B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE: RECHERCHES SUR LES DÉFORMATIONS

Volume LXIII, no. 1

ŁÓDŹ 2013

ŁÓDZKIE TOWARZYSTWO NAUKOWE
PL-90-505 Łódź, ul. M. Curie-Skłodowskiej 11
tel. (42) 66-55-459, fax (42) 66 55 464
sprzedaż wydawnictw: tel. (42) 66 55 448, http://sklep.ltn.lodz.pl
e-mail: biuro@ltn.lodz.pl; http://www.ltn.lodz.pl/

**Wydano z pomocą finansową Ministerstwa Nauki
i Szkolnictwa Wyższego**

PL ISSN 0459–6854

Wydanie 1.
Nakład 200 egz.
Skład komputerowy: Zofia Fijarczyk
Druk i oprawa: Drukarnia Wojskowa
Łódź, ul. Gdańska 130
tel. +48 42 6366171

The journal appears in the bases *Copernicus* and *EBSCOhost*

INSTRUCTION AUX AUTEURS

1. La présente Série du Bulletin de la Société des Sciences et des Lettres de Łódź comprend des communications du domaine des mathématiques, de la physique ainsi que de leurs applications liées aux déformations au sense large.
2. Toute communications est présentée à la séance d'une Commission de la Société par un des members (avec deux opinions de spécialistes designés par la Rédaction). Elle doit lui être adressée directement par l'auteur.
3. L'article doit être écrit en anglais, français, allemand ou russe et débuté par un résumé en anglais ou en langue de la communication présentée. Dans tous les travaux écrits par des auteurs étrangers le titre et le résumé en polonais seront préparés par la rédaction. Il faut fournir le texte original qui ne peut contenir plus de 15 pages (plus 2 copies).
4. Comme des articles seront reproduits par un procédé photographique, les auteurs sont priés de les préparer avec soin. Le texte tapé sur un ordinateur de la classe IBM PC avec l'utilisation d'un imprimante de laser, est absolument indispensable. Il doit être tapé préférablement en *AMS-TEX* ou, exceptionnellement, en *Plain-TEX* ou *LATEX*. Après l'acceptation de texte les auteurs sont priés d'envoyer les disquettes (PC). Quelle que soient les dimensions des feuilles de papier utilisées, le texte ne doit pas dépasser un cadre de frappe de 12.3×18.7 cm (0.9 cm pour la page courante y compris). Les deux marges doivent être le la même largeur.
5. Le nom de l'auteur (avec de prénom complet), écrit en italique sera placé à la 1ère page, 5.6 cm au dessous du bord supérieur du cadre de frappe; le titre de l'acticle, en majuscules d'orateur 14 points, 7.1 cm au dessous de même bord.
6. Le texte doit être tapé avec les caractères Times 10 points typographiques et l'interligne de 14 points hors de formules longues. Les résumés, les rénvois, la bibliographie et l'adresse de l'auteurs doivent être tapés avec le petites caractères 8 points typographiques et l'interligne de 12 points. Ne laissez pas de "blancs" inutiles pour respecter la densité du texte. En commençant le texte ou une formule par l'alinéa il faut taper 6 mm ou 2 cm de la marge gauche, respectivement.
7. Les texte des thèorémes, propositions, lemmes et corollaries doivent être écrits en italique.
8. Les articles cités seront rangés dans l'ordre alphabétique et précédés de leurs numéros placés entre crochets. Après les références, l'auteur indiquera son adress complète.
9. Envoi par la poste: protégez le manuscript à l'aide de cartons.
10. Les auteurs recevront une copie de fascicule correspondant à titre gratuit.

Adresse de la Rédaction de la Série:
Département de la Physique d'etat solide
de l'Université de Łódź
Pomorska 149/153, PL-90-236 Łódź, Pologne

Name and surname of the authors

TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

2. How to prepare a manuscript

To prepare the LaTeX 2e source file of your paper, copy the template file **instr.tex** with **Fig1.eps**, give the title of the paper, the authors with their affiliations/addresses, and go on with the body of the paper using all other means and commands of the standard class/style 'bull.cls'.

2.1. Example of a figure

Figures (including graphs and images) should be carefully prepared and submitted in electronic form (as separate files) in Encapsulated PostScript (EPS) format.

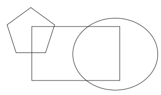

Fig. 1: The figure caption is located below the figure itself; it is automatically centered and should be typeset in small letters.

2.2. Example of a table

Tab. 1: The table caption is located above the table itself; it is automatically centered and should be typeset in small letters.

Description 1	Description 2	Description 3	Description 4
Row 1, Col 1	Row 1, Col 2	Row 1, Col 3	Row 1, Col 4
Row 2, Col 1	Row 2, Col 2	Row 2, Col 3	Row 2, Col 4

2.3. "Ghostwriting" and "guest authorship" are strictly forbiden

The printed version of an article is primary (comparing with the electronic version). Each contribution submitted is sent for evaluation to two independent referees before publishing.

3. How to submit a manuscript

Manuscripts have to be submitted in electronic form, preferably via e-mail as attachment files sent to the address **zofija@uni.lodz.pl**. If a whole manuscript exceeds 2 MB composed of more than one file, all parts of the manuscript, i.e. the text (including equations, tables, acknowledgements and references) and figures, should be ZIP-compressed to one file prior to transfer. If authors are unable to send their manuscript electronically, it should be provided on a disk (DOS format floppy or CD-ROM), containing the text and all electronic figures, and may be sent by regular mail to the address: **Department of Solid State Physics, University of Lodz, Bulletin de la Société des Sciences et des Lettres de Łódź, Pomorska 149/153, 90-236 Łódź, Poland**.

References

[1]

Affiliation/Address

TABLE DES MATIÈRES

Professor Yuri Zelinskiĭ commemorating our unforgetable friend
Professor Promarz Tamrazov at Będlewo in July 2012.
Photo by Prof. Z.-D. Zhang.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 9–22

In memory of
Professor Promarz M. Tamrazov

Pierre Dolbeault

ON QUATERNIONIC FUNCTIONS

Summary

Several sets of quaternionic functions are described and studied. Residue current of the right inverse of a quaternionic function is introduced in particular cases.

Keywords and phrases: Cauchy-Fueter, hyperholomorphy, residue

1. Introduction

We will work with the definition of quaternions using pairs of complex numbers and with a modified Cauchy-Fueter operator that have been introduced in [CLSSS 07]. We will only use right multiplication; the (right) inverse of a nonzero quaternion is defined. We will consider (for simplicity) C^∞ $\mathbb{H}$-valued quaternionic functions defined on an open set U of $\mathbb{H}$ containing 0. If such a function does not vanish over U, it has an (algebraic) inverse which is defined almost everywhere on U. Examples are given (Section 2).

The origin of this research is a tentative of extension to right inverse of a quaternionic function of the notion of residue current of a meromorphic differential 1-form of one complex variable, which will be developed in Section 5. In one complex variable, if the given function is holomorphic, with isolated zeros of finite multiplicity, its inverse is meromorphic, then holomorphic outside the set of poles; so it is natural to search when this property extends to hyperholomorphic functions.

In Section 3, we characterize the quaternionic functions which are hyperholomorphic and whose inverses are hyperholomorphic almost everywhere, on U, as the solutions of a system of two non linear PDE. We only find non trivial examples of a solution, showing that the considered space of functions is significant: we will call theses functions hypermeromorphic. This also defines a space of germs of functions at 0.

In Section 4, we try to describe a subspace $\mathcal{H}_U$ of hyperholomorphic and hypermeromorphic functions defined almost everywhere on U, having "good properties for addition and multiplication"; we obtain again systems of non linear PDE, and we give first results on, mainly unknown, spaces of functions.

In Section 5, we first recall Cauchy principal value and residue current in $\mathbb{C}$, locally at 0. Afterwards, we define and study, locally, Cauchy principal value and residue current for the inverse of a quaternionic function, in very particular cases, and in relation with the classical theory in two complex variables.

This paper is a first announcement of a more complete one in progress.

2. Quaternions. $\mathbb{H}$-valued functions [CLSSS 07]

2.1. Quaternions

If $q \in \mathbb{H}$, then $q = z_1 + z_2\mathbf{j}$ where $z_1, z_2 \in \mathbb{C}$, hence $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ as complex or real vector space. We have: $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$ (by computation in real coordinates); by definition, the *modulus* of q is

$$\| q \| = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}.$$

The *conjugate* of q is $\overline{q} = \overline{z}_1 - z_2\mathbf{j}$. Let $*$ denote the *(right) multiplication* in $\mathbb{H}$:

$$q * \overline{q} = (z_1 + z_2\mathbf{j}) * (\overline{z}_1 - z_2\mathbf{j}) = |z_1|^2 - z_1 z_2\mathbf{j} + z_2\mathbf{j}\overline{z}_1 - z_2\mathbf{j}z_2\mathbf{j} = |z_1|^2 + |z_2|^2,$$

then: the (*right*) *inverse* of $q = z_1 + z_2\mathbf{j}$ is:

$$(|z_1|^2 + |z_2|^2)^{-1}\overline{q} = (|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j}).$$

Moreover:

$$(|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j}) * (z_1 + z_2\mathbf{j}) = 1,$$

so the right inverse of q^{-1} is q.

2.2. Quaternionic functions

Let U be an open set of $\mathbb{H} \cong \mathbb{C}^2$ and $f \in C^\infty(U, \mathbb{H})$, then $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$. The complex valued functions f_1, f_2 will be called the *components* of f.

Remark that $\mathbb{H}$ is a real vector space in which real analysis is valid, in particular differential forms, distributions and currents are defined in $\mathbb{H}$.

Remark that

$$\frac{\partial f_1}{\partial \overline{z}_1}\mathbf{j} = \mathbf{j}\frac{\partial \overline{f}_1}{\partial z_1}$$

and analogous relations.

2.3. Modified Cauchy-Fueter operator $\mathcal{D}$. Hyperholomorphic functions [CLSSS 07, F 39]

For $f \in C^\infty(U, I\!\!H)$, with $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$,

$$\mathcal{D}f(q) = \frac{1}{2}\big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\big)f(q) = \frac{1}{2}\big(\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2}\big)(q) + \mathbf{j}\frac{1}{2}\big(\frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1}\big)(q).$$

A function $f \in C^\infty(U, I\!\!H)$ s said to be *hyperholomorphic* if $\mathcal{D}f = 0$.

Characterization of the hyperholomorphic function f on U:

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0, \text{ on } U.$$

The conditions:f_1 is holomorphic and: f_2 is holomorphic are equivalent. So *holomorphic functions* will be identified with hyperholomorphic functions f such that $f_2 = 0$.

Let

$$f' = f'_1 + f'_2\mathbf{j}, \qquad f" = f"_1 + f"_2\mathbf{j}$$

be two hyperholomorphic functions.

For every $\alpha \in I\!\!H$, $\mathcal{D}(f'\alpha) = 0$, $\mathcal{D}(f' + f") = \mathcal{D}f' + \mathcal{D}f" = 0$.

Proposition 1. *The set $\mathcal{H}$ of almost everywhere defined hyperholomorphic functions is an $I\!\!H$-right vector space.*

Proposition 2. *Let f', $f"$ be two hyperholomorphic functions. Then, their product $f' * f"$ satisfies:*

$$\mathcal{D}(f' * f") = \mathcal{D}f' * \mathbf{j}f" + \big(f'(\frac{\partial}{\partial \overline{z}_1}) + \overline{f}'\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\big)f"$$

Proof. $f' = f'_1 + f'_2\mathbf{j}$, $f" = f"_1 + f"_2\mathbf{j}$ be two hyperholomorphic functions.

