momentum (33) of the form has been introduced: $\vec{\boldsymbol{k}} = \vec{\boldsymbol{k}}_S + \vec{\boldsymbol{k}}_P$, which – after differentiation relative to time – is transformed into

$$\dot{\vec{\boldsymbol{k}}} = \dot{\vec{\boldsymbol{k}}}_S + \dot{\vec{\boldsymbol{k}}}_P. \tag{51}$$

This equation describes three vectors of moments lying in one plane.

From the theorem about increment of angular momentum [1] for the angular momentum of a body $\vec{\boldsymbol{k}}_S$ relative to the stationary pole S results that $\dot{\vec{\boldsymbol{k}}}_S = \vec{\boldsymbol{m}}_S$ where $\vec{\boldsymbol{m}}_S$ is the moment of outer forces acting on the body, calculated relative to the point S. Thus, from equation (51) one can conclude that the remaining two

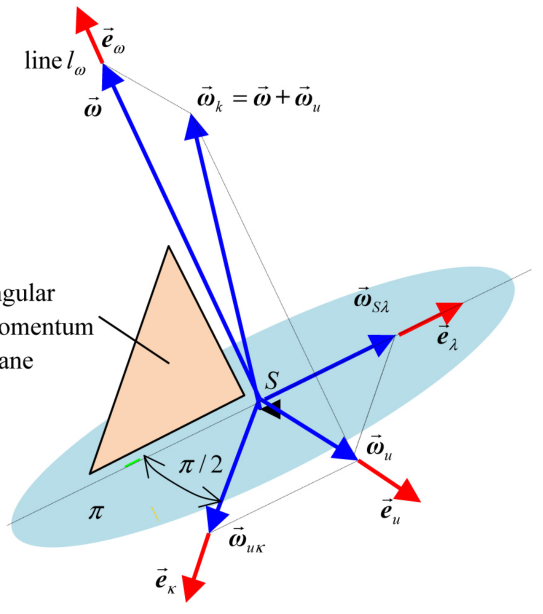

Fig. 7: Angular velocities of the angular momentum plane.

vectors of increment of the angular momentum are also vectors of moments of forces relative to this pole. Thus, by analogy with the theorem about increment of angular momentum

(52) $$\dot{\vec{k}}_S = \vec{m}_S, \quad \dot{\vec{k}}_p = \vec{m}_p, \quad \text{and their sum} \quad \dot{\vec{k}} = \vec{m}_d$$

were introduced.

The following was obtained

(53) $$\vec{m}_d = \vec{m}_S + \vec{m}_p,$$

where the vectors of increment of the angular momentum were called: $\vec{m}_p$ – *the vector of the plane moment*, $\vec{m}_d$ – *the vector of the dynamic moment of spherical motion*, respectively.

The vector of the moment of outer forces $\vec{m}_S$, the vector of the plane moment $\vec{m}_p$ and their resultant, the vector of the dynamic moment $\vec{m}_d$ form the plane of moments.

The vector derivative $\dot{\vec{r}}$ of any vector $\vec{r} = r\vec{e}_r$ is expressed by the formula

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + \vec{\omega}_r \times \vec{r}$$

where $\vec{\omega}_r$ is a vector of angular velocity of the vector $\dot{\vec{r}}$. Hence, the derivative of the vector of angular momentum of spherical motion $\dot{\vec{k}} = \dot{k}\vec{e}_\omega + \vec{\omega}_k \times \vec{k}$ – after the coordinates of the vector $\vec{\omega}_k$ (49); $\boldsymbol{\omega}_k = |\omega_{u\kappa}\omega_{u\lambda}\omega|^T$; and the vector $\vec{k}$; $k = |0\ 0\ k|^T$;

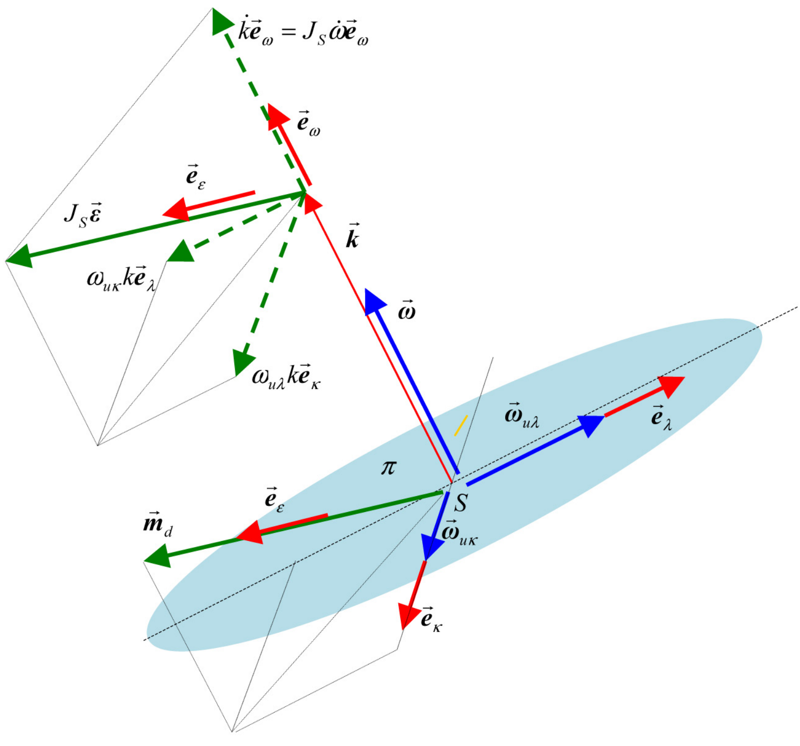

Fig. 8: A dynamic equation of spherical motion.

the system of axes $S\kappa\lambda\omega$ bound with the angular momentum plane, are taken into consideration – has the form

$$\dot{\vec{k}} = \dot{k}\vec{e}_\omega + \omega_{u\lambda}k\vec{e}_\kappa - \omega_{u\kappa}k\vec{e}_\lambda. \tag{54}$$

After substituting the value of $k = J_S\omega$ (40) of the resultant angular momentum $\vec{k}$ and taking into consideration $\dot{k} = J_S\dot{\omega}$ equation (50) assumes the form

$$\dot{\vec{k}} = J_S[\omega(\omega_{u\lambda}\vec{e}_\kappa - \omega_{u\kappa}\vec{e}_\lambda) + \dot{\omega}\vec{e}_\omega]. \tag{55}$$

The expression in brackets is an expansion of the expression $\vec{\varepsilon} = \dot{\vec{\omega}} = \dot{\omega}\vec{e}_\omega + \vec{\omega}_u \times \vec{\omega}$, which is the vector notation of the vector of angular acceleration $\vec{\varepsilon}$ of a rigid body in the non stationary system of axes $S\kappa\lambda\omega$,

$$\vec{\varepsilon} = \dot{\omega}\vec{e}_\omega + \omega\omega_{u\lambda}\vec{e}_\kappa - \omega\omega_{u\kappa}\vec{e}_\lambda. \tag{56}$$

It results from the comparison of the equations $\dot{\vec{k}} = \vec{m}_d$ (52), (55) and (56) that the vector dynamic equation for a rigid body (Fig. 8) moving with spherical motion around the stationary pole S of the form

$$J_S\vec{\varepsilon} = \vec{\boldsymbol{m}}_d. \tag{57}$$

The product of the vector of angular acceleration of a rigid body moving with spherical motion and the constant moment of inertia of this body, calculated relative to the pole – the centre of spherical motion – is equal to the vector of dynamic moment, calculated for this pole in the system of orthogonal axes suspended in the centre of spherical motion.

5. Summary

The moment of outer forces $\vec{\boldsymbol{m}}_S$ acting on a body moving with spherical motion causes a change in the vector of angular momentum $\dot{\vec{\boldsymbol{k}}}_S$ of this body, and this change results in a change in the vector of the plane angular momentum $\dot{\vec{\boldsymbol{k}}}_p$, which change can be called a vector of the plane moment, $\vec{\boldsymbol{m}}_p$. The sum of both vectors of moments yields a vector of dynamic moment, $\vec{\boldsymbol{m}}_d$, which in turn – proportionally to the constant value of the moment of inertia of a body, calculated relative to the stationary point, the centre of spherical motion S – forces a change in the vector of angular velocity $\vec{\boldsymbol{\omega}}$ in the form of the vector of angular acceleration of spherical motion

$$\vec{\varepsilon} = \dot{\vec{\boldsymbol{\omega}}} = \frac{1}{J_S}\vec{\boldsymbol{m}}_d,$$

which has the direction of the vector of dynamic moment; cf. [1, 4, 5].

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Division of Dynamics
Technical University of Łódź
Stefanowskiego 1/15 , PL 90-924 Łódź
Poland
e-mail: eia@thepolkadots.net

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MULTIILOCZYNY WEKTORÓW W OPISIE RUCHU SFERYCZNEGO II

RÓWNANIE DYNAMICZNE RUCHU SFERYCZNEGO

S t r e s z c z e n i e

W pracy pokazano możliwość zastosowania multiiloczynów wektorów oraz wersorów dla opisu ruchu sferycznego ciała sztywnego. Wykorzystano przy tym zarówno klasyczny zapis wektorowy, jak i odpowiadający mu zapis macierzowy, z użyciem iloczynu zewnętrznego wektorów to jest diady iloczynu skalarnego oraz diady iloczynu wektorowego dwóch wektorów. W opisie ruchu sferycznego użyto układu odniesienia, nazwanego tu parasolem, związanego z osią chwilowego obrotu. W drugiej części pracy dla układu parasola wyprowadzono równanie dynamiczne ruchu sferycznego.

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In memory of
Professor Roman Stanisław Ingarden

Emilia Fraszka-Sobczyk and Michał Marczak

THE ARBITRAGE PRICING OF A CALL OPTION IN THE RECURSIVE MODEL OF STOCK PRICES

Summary

The aim of this paper is connecting two theories: the discrete modeling in this the recursive sequences and the no arbitrage pricing of a call option. We consider two states in the model of stock prices. The price S_T^i in a scenario number i, $i = 1$ or $i = 2$, at the moment T is set by the linear recurrence $S_T^i = a_i \cdot S_{T-1}^i + b_i \cdot S_{T-2}^i$, $a_i, b_i \in R$ which depends on prices from two previous moments $T-1$ and $T-2$. There are two methods of the arbitrage pricing of a call option: the first method provides for replicating this option and the second method is a martingale approach. Two states of the model mean that we consider one period in the classical model of CRR.

1. Introduction

This paper begins with a reminder of some notions connected with the pricing of a call option.

Then we formulate the main problem.

1.1. The pricing of a call option in a one step model. Conditions for no arbitrage in the financial market

We shall use some non standard notations. We assume that there are only two securities in the market, the bank account and the stock. Let $S_0 > 0$ denote the stock price at the moment $t_0 = 0$, $S_{-1} > 0$ – the price in the previous day, S_T – the stock price at the moment T. We assume that $S_T > 0$ for any $T \in R$ and the stock price S_T at the moment T takes only one of two possible values:

$$S_T = \begin{cases} S_T^1 & \text{with prbability } p \\ S_T^2 & \text{with prbability } 1-p \end{cases},$$

and it should be $S_T^1 > K$, $S_T^2 \leq K$, for some number $K \in R_+$. The price of the European call option at the moment T totals $C_T = (S_T - K)^+$, that is

$$C_T = \begin{cases} S_T^1 - K =: C_T^1 & \text{with prbability } p \\ 0 =: C_T^2 & \text{with prbability } 1-p \end{cases}.$$

In the next two sections we present two methods of setting the arbitrage price of the European call option with the strike price K and the maturity time T. We assume that the interest rate of the bank account (or credit) for one period is $r \in R$. We denote $\tilde{r} := e^{rT}$.