We have:

$$f' * f" = (f'_1 + f'_2\mathbf{j})(f"_1 + f"_2\mathbf{j}) = f'_1 f"_1 - f'_2\overline{f}"_2 + (f'_1 f"_2 + f'_2\overline{f}"_1)\mathbf{j}.$$

Compute

$$\frac{1}{2}\big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\big)\big(f'_1 f"_1 - f'_2\overline{f}"_2 + (f'_1 f"_2 + f'_2\overline{f}"_1)\mathbf{j}\big).$$

By derivation of the first factors of the sum $f' * f"$, we get the first term:

$$\frac{1}{2}\big(\frac{\partial f'_1}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial \overline{z}_2}\big)(f"_1 + f"_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f'_2}{\partial \overline{z}_2}\big)\mathbf{j}\mathbf{j}(\overline{f}"_2 - \overline{f}"_1\mathbf{j})$$

$$= \frac{1}{2}\big(\frac{\partial f'_1}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial \overline{z}_2}\big)(f"_1 + f"_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2\mathbf{j}}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f'_2\mathbf{j}}{\partial \overline{z}_2}\big)\mathbf{j}(f"_2\mathbf{j} + f"_1) = \mathcal{D}f' * \mathbf{j}f".$$

By derivation in

$$\frac{1}{2}\big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\big)\big(f'_1 f"_1 + f'_2\mathbf{j}f"_2\mathbf{j} + (f'_1 f"_2\mathbf{j} + f'_2\mathbf{j}f"_1)\big)$$

of the second factors of the sum $f' * f$", we get the second term (up to factor $\frac{1}{2}$):

$$f_1'\frac{\partial f"_1}{\partial\overline{z}_1}+\overline{f}'_1\mathbf{j}\frac{\partial f"_1}{\partial\overline{z}_2}+f_1'\frac{\partial f"_2}{\partial\overline{z}_1}\mathbf{j}+\overline{f}'_1\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_2}\mathbf{j}+f_2'\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_1}\mathbf{j}+\overline{f}'_2\mathbf{j}\frac{\partial f"_2}{\partial\overline{z}_2}+f_2'\mathbf{j}\frac{\partial f"_1}{\partial\overline{z}_1}+\overline{f}'_2\mathbf{j}\mathbf{j}\frac{\partial f"_1}{\partial\overline{z}_2}$$

$$= (f_1'+f_2'\mathbf{j})(\frac{\partial}{\partial\overline{z}_1})(f"_1+f"_2\mathbf{j})+(\overline{f}'_1+\overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}(f"_1+f"_2\mathbf{j})$$

$$= \big((f_1'+f_2'\mathbf{j})(\frac{\partial}{\partial\overline{z}_1})+(\overline{f}'_1+\overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(f"_1+f"_2\mathbf{j})$$

$$= \big(f'(\frac{\partial}{\partial\overline{z}_1})+\overline{f}'\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)f".$$

If the components of f' and f" are real, the second term is:

$$\frac{1}{2}(f_1'+f_2'\mathbf{j})(\frac{\partial}{\partial\overline{z}_1}+\mathbf{j}\frac{\partial}{\partial\overline{z}_2})(f"_1+f"_2\mathbf{j}) = f' * \mathcal{D}f"$$

i.e.

Corollary 1. *The set $\mathcal{H}_{\mathbb{R}}$ of almost everywhere defined hyperholomorphic functions whose components are real is an $\mathbb{R}$-right algebra.*

2.4. Null set and inverse of a quaternionic function

We call *inverse* of a function $f : q \mapsto f(q)$, the function $f^{-1} : q \mapsto f(q)^{-1}$. Let $f = f_1 + f_2\mathbf{j}$ be a quaternionic function on U. The null set $Z(f)$ satisfies: $f_1 = 0; f_2 = 0$, then $Z(f)$ is of measure 0 in U.

Ex.: $f_1 = \overline{z}_1$; $f_2 = \overline{z}_2$, then $Z(f) = \{0\}$. Note that if f is holomorphic, then, $f_2 \equiv 0$ and $Z(f)$ is a complex hypersurface in $\mathbb{C}^2$.

Inversion and hyperholomorphy. The inverse of the quaternionic function f is the peculiar quaternionic function defined almost everywhere on U:

$$\frac{1}{f} = (|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1 - f_2\mathbf{j}) = |f|^{-1}\overline{f}$$

where $\overline{f}$ is the (quaternionic) conjugate of f.

Assume f to be hyperholomorphic and $Z(f) = \{0\}$, then $\frac{1}{f}$ is not necessarily hyperholomorphic outside $\{0\}$.

Ex.: $f = \overline{z}_1 + \overline{z}_2\mathbf{j}$, then

$$\frac{1}{f} = (z_1\overline{z}_1 + z_2\overline{z}_2)^{-1}(z_1 - \overline{z}_2\mathbf{j}); \qquad \mathcal{D}(\frac{1}{f}) \neq 0,$$

Example of a function hyperholomorphic outside 0.

$$H(q) = (z_1\overline{z}_1 + z_2\overline{z}_2)^{-2}(\overline{z}_1 - \overline{z}_2\mathbf{j})$$

is hyperholomorphic since:

$$\mathcal{D}H(q) = \frac{1}{2}(z_1\overline{z}_1+z_2\overline{z}_2)^{-3}(-2z_1\overline{z}_1+z_1\overline{z}_1+z_2\overline{z}_2-2z_2\overline{z}_2+(z_1\overline{z}_1+z_2\overline{z}_2)) = 0.$$

But $F = z_1 + \overline{z}_2\mathbf{j}$ *is not hyperholomorphic*: the conjugate of F is

$$\overline{F} = \overline{z}_1 - \overline{z}_2\mathbf{j}; \quad (z_1 + \overline{z}_2\mathbf{j}) * (\overline{z}_1 - \overline{z}_2\mathbf{j}) = z_1\overline{z}_1 + z_2\overline{z}_2.$$

So

$$F^{-1} = (\overline{z}_1 - \overline{z}_2\mathbf{j})(z_1\overline{z}_1 + z_2\overline{z}_2)^{-1},$$

and

$$H(q) = F^{-1}(z_1\overline{z}_1 + z_2\overline{z}_2)^{-1} = \frac{F^{-1}}{|F|}.$$

(H *is the Cauchy kernel* for the modified Cauchy-Fueter operator $\mathcal{D}$).

Inverse of a holomorphic function. Let $f = f_1 + 0\mathbf{j}$ be a hyperholomorphic function. Then $f^{-1} = f_1^{-1} + 0\mathbf{j}$ and f^{-1} is hyperholomorphic outside of the complex hypersurface $Z(f)$. Remark that $Z(f)$ is a subvariety of complex dimension 1, then of measure zero, in U.

We will consider almost everywhere defined hyperholomorphic functions on U. Ex.: holomorphic, meromorphic functions.

3. Hyperholomorphic functions whose inverses are hyperholomorphic almost everywhere

Proposition 3. *The following conditions are equivalent*

(i) *the function $f = f_1 + f_2\mathbf{j}$ and its right inverse are hyperholomorphic, when they are defined;*

(ii) *we have the equations:*

$$(\overline{f}_1 - f_1)\frac{\partial\overline{f}_1}{\partial z_1} - \overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2} = 0, \tag{3}$$

$$\overline{f}_2\frac{\partial f_1}{\partial z_1} + \frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) - f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2} = 0. \tag{4}$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ be a hyperholomorphic function and

$$g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f}_1 - f_2\mathbf{j})$$

its inverse; so

$$g_1 = |f|^{-1}\overline{f}_1; \qquad g_2 = -|f|^{-1}f_2,$$

where

$$|f| = (f_1\overline{f}_1 + f_2\overline{f}_2).$$

$$\mathcal{D}g(q) = \frac{1}{2}\Big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\Big)g(q) = \frac{1}{2}\Big(\frac{\partial g_1}{\partial\overline{z}_1} - \frac{\partial\overline{g}_2}{\partial z_2}\Big)(q) + \mathbf{j}\frac{1}{2}\Big(\frac{\partial g_1}{\partial\overline{z}_2} + \frac{\partial\overline{g}_2}{\partial z_1}\Big)(q)$$

$$\frac{\partial g_1}{\partial\overline{z}_1} = |f|^{-1}\frac{\partial\overline{f}_1}{\partial\overline{z}_1} - |f|^{-2}\overline{f}_1\Big(\frac{\partial f_1}{\partial\overline{z}_1}\overline{f}_1 + f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_1} + \frac{\partial f_2}{\partial\overline{z}_1}\overline{f}_2 + f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}\Big), \quad \text{etc}$$

$$(**)\qquad 2|f|^2\mathcal{D}g = (f_1\overline{f}_1 + f_2\overline{f}_2)(\frac{\partial\overline{f}_1}{\partial\overline{z}_1} + \frac{\partial\overline{f}_2}{\partial z_2}) - \overline{f}_1 f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_1} - \overline{f}_1 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}$$
$$-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_2}. - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2} - f_2\overline{f}_2\frac{\partial\overline{f}_2}{\partial z_2}$$
$$+\mathbf{j}\Big((f_1\overline{f}_1 + f_2\overline{f}_2)(\frac{\partial\overline{f}_1}{\partial\overline{z}_2} - \frac{\partial\overline{f}_2}{\partial z_1}) - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2} - \overline{f}_1 f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f}_1 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}$$
$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1} + \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} + f_2\overline{f}_2\frac{\partial\overline{f}_2}{\partial z_1}\Big)$$

f being hyperholomorphic, g hyperholomorphic is equivalent to:

$$(**)\qquad +f_1\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_2} + f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_1} - \overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_1} - \overline{f}_1 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1} - \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2} - \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2}$$
$$+\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_1}(f_1 - \overline{f}_1) = 0$$
$$+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2} - \overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f}_1 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2} + \overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f}_1) + \overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}$$
$$+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} = 0.$$

After conjugaison of the first equation, and using

$$(1)\qquad \frac{\partial f_1}{\partial\overline{z}_1} - \frac{\partial\overline{f}_2}{\partial z_2} = 0;\quad \frac{\partial f_1}{\partial\overline{z}_2} + \frac{\partial\overline{f}_2}{\partial z_1} = 0,$$

we get:

(**)

$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1} + f_1(\overline{f}_1 - f_1)\frac{\partial\overline{f}_1}{\partial z_1} - f_1\overline{f}_2\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) - f_1 f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2} - f_2 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2} = 0$$
$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1} + (f_1 - \overline{f}_1)\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1} + \overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1} + \overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f}_1) + f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}. - \overline{f}_1 f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2} = 0.$$

Assume $f_1 \neq 0$, $f_2 \neq 0$. After multiplication of the first equation by f_1 and of the second by $-f_2$, and sum, we get

$$(\overline{f}_1 - f_1)\frac{\partial\overline{f}_1}{\partial z_1} - \overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2} = 0.$$

By an analogous process, we get:

$$\overline{f}_2\frac{\partial f_1}{\partial z_1} + \frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1 - f_1) - f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2} = 0.$$

Corollary 2. *If f satisfies the conditions of the Proposition, the same is true for αf with $\alpha \in I\!\!R$.*

Let $f = f_1 + 0\mathbf{j}$ be an almost everywhere holomorphic function, then the condition (ii) of Proposition 3.1 is satisfied.

Now give another example of quaternionic function satisfying the conditions of Proposition 3.1:

Proposition 4. *Let* $f = f_1 + f_2\mathbf{j}$, *with*

$$f_1 = z_1 + \overline{z}_1 + z_2 + \overline{z}_2 + A, \qquad f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + B,$$

$A, B \in I\!R$, *then:* f *and* f^{-1} *outside the zero set of* f, *are hyperholomorphic The null set of* $f = f_1 + f_2\mathbf{j}$, *for* f_1, f_2 *as above, for* $A = B = 0$, *is:*

$$z_1 + \overline{z}_1 + z_2 + \overline{z}_2 = 0; -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 = 0$$

i.e, by difference and sum: $\overline{z}_1 + z_1 = 0; z_2 + \overline{z}_2 = 0$, *i.e.* $x_1 = 0; x_2 = 0$ *in* $I\!R^4$.

Proof. $f_1 = z_1 + \overline{z}_1 + z_2 + \overline{z}_2 + A$, A constant; then:

(3) $$-\overline{f}_2 \frac{\partial f_2}{\partial z_1} - f_2 \frac{\partial \overline{f}_1}{\partial \overline{z}_2} = 0, \quad \text{i.e.} \quad -\overline{f}_2 \frac{\partial f_2}{\partial z_1} - f_2 = 0.$$

Try: f_2 *real.* Then

$$\frac{\partial f_2}{\partial z_1} = -1 \quad \text{and} \quad f_2 = -z_1 + C(\overline{z}_1, z_2, \overline{z}_2) = -z_1 - \overline{z}_1 + C'(\overline{z}_1, z_2, \overline{z}_2),$$

with C' real and $\frac{\partial C'}{\partial z_1} = 0$.

From (4),

$$\frac{\partial f_2}{\partial \overline{z}_2} = \frac{\partial C'}{\partial \overline{z}_2} = 1,$$

and

$$C' = z_2 + \overline{z}_2 + C"(\overline{z}_1, z_2),$$

with $C"$ real.

$$f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + C"(\overline{z}_1, z_2),$$

with $C"$ real, and

$$\frac{\partial C"}{\partial \overline{z}_2} = 0, \quad \frac{\partial C"}{\partial z_1} = 0.$$

$C"$ being holomorphic in z_2 and $\overline{z}_1$ is a constant B.

Hence: $f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + B$, with $B \in I\!R$.