1.1.1. The call option replication

Let (α_t, β_t) be a composition of the investor's portfolio at the moment t, where

α_t – the number of investor's shares of the stock $\alpha_t \in R$,

β_t – the amount of the bank account or the amount of credit when $\beta_t < 0$, $\beta_t \in R$,

V_t – the value of the portfolio at the moment t.

The investor sell the call option and makes the portfolio for hedging the option. We assume that the portfolio setting at the moment 0 will not change until the moment T. The portfolio replicates the option when $V_T = C_T$. Then the value of the portfolio at the moment T satisfies the following conditions:

$$\begin{cases} \alpha_0 \cdot S_T^1 + e^{rT} \cdot \beta_0 = C_T^1, \\ \alpha_0 \cdot S_T^2 + e^{rT} \cdot \beta_0 = C_T^2. \end{cases} \tag{1.1.1}$$

Thus

$$\alpha_0 = \frac{S_T^1 - K}{S_T^1 - S_T^2},$$
$$\beta_0 = -\alpha_0 \frac{S_T^2}{\tilde{r}} = -\frac{S_T^1 - K}{S_T^1 - S_T^2} \cdot \frac{S_T^2}{\tilde{r}}.$$

The cost of the replication of the option, which is called its arbitrage price equals

$$C_0 = V_0 = \alpha_0 \cdot S_0 + \beta_0 = \frac{S_T^1 \cdot (\tilde{r} \cdot S_0 - S_T^2)}{S_T^1 - S_T^2} \cdot \frac{1}{\tilde{r}} - \frac{S_0 \tilde{r} - S_T^2}{S_T^1 - S_T^2} \cdot \frac{K}{\tilde{r}}.$$

1.1.2. The martingale approach

We search for a probability P^* of appearing the price $S_T = S_T^1$ or $S_T = S_T^2$ that discounting stock prices form the martingale with respect to the probability P^*. Let S^* be the random process of discounting stock prices

$$S_0^* = S_0, \quad S_T^* = \tilde{r}^{-1} \cdot S_T.$$

The martingale property of the process S^* is as follows:

$$S_0^* = E_{P^*}[S_T^*].$$

Then

$$S_0 = \tilde{r}^{-1} \cdot [p^* \cdot S_T^1 + (1-p^*) \cdot S_T^2].$$

Hence

$$p^* = \frac{S_0 \tilde{r} - S_T^2}{S_T^1 - S_T^2}.$$

The discounting arbitrage price of the call option $C_T = (S_T - K)^+$, which is replicated by the strategy (1.1.1), is also a martingale with respect to the probability P^*. It means that

$$C_0 = E_{P^*}[\tilde{r}^{-1} \cdot C_T] = [\tilde{r}^{-1} \cdot C_T^1] \cdot p^* + 0 \cdot [1-p^*]$$

$$= \frac{S_T^1 \cdot (\tilde{r} \cdot S_0 - S_T^2)}{S_T^1 - S_T^2} \cdot \frac{1}{\tilde{r}} - \frac{S_0 \tilde{r} - S_T^2}{S_T^1 - S_T^2} \cdot \frac{K}{\tilde{r}}.$$

1.2. The conditions for no arbitrage in the financial market

The financial market satisfies the famous no arbitrage conditions when

$$S_T^2 < S_0 \tilde{r} < S_T^1.$$

The fundamental result (Theorem 2.2.) gives the necessary and sufficient conditions for the absence of arbitrage in our one step model if only T is large. Such no arbitrage is described by

$$\underline{\lim}_{T\to\infty} \frac{S_T^1}{S_0 \tilde{r}} > 1 \ \wedge \overline{\lim}_{T\to\infty} \frac{S_T^2}{S_0 \tilde{r}} < 1.$$

2. The linear recurrence. The recursive model of stock prices

In this section we present the recursive model of stock prices and we get no arbitrage conditions as the fundamental result of this paper (Theorem 2.2.).

2.1. The linear recurrence, the example of the stock prices

In this paper we assume that possible stock prices S_T^1, S_T^2 are defined by a linear recurrence. Now we remind a definition and an explicit formula for a recursive sequences.

Let consider a linear dependence:

$$S_n = a \cdot S_{n-1} + b \cdot S_{n-2}, \qquad a, b \in R, \quad n \in N, \quad n \geq 2.$$

This

$$x^2 - a \cdot x - b = 0$$

is called the characteristic equation for this dependence.

Let us determine a formula for the n-th term of the sequence $(S_n)_{\substack{n\geq 2\\ n\in N}}$ when initial values S_0 and S_1 are given.

We have two cases:

1. $\Delta > 0$ or $\Delta < 0$, this is $a^2 \neq -4b$.

Then the formula for the n-th term of the sequence $(S_n)_{\substack{n\geq 2\\ n\in N}}$ is equal to

$$S_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n, \qquad c_1, c_2 \in C,$$

where

$$r_1 = \frac{a+\sqrt{a^2+4b}}{2}, \qquad r_2 = \frac{a-\sqrt{a^2+4b}}{2}.$$

We set c_1 and c_2 by the following system of equations:

$$\begin{cases} c_1 + c_2 = S_0, \\ c_1 \cdot r_1 + c_2 \cdot r_2 = S_1. \end{cases}$$

Finalny, we have

$$\begin{aligned} S_n = &\left[\frac{S_1}{\sqrt{a^2+4b}} - \frac{S_0}{\sqrt{a^2+4b}} \cdot \left(\frac{a-\sqrt{a^2+4b}}{2}\right)\right] \cdot \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n \\ &+ \left[\frac{S_0}{\sqrt{a^2+4b}} \cdot \left(\frac{a+\sqrt{a^2+4b}}{2}\right) - \frac{S_1}{\sqrt{a^2+4b}}\right] \cdot \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n. \end{aligned}$$

2. $\Delta = 0$ this is $a^2 = -4b$.

Then the formula for the n-th term of the sequence $(S_n)_{\substack{n\geq 2\\ n\in N}}$ is equal to

$$S_n = c_1 \cdot r_0^n + n \cdot c_2 \cdot r_0^n, \qquad c_1, c_2 \in C,$$

where

$$r_0 = \frac{a}{2}.$$

We set c_1 and c_2 by the following system of equations:

$$\begin{cases} c_1 = S_0, \\ c_1 \cdot r_0 + c_2 \cdot r_0 = S_1. \end{cases}$$

Thus

$$S_n = \begin{cases} 0 & \text{for } a = 0, \\ \left(S_0 + n \cdot \dfrac{2S_1 - a \cdot S_0}{a}\right) \cdot \left(\dfrac{a}{2}\right)^n & \text{for } a \neq 0. \end{cases}$$

In conclusion

$$\begin{aligned} S_n = &\left[\frac{S_1}{\sqrt{a^2+4b}} - \frac{S_0}{\sqrt{a^2+4b}} \cdot \left(\frac{a-\sqrt{a^2+4b}}{2}\right)\right] \cdot \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n \\ &+ \left[\frac{S_0}{\sqrt{a^2+4b}} \cdot \left(\frac{a+\sqrt{a^2+4b}}{2}\right) - \frac{S_1}{\sqrt{a^2+4b}}\right] \cdot \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n \end{aligned}$$

for $a^2 \neq -4b$,

$$S_n = \left(S_0 + n \cdot \frac{2S_1 - a \cdot S_0}{a}\right) \cdot \left(\frac{a}{2}\right)^n$$

for $a^2 = -4b \wedge a \neq 0$,

$$S_n = 0$$

for $a^2 = -4b \wedge a = 0$.

In this paper we do not allow for $\Delta < 0$, because imaginary value can not be value of money. We are not interested in the case $S_n = 0$.

Let consider the following recursive model of stock prices:

$$S_t = \begin{cases} a_1 \cdot S^1_{t-1} + b_1 \cdot S^1_{t-2} =: S^1_t & \text{with probablity } p, \\ c_1 \cdot S^2_{t-1} + d_1 \cdot S^2_{t-2} =: S^2_t & \text{with probablity } 1-p, \end{cases}$$

where

$$(a_1, b_1) \in W_1 := \left\{(a,b) \in R^2 : S_T > K\right\},$$
$$(c_1, d_1) \in W_2 := \left\{(c,d) \in R^2 : 0 < S_T \leq K\right\}, \quad K \in R_+.$$

Then the arbitrage price is equal to

$$C_0 = \frac{S^1_T \cdot (\tilde{r} \cdot S_0 - S^2_T)}{S^1_T - S^2_T} \cdot \frac{1}{\tilde{r}} - \frac{S_0 \tilde{r} - S^2_T}{S^1_T - S^2_T} \cdot \frac{K}{\tilde{r}}.$$

where

$$S^1_T = \left[\frac{S_0}{\sqrt{a_1^2 + 4b_1}} - \frac{S_{-1}}{\sqrt{a_1^2 + 4b_1}} \cdot \left(\frac{a_1 - \sqrt{a_1^2 + 4b_1}}{2}\right)\right] \cdot \left(\frac{a_1 + \sqrt{a_1^2 + 4b_1}}{2}\right)^T +$$
$$+ \left[\frac{S_{-1}}{\sqrt{a_1^2 + 4b_1}} \cdot \left(\frac{a_1 + \sqrt{a_1^2 + 4b_1}}{2}\right) - \frac{S_0}{\sqrt{a_1^2 + 4b_1}}\right] \cdot \left(\frac{a_1 - \sqrt{a_1^2 + 4b_1}}{2}\right)^T$$

(2.1.1) for $a_1^2 > -4b_1$,

$$S^1_T = \left(S_{-1} + T \cdot \frac{2S_0 - a_1 \cdot S_{-1}}{a_1}\right) \cdot \left(\frac{a_1}{2}\right)^T \quad \text{for} \quad a_1^2 = -4b \wedge a_1 \neq 0,$$

$$S^2_T = \left[\frac{S_0}{\sqrt{c_1^2 + 4d_1}} - \frac{S_{-1}}{\sqrt{c_1^2 + 4d_1}} \cdot \left(\frac{c_1 - \sqrt{c_1^2 + 4b_1}}{2}\right)\right] \cdot \left(\frac{c_1 + \sqrt{c_1^2 + 4b_1}}{2}\right)^T +$$
$$+ \left[\frac{S_{-1}}{\sqrt{c_1^2 + 4d_1}} \cdot \left(\frac{c_1 + \sqrt{c_1^2 + 4b_1}}{2}\right) - \frac{S_0}{\sqrt{c_1^2 + 4d_1}}\right] \cdot \left(\frac{c_1 - \sqrt{c_1^2 + 4d_1}}{2}\right)^T$$

(2.1.2) for $c_1^2 > -4d_1$,

$$S^2_T = \left(S_{-1} + T \cdot \frac{2S_0 - c_1 \cdot S_{-1}}{c_1}\right) \cdot \left(\frac{c_1}{2}\right)^T \quad \text{for} \quad c_1^2 = -4d_1 \wedge c_1 \neq 0.$$

Example. Suppose that a stock S has price 100 on a month and 120 on the previous month. We assess that the probability of increasing the price p is 0.4 and decreasing the price is 0.6. We set the arbitrage price of the call option with the exercise price $K = 100$ and the maturity time totals 6 month. The only trading dates are 0 and T, so that the portfolio fixed at time 0 is held until time T. We assume that the interest rate for 6 month r amount to 10 % and it does not change. Let

$$S_t = \begin{cases} S^1_{t-1} + \frac{1}{4} \cdot S^1_{t-2} =: S^1_t & \text{with probability } 0.4, \\ S^2_{t-1} - \frac{1}{4} \cdot S^2_{t-2} =: S^2_t & \text{with probability } 0.6. \end{cases}$$