4. On the space of hyperalgebraic functions

Definition 4.1. Let U be an open neighborhood of 0 in $I\!H \cong \mathbb{C}^2$. *From now on,we will only consider the quaternionic functions* $f = f_1 + f_2\mathbf{j}$ *having the following properties*:

(i) when f_1 and f_2 are not holomorphic, the set $Z(f_1) \cap Z(f_2)$ is *discrete* on U;

(ii) for every $q \in Z(f_1) \cap Z(f_2)$, $J_q^\alpha(.)$ denoting the *jet of order* α *at* q [M 66], let

$$m_i = \sup_{\alpha_i} J_q^{\alpha_i}(f_i) = 0; \qquad m_i, \ i = 1, 2,$$

is finite.

Define: $m_q = \inf_i m_i$.

Remark that, in this paper, the considered peculiar examples of quaternionic functions f satisfy: the set $Z(f_1) \cap Z(f_2)$ is reduced to one point and that $\alpha_i = 1$.

Let $f = f_1 + f_2\mathbf{j}$ be a quaternonic function on U and $g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f}_1 - f_2\mathbf{j})$ its inverse; so

$$g_1 = |f|^{-1}\overline{f}_1; \quad g_2 = -|f|^{-1}f_2,$$

where $|f| = (f_1\overline{f}_1 + f_2\overline{f}_2)$.

The right inverse of g is $h = h_1 + h_2\mathbf{j}$, with

$$h_1 = |g|^{-1}\overline{g}_1; \quad h_2 = -|g|^{-1}g_2; \quad |g| = |f|^{-2}(\overline{f}_1 f_1 + \overline{f}_2 f_2) = |f|^{-1};$$

then: $h_1 = |g|^{-1}\overline{g}_1 = f_1$, ... So *the right inverse of g is f.*

Definition 4.2. We will call *hypermeromorphic function*, on U, any almost everywhere defined hyperholomorphic function whose right inverse is hyperholomorphic almost everywhere.

Thanks to Definition 4.1, meromorphic functions in one complex varable are hypermeromorphic.

From Proposition 3.3, the set $\mathcal{M}$ of hypermeromorphic functions is not reduced to the space of meromorphic functions in one complex variable.

Let $\mathcal{M}_0$ be the set of elements f of $\mathcal{M}$ described in Proposition 3.3. $\mathcal{M}_0$ is an $I\!R$-vector space; it also contains f^{-1} and the products $f * f^{-1} = 1$.

Proposition 5. *Let f, g be two hypermeromorphic functions on U, then the following conditions are equivalent:*

(i) *the product $f * g$ is hypermeremorphic;*

(ii) *f and g satisfy the system of PDE:*

$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}\Big) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_1} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}\Big) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_2} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0.$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ and $g = g_1 + g_2\mathbf{j}$ two hypermeromorphic functions and

$$f * g = f_1g_1 - f_2\overline{g}_2 + (f_1g_2 - f_2\overline{g}_1)\mathbf{j}$$

their product, then

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \qquad \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0$$

and..., on U and, the conditions for the product to be hyperholomorphic are:

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_1} - \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2 g_1)}{\partial z_2} =$$

$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}\Big) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0,$$

$$\frac{\partial(f_1g_1 - f_2\overline{g}_2)}{\partial\overline{z}_2} + \frac{\partial(\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_1} =$$

$$g_1(\frac{\partial f_1}{\partial\overline{z}_2} - \frac{\partial\overline{f}_2}{\partial z_1}) + f_1\frac{\partial g_1}{\partial\overline{z}_2} + \overline{f}_1\frac{\partial\overline{g}_2}{\partial z_1} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial\overline{g}_2}{\partial\overline{z}_2} = 0.$$

Corollary 3. *Let f, g be two hypermeromorphic functions on U, whose components are real, then the following conditions are equivalent:*
(i) *the product $f * g$ is hypermeremorphic;*
(ii) *f and g satisfy te system of PDE:*

$$g_1(\frac{\partial f_1}{\partial\overline{z}_1} + \frac{\partial f_2}{\partial z_2}) + f_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial g_2}{\partial\overline{z}_1} = 0$$

$$g_1(\frac{f_1}{\partial\overline{z}_2} - \frac{\partial f_2}{\partial z_1}) - f_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial g_2}{\partial\overline{z}_2} = 0.$$

Definition 4.3. We will call *hyperalgebraic* the hypermeromorphic functions whose sum and product are hypermeromorphic: see Section 4.2.

Proposition 6. *The set $\mathcal{M}$ of hypermeromorphic functions on U is a subalgebra of the algebra of quaternionic functions.*

Proposition 7. *The set $\mathcal{A}$ of hyperalgebraic functions on U is a "field" with only associativity of the multiplication.*

5. About residue current in quaternionic analysis: particular cases

5.1. Residue current in an open set of $\mathbb{C}$ [D10a, D10b]

Let $\omega = g(z)dz$ be a meromorphic 1-form on a small enough open set $0 \in U \subset \mathbb{C}$ having 0 as unique pole, with multiplicity k:

$$g = \sum_{l=1}^{k} \frac{a_{-l}}{z^l} + \text{holomorphic function.}$$

Note that ω is d-closed.

Let $\psi = \psi_0 d\overline{z} \in \mathcal{D}^1(U)$ be a 1-test form. In general $g\psi$ is not integrable, but the Cauchy principal value

$$Vp[\omega](\psi) = \lim_{\epsilon\to 0}\int_{|z|\geq\epsilon} \omega \wedge \psi$$

exists as a current, and $dVp[\omega] = d"Vp[\omega] = \mathrm{Res}[\omega]$ is the residue current of ω. For any test function φ on U,

$$\mathrm{Res}[\omega](\varphi) = \lim_{\epsilon\to 0}\int_{|z|=\epsilon} \omega\wedge\varphi.$$

Then

$$\mathrm{Res}[\omega] = 2\pi i\ \mathrm{res}_0(\omega)\delta_0 + dB = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}\delta_0$$

where $\mathrm{res}_0(\omega) = a_{-1}$ is the Cauchy residue. We remark that δ_0 is the integration current on the subvariety $\{0\}$ of U, that

$$D = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}$$

with $b_j = \lambda_j a_{-j}$ where the λ_j are universal constants.

Conversely, given the subvariety $\{0\}$ and the differential operator D, then the meromorphic differential form ω is equal to gdz, up to holomorphic form; hence the residue current $\mathrm{Res}[\omega] = D\delta_0$, can be constructed.

5.2. Cauchy principal value of a quaternionic 1-form

Let f be a quaternionic function on an open neighborhood U of 0 in $\mathbb{H}$ satisfying the conditions of Definition 4.1, with $m_i = 1$, $i = 1, 2$.

Let

$$\omega = \frac{df}{f} = \frac{1}{f}(df_1 + df_2\mathbf{j}).$$

We want to extend ω into a current of degree 1; first, consider the part of type (1,0). Let

$$\psi = \psi_1 d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \psi_2 dz_1 \wedge d\overline{z}_1 \wedge d\overline{z}_2$$

be a test form.

Define:

$$Vp[\omega](\psi)$$

$$= \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon} (|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1 - f_2\mathbf{j})(df_1+df_2\mathbf{j})\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2).$$

We have to prove the existence of $Vp[\omega]$ at least if f is hyperholomorphic (Check the proof in the classical case where f is holomorphic in one complex variable or in two complex variables where the proof is less easy, but don't need the resolution of singularities).

$$(df_1 + df_2\mathbf{j}) \wedge (\psi_1 d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \psi_2 dz_1 \wedge d\overline{z}_1 \wedge d\overline{z}_2)$$
$$= \Big(\frac{\partial f_1}{\partial z_1}\psi_1 + \frac{\partial f_1}{\partial z_2}\psi_2 - (\frac{\partial f_2}{\partial \overline{z}_1}\overline{\psi}_1 - \frac{\partial f_2}{\partial \overline{z}_2}\overline{\psi}_2)\mathbf{j}\Big) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2.$$

Take polar coordinates:

$$\lambda = \| q \| = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$$

and the spherical coordinates on $\lambda\mathbf{S}^3$. Let $d\sigma$ be the volume element on $\mathbf{S}^3$, and K a convenient universal constant, then:

$$dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 = K\lambda d\lambda \wedge d\sigma,$$

$$Vp[\omega](\psi)$$

$$= \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|f_1|^2+|f_2|^2)^{-1}(\overline{f}_1-f_2\mathbf{j})\big(\frac{\partial f_1}{\partial z_1}\psi_1+\frac{\partial f_1}{\partial z_2}\psi_2-(\frac{\partial f_2}{\partial \overline{z}_1}\overline{\psi}_1-\frac{\partial f_2}{\partial \overline{z}_2}\overline{\psi}_2)\mathbf{j}\big)K\lambda d\lambda\wedge d\sigma.$$

Same result for the part of type ((0,1) of $Vp[\omega]$.

We will prove the existence of $Vp[\omega]$ in a particular case:

5.2.2. *Particular case*: $f_1 = \overline{z}_1$; $f_2 = \overline{z}_2$. Then: $\mathcal{D}f = 0$.

$$Vp[\omega](\psi)$$

$$= \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|\overline{z}_1|^2+|\overline{z}_2|^2)^{-1}(\overline{z}_1-z_2\mathbf{j})(d\overline{z}_1+d\overline{z}_2\mathbf{j})\wedge(\psi_1 d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2+\psi_2 dz_1\wedge d\overline{z}_1\wedge d\overline{z}_2)$$

$$Vp[\omega](\psi) = \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}(|\overline{z}_1|^2 + |\overline{z}_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j})(\overline{\psi}_2 dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2\mathbf{j})$$

$$Vp[\omega](\psi) = \lim_{\varepsilon\to 0}\int_{\lambda\geq\varepsilon}\lambda^{-2}(\overline{z}_1 - z_2\mathbf{j})(K\overline{\psi}_2\lambda d\lambda \wedge d\sigma\mathbf{j}).$$

Same result for the part of type ((0,1) of $Vp[\omega]$.

This defines a current of order 0 on $\mathbb{H}$.

5.2.3. If $f_2 = 0$ and $f = f_1$ is *holomorphic*, then

$$Vp[\omega](\psi) = \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}\frac{df}{f}\wedge(\psi_1 d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \psi_2 dz_1 \wedge d\overline{z}_1 \wedge d\overline{z}_2)$$

$$= Vp[\omega](\psi) = \lim_{\varepsilon\to 0}\int_{|f|\geq\varepsilon}\frac{1}{f}(\frac{\partial f}{\partial z_1}\psi_1 + \frac{\partial f}{\partial z_2}\psi_2)dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2.$$

5.3. Residue

5.3.1. Assume: $Z(f) = \{0\}$.

We want to define a current, $\mathrm{Res}[\omega]$, and first its part of type $(1,1)$, on a test form

$$\varphi = \varphi_{11}dz_1 \wedge d\overline{z}_1 + \varphi_{12}dz_1 \wedge d\overline{z}_2 + \varphi_{21}dz_2 \wedge d\overline{z}_1 + \varphi_{22}dz_2 \wedge d\overline{z}_2,$$

as follows:

$$\mathrm{Res}[\omega](\varphi) = \lim_{\varepsilon\to 0}\int_{|f|=\varepsilon}(|f_1|^2 + |f_2|^2)^{-1}(\overline{f}_1 - f_2\mathbf{j})(df_1 + df_2\mathbf{j})(\varphi) \tag{5}$$

$$\begin{aligned}
&(df_1 + df_2\mathbf{j})(\varphi) = \\
&= \Big(\frac{\partial f_1}{\partial z_1}dz_1 + \frac{\partial f_1}{\partial z_2}dz_2 - (\frac{\partial f_2}{\partial \overline{z}_1}d\overline{z}_1 - \frac{\partial f_2}{\partial \overline{z}_2}d\overline{z}_2)\mathbf{j}\Big) \times \\
&\times(\varphi_{11}dz_1 \wedge d\overline{z}_1 + \varphi_{12}dz_1 \wedge d\overline{z}_2 + \varphi_{21}dz_2 \wedge d\overline{z}_1 + \varphi_{22}dz_2 \wedge d\overline{z}_2) \\
&= \frac{\partial f_1}{\partial z_1}dz_1 \wedge (\varphi_{21}dz_2 \wedge d\overline{z}_1 \\
&+\varphi_{22}dz_2 \wedge d\overline{z}_2) + \frac{\partial f_1}{\partial z_2}dz_2 \wedge (\varphi_{11}dz_1 \wedge d\overline{z}_1 + \varphi_{12}dz_1 \wedge d\overline{z}_2) \\
&-\frac{\partial f_2}{\partial \overline{z}_1}d\overline{z}_1\mathbf{j} \wedge (\varphi_{21}dz_2 \wedge d\overline{z}_1 \\
&+\varphi_{22}dz_2 \wedge d\overline{z}_2) + \frac{\partial f_2}{\partial \overline{z}_2}d\overline{z}_2\mathbf{j} \wedge (\varphi_{11}dz_1 \wedge d\overline{z}_1 + \varphi_{12}dz_1 \wedge d\overline{z}_2) \\
&= (-\frac{\partial f_1}{\partial z_1}\varphi_{21} + \frac{\partial f_1}{\partial z_2}\varphi_{11})dz_1 \wedge d\overline{z}_1 \wedge dz_2 + (\frac{\partial f_1}{\partial z_1}\varphi_{22} - \frac{\partial f_1}{\partial z_2}\varphi_{12})dz_1 \wedge dz_2 \wedge d\overline{z}_2 \\
&+\big(\frac{\partial f_2}{\partial \overline{z}_1}\overline{\varphi}_{21} - \frac{\partial f_2}{\partial \overline{z}_2}\overline{\varphi}_{11}\big)\mathbf{j}dz_1 \wedge d\overline{z}_1 \wedge dz_2 + \big(-\frac{\partial f_2}{\partial \overline{z}_1}\overline{\varphi}_{22} - \frac{\partial f_2}{\partial \overline{z}_2}\overline{\varphi}_{12}\big)\mathbf{j}dz_1 \wedge dz_2 \wedge d\overline{z}_2 \\
&= (A + B\mathbf{j})dz_1 \wedge d\overline{z}_1 \wedge dz_2 + (C + D\mathbf{j})dz_1 \wedge dz_2 \wedge d\overline{z}_2.
\end{aligned}$$

Take polar coordinates: $\lambda = \parallel q \parallel = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$ and the spherical coordinates on $\mathbf{S}^3$. Let $d\sigma$ the volume element on $\mathbf{S}^3$, and K_j, $(j = 1, 2)$ a convenient universal constant:

$$dz_1 \wedge d\overline{z}_1 \wedge dz_2|_{\mathbf{S}^3} = K_1 d\lambda \wedge d\sigma,$$
$$dz_1 \wedge dz_2 \wedge d\overline{z}_2|_{\mathbf{S}^3} = K_2 d\lambda \wedge d\sigma.$$

Same result for the part of any type of $\mathrm{Res}[\omega]$.

5.3.2. *Particular case*

$f_1 = \overline{z}_1$; $f_2 = \overline{z}_2$. Then: $\mathcal{D}f = 0$.

$$\begin{aligned}
(d\overline{z}_1 + d\overline{z}_2\mathbf{j})(\varphi) &= \varphi_{22}d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + d\overline{z}_2\mathbf{j} \wedge \varphi_{12}dz_1 \wedge d\overline{z}_2 \\
&= \varphi_{22}d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \overline{\varphi}_{12}d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Res}[\omega](\varphi) \\
&= \lim_{\varepsilon\to 0}\int_{|f|=\varepsilon}(|z_1|^2 + |z_2|^2)^{-1}(z_1 - \overline{z}_2\mathbf{j})(\varphi_{22}d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 + \overline{\varphi}_{12}d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2\mathbf{j}) \\
&= \lim_{\varepsilon\to 0}\int_{\|f\|=\varepsilon}(|z_1|^2 + |z_2|^2)^{-1}(z_1 - \overline{z}_2\mathbf{j})(\varphi_{22} - \overline{\varphi}_{12}\mathbf{j})d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2) \\
&= \lim_{\varepsilon\to 0}\int_{\mathbf{S}^3}(|z_1|^2 + |z_2|^2)^{-1}(z_1 - \overline{z}_2\mathbf{j})(\varphi_{22} - \overline{\varphi}_{12}\mathbf{j})\varepsilon d\sigma \\
&= \mathrm{res}[\omega](\varphi_{22} - \overline{\varphi}_{12}\mathbf{j})(0) = \delta_0(\varphi_{22} - \overline{\varphi}_{12}\mathbf{j})
\end{aligned}$$

$\int_{\|f\|=\varepsilon}$ means the integration on $\varepsilon\mathbf{S}^3$, where $\mathbf{S}^3$ is the unit 3-sphere and $d\sigma$ the volume element on $\mathbf{S}^3$, $\mathrm{res}[\omega]$, a constant playing the part of the Cauchy residue, and δ_0 the Dirac measure at 0.

Same result for the part of any type of $\mathrm{Res}[\omega]$.

5.3.3. f holomorphic

If $f_2 = 0$ and $f = f_1$ is holomorphic,

$$\mathrm{Res}[\omega](\varphi) = \lim_{\varepsilon\to 0}\int_{|f_1|=\varepsilon}\frac{df_1}{f_1}(\varphi).$$

Assume $f_1 = z_1$, then: $z_1 = \varepsilon^{i\theta}$, and

$$\mathrm{Res}[\omega](\varphi) = \lim_{\varepsilon\to 0}\int_0^{2\pi} id\theta\varphi_{2,2}(\varepsilon e^{i\theta}, z_2)dz_2\wedge d\overline{z}_2 = 2\pi i[Z(z_1)](\varphi).$$

Same result for any holomorphic f_1 since $f_1 = \varepsilon e^{i\theta}$.

References

[CSSS 04] F. Colombo, I. Sabadini, F. Sommen, and D. C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Progress in Math. Physics **39**, Birkhäuser (2004).

[CLSSS 07] F. Colombo, E. Luna-Elizarrarás, I. Sabadini, M. V. Shapiro, and D. C. Struppa, *A new characterization of pseudoconvex domains in* $\mathbb{C}^2$, Comptes Rendus Math. Acad. Sci. Paris **344** (2007), 677–680.

[D10a] P. Dolbeault, *About the characterization of some residue currents*, Complex Analysis and Digital Geometry, Proceedings from the Kiselmanfest, 2006, Acta Universitatis Upsaliensis, C. Organisation och Historia, 86, Uppsala University Library (2009), 147–157.

[D10b] P. Dolbeault, *About the characterization of some residue currents*, arXiv:1002.3025v1

[F 39] R. Fueter, *Über einen Hartogs'schen Satz*, Comm. Math. Helv. **12** (1939), 75–80.

[M 66] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, Oxford 1966.

Université Pierre et Marie Curie – Paris 6
UMR 7586, I.M.J.
F-75005 Paris
France

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 13, 2012

O FUKCJACH KWATERNIONOWYCH

S t r e s z c z e n i e

Scharakteryzowane są i zbadane rozmaite rodziny funkcji kwaternionowych. W szczególnych przypadkach wprowadzony jest prąd residualny prawego odwrócenia funkcji kwaternionowej.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 23–31

In memory of
Professor Promarz M. Tamrazov

Yuri B. Zelinskiĭ, Maksim V. Tkachuk, and Bogdan A. Klishchuk

INTEGRAL GEOMETRY AND MIZEL'S PROBLEM

Summary

The solution of a Zamfirescu's problem was obtained. The unsolved questions related to Mizel's problem were discussed.

Keywords and phrases: Jordan curve, circle, sphere, convex set, rectangle

1. Introduction

The subject of this article is to combine in one bundle some questions of complex analysis, geometry and probability theory. First investigations of geometric probability start with the well-known Buffoon's needle problem and related paradoxes. A needle is considered as the real line and then the problem reduces to finding an invariant measure of a set relative to a movement (R. V. Ambartsumyan, G. Matheron, L. Santalo [1, 9, 13]).

Many questions of integral geometry are reduced to the estimation of a measure of linear spaces crossing a convex set. Finding such a measure we obtain probabilistic estimations. Other problems, more close to geometry, are the estimations of properties of the set under investigation if properties of its intersections with families of some sets are well known:

1) with planes of fixed dimension:
   a) the real case (G. Aumann, A. Kosiński, E. Shchepin [2, 7, 14]);
   b) the complex case (Yu. Zelinskii [19]);

2) with a set of vertices of an arbitrary rectangle (A. S. Besicovitch, L. Danzer, M. Tkachuk, T. Zamfirescu [3, 4, 15, 18]).

The first problem is connected with the well-known Ulam's problem [10].

Ulam's problem. Let M^n be an n-dimensional manifold and every section of M^n by arbitrary hyperplane L be homeomorphic to an $(n-1)$-dimensional sphere S^{n-1}. Is it true that M^n is an n-dimensional sphere?

In the real case A. Kosiński solved this problem in 1962 [7]. L. Montejano obtained the same result in 1990 [11]. In the complex case Yu. Zelinskii got a similar result in 1993 [19].

The second problem is known in literature as **Mizel's problem** (a characterization of the circle): *A closed convex curve such that, if three vertices of any rectangle lie on it, so does the fourth, must be a circle.*

In 1961 A. S. Besicovitch [3] solved this problem. Later, a modified proof of this statement was presented by L. W. Danzer, W. H. Koenen, C. St. J. A. Nash-Williams, A. G. D. Watson [4, 6, 12, 17].

In 1989 T. Zamfirescu [18] proved a similar result for a Jordan curve (not convex à priori) and for a rectangle with the infinitesimal relation between its sides:

$$\left|\frac{a}{b}\right| \le \varepsilon, \quad \varepsilon > 0,$$

where a and b are side lengths of a rectangle.

In 2006 M. Tkachuk [15] obtained the most general result in this area for any arbitrary compact set $C \subset \mathbb{R}^2$, where the complement $\mathbb{R}^2 \backslash C$ is not connected.

It is obvious that the requirement of the compactness is necessary, otherwise the straight line and some other sets noted below will satisfy the stated requirement. But if the partition of the plane is not required, the ensembles below will satisfy, for instance, that condition of a rectangle: an ensemble from three points of a plane such that the triangle with vertices in these points will not be rectangular, a proper arc of a semicircle, a set of points of a plane with rational (irrational) coordinates.

T. Zamfirescu proved that *every analytic curve of constant width satisfying the infinitesimal rectangle property is a circle.* Our aim is to prove the theorem for a convex curve without the analyticity condition (Theorem 1) and to discuss new unsolved problems concerning Mizel's problem.

2. The main theorem

Definition 2.1. A set in $\mathbb{R}^2$ is called *convex* if it contains, with any two distinct points x and y, the (closed) line segment between x and y.

Definition 2.2. A width of a convex curve *in the direction* is the distance between two parallel support lines of this curve perpendicular to this direction. A curve has *a constant width* if its width is equal in all directions.

Definition 2.3. A set in $\mathbb{R}^2$ has *the infinitesimal rectangle property* if there is some $\varepsilon > 0$ such that no rectangle with side lengths ratio at most ε has exactly three vertices in the set (if three vertices of any such rectangle lie on the set, so does the fourth).

We introduce the following notation: let Γ be a convex curve of constant width satisfying the infinitesimal rectangle property and for each point $x \in \Gamma$ the circle having with the curve Γ a common tangent at the point x be denoted by $\partial\Delta_x$ in some neighbourhood of a point of the curve Γ we assume that the upward direction is the direction along the inner normal and thus we consider the right direction and the left direction along the curve Γ; $U_\varepsilon(x, \Gamma)$ is the ε-neighborhood of x on the curve Γ, $U^l_\varepsilon(x, \Gamma)$ is the left ε-neighborhood of x on the curve Γ, i.e. the subset of $U_\varepsilon(x, \Gamma)$ each point of which lies to the left of x; $U^r_\varepsilon(x, \Gamma)$ is the right ε-neighborhood of x on Γ.

Definition 2.4. A point $y \in \Gamma$ is called *opposite* to a point $x \in \Gamma$ if the straight line xy is perpendicular to the straight line T_x that is supporting for Γ and passes through the point x. The straight line xy is called *a normal* of a convex curve at the point x. A straight line that is a normal of a convex curve at two points of their intersection is called *a binormal.*

Theorem 2.1. *Any convex curve of constant width satisfying the infinitesimal rectangular condition is a circle.*

Proof. We are only interested in rectangles with the diagonal length d. Let a and b be sides of the rectangle. By the infinitesimal rectangular property:

$$\frac{a}{b} < \delta.$$

From the Pythagorean theorem we have:

$$b = \sqrt{d^2 - a^2}.$$

Hence

$$\frac{a}{\sqrt{d^2 - a^2}} < \delta,$$
$$a^2 < \delta^2(d^2 - a^2),$$
$$a^2 < \frac{\delta^2 d^2}{1 + \delta^2}.$$

Let $\delta d/\sqrt{1 + \delta^2} = \varepsilon$; then $a < \varepsilon$, $\varepsilon > 0$. Hence the infinitesimal condition may be replaced by the requirement that the smaller rectangle side has length less than some ε for the fixed diagonal length d.

In [15] it was proved that the curve Γ has a continuous tangent.

We prove that Γ does not contain the points through which several different straight lines supporting Γ may pass. In what follows, we call these points *angular* and the other points *regular.* The proof of the statement that every normal of a convex curve that satisfies the rectangle condition (if three vertices of a rectangle belong to Γ then its fourth vertex also belongs to Γ) is a binormal is given in [3] and can also be applied to Γ in the case considered (this was proved by T. Zamfirescu).