The stock price in the recursive model which is given above at the moment $T = 6$ is:

1. for the increasing model of stock prices

$$S^1_T = \left[50\sqrt{2} - 30\sqrt{2}\left(1 - \sqrt{2}\right)\right] \cdot \left(\frac{1+\sqrt{2}}{2}\right)^6 + \left(60\frac{1+\sqrt{2}}{\sqrt{2}} - 50\sqrt{2}\right)\left(\frac{1-\sqrt{2}}{2}\right)^6$$

$$= \left[20\sqrt{2} + 60\right] \cdot \left(\frac{1+\sqrt{2}}{2}\right)^6 + \left(60 - 20\sqrt{2}\right) \cdot \left(\frac{1-\sqrt{2}}{2}\right)^6.$$

2. for the decreasing model of stock prices

$$S^2_T = [120 + 6(200 - 120)] \cdot \left(\frac{1}{2}\right)^6 = 9.375.$$

Then

$$C_T = \begin{cases} S^1_T - 100 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$

When we solve the following system of equations:

$$\begin{cases} \alpha_0 \cdot S^1_T + 1.1 \cdot \beta_0 & = & S^1_T - 100, \\ \alpha_0 \cdot S^2_T + 1.1 \cdot \beta_0 & = & 0, \end{cases}$$

we get

$$\alpha_0 = \frac{\frac{5\sqrt{2}+15}{16}(3 + 2\sqrt{2})^3 + \frac{15-5\sqrt{2}}{16}(3 - 2\sqrt{2})^3 - 100}{\frac{5\sqrt{2}+15}{16}(3 + 2\sqrt{2})^3 + \frac{15-5\sqrt{2}}{16}(3 - 2\sqrt{2})^3 - 9.375},$$

$$\beta_0 = -\frac{375}{44}\alpha_0.$$

Thus the arbitrage price is equal to

$$C_0 = \frac{4025}{44}\alpha_0 \approx 60.05.$$

Now we set the arbitrage price by using a matingale approach. We have

$$S_0 = (1 + r)^{-1} \cdot [p^* \cdot S^1_T + (1 - p^*) \cdot S^2_T].$$

Then

$$p^* = \frac{S_0(1 + r) - S^2_T}{S^1_T - S^2_T}, \qquad C_0 = \frac{4025}{44}\alpha_0 \approx 60.05.$$

2.2. The conditions for no arbitrage in the financial market when lower and upper prices are defined by the recurrences

In this section we set conditions on $a_1, b_1, c_1, d_1, S_0, S_{-1}$, for which the financial market is without arbitrage for large T. We must prove for which $a_1, b_1, c_1, d_1, S_0, S_{-1}$ we get

$$\underline{\lim}_{T\to\infty} \frac{S_T^1}{S_0\tilde{r}} > 1 \ \wedge \overline{\lim}_{T\to\infty} \frac{S_T^2}{S_0\tilde{r}} < 1,$$

where S_T^1, S_T^2 are given by (2.1.1) and (2.1.2).

In the future we assume that

$$(*) \qquad a_1 > 0, \quad b_1 > 0, \quad c_1 > 0, \quad d_1 > 0, \quad S_0 > S_{-1} \cdot e^r.$$

This assumption gives us reasonable simplification.

We consider only two cases.

I. $a_1^2 = -4b_1, \qquad c_1^2 = -4d_1.$

The equality $a_1^2 = -4b_1$ always implies

$$\underline{\lim}_{T\to\infty} \frac{S_T^1}{S_0\tilde{r}} = \underline{\lim}_{T\to\infty} \frac{S_{-1} + A\cdot T}{S_0}\left(\frac{a_1}{2e^r}\right)^T,$$

where

$$A := \frac{2S_0 - a_1\cdot S_{-1}}{a_1}.$$

Let notice that the financial market is without arbitrage when

$$\begin{cases} \frac{a_1}{2e^r} > 1 \\ A \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} \frac{a_1}{2e^r} = 1 \\ A > 0 \end{cases}.$$

Provide for (*) we have

$$e^r \leq \frac{a_1}{2} \leq \frac{S_0}{S_{-1}}.$$

Analogously the equality $c_1^2 = -4d_1$ implies

$$\overline{\lim}_{T\to\infty} \frac{S_T^2}{S_0\tilde{r}} = \overline{\lim}_{T\to\infty} \frac{S_{-1} + B\cdot T}{S_0}\left(\frac{c_1}{2e^r}\right)^T,$$

where

$$B := \frac{2S_0 - c_1\cdot S_{-1}}{c_1}.$$

We have no arbitrage in the financial market if only

$$\frac{c_1}{2e^r} < 1 \quad \text{or} \quad \begin{cases} \dfrac{c_1}{2e^r} = 1 \\ B = 0 \\ \dfrac{S_{-1}}{S_0} < 1 \end{cases}.$$

Providing (*), we get

$$\frac{c_1}{2} < e^r.$$

Now we consider the second case.

II. $a_1^2 > -4b_1, \quad c_1^2 > -4d_1.$

The inequality $a_1^2 > -4b_1$ always implies

$$\varliminf_{T\to\infty} \frac{S_T^1}{S_0 \tilde{r}} = \varliminf_{T\to\infty} \left[\frac{1}{S_0}\left(C \cdot \left(\frac{E}{e^r}\right)^T + D \cdot \left(\frac{F}{e^r}\right)^T\right)\right],$$

where

$$\begin{aligned}
C &:= \frac{S_0}{\sqrt{a_1^2+4b_1}} - \frac{S_{-1}}{\sqrt{a_1^2+4b_1}} \cdot \left(\frac{a_1 - \sqrt{a_1^2+4b_1}}{2}\right),\\
D &:= \frac{S_{-1}}{\sqrt{a_1^2+4b_1}} \cdot \left(\frac{a_1 + \sqrt{a_1^2+4b_1}}{2}\right) - \frac{S_0}{\sqrt{a_1^2+4b_1}},\\
E &:= \frac{a_1 + \sqrt{a_1^2+4b_1}}{2},\\
F &:= \frac{a_1 - \sqrt{a_1^2+4b_1}}{2}.
\end{aligned}$$

We get no arbitrage in the financial market when

$$\begin{cases} E > e^r \\ C > 0 \end{cases} \quad \text{or} \quad \begin{cases} C = 0 \\ F > e^r \\ D > 0, \end{cases} \quad \text{or} \quad \begin{cases} E = e^r \\ C > S_0 \end{cases}$$

Providing (*), we have

$$\begin{cases} \dfrac{a_1 + \sqrt{a_1^2+4b_1}}{2} > e^r \\ \dfrac{a_1 - \sqrt{a_1^2+4b_1}}{2} < \dfrac{S_0}{S_{-1}} \end{cases}$$

or

$$\frac{a_1 - \sqrt{a_1^2+4b_1}}{2} = \frac{S_0}{S_{-1}} < \frac{a_1 + \sqrt{a_1^2+4b_1}}{2}$$

or

$$\begin{cases} \dfrac{a_1 + \sqrt{a_1^2+4b_1}}{2} = e^r \\ \dfrac{a_1 - \sqrt{a_1^2+4b_1}}{2} < \dfrac{S_0}{S_{-1}}\left(1 - \sqrt{a_1^2+4b_1}\right). \end{cases}$$

The inequality $c_1^2 > -4d_1$ implies

$$\varlimsup_{T\to\infty} \frac{S_T^2}{S_0 \tilde{r}} = \varlimsup_{T\to\infty} \left[\frac{1}{S_0}\left(\tilde{C} \cdot \left(\frac{\tilde{E}}{e^r}\right)^T + \tilde{D} \cdot \left(\frac{\tilde{F}}{e^r}\right)^T\right)\right],$$

where

$$\tilde{C} := \frac{S_0}{\sqrt{c_1^2+4d_1}} - \frac{S_{-1}}{\sqrt{c_1^2+4d_1}} \cdot \left(\frac{c_1 - \sqrt{c_1^2+4d_1}}{2} \right),$$

$$\tilde{D} := \frac{S_{-1}}{\sqrt{c_1^2+4d_1}} \cdot \left(\frac{c_1 + \sqrt{c_1^2+4d_1}}{2} \right) - \frac{S_0}{\sqrt{c_1^2+4d_1}},$$

$$\tilde{E} := \frac{c_1 + \sqrt{c_1^2+4d_1}}{2},$$

$$\tilde{F} := \frac{c_1 - \sqrt{c_1^2+4d_1}}{2}.$$

In this case the financial market is without arbitrage when

$$\frac{\tilde{E}}{e^r} < 1 \qquad \text{or} \qquad \begin{cases} \dfrac{\tilde{E}}{e^r} = 1 \\ \\ \dfrac{C}{S_0} \in [0,1) \end{cases}.$$

Providing (*), we get

$$\frac{c_1 + \sqrt{c_1^2+4d_1}}{2} < e^r \quad \text{or} \quad \begin{cases} \dfrac{c_1 + \sqrt{c_1^2+4d_1}}{2} = e^r \\ \dfrac{S_0}{S_{-1}} \left(1 - \sqrt{c_1^2+4d_1} \right) < \dfrac{c_1 - \sqrt{c_1^2+4d_1}}{2} \end{cases}$$

In conclusion, we have the following theorem.

Theorem 2.2. *If the final price of the stock is defined by the linear recurrence* (2.1.1)–(2.1.2)*, then the following equivalences are set.*

I. If the discriminants of characteristic equations of the recurrence are equal to 0, thus if $a_1^2 = -4b_1$, $c_1^2 = -4d_1$, then the following conditions are equivalent:

(α) there is no arbitrage for large T

$$\underline{\lim}_{T\to\infty} \frac{S_T^1}{S_0 \tilde{r}} > 1 \ \wedge \overline{\lim}_{T\to\infty} \frac{S_T^2}{S_0 \tilde{r}} < 1;$$

(β) $\dfrac{c_1}{2} < e^r \leq \dfrac{a_1}{2} \leq \dfrac{S_0}{S_{-1}}$.