Let x be an angular point of Γ. Consider the set λ of points $z \in \Gamma$ opposite to the point x. Consider a mapping that associates every point $z \in \lambda$ with the straight line p perpendicular to xz, $z \in p$, that is supporting for Γ at the point z. We obtain a continuous mapping of the arc λ onto the set of elements supporting for the arc λ of the convex curve Γ [8]. Thus the arc λ cannot contain angular points because the mapping $p(z)$ is continuous.

Let us prove that the supporting straight line $p(z)$ is tangent to λ at a point z in the classical sense of this word (the limit location of a secant). It is known that one can represent a convex curve in the neighborhood of an arbitrary point of it as the graph of a convex function by choosing corresponding coordinate axes. A convex function has right-hand and left-hand derivatives everywhere but they may be different at certain points [8]. We prove that the tangents to λ from the right and from the left at all points coincide with supporting straight lines. For an arbitrary point $z_0 \in \lambda$ we take a sequence of points $z_n \in \lambda$ such that $z_n \to z_0$, from the right with respect to the chosen orientation of axes. The straight line $z_n z_0$ tends to the right tangent at the point z_0. Since the mapping $p(z)$ is continuous, there exists a point $t_n \in \bigcup z_0 z_n$ such that $p(t_n)$ is parallel to $z_0 z_n$. Since $z_n \to z_0$ and $t_n \in \bigcup z_0 z_n$, we have $t_n \to z_0$, and, hence, $p(t_n)$ tends to the right tangent at the point z_0 that coincides with $p(z_0)$. For the left tangent the arguments are analogous.

Thus, the arc λ is smooth, i.e., a diffeomorphism of an open segment onto the plane. Using differential calculus and taking into account the condition $p(z)$ perpendicular to xz, we prove that λ is an arc of a circle. For an arbitrary point of a circle, there exists its neighbourhood on the arc symmetric relative to the normal to the arc at this point, and the neighborhood of the angular point x can be symmetric only relative to the bisector of the supporting angle (the supporting straight lines have also to be symmetric). We draw a binormal different from the bisector of the supporting angle through the point x and an interior point of the arc of the circle and construct a rectangle with two points of the arc of the circle symmetric relative to the binormal and the third vertex in the neighborhood of the point x. The fourth vertex, which is symmetric to the third one, belongs to Γ. Thus, there exists a neighborhood of the angular point x that is symmetric relative to this binormal, which is impossible. Therefore Γ does not contain angular points and is a smooth curve.

Let $A \subset R^2$ be some set of points and ΛA linear measure of A.

A. S. Besicovitch proved the following four lemmas:

Lemma 2.1. *If E, $\Lambda E = 0$, is the set of the end points of a set of normals of Γ and U the set of mid-points of the normals, then $\Lambda U = 0$.*

Lemma 2.2. *The set of centres of curvature of* Γ *at the points, at which the curvature is equal to* $2/d$, *is of linear measure 0.*

Lemma 2.3. *Given a set* E, $\Lambda E > 0$, *of points of* Γ, *at which the radius of curvature is different from* $\frac{1}{2}d$ *the set* U *of mid-points of the normals at* E *is also of positive measure and at almost all points of* U *the tangent to* U *exists and is perpendicular to the normal to which the point belongs.*

Lemma 2.4. *If at almost all points of an arc* $\smile AB$ *of* Γ, *and of the opposite arc* $\smile A^{'}B^{'}$ *the radius of curvature is* $\frac{1}{2}d$ *then* $\smile AB$ *and* $\smile A^{'}B^{'}$ *are circular arcs.*

Notice that these lemmas are valid for the infinitesimal rectangle property since they have a local character. But the result following from these lemmas changes: if at some point $x \in \Gamma$ the circle $\partial\Delta_x$ intersects Γ at y and the distance between x and y is less than ε; then in some neighbourhood of x (and also of the opposite point x^*) the curve Γ is an arc of the circle $\partial\Delta_x$.

Consequently, all points of the curve Γ can be divided into five disjoint sets:

$$A = \{x \in \Gamma | U_\varepsilon(x, \Gamma) \backslash \{x\} \subset \Delta_x\}$$

is the set of points of the curve Γ such that the curve Γ lies in the open circular disk Δ_x in ε-neighbourhood of $x \in A$ except the point x itself;

$$B = \{x \in \Gamma | U_\varepsilon(x, \Gamma) \backslash \{x\} \subset \mathbb{R}^2 \backslash \overline{\Delta_x}\}$$

is the set of points of the curve Γ such that in their $\varepsilon-$neighborhood the curve Γ is situated outside of the closed circular disk $\overline{\triangle_x}$ except for the point x itself;

$$AB = \{x \in \Gamma | U^l_\varepsilon(x, \Gamma) \subset \Delta_x, U^r_\varepsilon(x, \Gamma) \subset \mathbb{R}^2 \backslash \overline{\Delta_x}\}$$

is the set of points of the curve Γ such that in their ε-neighbourhood the curve Γ lies in the open circular disk $\triangle_x$ to the left of x and outside of the closed circular disk $\overline{\triangle_x}$ to the right of x except the point x itself;

$$BA = \{x \in \Gamma | U^l_\varepsilon(x, \Gamma) \subset \mathbb{R}^2 \backslash \overline{\Delta_x}, U^r_\varepsilon(x, \Gamma) \subset \Delta_x\}$$

is the set of points of the curve Γ such that in their ε-neighbourhood the curve Γ lies in the open circular disk $\triangle_x$ to the right of x and outside of the closed circular disk $\overline{\triangle_x}$ to the left of x except for the point x itself;

C is the set of points of the curve Γ each of which has the neighbourhood where the curve coincides with an arc of the circle $\partial\Delta_x$.

Hence

$$\Gamma = A \bigcup B \bigcup AB \bigcup BA \bigcup C.$$

Suppose that $x_n \to x$, $x_n \in \Gamma$ and let all x_n be in ε-neighbourhood of x to the left of it. Assume that $x_n \in A \bigcup BA$ and $x \in AB$ then in some neighbourhood of x the curve Γ lies above each circle $\partial\Delta_{x_n}$. Γ is inside the circular disk $\overline{\triangle_x}$. This fact follows from the continuity of a tangent to Γ. We obtain a contradiction with $x \in AB$. So in some neighbourhood of x to the left of it there are no points of the sets A and BA. In this neighbourhood we consider a similar sequence of points $x_n \in B \bigcup AB$ and as a consequence we find that in some neighbourhood of x the curve Γ lies under every circle $\partial\Delta_{x_n}$ and therefore outside of the circular disk Δ_x which is impossible. So in some left half-neighborhood of x our curve is an arc of a circle $\partial\Delta_x$ which is also not compatible with the condition $x \in AB$. We conclude that $AB = \emptyset$. Similarly, $BA = \emptyset$.

Thus
$$\Gamma = A \bigcup B \bigcup C.$$
Suppose that $C = \emptyset$, $\Gamma = A \bigcup B$. Consider a sequence of points $x_n \in A$, $x_n \to x$. If $x \in B$ then as we know the contradiction is obtained. Hence x belongs to A and A is a closed set. Similarly, B is closed. The curve Γ is connected and therefore either A or B is empty. It means that $\Gamma = A$ or $\Gamma = B$, which implies that the curve Γ either contains a circular disk Δ_x or is contained in a circular disk Δ_x, which contradicts the fact that Γ is the convex curve of constant width and has a length πd.

Hence $C \neq \emptyset$ and there exists a point $x \in C$. Some neighbourhood of x is also contained in the set C. We shall move along the curve Γ to the left from the point x to the first point y that does not belong to C. It is obvious that y can belong neither to A nor to B and therefore $\Gamma = C$. Then by applying the Heine-Borel lemma we conclude that Γ is a circle and Theorem 1 is proved.

Combining this result with the result of Zamfirescu [18], we obtain:

Corollary 2.1. *Every Jordan curve satisfying the infinitesimal rectangle condition is a circle.*

3. Open Problems

A row of similar open problems in the plane and in an n-dimensional case appears in connection with Mizel's problem.

Problem 3.1. Let C be a closed Jordan curve in $\mathbb{R}^2$, and for an arbitrary algebraic closed curve L of the order n from the property that the intersection $C \cap L$ contains m points follows that $C \cap L$ contains no less than $m + 1$ points. Does there exist a number m such that from the property above follows that C is an algebraic curve of the order n?

Problem 3.2. In the previous question let L be a circle and $m = 3$. Is it true that also C is a circle?

Problem 3.3. In Problem 3.1 let L be an ellipse and $m = 5$. Is it true that also C is an ellipse?

Problem 3.4. Will a compact C be a sphere in $\mathbb{R}^n$ if C divides the space and if from the belonging $n+1$ tops of the arbitrary rectangular parallelepiped to a compact C, it follows that one more top lies in C too?

The latter question is interesting even if C is an $(n-1)$-dimensional manifold or a boundary of a convex set.

Problem 3.5. Let C be an $(n-1)$-dimensional manifold (or a boundary of a convex domain) in $\mathbb{R}^n$ and there does not exist an $(n-1)$-dimensional sphere S^{n-1} such that the intersection $C \cap S^{n-1}$ contains exactly $n+1$ points. Is it true that C is an $(n-1)$-dimensional sphere?

Problem 3.6. Does the result of [18] remain valid if we consider a compact set $C \subset \mathbb{R}^2$ where the complement $\mathbb{R}^2 \backslash C$ is not connected?

Problem 3.7. Are the cited results and Problems 3.1-3.5 true, if one point (top) on C is fixed?

Problem 3.8. Will the Problems 3.1-3.6 be true if one of the intersection points belongs to some selected subset $E \subset C$ that is not everywhere dense in C? What is the minimum power of this set for the correctness of problems quoted?

Example 3.1. Let $C = F \cup S^1$, where S^1 is a unit circle in the plane and F is any compact subset of y-axis. Let all points of F be at the distance more than one unit from the origin. If the selected set consists of the pair of points $E = (0,1) \cup (0,-1)$, it is obvious that the compact C satisfies the Mizel's problem for the vertices of a rectangle withh one vertex lying on the set E, but C is not a circle.

This example shows that the selected set of Problem 3.8 should contain at least three points or we restrict ourselves to study compacts that are Jordan curves.

The example below will show that in Problem 3.2 it is impossible to consider a compact set dividing the plane instead of a curve, in analogy Tkachuk's result.

Example 3.2. Consider the domain D on the plane bounded by the circle S^1. Next, remove from the interior of the domain D the infinite ensemble of opened balls D_i everywhere dense in D which do not intersect pairwise even on the border and also do not intersect the circle S^1.

Then we obtain the compact fractal set $K = \bar{D} \backslash \cup D_i$ without interior points that divides the plane into a countable set of components. Yet, it is easy to see that the intersection of an arbitrary circle and K is only one point or an infinite number of points (see Figure 1).

We receive other examples if domains $D(D_i)$ are domains bounded by ellipses or squares. In this case intersections with a circle can contain one, two, four or infinitely many points but never three points. These examples give a negative answer to Problem 8 from [20].

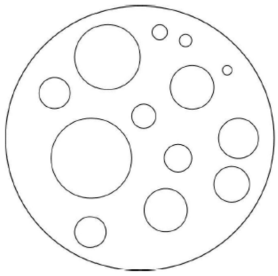

Fig. 1: Construction for Example 3.2.

Corollary 3.1. *On the two-dimensional plane $\mathbb{R}^2$ there exist compact sets that divide the plane and such that there are no circles having with them precisely three cross points.*

In particular, a class of similar sets includes the Shottky sets and Sierpiński's carpet.

Other similar problems can be found in the work of Grünbaum [5].

Acknowledgment

This investigation was partially supported by Tubitek-NASU grant number 110T558.

References

[1] R. V. Ambartsumyan, I. Mekke, D. Shtoiyan, *Introduction in stochastic geometry*, Nauka, Moscow 1989, 400 pp. (in Russian).

[2] G. Aumann, *On a topological characterization of compact convex point sets*, Annals of Mathematics, 2nd Ser. **37**, no. 2 (1936), 443–447.

[3] A. S. Besicovitch, *A problem on a circle*, J. London Math. Soc. **36** (1961), 241–244.

[4] L. W. Danzer, *A characterization of the circle*, Amer. Math. Soc., Providence, R. I. Convexity, Proc. Symposia in Pure Math. **7** (1963), 99–100.

[5] B. Grünbaum, *Characterization of circles and spheres*, Amer. Math. Soc., Providence, R.I. Convexity, Proc. Symposia in Pure Math. **7** (1963), 497.

[6] W. Koenen, *Characterizing the circle*, Amer. Math. Monthly **78** (1971), 993–996.

[7] A. Kosiński, *A theorem on families of acyclic sets and its applications*, Pacific J. Math. **12** (1962), 317–325.

[8] K. Leichtweiss, *Konvexe Mengen*, Springer-Verlag, Berlin-New York 1980.