II. If the discriminants of characteristic equations of the recurrence are strictly positive, i.e. $a_1^2 > -4b_1$, $c_1^2 > -4d_1$, then the following conditions are equivalent:

(α) there is no arbitrage for large T

$$\underline{\lim}_{T\to\infty} \frac{S_T^1}{S_0 \tilde{r}} > 1 \ \wedge \overline{\lim}_{T\to\infty} \frac{S_T^2}{S_0 \tilde{r}} < 1;$$

$$(\beta)\quad \begin{cases} \dfrac{c_1+\sqrt{c_1^2+4d_1}}{2} < e^r < \dfrac{a_1+\sqrt{a_1^2+4b_1}}{2} \\[2ex] \dfrac{a_1-\sqrt{a_1^2+4b_1}}{2} < \dfrac{S_0}{S_{-1}} \end{cases}$$

or

$$\begin{cases} \dfrac{a_1-\sqrt{a_1^2+4b_1}}{2} < \dfrac{S_0}{S_{-1}} \\[2ex] \quad \dfrac{S_0}{S_{-1}}\left(1-\sqrt{c_1^2+4d_1}\right) < \dfrac{c_1-\sqrt{c_1^2+4d_1}}{2} < e^r \\[2ex] = \dfrac{c_1+\sqrt{c_1^2+4d_1}}{2} < \dfrac{a_1+\sqrt{a_1^2+4b_1}}{2} \end{cases}$$

or

$$\frac{c_1+\sqrt{c_1^2+4d_1}}{2} < e^r < \frac{S_0}{S_{-1}} = \frac{a_1-\sqrt{a_1^2+4b_1}}{2} < \frac{a_1+\sqrt{a_1^2+4b_1}}{2}$$

or

$$\begin{aligned} &\frac{S_0}{S_{-1}}\left(1-\sqrt{c_1^2+4d_1}\right) < \frac{c_1-\sqrt{c_1^2+4d_1}}{2} < e^r \\ &= \frac{c_1+\sqrt{c_1^2+4d_1}}{2} < \frac{S_0}{S_{-1}} \\ &= \frac{a_1-\sqrt{a_1^2+4b_1}}{2} < \frac{a_1+\sqrt{a_1^2+4b_1}}{2} \end{aligned}$$

or

$$\begin{cases} \dfrac{c_1+\sqrt{c_1^2+4d_1}}{2} < e^r = \dfrac{a_1+\sqrt{a_1^2+4b_1}}{2} \\[2ex] \dfrac{a_1-\sqrt{a_1^2+4b_1}}{2} < \dfrac{S_0}{S_{-1}}\left(1-\sqrt{a_1^2+4b_1}\right) \end{cases}$$

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Faculty of Mathematics and Informatics
University of Łódź
Banacha 22, PL-90-238 Łódź
Poland
e-mail: emilaf@math.uni.lodz.pl

Faculty of Organization and Management
Technical University of Łódź
Piotrkowska 266, PL-90-924 Łódź
Poland
e-mail: marczak_m@wp.pl

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ARBITRAŻOWA WYCENA OPCJI W OPARCIU O REKURENCYJNY MODEL CEN AKCJI

S t r e s z c z e n i e

Celem pracy jest połączenie dwóch teorii: modelowania dyskretnego, w tym ciągami rekurencyjnymi oraz wyceny bezarbitrażowej opcji. Rozważono dwustanowy model cen akcji. Cena akcji S_T^i w i-tym scenariuszu, $i = 1$ lub $i = 2$, w chwili T została wyznaczona w oparciu o liniową rekurencję $S_T^i = a_i \cdot S_{T-1}^i + b_i \cdot S_{T-2}^i$, $a_i, b_i \in R$ zależną od cen akcji z dwóch poprzednich okresów, tj. w chwili $T-1$ i chwili $T-2$. Przedstawiono dwa sposoby arbitrażowej wyceny opcji kupna: jeden uwzględnia pojęcie replikacji opcji, drugi zaś metodę martyngałową. Dwustanowość modelu oznacza rozpatrywanie klasycznego modelu jednookresowego CRR. Jedynie górna i dolna zmiana cen akcji jest modelowana w bardziej skomplikowany sposób.

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In memory of
Professor Roman Stanisław Ingarden

Iryna V. Denega

SOME EXTREMAL PROBLEMS ON NON-OVERLAPPING DOMAINS WITH FREE POLES

Summary

Paper is devoted to extremal problems of geometric function theory with estimates of functionals defined on systems of non-overlapping domains. In particular, the focus of investigation is a well-known problem of V. N. Dubinin and generalization of some results in this problem.

1. Introduction

In geometric function theory of a complex variable extremal problems on non-overlapping domains form a well-known classic direction and have a rich history (see [1–14]). Paper [1] was the initial impetus for such direction, in which, it was first proposed and solved the problem of maximizing the product of conformal radii for two non-overlapping simply connected domains. Further, themes connected with the study of problems on non-overlapping domains were developed in papers [1–14]. This paper summarizes some results obtained in [**?**, **?**, **?**].

Let $\mathbb{N}$, $\mathbb{R}$ be the set of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be the one point compactification od $\mathbb{C}$, and $\mathbb{R}^+ = (0, \infty)$.

Let $r(B, a)$ be the inner radius of domain $B \subset \overline{\mathbb{C}}$, with respect to a point $a \in B$ (see [**?**, **?**, ?]) and $\chi(t) = \frac{1}{2}(t + t^{-1})$.

Let $n \in \mathbb{N}$. A set of points

$$A_n := \left\{ a_k \in \mathbb{C} : \; k = \overline{1, n} \right\},$$

is called *n–radial* system iff

$$|a_k| \in \mathbb{R}^+, \quad k=\overline{1,n}, \quad \text{and} \quad 0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi.$$

Denote

$$P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\},$$

$$\theta_k := \arg a_k,\ a_{n+1} := a_1,\ \theta_{n+1} := 2\pi,$$

$$\alpha_k := \frac{1}{\pi} \arg \frac{a_{k+1}}{a_k},\ \alpha_{n+1} := \alpha_1,\ k=\overline{1,n}.$$

This work is based on application of the piecewise-separating transformation developed in [4–6]. For specific use of this method we consider a special system of conformal mappings. By $\zeta = \pi_k(w), \quad k=\overline{1,n}$ we denote the unique branch of multi-valued analytic function $-i\left(e^{-i\theta_k}w\right)^{1/\alpha_k}$, which performs univalent and conformal mappings $P_k(A_n)$ onto the right half-plane $\mathrm{Re}\,\zeta > 0$.

For an arbitrary n-radial system of points $A_n = \{a_k\}$ and $\gamma \in \mathbb{R}^+ \cup \{0\}$ we assume that

$$\mathcal{L}^{(\gamma)}(A_n) := \prod_{k=1}^{n} \left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{1-\frac{1}{2}\gamma\alpha_k^2} \cdot \prod_{k=1}^{n} |a_k|^{1+\frac{1}{4}\gamma(\alpha_k+\alpha_{k-1})}.$$

The class of n-radial systems of points for which $\mathcal{L}^{(\gamma)}(A_n) = 1$ automatically includes all systems with n different points, located on the unit circle.

The main purpose of this work is to obtain exact upper estimates for the functionals:

(1) $$J_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

(2) $$I_n(\gamma) = \left[r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right]^{\gamma} \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $\gamma \in \mathbb{R}^+$, $A_n = \{a_k\}_{k=1}^n$ is an n-radial system of points, $a_0 = 0$, and $\{B_k\}_{k=0}^n$ is a system of non-overlapping domains (e.i. $B_p \cap B_j = \emptyset$ if $p \neq j$) such that $a_k \in B_k$, $a_\infty \in B_\infty$, $k = \overline{0,n}$.

2. Main results

V. N. Dubinin in paper ([?], p. 68, 9.2) and his monograph ([**?**], p. 381, no. 16) formulated the following

Problem. Prove that the maximum of functional (**??**) is attained for some domains that have n-tuple symmetry, where B_0, B_1, B_2,..., B_n, $n \geq 2$ are non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $r(B_j, a_j)$ is a inner radius of the domain B_j in point a_j, $(a_j \in B_j)$, $j = \overline{0,n}$, and $\gamma \leq n$.

This problem caused great interest and has been studied in different directions (see, for example, [?, **?**, ?]). In 1988 Dubinin [?] completely solved problem for $\gamma = 1$, $n \geq 2$ in the case when the points lie on the unit circle $|a_k| = 1$, but the result is also true for $\gamma \in (0,1]$ (this is implied from his method). Further, G. V. Kuz'mina repeated this result for simply connected domains by another method. In 1996 Kovalev [?] obtained solution to this problem, however not for an arbitrary system of points, but for a subclass of systems satisfying the condition $0 < \alpha_k \leq 2\pi/\sqrt{\gamma}$, $k = \overline{1,n}$. Then Bakhtin in his monograph [?] extended the ideas and methods of [?], and thus proved that the hypothesis is true for an arbitrary $\gamma \in \mathbb{R}^+$, but starting with some number $n_0(\gamma)$. Further, Bakhtin, Bakhtina, and Podvysotskii [**?**] first showed that for $n \geq 5$ we can get stronger results and confirmed that the problem is valid for some $\gamma > 1$. We shall prove

Theorem 1. *Let*

$$n \in \mathbb{N}, \quad n \geq 2, \quad \gamma \in (0, \gamma_n], \quad \gamma_n = \begin{cases} \sqrt[8]{n}, & n = \overline{2,7} \\ \sqrt[4]{n}, & n \geq 8 \end{cases}.$$

Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}^{(\gamma)}(A_n) = 1$, $\mathcal{L}^{(0)}(A_n) \leq 1$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $a_0 = 0 \in B_0$, $(k = \overline{1,n})$ we have the inequality

$$J_n(\gamma) \leq r^{\gamma}(D_0, 0) \prod_{k=1}^{n} r(D_k, d_k),$$

where D_k, d_k, $k = \overline{0,n}$, $d_0 = 0$ are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2. \tag{3}$$

Theorem 1 generalizes the result of paper [**?**] on more general systems of points of the complex plane.

Corollary 1. *Let $n \in \mathbb{N}$, $n \geq 2$ $\gamma \in (0,1]$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ and any system of non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, we have the inequality*

$$J_n(\gamma) \leq \frac{4^{n+\gamma/n}\gamma^{\gamma/n} n^n}{(n^2-\gamma)^{n+\gamma/n}} \left(\frac{n-\sqrt{\gamma}}{n+\sqrt{\gamma}}\right)^{2\sqrt{\gamma}} R^{n+\gamma}.$$

Equality in this inequality is attained when a_k and B_k, $k = \overline{0,n}$ are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + R^n\gamma}{w^2(w^n-R^n)^2}\, dw^2,$$

where $R^{n+\gamma} = \mathcal{L}^{(\gamma)}(A_n)$.

In [?] Dubinin obtained an estimate for the functional (**??**) if $\gamma = \frac{1}{2}$ and $n \geq 2$ ($|a_k| = 1$) by the method of symmetrization. Kuz'mina [?] used extremal-metric approach and obtained estimate for (**??**) if $\gamma \in \left(0, \frac{1}{8}n^2\right]$ and $n \geq 2$. In [?] Kuz'mina also emphasized that the upper bound for γ is not the best possible. And the question about the exact upper bounds for γ is still open. Note that when $n = 2$ evaluation for the functional (**??**) in [?] coincides exactly with the estimate of work [?]. We improved estimates for the functional (**??**) for $n = 2, 3$ on more general systems of points.

Theorem 2. *Let*

$$0 < \gamma \leq \gamma_2, \quad \gamma_2 = \frac{3}{5}.$$

Then for any 2*-radial system of points* $A_2 = \{a_k\}_{k=1}^{2}$ *such that* $\mathcal{L}^{(0)}(A_2) = 1$ *and any system of non-overlapping domains* B_0, B_1, B_2, B_∞ ($a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$) *we have the inequality*

$$\begin{aligned} & [r(B_0, 0)\, r(B_\infty, \infty)]^\gamma\, r(B_1, a_1)\, r(B_2, a_2) \\ \leq\ & [r(\Lambda_0, 0)\, r(\Lambda_\infty, \infty)]^\gamma\, r(\Lambda_1, \lambda_1)\, r(\Lambda_2, \lambda_2), \end{aligned} \tag{4}$$

where domains Λ_0, Λ_∞, Λ_1, Λ_2 *and points* 0, ∞, λ_1, λ_2 *are, respectively, circular domains and poles of the quadratic differential*

$$Q(w)dw^2 = -\frac{\gamma w^4 + (4 - 2\gamma)w^2 + \gamma}{w^2(w^2 - 1)^2}\, dw^2. \tag{5}$$

Theorem 3. *Let*

$$0 < \gamma \leq \gamma_3, \quad \gamma_3 = \frac{6}{5}.$$

Then for any 3*-radial system of points* $A_3 = \{a_k\}_{k=1}^{3}$ *such that* $\mathcal{L}^{(0)}(A_3) = 1$ *and any system of non-overlapping domains* B_0, B_1, B_2, B_3, B_∞ ($a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,3}$) *we have the inequality*

$$\begin{aligned} & [r(B_0, 0)\, r(B_\infty, \infty)]^\gamma \prod_{k=1}^{3} r(B_k, a_k) \\ \leq\ & [r(\Lambda_0, 0)\, r(\Lambda_\infty, \infty)]^\gamma \prod_{k=1}^{3} r(\Lambda_k, \lambda_k), \end{aligned} \tag{6}$$

where domains Λ_0, Λ_∞, Λ_1, Λ_2, Λ_3 *and points* 0, ∞, λ_1, λ_2, λ_3 *are, respectively, circular domains and poles of the quadratic differential*

$$Q(w)dw^2 = -\frac{\gamma w^6 + (9 - 2\gamma)w^3 + \gamma}{w^2(w^3 - 1)^2}\, dw^2. \tag{7}$$

From Theorem 2 we have the following corollaries:

Corollary 2. *Under the conditions of Theorem 2 we have the estimate*

$$(8)\qquad [r\,(B_0,0)\,r\,(B_\infty,\infty)]^\gamma\, r\,(B_1,a_1)\,r\,(B_2,a_2) \le \frac{4\cdot\gamma^\gamma}{|1-\gamma|^{1+\gamma}}\cdot\left|\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}\right|^{2\sqrt{\gamma}}.$$

Equality in (**??**) *is attained when domains* B_0, B_∞, B_1, B_2 *and points* 0, ∞, a_1, a_2 *are, respectively, poles and circular domains of the quadratic differential* (**??**).

Corollary 3. *Let*

$$0<\gamma\le\gamma_2,\quad \gamma_2=\frac{3}{5}.$$

Then for any 2*-radial system of points* $A_2=\{a_k\}_{k=1}^2$ *such that* $|a_k|=1$, $k\in\{1,2\}$ *and any system of non-overlapping domains* B_0, B_1, B_2, B_∞ ($a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_1\in B_1\subset\overline{\mathbb{C}}$, $a_2\in B_2\subset\overline{\mathbb{C}}$)*, we have inequality* (**??**)*. Equality is attained when domains* Λ_0, Λ_∞, Λ_1, Λ_2 *and points* 0, ∞, λ_1, λ_2 *are, respectively, poles and circular domains of the quadratic differential* (**??**).

The estimate in Corollary 3 is new for $\gamma\in\left(\frac{1}{2},\frac{3}{5}\right]$.

From Theorem 3 we can easy obtain the following statements:

Corollary 4. *Under the conditions of Theorem 3 we have the estimate*

$$(9)\qquad [r\,(B_0,0)\,r\,(B_\infty,\infty)]^\gamma \prod_{k=1}^{3} r\,(B_k,a_k) \le \frac{4^{3+2\gamma/3}\cdot\gamma^{2\gamma/3}}{|9-4\gamma|^{3/2+2\gamma/3}}\cdot\left|\frac{3-2\sqrt{\gamma}}{3+2\sqrt{\gamma}}\right|^{2\sqrt{\gamma}}.$$

Equality in (**??**) *is attained when domains* B_0, B_∞, B_1, B_2, B_3 *and points* 0, ∞, a_1, a_2, a_3 *are, respectively, poles and circular domains of the quadratic differential* (**??**).

Corollary 5. *Let*

$$0<\gamma\le\gamma_3,\quad \gamma_3=\frac{6}{5}.$$

Then for any 3*-radial system of points* $A_3=\{a_k\}_{k=1}^3$ *such that* $|a_k|=1$, $k\in\{1,2,3\}$ *and any system of non-overlapping domains* B_0, B_1, B_2, B_3, B_∞ ($a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,3}$)*, we have inequality* (**??**)*. Equality is attained when domains* Λ_0, Λ_∞, Λ_1, Λ_2, Λ_3 *and points* 0, ∞, λ_1, λ_2, λ_3 *are, respectively, poles and circular domains of the quadratic differential* (**??**).

The estimate in Corollary 6 is new for $\gamma\in(1,125;1,2]$.

Proof of Theorem 1. Validity of this theorem for $\gamma\in(0,1]$ follows from the works [?, ?]. Consider first the case $\gamma=\sqrt[4]{n}$. We use the method due to Bakhtin [**?**, ?], and properties of separating transformation (see [**?**, ?, ?, ?, ?]). We make separating transformation of domains $\{B_k\}_{k=1}^n$. Suppose

$$P_k := P_k(A_n) := \{w : \theta_k < \arg w < \theta_{k+1}\}.$$

Consider the introduced system of functions $\zeta = \pi_k(w) = -i\left(e^{-i\theta_k}w\right)^{1/\alpha_k}$, $k = \overline{1,n}$.

Let $\Omega_k^{(1)}$, $k = \overline{1,n}$ be a domain of the plane $\mathbb{C}_\zeta$ obtained by combining the connected component $\pi_k(B_k \bigcap \overline{P}_k)$ containing a point $\pi_k(a_k)$, with its symmetrical reflection with respect to the imaginary axis. By $\Omega_k^{(2)}$, $k = \overline{1,n}$, we denote the domain of the plane $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_{k+1} \bigcap \overline{P}_k)$, containing the point $\pi_k(a_{k+1})$, with its symmetrical reflection with respect to the imaginary axis, $B_{n+1} := B_1$, $\pi_n(a_{n+1}) := \pi_n(a_1)$. Besides, by $\Omega_k^{(0)}$ we denote the domain of $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_0 \bigcap \overline{P}_k)$, containing the point $\zeta = 0$, with its symmetrical reflection with respect to the imaginary axis. Denote $\pi_k(a_k) := \omega_k^{(1)}$, $\pi_k(a_{k+1}) := \omega_k^{(2)}$, $k = \overline{1,n}$, $\pi_n(a_{n+1}) := \omega_n^{(2)}$.

The definition of π_k implies that

$$|\pi_k(w) - \omega_k^{(1)}| \sim \frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot |w - a_k|, \quad w \to a_k, \quad w \in \overline{P_k},$$

$$|\pi_k(w) - \omega_k^{(2)}| \sim \frac{1}{\alpha_k}|a_{k+1}|^{\frac{1}{\alpha_k}-1} \cdot |w - a_{k+1}|, \quad w \to a_{k+1}, \quad w \in \overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to 0, \quad w \in \overline{P_k}.$$

Then, using results of papers [?]– [?], [?] we obtain inequalities

$$r\left(B_k, a_k\right) \leq \left[\frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot \frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right]^{\frac{1}{2}}, \tag{10}$$

$$k = \overline{1,n}, \quad \Omega_0^{(2)} := \Omega_n^{(2)}, \quad \omega_0^{(2)} := \omega_n^{(2)},$$

$$r\left(B_0, 0\right) \leq \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{1}{2}}. \tag{11}$$

Repeating arguments given in [?] in the proof of Theorem 5.2.1 and taking into account the introduced sets of domains $\{P_k\}_{k=1}^n$, functions $\{\pi_k\}_{k=1}^n$, and numbers $\{\theta_k\}_{k=1}^n$, we obtain an inequality for the investigated functional (**??**):

$$J_n(\gamma) \leq \prod_{k=1}^{n}\left[r\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{\alpha_k^2}{2}\gamma} \cdot \prod_{k=1}^{n}\left[\frac{r\left(\Omega_{k-1}^{(2)}, \omega_{k-1}^{(2)}\right) r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right)}{\frac{1}{\alpha_{k-1}\cdot\alpha_k}|a_k|^{\frac{1}{\alpha_{k-1}}-1} \cdot |a_k|^{\frac{1}{\alpha_k}-1}}\right]^{\frac{1}{2}} =$$

(12)

$$= \prod_{k=1}^{n}\alpha_k \cdot \prod_{k=1}^{n}\frac{|a_k|}{|a_k a_{k+1}|^{\frac{1}{2\alpha_k}}} \cdot \left[\prod_{k=1}^{n} r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\prod_{k=1}^{n} r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)\right]^{\frac{1}{2}}.$$

Expression (**??**) in parentheses of the latter formula is a product of the functional $r^{\beta^2}(\Omega_k^{(0)},0)r(\Omega_k^{(1)},\omega_k^{(1)})r(\Omega_k^{(2)},\omega_k^{(2)})$ on triples of domains $\left(\Omega_k^{(0)},\Omega_k^{(1)},\Omega_k^{(2)}\right)$ of the plane $\mathbb{C}_\zeta$.

It is known [**?**] that the functional

$$Y_3(\sigma_1,\sigma_2,\sigma_3)=\frac{r^{\sigma_1}(D_1,d_1)\cdot r^{\sigma_2}(D_2,d_2)\cdot r^{\sigma_3}(D_3,d_3)}{|d_1-d_2|^{\sigma_1+\sigma_2-\sigma_3}\cdot|d_1-d_3|^{\sigma_1-\sigma_2+\sigma_3}\cdot|d_2-d_3|^{-\sigma_1+\sigma_2+\sigma_3}},$$

$\sigma_k\in\mathbb{R}^+,\ d_k\in D_k\subset\overline{\mathbb{C}},\ D_k\bigcap D_p=\varnothing,\ k=1,2,3,\ p=1,2,3,\ k\neq p$, is invariant under all conformal automorphisms of the complex plane $\overline{\mathbb{C}}$.

With this relation in mind, the following estimate holds:

$$J_n(\gamma)\leq\left(\prod_{k=1}^n\alpha_k\right)\cdot\prod_{k=1}^n\frac{|a_k|}{|a_ka_{k+1}|^{\frac{1}{2\alpha_k}}}\times$$

$$\times\left\{\prod_{k=1}^n\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}\times$$

$$\times\left[\prod_{k=1}^n|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}\right]^{\frac{1}{2}}.$$

Note that $|\omega_k^{(1)}|=|a_k|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(2)}|=|a_{k+1}|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(1)}-\omega_k^{(2)}|=|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}$. Taking into account these equalities we obtain

$$J_n(\gamma)\leq\left(\prod_{k=1}^n\alpha_k\right)\cdot\prod_{k=1}^n\frac{|a_k|}{|a_ka_{k+1}|^{\frac{1}{2\alpha_k}}}\times$$

$$\times\left(\prod_{k=1}^n|\omega_k^{(1)}-\omega_k^{(2)}|\right)\left(\prod_{k=1}^n\frac{|\omega_k^{(1)}\cdot\omega_k^{(2)}|}{|\omega_k^{(1)}-\omega_k^{(2)}|}\right)^{\frac{\gamma\alpha_k^2}{2}}\times$$

$$\times\left\{\prod_{k=1}^n\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}=$$

$$=2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^n\alpha_k^2}\cdot\left(\prod_{k=1}^n\alpha_k\right)\cdot\prod_{k=1}^n\left[\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\right]^{1-\frac{\gamma\alpha_k^2}{2}}\times$$

$$\times\prod_{k=1}^n|a_k|^{1+\frac{1}{4}\gamma(\alpha_k+\alpha_{k-1})}\times$$

$$\times\left\{\prod_{k=1}^n\frac{r^{\gamma\alpha_k^2}\left(\Omega_k^{(0)},0\right)\cdot r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{|\omega_k^{(1)}\cdot\omega_k^{(2)}|^{\gamma\alpha_k^2}|\omega_k^{(1)}-\omega_k^{(2)}|^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}.$$

For each $k=\overline{1,n}$ we can easily define the conformal automorphism $\zeta=T_k(z)$ of complex numbers of the plane $\overline{\mathbb{C}}$ such that

$$T_k(0)=0,\ T_k\left(\omega_k^{(s)}\right)=(-1)^s\cdot i,\ G_k^{(q)}:=T_k\left(\Omega_k^{(q)}\right),\ k=\overline{1,n},\ s=1,2,\ q=0,1,2.$$