[9] G. Matheron, *Random sets and integral geometry*, Wiley Series in Probability and Mathematical Statistics, New York-London-Sydney 1975.

[10] R. D. Mauldin, *The Scottish Book*, Birkhäuser Verlag, Boston 1981.

[11] L. Montejano, *About a problem of Ulam concerning flat sections of manifolds*, Comment. Math. Helvetici **65**, no. 3 (1990), 462–473.

[12] C. St. J. A. Nash-Williams, *Plane curves with many inscribed rectangles*, J. London Math. Soc., 2nd Ser. **5** (1972), 417–418.

[13] L. A. Santalo, *Integral Geometry and Geometric Probability, Vol. 1, Encyclopedia of Mathematics and Its Applications*, Ed. G.-C.Rota, Addison-Wesley Publishing Company, Massachusetts 1976.

[14] E. Shchepin, *Convexity criterion of open set*, III Tiraspol Symposium on General Topology and Application, Shtyyntsa, Kishyniev 1973, p. 149 (in Russian).

[15] M. V. Tkachuk, *Characterization of the circle of the type Besicovitch-Danzer*, Transaction of the Institute of Mathematics NAS of Ukraine **3**, no. 4 (2006), 366–373 (in Ukrainian).

[16] M. V. Tkachuk, *Besicovitch-Danzer-type characterization of a circle*, Ukrainian Math. J. **60**, no. 6 (2008), 1009–1011.

[17] A. G. D. Watson, *On Mizel's problem*, J. London Math. Soc. **37** (1962), 307–308.

[18] T. Zamfirescu, *An infinitesimal version of the Besicovitch-Danzer characterization of the circle*, Geom. Dedicata **27**, no. 2 (1988), 209–212.

[19] Yu. B. Zelinskii, *Multivalued mappings in the analysis*, Naukova dumka, Kyiv 1993, 264 pp. (in Russian).

[20] Yu. Zelinskii, *Integral complex geometry*, Bull. Soc. Sci. Lett. Łódź, Sér. Rech. Déform. **60**, no. 3 (2010), 73–80.

[21] Yu. B. Zelinskii, *Some unsolved problems in complex linear convex analysis*, Matematychni Studii **30**, no. 2 (2008), 195–197 (in Ukrainian).

Institute of Mathematics
National Academy of Sciences of Ukraine
Tereshchenkivs'ka vul. 3, UA-01601 Kyiv
Ukraine
e-mail: zel@imath.kiev.ua
max@imath.kiev.ua
bogdanklishchuk@mail.ru

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 29, 2012

GEOMETRIA CAŁKOWA I PROBLEM MIZELA

S t r e s z c z e n i e

Uzyskujemy rozwiązanie problemu Zamfirescu. Dyskutujemy też nierozwiązane pytania z problemem Mizela.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 33–41

In memory of
Professor Promarz M. Tamrazov

Claude Surry

HISTORICAL DEVELOPMENTS OF COMPUTING OF THE INHOMOGENEOUS DIRICHLET PROBLEM IN BIDIMENSIONAL OR MULTIDIMENSIONAL DOMAINS

Summary

We present the question of solving and computing nonhomogeneous *Dirichlet problems* in domains in $\mathbb{R}^2$ or $\mathbb{R}^n$ ($n \geq 2$). Using complex analysis we present the *Kutta-Joukowski* method of computing a bidimesional flow around a profile. In the case [1, 2] of a three-dimensional flow around a cylindrical profile, we determine Sobolev spaces concerned and calculate by optimization methods an approximation of the solution by the use of Galerkin approximations [5–7]. This problem arises in engineering science, thermal physics or dynamics of flows in porous media [6, 9].

Keywords and phrases: Dirichlet problem, flow profile, Dirichlet beam

1. Flows past a Jordan curve profile

It is well known that a continuous differentiable plane flow $u+iv$ defined in a simply connected domain G is irrotational and only solenoidal if and if

$$u + iv = \overline{f'}(z) \quad \text{with} \quad z = x + iy, \tag{1}$$

where $f(z)$, an analytic function defining the complex potential of the flow $u + iv$, is solenoidal and irrotational. Let

$$w = u + iv. \tag{2}$$

Define a *Jordan curve* c and observe that

$$\int_c \overline{w}dz = \int_c \left(\overline{u} + \overline{iv}\right) \ (dx + idy) = \int_c (udx + vdy) + i \int_c (-vdx + udy) \tag{3}$$

and

$$\int_c udx + vdy = \mathrm{Re} \int_c \overline{w}dz, \tag{4}$$

$$\int_c -vdx + udy = \mathrm{Im} \int_c \overline{w}dz. \tag{5}$$

We have

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}, \tag{6}$$

$$-v = \frac{\partial \psi}{\partial x}, \qquad u = \frac{\partial \psi}{\partial y} \tag{7}$$

$$f'(z) = \frac{\partial \varphi}{\partial x} + i \frac{\partial \phi}{\partial x} = u - iv$$

and we get (1).

Conversely, take $u + iv$ to satisfy (1), where $f(z)$ in analytic in G. Take a smooth closed Jordan curve c contained in G. We get

$$\int_c \overline{w}dz = \int_c f'(z)dz = 0.$$

We use the Cauchy integral theorem – the derivative of an analytic function is itself analytic:

$$\int_c \overline{w}dz = \int_c f'(z)dz = 0.$$

Then, the flow $u + iv$ is irrotational and solenoidal by (4) and (5). φ and ψ define the velocity potential and the stream function. Equipotential and stream lines correspond to

$$\varphi(x, y) = \mathrm{cts}, \qquad \phi(x, y) = \mathrm{cts}'. \tag{8}$$

The mapping $\zeta = \xi + i\eta = f(z)$ is the complex potential; it is conformal at every point of G except at points where $f'(z)$ vanishes. There are also points, stagnation points where the velocity $u + iv$ vanishes and $\zeta = f(z)$ maps the curve (8) into the curves:

$$\xi = \mathrm{cts}, \qquad \eta = \mathrm{cts}', \tag{9}$$

and every curve $\xi = \mathrm{cts}$ is orthogonal to every curve $\eta = \mathrm{cts}'$ and vice versa. The curves (8) form an orthogonal system except at stagnation points. By (8), we get the equipotential lines

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = udx + vdy = 0 \tag{10}$$

and the streamlines

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{dy} = -vdx + udy = 0.$$

The velocity is normal to the equipotential line through (x, y) and tangential to the streamline through (x, y). The streamlines are the actual trajectories of the moving fluid elements. Every curve making the boundary Γ of the flow domain G has to be a part of the streamline $\psi(x, y)$ and, if $f(z)$ is to be the complex potential, then $\psi(x, y) = \mathrm{Im} f(z)$ has to be constant on every curve making up Γ.

2. Flow past a circular cylinder of radius R

Suppose that$f(z)$ is the complex potential and $f'(z)$ is an analytic function in the domain $|z| > R$ taking the value $\overline{w}_\infty$ at infinity. By Laurent expansion, $f'(z)$ has the form

$$f'(z) = \overline{w}_\infty + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \dots \tag{11}$$

and $f(z)$ is

$$f(z) = \overline{w}_\infty z + c_1 \ln z - \frac{c_2}{z} - \frac{c_3}{2z^2}. \tag{12}$$

Integration has no effect on the velocity field. In polar coordinates $\psi(r, \theta) = \mathrm{Im} f(z)$, we have with $w_\infty = u_\infty + iv_\infty$, $z = re^{i\theta}$ and

$$c_1 = a_1 + ib_1, \qquad c_2 = a_2 + ib_2 \qquad c_3 = a_3 + ib_3,$$

the formula

$$\begin{aligned}\psi(r, \theta) = a_1\theta + b_1 \ln r + \frac{a_2 + r^2 u_\infty}{r} \sin\theta \\ - \frac{b_2 + r^2 v_\infty \cos\theta}{r} + \frac{a_3}{2r^2} \sin 2\theta - \frac{b_3}{2r^2} \cos 2\theta.\end{aligned} \tag{13}$$

Let Γ be the circle $|z| = R$; Γ must be one of the streamlines of the flow. The function $\psi(r, \theta)$ has to be constant for $r = R$ and arbitrary θ. This is satisfied if we make the following choice:

$$a_1 = 0, \quad b_2 + Rv_\infty = 0, \quad a_2 + R^2 u_\infty = 0 \quad a_3 = b_3 = 0,$$

and we have

$$b'(z) = \overline{w}_\infty + \frac{ib_1}{z} - \frac{R^2 w_\infty}{z^2} \tag{14}$$

$$f(z) = \overline{w}_\infty z + ib_1 \ln z + \frac{R^2}{z} w_\infty. \tag{15}$$

We get the circulation around Γ:

$$\kappa = \operatorname{Re} \int\limits_{\Gamma} f'(z)dz = \operatorname{Re} \int\limits_{|z|=r} f'(z)dz = \operatorname{Re} \int\limits_{|z|=r} \overline{w}_\infty + \frac{ib_1}{z} - \frac{R^2 w_\infty}{z^2} = -2\pi b_1 \tag{16}$$

if $r > R$. Eq. (14) and (15) become:

$$f'(z) = \overline{w}_\infty - \frac{\kappa}{2i\pi z} - \frac{R^2 w_\infty}{z^2}, \tag{17}$$

$$f(z) = \overline{w}_\infty z + \frac{\kappa}{2i\pi} \ln z + \frac{R^2 w_\infty}{z}. \tag{18}$$

The second term of (18) is a pure circulatory flow. This holds also when Γ is an arbitrary closed Jordan curve instead of the circle $|z| = R$. In fact κ is the circulation around Γ and take $|z| = r$ to be any circle surrounding Γ. Eq. (16) is changed to

$$\kappa = \operatorname{Re} \int\limits_{|z|=r} \left(\overline{w}_\infty + \frac{c_1}{z} + \frac{c_2}{z^2 + \ldots} \right) dz = \operatorname{Re}(2i\pi c_1), \quad |z| = r.$$

The flux through Γ, equal to $2i\pi c_1$, must vanish, the flow being solenoidal in G. Thus, c_1 is purely imaginary and again we have

$$c_1 = \frac{\kappa}{2i\pi},$$

$$f'(z) = \overline{w}_\infty + \frac{\kappa}{2i\pi z} + \frac{c_2}{z^2}.$$

We assume that $w - \infty = u_\infty > 0$ and with this last expression of $f'(z)$ we get stagnation points of the flow ($u = v = 0$) and

$$z^2 + \frac{\kappa}{2\pi i u_\infty} z - R^2 = 0,$$

$$z_{1,2} = \frac{i\kappa}{4\pi u_\infty} \pm \sqrt{R^2 - \left(\frac{\kappa}{4\pi u_\infty} \right)^2}.$$

a) If $|\kappa| \geq 4\pi R u_\infty$, z_1 and z_2 are purely imaginary. With the relation $z_1 z_2 = -R^2$, one of the points lies outside the circle $|z| = R$, in the flow domain;

b) If $|\kappa| = 4\pi R u_\infty$ only there is one stagnation point, at one of the points in which the imaginary axis intersects the circle $z = R$;

c) If $|\kappa| < 4\pi R u_\infty$, there are two stagnation points z_1 and z_2 on the circle $|z| = R$ which are symmetric with respect the imaginary axis.

If $\kappa = 0$, there are two stagnation points at the points $\pm R$ in which the real axis intersects the circle $|z| = R$.