Then, using results of [7–11] we obtain

$$J_n(\gamma)\leq 2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2}\cdot\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\mathcal{L}^{(\gamma)}\left(A_n\right)\times$$

$$\times\prod_{k=1}^{n}\left\{\frac{r^{\alpha_k^2\gamma}\left(G_k^{(0)},0\right)\cdot r\left(G_k^{(1)},-i\right)\cdot r\left(G_k^{(2)},i\right)}{2^{2-\gamma\alpha_k^2}}\right\}^{\frac{1}{2}}=$$

$$=2^{n-\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2}\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\mathcal{L}^{(\gamma)}(A_n)\cdot 2^{-n+\frac{\gamma}{2}\sum\limits_{k=1}^{n}\alpha_k^2}\times$$

$$\times\left[\prod_{k=1}^{n}r^{\alpha_k^2\gamma}\left(G_k^{(0)},0\right)\cdot r\left(G_k^{(1)},-i\right)\cdot r\left(G_k^{(2)},i\right)\right]^{\frac{1}{2}}.$$

Hence

$$J_n(\gamma)\leq\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\left[\prod_{k=1}^{n}r^{\alpha_k^2\gamma}\left(G_k^{(0)},0\right)\cdot r\left(G_k^{(1)},-i\right)\cdot r\left(G_k^{(2)},i\right)\right]^{\frac{1}{2}}. \tag{13}$$

As a result of the calculations the initial problem is reduced to an upper estimate of the functional $r^{x^2}(B_0,0)r(B_1,i)r(B_2,-i)$ in the class of triples of disjoint domains $\{B_0,B_1,B_2\}$ such that $0\in B_0$, $i\in B_1$, $-i\in B_2$, $B_k\subset\overline{\mathbb{C}}$, $k=0,1,2$.

Following the paper [?] we have

$$r^{x^2}(B_0,0)r(B_1,i)r(B_2,-i)\leq F(x)=$$
$$=2^{x^2+6}\cdot x^{x^2}(2-x)^{-\frac{1}{2}(2-x)^2}\cdot(2+x)^{-\frac{1}{2}(2+x)^2},\quad x\in[0,2].$$

Kovalev [?] proved that inequality (**??**) is true if $\alpha_k\sqrt{\gamma}\leq 2$, $k=1,n$ and $n\geq 5$. Therefore it remains to prove that it holds under the condition $\alpha_0\sqrt{\gamma}>2$, where $\alpha_0=\max\limits_k\alpha_k$. Further we use the method proposed in [?] (p. 255) by Bakhtin. From Theorem 5.2.3 in [?] if $\alpha_0\sqrt{\gamma}>2$ there is a chain of inequalities

$$J_n(\gamma)\leq\prod_{k=1}^{n}\left[r(B_0,0)r(B_k,a_k)\right]^{\frac{\gamma}{n}}\left[\prod_{k=1}^{n}r(B_k,a_k)\right]^{1-\frac{\gamma}{n}}\leq$$

$$\leq\left[\prod_{k=1}^{n}|a_k|^2\right]^{\frac{\gamma}{n}}\cdot\left[2^n\prod_{k=1}^{n}\alpha_k\cdot\mathcal{L}^{(0)}(A_n)\right]^{1-\frac{\gamma}{n}}\leq\left[2^n\prod_{k=1}^{n}\alpha_k\right]^{1-\frac{\gamma}{n}}\leq$$

$$\leq\left[2^n\alpha_0\left(\frac{2-\alpha_0}{n-1}\right)^{n-1}\right]^{1-\frac{\gamma}{n}}=\left[2^n\alpha_0(2-\alpha_0)^{n-1}(n-1)^{-(n-1)}\right]^{1-\frac{\gamma}{n}},$$

where

$$\alpha_0 = \max_k \alpha_k, \quad \text{and} \quad \alpha_0 \geq \frac{2}{\sqrt{\gamma}}.$$

On the other side, from the results of [?] (p. 257) and properties of separating transformation we obtain

$$J_n^0(\gamma) = r^{\gamma}\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right) = \left(\frac{4}{n}\right)^n \cdot \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \cdot \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}},$$

where D_k, d_k, $k = \overline{0,n}$, $d_0 = 0$, are, respectively, poles and circular domains of the quadratic differential (**??**). Estimate the value

$$O_n(\gamma) = \frac{r^{\gamma}(B_0, 0) \prod\limits_{k=1}^{n} r(B_k, a_k)}{r^{\gamma}(D_0, 0) \prod\limits_{k=1}^{n} r(D_k, d_k)} \leq$$

$$\leq \frac{\left[2 \cdot 2^{n-1} \cdot \alpha_0 (2 - \alpha_0)^{n-1} (n-1)^{-(n-1)}\right]^{1-\frac{\gamma}{n}}}{\left(\frac{4}{n}\right)^{n-1-\gamma\left(1-\frac{1}{n}\right)} \cdot \left(\frac{4}{n}\right)^{\gamma+1-\frac{\gamma}{n}} \cdot \left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}} \cdot \left(1 - \frac{\gamma}{n^2}\right)^{-n-\frac{\gamma}{n}} \cdot \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}} \leq$$

$$\leq \left[\frac{n}{4}\right]^{\gamma+1} \cdot \left[1 - \frac{1}{\sqrt{\gamma}}\right]^{n-1-\gamma\frac{n-1}{n}} \cdot \left(\frac{n}{\gamma}\right)^{\frac{\gamma}{n}} \cdot \left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}} \times$$

$$\times \left(\frac{1 + \frac{\sqrt{\gamma}}{n}}{1 - \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}} \cdot \left(\frac{4}{\sqrt{\gamma}}\right)^{1-\frac{\gamma}{n}} \cdot \left(\frac{n}{n-1}\right)^{n-1-\gamma\frac{n-1}{n}}$$

if $\gamma = \sqrt[4]{n}$ for $n \geq 3$.

Uncomplicated estimates, as in paper [?], show that $O_n(\sqrt[4]{n}) < 1$, $n \geq 8$ and $O_n(\sqrt[8]{n}) < 1$, $n = \overline{2,7}$. It is not difficult to show by standard methods that the function $Q_n(\gamma)$ on interval $\gamma \in (1; \sqrt[4]{n}\ ]$ is monotonically increasing with respect to γ. It follows that for these configurations maximum is not attained that is the assertion of Theorem 1 if $\alpha_0\sqrt{\gamma} > 2$ is proved. Thus it remains to consider the case $\alpha_0\sqrt{\gamma} \leq 2$.

Then according to the method of works [?, ?], we turn to the function $F(x)$ and from these works as a result of the calculations we obtain the inequality Theorem 1 for the functional (**??**). Theorem 1 is proved.

Proof of Theorem 2. We retain all notation for separating transformation of domains introduced in the proof of Theorem 1 for domains B_k, $k = \overline{0,n}$. By $\Omega_k^{(\infty)}$ we denote the domain of plane $\mathbb{C}_\zeta$, obtained by combining the connected component $\pi_k(B_\infty \bigcap \overline{E}_k)$ containing the point $\zeta = \infty$ with its symmetrical reflection with respect to the imaginary axis. The family

$$\left\{\Omega_k^{(\infty)}\right\}_{k=1}^{n}$$

is a result of separating transformation of an arbitrary domain B_∞, $\infty \in B_\infty \subset \overline{\mathbb{C}}$ with respect to families $\{E_k\}_{k=1}^n$ and $\{\pi_k\}_{k=1}^n$ at the point $\zeta = \infty$.

By Theorem 2 in [?] we have

$$r(B_\infty, \infty) \leq \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(\infty)}, \infty\right)\right]^{\frac{1}{2}}. \tag{14}$$

Using (**??**), (**??**), (**??**), we obtain

$$J_2(\gamma) \leq \prod_{k=1}^{2}\left(r\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(\infty)}, \infty\right)\right)^{\frac{\gamma\alpha_k^2}{2}} \times$$

$$\times\left(\frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\left(\frac{1}{\alpha_k}\right)^2\left(|a_k||a_{k+1}|\right)^{\frac{1}{\alpha_k}-1}}\right)^{\frac{1}{2}}.$$

Further, considering the methods of works [?, ?] from the latter relation we have

$$J_2(\gamma) \leq 4\left(\prod_{k=1}^{2}\alpha_k\right)\cdot\prod_{k=1}^{2}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|\times$$

$$\times\prod_{k=1}^{2}\left\{\frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\left(r\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2}\right\}^{\frac{1}{2}},$$

$$|\omega_k^{(1)}| = |a_k|^{\frac{1}{\alpha_k}},\ |\omega_k^{(2)}| = |a_{k+1}|^{\frac{1}{\alpha_k}},\ |\omega_k^{(1)}-\omega_k^{(2)}| = |a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}.$$

Each expression in the braces of the last inequality is the value of the functional

$$K_\tau = \left[r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right]^{\tau^2}\cdot\frac{r\left(B_1, a_1\right) r\left(B_2, a_2\right)}{|a_1-a_2|^2} \tag{15}$$

on the system of non-overlapping domains $\{\Omega_k^{(0)}, \Omega_k^{(1)}, \Omega_k^{(2)}, \Omega_k^{(\infty)}\}$ and corresponding system of points $\{0, \omega_k^{(1)}, \omega_k^{(2)}, \infty\}$ $(k \in \{1, 2\})$. Estimate of functional (**??**) in the case of fixed poles was first obtained by Dubinin [?].

Basing on Theorem 4.1.1 in [?] and invariance of the functional (**??**) we obtain an estimate

$$K_\tau \leq \Phi(\tau), \quad \tau \geq 0,$$

where $\Phi(\tau) = \tau^{2\tau^2}|1-\tau|^{-(1-\tau)^2}(1+\tau)^{-(1+\tau)^2}$. Then

$$J_2(\gamma) \leq \frac{4}{\gamma}\cdot\left[\prod_{k=1}^{2}\left(\tau_k^{2\tau_k^2+2}\cdot|1-\tau_k|^{-(1-\tau_k)^2}\cdot(1+\tau_k)^{-(1+\tau_k)^2}\right)\right]^{\frac{1}{2}}, \tag{16}$$

where $\tau_k = \sqrt{\gamma}\cdot\alpha_k$, $k = \overline{1, 2}$.

Consider in detail the function

$$\Psi(x) = x^{2x^2+2}|1-x|^{-(1-x)^2}(1+x)^{-(1+x)^2}.$$

$\Psi(x)$ is logarithmically convex on the interval $[0, x_0]$, where $x_0 \approx 0,88441$, $\Psi(x_0) = 0,07002$. On interval $[0, x_1]$ ($x_1 \approx 0,58142$ is the maximum of function $\Psi(x)$, $\Psi(x_1) \approx 0,08674$) the function is increasing from $\Psi(0) = 0$ to $\Psi(x_1)$, and decreases on the interval $(x_1, \infty]$.

Consider case $\gamma = \gamma_2$. We shall show that for any τ_1, τ_2 such that $\tau_1 + \tau_2 = 2\sqrt{\gamma_2}$, the following inequality holds:

$$\Psi(\tau_1) \cdot \Psi(\tau_2) \le \Psi^2(\sqrt{\gamma_2}). \tag{17}$$

For $\tau_1, \tau_2 \in (0, x_0]$ the statement (**??**) follows from the logarithmic convexity of the function $\Psi(x)$.