3. Kutta-Joukowski theorem – circulation of the flow around a Jordan curve Γ

Define $\zeta = g(z)$ the unique conformal mapping of the exterior of Γ onto the exterior of the unit circle $|\zeta| = 1$ such that $g(\infty) = \infty$ and $g'(\infty)$ ia a positive number. The Laurent expansion of $g(z)$ at infinity has the form

$$\zeta = g(z) = cz + c_0 + \frac{c_1}{z} + \dots, \qquad 0 < c < +\infty. \tag{19}$$

The complex flow past the circle $|\zeta| = 1$ with velocity A at infinity is

$$\Phi(\zeta) = \frac{\kappa}{2i\pi} L_n \zeta + \overline{A}\zeta + \frac{A}{\zeta}, \tag{20}$$

where A can be chosen so that, with $\zeta = g(z)$,

$$f(z) = \Phi(g(z)) = \frac{\kappa}{2i\pi} L_n g(z) + \overline{A} g(z) + \frac{A}{g(z)}. \tag{21}$$

On Γ, we have a constant imaginary part for $f(z)$ and its derivative is analytic outside Γ. At infinity we have

$$\overline{w}_\infty = f'(\infty) = \Phi'(\infty) g'(\infty) = \overline{A} c.$$

Here A can be expressed as

$$A = \frac{w_\infty}{c} = \frac{w_\infty}{g'(\infty)}$$

with $f(z)$ given by

$$f(z) = \frac{\kappa}{2i\pi} \ln g(z) + \frac{\overline{w}_\infty}{g'(\infty)} g(z) + \frac{w_\infty}{g'(\infty) g(z)}. \tag{21'}$$

The force exerted on a cylinder of cross section Γ is an application of Bernoulli law

$$P + \frac{1}{2} p |w|^2$$

which is constant along streamlines and hence along the cross section Γ. With (21'), A being positive, we get

$$P = A - \frac{1}{2} \rho |w|^2.$$

Take an element of Γ, $dz = dx + idy$. The force acting on dz due to the pressure is

$$P i dz = A i dz - \frac{1}{2} \rho i |w|^2 dz,$$

$$F = X + iY = \int_\Gamma P i dz = A i \int_\Gamma dz - \frac{1}{2} \rho i \int_\Gamma |w|^2 dz, \tag{22}$$

$$X + iY = -\frac{1}{2} \rho i \int_\Gamma |w|^2 dz. \tag{23}$$

The first integral of (22) vanishes. Γ is a streamline, $f(z)$ stands for the complex potential

$$w = \overline{f'(z)} = |w|e^{i\theta},$$

$\theta = \arg dz$, and we have

$$|w| = \overline{f'(z)}e^{-i\theta}. \tag{24}$$

Putting (24) in (23) we have

$$X + iY = -\frac{1}{2}\rho i \int\limits_{\Gamma} |\overline{f'(z)}|^2 e^{-2i\theta} dz = -\frac{1}{2}\rho i \int\limits_{\Gamma} |\overline{f'(z)}|^2 \times d\overline{z}. \tag{25}$$

Take the complex conjugate of (25):

$$X - iY = \frac{1}{2}\rho i \int\limits_{\Gamma} |f'(z)|^2 dz = \frac{1}{2}\rho i \int\limits_{|z|=r} |f'(z)|^2 dz. \tag{26}$$

We replace Γ by any circle $|z| = r$. Put (17) in (26). We get

(27)
$$X - iY = \frac{1}{2}\int\limits_{|z|=r} \rho i \left(\overline{w_\infty} + \frac{\kappa}{2i\pi z} + \frac{c_1}{z^2} + \dots\right) dz = \frac{1}{2}\rho i \frac{2\kappa w_\infty}{2i\pi}\int\limits_{c} \frac{dz}{z} = \rho\kappa\overline{w_\infty} i,$$

so $X = 0$, $Y = -\rho\kappa w_\infty$.

4. Dirichlet beam problem of $\mathbb{R}^n$ in a cylinder

A certain number of problems needs to be solved in a beam $(x_1, \dots, x_{n-1})$ $(x_n = +h, x_n = -h)$ on the boundary Γ of the beam; we have $u = 0$ for $x_n = +h$ and $x_n = -h$. For $x_1 = 0$ we have $u = 1$. In the case of $\mathbb{R}^2$, the beam is infinite in the x_1-direction. We have

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1^2} = 0, \tag{28}$$

$$\begin{cases} u = 0, & x_1 = 1 \text{ and } x_1 = -1 \\ u = 1, & x_2 \\ u \to 0, & x_2 \to \infty \end{cases} \qquad \text{on} \quad \Gamma \tag{29}$$

in the case of κ, $(n = 2)$. Let

$$\Omega =]-1, +1[\times \omega,$$

$$\omega =]0, +\infty] \quad (x_2 > 0),$$

$$\Gamma_1 =]-1, +1[\cup[x_2 = 0],$$

$$\Gamma_2 =]0, +\infty] \cup \{x_2 > 0\} \cup \{|x_1| = 1\}.$$

We are going to solve

$$\Delta u = 0 \quad \text{on} \quad \Omega$$

and

$$\begin{cases} u = 0 & \text{on} \quad \Gamma_2, \\ u = 1 & \text{on} \quad \Gamma_1, \end{cases}$$

with $w = u + iv$, $z = x + iy$, the function $w = \sin\left(\frac{\pi}{2}z\right)$ maps the slab Ω on the half-plane $v > 0$ ($\operatorname{Im} w > 0$).

We have to solve a Dirichlet problem in the half-plane $(u, v; v > 0)$; u is equal to 1 on $\Gamma_1 =]-1, +1[$. The solution disappears on $]-\infty, -1]$ and $[1, +\infty[$:

$$\begin{aligned} u_0 &= \frac{1}{\pi}\int_{-1}^{+1} \frac{v}{(\xi - u)^2 + v^2} d\xi \\ &= \frac{1}{\pi}\left[\arctan\left(\frac{1-u}{v}\right) - \arctan\left(\frac{-1-u}{v}\right)\right] \\ &= \frac{1}{\pi}\arctan\left(\frac{w-1}{w+1}\right) = \frac{2}{\pi}\arctan\left(\cos\frac{1}{2}\pi x \times \frac{1}{\sinh\frac{1}{2}\pi y}\right). \end{aligned} \tag{30}$$

This gives the behaviour of u_0 close to the boundary [9].

Define now Ω as the cylinder of $\mathbb{R}^n$:

$$\Omega = (-1, +1) \times \omega,$$

where ω is a Lipschitz domain of $\mathbb{R}^{n-1}$. Define

$$x_1 = \{x_0, x_1, \ldots, x_{n-1}\}, \qquad x_2 = x_n,$$

$$\begin{aligned} \Gamma &=]-\infty, -1]^{n-1} \cup [1, +\infty[^{n-1} \cup [x_n = 0,\ x_1 \in \omega], \\ \Gamma_1 &= \left\{\left[x_1 \in]-\infty, -1]^{n-1}\right] \cup \left[x_1 \in]1, +\infty]^{n-1}\right]\right\}, \\ \Gamma_2 &= \{x_n = 0 \cup \omega\}. \end{aligned} \tag{31}$$

We are interested in solving the problem related with

$$\Delta u = 0 \quad \text{on} \quad \Omega, \tag{32}$$

$$\begin{cases} u = 0 & \text{on} \quad \Gamma_1, \\ u = 1 & \text{on} \quad \Gamma_2. \end{cases} \tag{33}$$

Several problems of determination of the harmonic function in $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with nonhomogeneous conditions on Γ are met in dynamics of flows in porous media, in thermal physics, or hydrodynamic flows [3, 4, 6]. This type of problem, especially when Ω in a three dimensional domain ($\mathbb{R}^3 \supset \Omega$), requires functional analysis applications [5, 7]. The use of Sobolev spaces with Galerkin discrete spaces [5, 7] is needed (finite elements).

In $\mathbb{R}^3$, it is the problem in a half-cylinder of square section parallel to the plane (x_1, x_2). The cylinder is infinite in the direction $(x_3 > 0)$. No manual computing method is used for this type of problem. We use Sobolev spaces and Lax-Milgram theorem [5]. Define

$$H^1(\Omega) = \left\{ v \int\limits_{\Omega} \left(\overrightarrow{\text{grad}} v \right)^2 d\Omega < +\infty \right\},$$

$$\|v\|^2_{H^1(\Omega)} = a(v,v) = \int\limits_{\Omega} \left(\overrightarrow{\text{grad}} v \right)^2 d\Omega,$$

$$K = \left\{ v \in H^1(\Omega) | v = 0 \quad \text{on} \quad \Gamma_1; \; v = 1 \quad \text{on} \quad \Gamma_2 \right\},$$

K is a closed convex set and we can define [5] the trace of $v \in H^1(\Omega)$ on v on Γ_2, $v \in L^2(\Gamma_2)$. By the Lax-Milgram theorems [5] it exists $u \in K$ such that for every $v \in K$ we have

$$a(u, v-u) \geq L(v-u),$$

$$L(v) = \int\limits_{\Gamma_2} v d\Gamma, \qquad u = 1 = v \quad \text{on} \quad \Gamma_2.$$

We construct $K_h \subset K$, the space of continuous functions, K_h being of finite dimension [7] and we have to solve (Galerkin method) the problem related with

$$K_h \subset K, \qquad \exists u_h \in K_h \quad \forall v_h \in K_h$$

$$a(u_h, v_h - u_h) \geq L(v_h - u_h).$$

We have suggested that in the case of Dirichlet problem with non-homogeneous boundary values, in the following cases:

– $n = 2$ (in $\mathbb{R}^2$); we can get manual computing solutions using applied complex analysis [8];

– $n = 3$ (in $\mathbb{R}^3$); we can get numerical solutions using Galerkin approximations in Sobolev spaces by Lax-Milgram theorem and the use of optimization methods.

The analytic solution of this bidimensional problem can be used as test function for computing (in the case of bidimensional finite element problem) a compared solution.

References

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge 1967.

[2] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1987.

[3] C. Baiocchi, *Problémes à frontière libre et inégalités variationelles*, C. R. Acad. Sci. Paris **283** (1976), 29–32.

[4] H. Brezis, *Analyse fonctionelle*, Masson, Paris 1983.

[5] M. Chipot, *Variational Inequalities and Flow in Porous Media*, Springer Verlag, New York 1984.

[6] M. Chipot, *Elements of Nonlinear Analysis*, Birhäuser Advanced Texts, Basel 2000.

[7] W. Rudin, *Real and Complex Analysis*, Dunod, Paris 1998.
[8] R. A. Silverman, *Complex Analysis with Applications*, Dover Publ., New York 1984.

Laboratoire Félix Trombe
Institut de Sciences et de Génie
de Matériaux et Procédes
Centre National de la Recherche Scientifique
B.P.5, Odeillo, F-66 125 Font Romeu Cédex
France

Presented by Claude Surry at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 29, 2012

HISTORYCZNY ROZWÓJ WYLICZEŃ NIEJEDNORODNEGO PROBLEMU DIRICHLETA W OBSZARACH DWU- I WIELOWYMIAROWYCH

S t r e s z c z e n i e

Omawiamy zadanie rozwiązania i wyliczeń niejednorodnego problemu Dirichleta w obszarach z $\mathbb{R}^2$ względnie $\mathbb{R}^n$ $(n \geq 2)$. W oparciu o analizę zespoloną przedstawiamy metodę Kutty-Żukowskiego wyznaczania dwuwymiarowych przepływów wokół profilu. W przypadku trójwymiarowego przepływu wokół profilu cylindrycznego, wyznaczamy stosowne przestrzenie Sobolewa i przy użyciu metod optymizacji aproksymujemy rozwiązanie przy użyciu aproksymacji Galerkina. Wskazujemy, że problem może powstać na gruncie rozważań technicznych, termodynamicznych i w zakresie dynamiki przepływów w ciałach porowatych.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 43–47

In memory of
Professor Promarz M. Tamrazov

O. V. Kostiuk and M. V. Zabolotskyi

REGULAR GROWTH OF ENTIRE FUNCTIONS OF ORDER ZERO

Summary

We establish a relationship between strongly regular growth of entire functions of order zero and regular growth of their logarithm of modulus and argument in $L^p[0, 2\pi]$-metrics.

Keywords and phrases: entire function; order zero; angular density of zeros; function of strongly regular growth

1. Introduction

Suppose that f is an entire function of order zero, $f(0) = 1$; $n(r, \alpha, \beta)$ is a counting function of zeros a_j of the function f in the sector $\{z : |z| \leq r, \alpha < \arg z \leq \beta\}$, $n(r) = n(r, 0, 2\pi)$ is a counting function of zeros of f. By $\lambda(r)$ we denote the refined order of counting function $n(r)$, i.e.,

1) $\lambda(r)$ is a continuously differentiable function on $[0, +\infty)$;
2) $\lambda(r) \to 0$ as $r \to \infty$;
3) $\lambda(r) + r\lambda'(r) \ln r \searrow 0$ as $r \to \infty$;
4) $0 < \overline{\lim\limits_{r\to\infty}} \dfrac{n(r)}{V(r)} < +\infty$, where $V(r) = r^{\lambda(r)}$.

We denote the set of such functions f by $\mathcal{H}_0(\lambda(r))$.

We say [1] that the set of zeros of a function $f \in \mathcal{H}_0(\lambda(r))$ has an angular density, if the limit

$$\lim_{r\to+\infty} \frac{n(r, \alpha, \beta)}{V(r)} = \Delta(\alpha, \beta)$$

exists for all $0 \le \alpha < \beta < 2\pi$, perhaps, with the exception of α or β belonging to a countable set.

The equality $\Delta(\beta) - \Delta(\alpha) = \Delta(\alpha, \beta)$ determines for fixed α, up to a constant term, a nondecreasing function $\Delta(\beta)$, which is called *angular density.*

Rays $\{z : \arg z = \theta\}$ satisfying the condition

$$\lim_{\varepsilon \to 0_+} \left\{ \overline{\lim_{r \to +\infty}} \frac{n(r, \theta - \varepsilon, \theta + \varepsilon)}{V(r)} \right\} = 0$$

are said to be *ordinary* for the function $f \in \mathcal{H}_0(\lambda(r))$.