Let now $\tau_2 \in (x_0, \infty)$, $\tau_1 \in (0, x_0]$; then

$$\Psi(\tau_2) \cdot \Psi(\tau_1) \le \Psi(x_0) \cdot \Psi(x_1) < \Psi^2(\sqrt{\gamma_2})$$

(because $\Psi(x_0) \cdot \Psi(x_1) \approx 6,0735 \cdot 10^{-3}$ and $\Psi^2(\sqrt{\gamma_2}) \approx 6,123 \cdot 10^{-3}$).

From this follows that the statement (**??**) is true for all τ_1, τ_2. Taking into account the above considerations we conclude that (**??**) is also true for $\gamma \in (0, \gamma_2]$. Together with the inequalities (**??**), (**??**), (**??**) and (**??**) we obtain the inequality (**??**). Theorem 2 is proved.

Proof of Theorem 3 repeats the arguments presented in the proof of Theorem 2, taking into account some peculiarities in the case $n = 3$.

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Institute of Mathematics
National Academy of Sciences of Ukraine
Tereshchenkivs'ka vul. 3
UA-01 601 Kyiv
Ukraine
e-mail: iradenega@yandex.ru

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PEWNE ZAGADNIENIA EKSTREMALNE NA NIEZACHODZĄCYCH NA SIEBIE OBSZARACH ZE SWOBODNYMI BIEGUNAMI

S t r e s z c z e n i e

Praca jest poświęcona zagadnieniom ekstremalnym w geometrycznej teorii funkcji z oszacowaniami funkcjonałów określonych na układach niezachodzących na siebie obszarów. W szczególności, kładziemy nacisk na znany problem V. N. Dubinina i uogólnienia pewnych wyników w zakresie tego problemu.

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In memory of
Professor Roman Stanisław Ingarden

Agnieszka Niemczynowicz

THE DIAGONAL FORM OF THE HAMILTONIAN IN A ZWANZIG-TYPE CHAIN

Summary

In this paper we review the general procedure of the diagonalisation of Hamiltonian in the model of ferromagnetic thin films. In our work, we concentrate on the case when the considered sample is a simple linear chain of atoms, cutted from the ferromagnetic thin film structure in the direction perpendicular to the surface. The Hamiltonian of the system under study is written in the approximate second quantization approach.

1. Preliminaries

The theoretical and experimental study of spin wave resonance (SWR) in ferromagnetic thin films started nearly 60 years ago. The first who predicted the possibility of observing the SWR in such a structure was Kittel [1]. In 1958 Seavey and Tannenwald [2] experimentally confirmed the theory of Kittel (resonance standing spin waves). Next, research of many authors got the basic characteristic of SWR in thin samples of pure ferromagnetic metals e.g. Fe, Ni or Co and ferrites e.g. $NiFe_2O_4$ [3]. During the last half of century the theory of SWR was intensively investigated, complemented and corrected in many kind of materials.

In the process of SWR an important role is played mesaurement of power adsorption function. The shape of adsorbed power function $P(\omega)$ fitting to the experimental data can give us the information about the surface by possibility of finding the values of such parameters as e.g. the demagnetizing factor and its dispersion (the theory includes the inhomogeneity of demagnetizing fields).

The ferromagnetic sample is submerged in a static magnetic film $H = (0, 0, H^z)$ and the energy is absorbed from an external magnetic field $h^x(t)$ oscillating perpendicularly to $H = (0, 0, H^z)$. The nature of the shape of curve of power adsorption function is determined by the peaks. Each peak corresponds to an excitation of a distinct spin wave. The first who pointed out the theoretical possibility of the occurence of a surface peak of SWR was Wolf [4]. Sokolov *et. all* [5, 6], Puszkarski [7, 8] independently researched a method of identifying such a peak in the SWR spectrum. In theoretical considerations concerned with very thin films they proved that positions of peaks are independent to thickness, whereas the thinner ones it should shift towards growing field strengths with decreasing thickness. Important influence for the theoretical study of SWR have the models with assumptions regarding the surface anisotropy in the magnetic field H, studied by many researchers e.g. [9,10, 11]. From the experimental point of view that models was discussed and their properties are reviewed [12]. Various experiments on SWR show that the resonance spectrum depends on the crystallographic structure of the sample and its surface roughness.

From the viewpoint of the vibrational problem of a thin film the specific aspects can be describing in terms of coupled oscillators in relation to their boundary conditions. The fundamentals of the theory of oscillators in various applications in different aspects have been reviewed in the available literature [e.g. 13, 14].

The propose of this article is to review some of the key of the method used for the diagonalisation of Hamiltonian in the model in the ferromagnetic thin films. We shall consider a simple linear chain of atoms, which is cutted from the ferromagnetic thin films in the perpendicular to the surface, with assumptions proposed in [15].

2. Linear harmonic Zwanzig's chain

Let us consider the sample which is a ferromagnetic thin film interacting with the rf magnetic field. We divide the sample of thickness $d = Na$ into N monoatomic, two-dimensional layers parallel to the surface planes of sample, which shall be numbered by ν, $(\nu = 1, 2, ..., N)$. The position of each atom localized at the crystallographic lattice site is determined by the vector $\overrightarrow{j}$. The sample is characterized by the magnetization $M(t, z)$ in the plane of the surface with respect to the easy magnetisation axes parallel to the quantization direction. The rf magnetic field $h^x(t)$ is perpendicular to the constant magnetic field $H = (0, 0, H^z)$. We will restrict our considerations to interactions between nearest neighbours (Fig.1).

The Hamiltonian of above system takes the form

$$\mathcal{H} = \frac{1}{2}\sum_{\nu} \frac{p_{\nu}^{x}}{M} + \frac{1}{2}\sum_{(\nu,\nu\prime)} K_{\nu}\left(x_{\nu\prime} - x_{\nu}\right)^2, \tag{1}$$

where K_ν denote the harmonic forces and M is mass of the atom localized in the position ν, the symbol $\sum_{(\ ,\ )}$ denotes a sum containing each pair of atoms once only.

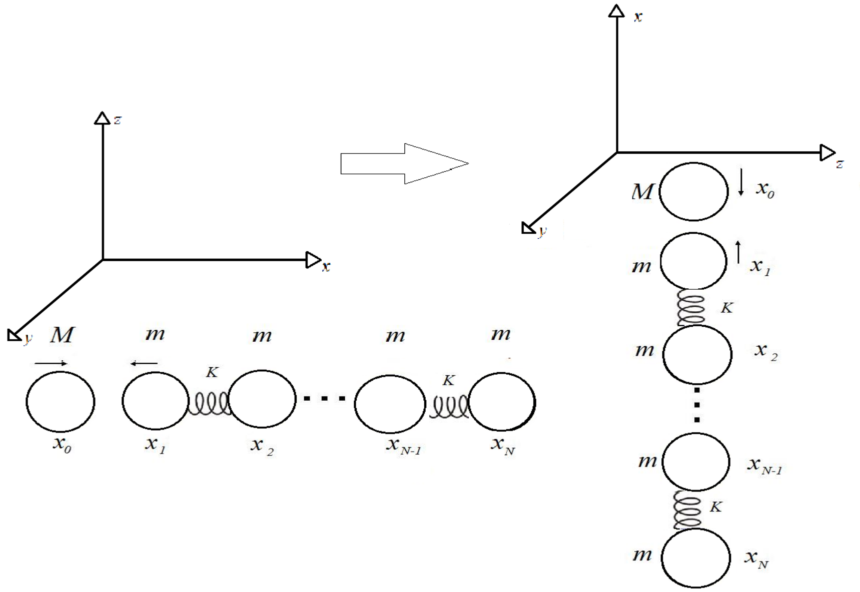

Fig. 1: The way of form one dimensional chain of atoms cutted from the ferromagnetic sample in accordance with assumptions of Zwanzig.

The equations of motions

$$\dot{p}^x_\nu = -\frac{\partial}{\partial x_\nu}\mathcal{H}, \qquad \dot{x}^x_\nu = -\frac{\partial}{\partial p^x_\nu}\mathcal{H} \tag{2}$$

read

$$\dot{p}^x_\nu = K_\nu \sum_{\nu\prime} \left(x_{\nu\prime} - x_\nu\right), \qquad \dot{x}_\nu = \frac{1}{M}\dot{p}^x_\nu \tag{3}$$

and, consequently

$$\ddot{x}_\nu = \frac{K_\nu}{M} \sum_{\nu\prime\in\nu} \left(x_{\nu\prime} - x_\nu\right). \tag{4}$$

Now, assuming the boundary conditions and the effective external force κ_ν, the Hamiltonian (1) reads

$$\mathcal{H} = \frac{1}{2}\sum_\nu \frac{p^x_\nu}{M} + \frac{1}{2}\sum_{(\nu,\nu\prime)} K_\nu \left(x_{\nu\prime} - x_\nu\right)^2 + \sum_\nu \kappa_\nu x_\nu. \tag{5}$$

In order to describe the magnons properties we use the spin operator varialbles $S^x_\nu, S^y_\nu, S^z_\nu$. The use of Zwanzig approach to the spin waves resonance is based on the analogy between spin operators described in the harmonic approximation and the harmonic operators which refer to the model of lattice vibrations.

Using this analogy, we can see that the spin operators (according to the Holstein-Primakoff transformation in the harmonic approximation)

$$S_\nu^x = \sqrt{2S}(a_\nu^+ + a_\nu^-), \quad S_\nu^y = \sqrt{2S}(a_\nu^+ - a_\nu^-)$$

express by the magnon creation a_ν^+ and annihilation a_ν^- operators in the harmonic approximation, correspond to the lattice vibration operators, it means, the position operator denoting the displacement of the considered atom from its equilibrium position related to the lattice site j on the layer ν

$$X_\nu = \frac{1}{2}(a_\nu^+ + a_\nu^-),$$

and momentum operator

$$P_\nu = \frac{1}{2i}(a_\nu^+ - a_\nu^-),$$

which is canonically conjugated to X_ν. The quantum-mechanical equations of motion for the spin vibrations considered in the direction perpendicular to the chain axes take the following form

$$i\hbar\dot{S}_r^x = [S_r^x, \mathcal{H}], \ -i\hbar\dot{S}_r^y = [S_r^y, \mathcal{H}], \tag{6}$$

Here we assume that $\langle S_\nu^x\rangle \Leftrightarrow X_\nu, \langle S_\nu^y\rangle \Leftrightarrow P_\nu, \langle S_\nu^z\rangle \Leftrightarrow S.$

In the original consideration performed by Zwanzig [7] we can recall his Hamiltonian to the form (5) and his equations of motions for the phonon operator $X_\nu,\ P_\nu$. We obtain

$$\frac{dX_\nu}{d\tau} = \frac{1}{2}u_{2\nu}, \quad R_\nu - R_{\nu+1} = u_{2\nu+1} \ \text{ for } \ \nu = 0, 1, 2, ..., \tag{7}$$

with

$$\frac{dX_0}{d\tau} = \frac{1}{2\mu}F(X_1), \quad \frac{dX_2}{d\tau} = -\frac{1}{2\mu}\left(F(X_1) + X_3\right) \tag{8}$$

and the resolvent function

$$\begin{aligned}\Theta(z,\tau) \quad &= \left[\exp\frac{1}{2}\left(z - z^{-1}\right)\tau\right]\Theta(z,0) + \frac{1}{2}\int_0^\tau \left[\exp\frac{1}{2}\left(z - z^{-1}\right)(\tau - s)\right] ds \,\times \\ &\times \left\{(1 - z^2)u_1(s) + zu_0(s) - z^2F[u_1(s)]\right\},\end{aligned} \tag{9}$$

where $\mu,\ \tau$ and F denote the reduced mass, reduced time and reduced force, respectively.

3. Diagonalisation of the Hamiltonian

In the case of application of Zwanzig model we start from the Hamiltonian contains three parts

$$\mathcal{H} = \mathcal{H}_{\text{ex}} + \mathcal{H}_{\text{anis}} + \mathcal{H}_{\text{Z}}. \tag{10}$$

The first term denotes the exchange term, namely

$$\mathcal{H}_{\mathrm{ex}} = -J \sum_{(\nu,\nu')} \vec{S_{\nu j}} \vec{S_{\nu' j}}, \tag{11}$$

where J is twice the exchange integral corresponding to two nearest neighbours. The anisotropy term of the Hamiltonian takes the form

$$\mathcal{H}_{\mathrm{anis}} = -\sum_{\nu j} A^0_{\nu j} S^z_{\nu_j} S^z_{\nu_j} - \sum_{\nu j} A^s_{\nu j} S^z_{\nu j}, \tag{12}$$

where $A^0_{\nu j}$ corresponding to the homogenous volume anisotropy and $A^s_{\nu j}$ corresponding to the surface anisotropy. As for the Zeeman term, it can be written

$$\mathcal{H}_{\mathrm{Z}} = -\mu_B H \sum_{\nu j} S^z_{\nu j}, \tag{13}$$

where $H = H^z$ is the component of the magnetic field $H = (0, 0, H^z)$ in the direction od easy magnetization axes. Taking into account, according to the Holstein-Primakoff theory, the spin operators are related to the creation and annihilation operators by the relations

$$S^+_r = \sqrt{2S} a^-_r, \quad S^-_r = \sqrt{2S} a^+_r, \quad S^z_r = S - a^+_r a^-_r, \text{ for } r = (\nu \vec{j})$$

or, in more general case

$$S^{\pm}_r = \sqrt{2S} f a^{\pm}_r \quad \text{with} \quad f = \sqrt{1 - \frac{a^+_r a^-_r}{2S}}.$$

In the harmonic approximation, f may be replaced by 1. Next, according to the procedure of Corciovei [16], the Hamiltonian become

$$\mathcal{H}_0 = -J \sum_{(r,r')} \left(S^z_r S^z_{r'} + \frac{1}{2}(S^+_r S^-_{r'} + S^-_r S^+_{r'}) \right) - \sum_{\nu_j} A^0_\nu S^z_r S^z_r - \sum_{\nu_j} (A^S_\nu + \mu_B H)(S - a^+_r a^-_r)$$

and, in the terms of creation and annihilation operators

$$\mathcal{H}_0 = -J \sum_{r,r'} \left(S - a^+_r a^-_r\right)\left(S - a^+_r a^-_r\right) - J1/2 \sum_{r,r'} 2S^2 (a^-_r a^+_{r'} + a^+_r a^-_{r'})$$
$$- \sum_r A^0_r (S - a^+_r a^-_r)(S - a^+_r a^-_r) - \sum_r (A^S_r + \mu_B H)(S - a^+_r a^-_r).$$

Further, by easy calculations, we obtain

$$\mathcal{H}_0 = -J \sum_{r,r'} \left(S^2 - S(a^+_r a^-_r + a^+_{r'} a^-_{r'})\right) - JS \sum_{r,r'} (2a^+_r a^-_{r'})$$
$$- \sum_r A^0_r \left(S^2 - 2S a^+_r a^-_r\right) - \sum_r \left(A^S_r + \mu_B H\right)\left(S - a^+_r a^-_r\right).$$

Finally,

$$\mathcal{H}^1_0 = \sum_r \left(\mu_B H + A^S_r + 2SA^0_r + 2JS\mathrm{z}(r)\right) \sum_{q,q'} T_{qr} T_{q'r} a^+_q a^-_{q'}$$
$$-2JS \sum_{r,r',r\neq r'} \sum_{qq'} T_{qr} T_{q'r'} a^+_q a^-_{q'}, \tag{14}$$

where $\mathrm{z}(r)$ is the number of nearest neighbours of any atom in the same layer. Here we introduced the following notations

$$|1_q\rangle = a_q^+|0\rangle = \sum_r T_{qr} a_r^+|0\rangle$$

$$a_q^+ = \sum_r T_{qr} a_r^+,$$

where $|1_q\rangle$ is defined in the space $[q = (\tau, h)]$ of quantum numbers τ, h by means of the linear combination of the localized states

$$|1_{\nu_j}\rangle = |\uparrow\uparrow\downarrow \dots \uparrow\rangle$$

with the following commutation relations

$$a_q^+ a_{q'}^- - a_{q'}^- a_q^+ = \delta_{qq'}, \quad \langle 1_q|1_{q'}\rangle = \delta_{qq'}, \quad \langle 1_r|1_{r'}\rangle = \delta_{rr'}.$$

Performing the calculations, the Hamiltonian (14) takes the shape

$$\begin{aligned} \mathcal{H}_0^1 &= \sum_q \sum_{q'} \Bigg[\sum_r T_{qr} \left(\mu_B H + A_r^S + 2SA_r^0 + 2JS\mathrm{z}(r)\right) T_{q'r} \\ &- 2JS \sum_{r,r',r\neq r'} T_{q'r'} \Bigg] a_q^+ a_{q'}^-, \end{aligned} \tag{15}$$

If we introduce the following notation

$$\left(\mu_B H + A_r^S + 2SA_r^0 + 2JS\mathrm{z}(r)\right) T_{q'r} - 2JS \sum_{r\neq r'} T_{q'r'} = \omega_{q'} T_{q'r} \tag{16}$$

the Hamiltonian can be written in the form

$$\mathcal{H}_0^1 = \sum_q \sum_{q'} \sum_r T_{qr} \omega_{q'} T_{q'r} a_q^+ a_{q'}^-.$$

Taking into account the fact that $\sum_r T_{qr} T_{q'r} = \delta_{qq'}$ we see that $\mathcal{H}_0^1 = \sum_q \omega_q a_q^+ a_{q'}^-$.

The classical approach consists in the digitalization procedure of the Hamiltonian [16, 17] by means of the transformation

$$a_r^\pm = \sum_\tau T_{\tau r} a_\tau^\pm \tag{17}$$

determining the spectrum of eigenvalues for the magnons in the space of the wave vector τ. The transformation coefficients $T_{\tau r}$ play the role of the spin waves amplitudes. They can satisfy the equation applied by the diagonalisation of equation (16), namely

$$\Omega_r T_{r\tau} - 2JS \sum_{r'} T_{r'\tau} = \omega_\tau T_{r\tau} \tag{18}$$

where ω_τ are the eigenfrequencies of magnons with the wave vectors characterised by τ and

$$\Omega_r = \mu_B H + A_r^S + 2SA_r^0 + 2JS\mathrm{z}(r).$$

Taking into account the temporal behaviour of $\langle a_\tau \rangle$ (c.g. [17]) we obtain the adsorption power proportional to the expresion

$$P(\omega) \sim \sum_\tau T_\tau \delta(\omega - \omega_\tau) \tag{19}$$

where the eigenfrequencies ω_τ are given by (16) and

$$T_\tau = \left(\sum_r T_{r\tau} \right)^2 \tag{20}$$

determines the power adsorbed by the mode τ.

In the case of Zwanzig's model we calculate directly the value of the temporal derivative of the magnetisation deviation $M_r^x(t)$. In order to apply this model we introduce the canonically conjugated operators P_r and Q_r which show the coincidence with the spin component operators

$$P_r \Longleftrightarrow S_r^y, \; Q_r \Longleftrightarrow S_r^x. \tag{21}$$

The Hamiltonian (10) takes its form

$$\mathcal{H} = \frac{1}{2} \sum_r \Omega_r P_r^2 - JS \sum_{r,\, r'} P_r P_{r'} + \frac{1}{2} \sum_r \Omega_r Q_r^2 - JS \sum_{r,\, r'} Q_r Q_{r'} - \mu_B h \sqrt{S} \sum_r Q_r \tag{22}$$

which is convenient for consideration of the solutions for Q_r in terms of Zwanzig's approach equivalent to the magnetization component S_r^x appearing in the formula (19).

The effective solution is discussed in the model in which the off-diagonal terms $P_r P_{r'}$ for $r' \neq r$ are neglected. (The general case where the terms mentioned are not neglected will be studied in a subsequent paper. They will cause the appearance of terms with $\langle S_r^x \rangle^2$ in the differential equation (7) below, complicating considerably the method used by a necessity of using a proper perturbation procedure.)

Therefore the effective Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2} \sum_r \Omega_r P_r^2 - JS \sum_{r,\, r'} Q_r Q_{r'} + \frac{1}{2} \sum_r \Omega_r Q_r^2 - \mu_B h \sqrt{S} \sum_r Q_r, \tag{23}$$

from which the equation of motion can be written as

$$\frac{d^2}{dt^2} \langle S_r^x \rangle = \Omega_r \left(\Omega_r \langle S_r^x \rangle - JS \sum_{r'} \langle S_{r'}^x \rangle \right) - Q_r \mu_B h \sqrt{S} \tag{24}$$

and it allows us to apply the procedure proposed by Zwanzig and used in the paper [18, 19]. According to the considerations of the extended method [17] we can see that

$$\frac{d}{dt} \langle S_r^x \rangle = u_{2r} \tag{25}$$

where the solution for u_{2r} found in [20, 21] can be applied to the formula (19) in the present paper.

4. Conclusions

An essential aim of this article was the review the method used for the diagonalisation of the Hamiltonian. We applied usual steps of diagonalisation procedure described in the many papers [e.g. 7, 8, 16] for the case of thin films with the Zwanzig's assumptions [15]. The procedure contains the following two important steps:

1) we introduced the creation and annihilation operators of spin waves in the Holstein - Promakoff approximation,

2) we transform the creation and annihilation operators by relation (17).

The coefficient $T_{\tau r}$ are determined by the following difference equation [16] (structure is assumed with orientation (100))

$$-x_\tau T_{\tau r} + T_{\tau+1,r} + T_{\tau-1,r} = 0,$$

with boundary conditions

$$(1 - x - \tau)T_{1r} + T_{2r} = 0,$$
$$(1 - x - \tau)T_{nr} + T_{n-1,r} = 0.$$

In this meaning, we can consider the the power function $P(\omega)$ for adsorption of the magnetic field in the terms of functions $T_{\tau r}$, namely [20]

$$P(\omega) = \frac{\mu_B}{\pi}\omega \sum_r h^0 \left(\sum_q T_{\tau r}\right) \frac{1}{T} \int_{-T/2}^{T/2} \frac{d\langle S_r^x \rangle}{dt} \cos(\omega t) dt,$$

where the brackets $\langle\ \rangle$ denotes the statistical average value of the spin component operator S_r^x in lattice site while μ_B stands for the Bohr magneton multiplied by the gyromagnetic factor.

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Department of Relativity Physics
University of Warmia and Mazury in Olsztyn
Słoneczna 54, PL-10-710 Olsztyn,
Poland

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DIAGONALNA POSTAĆ HAMILTONIANU W ŁAŃCUCHACH TYPU ZWANZIGA

S t r e s z c z e n i e

Praca przedstawia przegląd metody diagonalizacji Hamiltonianu zaproponowanej w pracy Corcioveia (1963) w przypadku kiedy rozważamy cienkie warstwy, w szczególności z uwzględnieniem założeń Zwanziga [15]. W pierwszym kroku wyrażamy Hamiltonian układu za pomocą operatorów kreacji i anihilacji stosując przekształcenia Holsteina-Primakoffa. Następnie w celu otrzymania postaci diagonalnej Hamiltonianu, to znaczy jego postaci jako sumy Hamiltonianw opisujących niezależne oscylatory wprowadzamy konieczne przekształcenia zgodnie z [16].

Rapporteurs – Referees

CONTENU DU VOLUME LXII, no. 1