We denote by $\ln f$ the unique branch of the multivalued function

$$\mathrm{Ln} f = \ln |f(z)| + i \mathrm{Arg} f(z)$$

in the domain

$$G = \mathbb{C} \setminus \bigcup_{j=1}^{\infty} \{z : |z| \ge |a_j|, \ \arg z = \arg a_j\} \quad \text{with} \quad \ln f(0) = 0.$$

We say [1] that an entire function $f \in \mathcal{H}_0(\lambda(r))$ is of *strongly regular growth* (s.r.gr.) on the ordinary ray $\{z : \arg z = \theta\}$ if the following limit

$$\lim_{r \to +\infty}{}^{*} \frac{\ln f(re^{i\theta}) - N(r)}{V(r)} = H(\theta, f)$$

exists, where $\lim^{*}$ indicates that r tends to infinity outside some E_0-set.

If an entire function $f \in \mathcal{H}_0(\lambda(r))$ is of s.r.gr. on all ordinary rays $\arg z = \theta, 0 \le \theta < 2\pi$, then f is called a *function* of s.r.gr. We denote the set of such functions by $\mathcal{H}_0^{\star}(\lambda(r))$.

Theorem A [1]. *If $f \in \mathcal{H}_0(\lambda(r))$ and the set of its zeros has an angular density, then $f \in \mathcal{H}_0^{\star}(\lambda(r))$ and*

$$H(\theta, f) = i \int_{\theta - 2\pi}^{\theta - 0} (\theta - \psi - \pi) d\Delta(\psi).$$

Conversely, if $f \in \mathcal{H}_0^{\star}(\lambda(r))$ and zeros of f are arranged on a finite system of rays, then the set of these zeros has an angular density.

In Theorem A it is essential that zeros are located on finitely many rays In the general case [2], strongly regular growth of an entire function f does not imply the existence of angular density of its zeros.

Suppose that

$$n_k(r) = \sum_{|a_j| \le r} e^{-ik \arg a_j}, \quad k \in \mathbb{Z}.$$

From Carathéodory-Levy theorem (see, for example, [3, p. 98]) it follows that zeros of $f \in \mathcal{H}_0(\lambda(r))$ have an angular density if and only if for all $k \in \mathbb{Z}$ there exists a finite limit

$$\lim_{r \to +\infty} \frac{n_k(r)}{v(r)} = \delta_k, \quad \delta_k \in \mathbb{C}.$$

In [4] necessary and sufficient conditions of regular growth for functions $\ln|f|$ and $\arg f$ in $L^p[0, 2\pi]$-metrics were established.

We denote by $\|\cdot\|$ – p-norm in the space $L^p[0, 2\pi]$,

$$G(\theta, f) = i\sum_{k\neq 0}\frac{\delta_k}{k}\,e^{ik\theta}, \quad V_1(r) = \int\limits_0^r \frac{V(t)}{t}dt.$$

Theorem B [4]. *If zeros of a function $f \in \mathcal{H}_0(\lambda(r))$ have an angular density, then for all $p \in [1, +\infty)$ conditions*

$$\left\|\frac{\ln|f(re^{i\theta})|}{V_1(r)} - \delta_0\right\|_p \to 0, \quad \left\|\frac{\arg f(re^{i\theta})}{V(r)} - G(\theta, f)\right\|_p \to 0, \quad r \to +\infty,$$

hold.

Conversely, suppose that $f \in \mathcal{H}_0(\lambda(r))$ and for some numbers $p \in [1, +\infty)$, $b_0 \in \mathbb{R}$ and a function $g \in L^p[0, 2\pi]$ conditions

$$\left\|\frac{\ln|f(re^{i\theta})|}{V_1(r)} - b_0\right\|_p \to 0, \quad r \to +\infty, \tag{1}$$

$$\left\|\frac{\arg f(re^{i\theta})}{V(r)} - g(\theta)\right\|_p \to 0, \quad r \to +\infty, \tag{2}$$

hold. Then zeros of f have an angular density, $b_0 = \delta_0$, $g(\theta) = G(\theta, f)$ for almost all $\theta \in [0, 2\pi]$.

2. Results

We establish such results concerning the relationship between strongly regular growth of entire functions of order zero and regular growth of their logarithm of modulus and argument in $L^p[0, 2\pi]$-metrics.

Theorem 1. *Suppose that $f \in \mathcal{H}_0(\lambda(r))$ and for some numbers $p \in [1, +\infty)$, $b_0 \in \mathbb{R}$ and function $g \in L^p[0, 2\pi]$ conditions* (1) *and* (2) *hold. Then $f \in \mathcal{H}_0^\star(\lambda(r))$, $b_0 = \delta_0$, $g(\theta) = -iH(\theta, f)$ for almost all $\theta \in [0, 2\pi]$.*

Proof. By Theorem B part 2 we get, that zeros of f have angular density. Using Theorem B once again (part 1) we obtain, that $b_0 = \delta_0$ and $g(\theta) = G(\theta, f)$. From Theorem A it follows that $f \in \mathcal{H}_0^\star(\lambda(r))$.

Let $\gamma = \gamma_0\hat{}\,- 2\pi$ be the periodic continuation on $\mathbb{R}$ of the function

$$\gamma_0(\theta) = i\sum_{k\neq 0}\frac{1}{k}\,e^{ik\theta} = -2\sum_{k=1}^{+\infty}\frac{1}{k}\sin(k\theta) = -2\frac{\pi-\theta}{2} = \theta - \pi, \quad 0 < \theta < 2\pi;$$

$\Delta(\theta)$ being the angular density of zeros of f, and $\gamma * d\Delta$ denoting convolution of 2π-periodic function γ with measure $d\Delta$. Then (see, for example, [3, p. 109]) we obtain

$$G(\theta, f) = i\sum_{k\neq 0} \frac{\delta_k}{k}\, e^{ik\theta} = \gamma * d\Delta.$$

Taking into account Theorem A, for $a \in \mathbb{R}$ we have

$$G(\theta, f) = \int_a^{a+2\pi-0} \gamma(\theta-\psi)d\Delta(\psi) = \int_{\theta-2\pi}^{\theta-0} (\theta-\psi-\pi)\hat{}\, d\Delta(\psi)$$
$$= \int_{\theta-2\pi}^{\theta-0} (\theta-\psi-\pi)d\Delta(\psi) = -iH(\theta, f).$$

This proves Theorem 1.

Let

$$\Gamma_m = \bigcup_{j=1}^{m} l_{\theta_j},\ \ l_{\theta_j} = \{z\colon \arg z = \theta_j\},\ 0 \le \theta_1 < \ldots < \theta_m < 2\pi,$$

be a finite system of rays.

Theorem 2. *If $f \in \mathcal{H}_0^{\star}(\lambda(r))$ and its zeros are arranged on a finite system of rays Γ_m, then for arbitrary $p \in [1, +\infty)$*

$$\left\| \frac{\ln|f(re^{i\theta})|}{V_1(r)} - \delta_0 \right\|_p \to 0, \quad r \to +\infty \tag{3}$$

and

$$\left\| \frac{\arg f(re^{i\theta})}{V(r)} + iH(r,\theta) \right\|_p \to 0, \quad r \to +\infty \tag{4}$$

holds, then $H(\theta, f) = iG(\theta, f)$ for almost all $\theta \in [0, 2\pi]$.

Proof. Since the conditions of Theorem A hold, then zeros of f have an angular density. Taking into account Theorem B, we obtain relations (3) and

$$\left\| \frac{\arg f(re^{i\theta})}{V(r)} - G(\theta, f) \right\|_p \to 0, \quad r \to +\infty.$$

Further, using Theorem 1, for almost all $\theta \in [0, 2\pi]$ we get $G(\theta, f) = -iH(\theta, f)$, and therefore, (4) holds. The theorem is proved.

References

[1] M. V. Zabolotskyi, *Strongly regular growth of entire functions of order zero*, Mat. Zametky **63**, no. 2 (1998), 196–208 (in Russian).

[2] M. V. Zabolotskyi, *An example of entire function of strongly regular growth*, Mat. Studii **13**, no. 2 (2000), 145–148.

[3] A. A. Kondratjuk, *Fourier series and meromorphic functions*, Lvov, Vyshcha Shkola, 1988 (in Russian).

[4] O. V. Bodnar and M. V. Zabolotskyi, *The criteria of regular growth of logarithm of modulus and argument of an entire function*, Ukr. Mat. Zh. **62**, no. 7 (2010), 885–893 (in Ukrainian).

Institute of Mathematics
National Academy of Sciences of Ukraine
Tereshchenkivs'ka vul. 3, UA-01601 Kyiv
Ukraine
e-mail: zel@imath.kiev.ua
max@imath.kiev.ua
bogdanklishchuk@mail.ru

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 13, 2012

O REGULARNYM WZROŚCIE FUNKCJI CAŁKOWITYCH RZĘDU ZERO

S t r e s z c z e n i e

Wyznaczamy związek między silnie regularnym wzrostem funkcji całkowitych rzędu zero a regularnym wzrostem ich logarytmu modułu i argumentu w metrykach przestrzeni $L^p[0, 2\pi]$.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 49–56

In memory of
Professor Promarz M. Tamrazov

Marta Tymofiivna Bordulyak

ON l-INDEX BOUNDEDNESS OF THE WEIERSTRASS σ-FUNCTION

Summary

We prove Sheremeta conjecture concerning l-index boundedness of the Weierstrass σ-function with $l(r) = r$, $r \geq 1$.

Keywords and phrases: entire function of bounded l-index; Weierstrass σ-function; Weierstrass ζ-function

Let Λ be a class of positive continuous functions l on $[0, +\infty)$ and Q be a class of functions $l \in \Lambda$ such that

$$l(x + O(1/l(x))) = O(l(x)) \qquad (x \to +\infty).$$

For $l \in \Lambda$ an entire function f is said [1] to be *of bounded l-index* if there exists $N \in \mathbb{Z}_+$ such that

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \leq \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)} : 0 \leq k \leq N\right\}$$

for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$. The least such integer N is called *l-index* of f. If $l(r) \equiv 1$ we obtain the definition [2] of entire function of bounded index.

It is known ([1], p. 71) that if $l \in Q$ and f is of bounded l-index then

$$\ln M_f(R) = O\left(\int_0^R l(r)dr\right), \quad R \to \infty,$$

where $M_f(R) = \max\{|f(z) : |z| = R\}$. Among the functions of bounded l-index with $l(r) = r^{\rho-1}$, $r \geq 1$, we may mention the Mittag-Leffler function $E_\rho(z)$ ([4]; [1], p. 94; [5]) and a function of Mittag-Leffler type $E_\rho(z;\mu)$ [6]. The boundedness of l-index of entire functions from Laguerre-Polya class is investigated in [7]. M. M. Sheremeta in 1996 conjectured that *the Weierstrass σ-function is of bounded l-index with $l(r) = r$, $r \geq 1$.* (About the Weierstrass σ-function see [3].) In this paper we give a proof of Sheremeta conjecture.

We need the following criterion. For entire function f with zeros (a_k) let

$$n(r, z_0, 1/f) = \sum_{|a_k - z_0| \leq r} 1, \quad r > 0,$$

and

$$G_q(f) = \bigcup_k \{z : |z - a_k| \leq \frac{q}{l(|a_k|)}\}, \quad q > 0.$$

Lemma 1. [1] *Let $l \in Q$. An entire function f is of bounded l-index iff:*

1) *for every $q > 0$ there exists $n^*(q) \in \mathbb{N}$ such that*

$$n\left(\frac{q}{l(|z_0|)}, z_0, \frac{1}{f}\right) \leq n^*(q)$$

for all $z_0 \in \mathbb{C}$ and

2) *for every $q > 0$ there exists $P(q) > 0$ such that*

$$\left|\frac{f'(z)}{f(z)}\right| \leq P(q) l(|z|)$$

for all $z \in \mathbb{C} \setminus G_q(f)$.

Theorem 1. *The Weierstrass σ-function*

$$\sigma(z) = \sigma(z|L) = z \prod_{a \in L \setminus \{0\}} \left(1 - \frac{z}{a}\right) \exp\left\{\frac{z}{a} + \frac{z^2}{2a^2}\right\},$$

where

$$L = \{a_{kj} : a_{kj} = 2k\omega_1 + 2j\omega_2,\ k, j \in \mathbb{Z}\},$$

$$\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}, \quad \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0,$$

is of bounded l-index with $l(r) = \max\{1, r\}$.

Proof. Since $l(r)$ is an increasing function for $r \geq 1$, condition 1) of Lemma 1 for lattice L holds.

Let us show the validity of condition 2), that is for all $z \in \mathbb{C} \setminus G_q(\sigma)$, $|z| \geq 1$,

$$\left|\frac{\sigma'(z)}{z\sigma(z)}\right| = \left|\frac{\zeta(z)}{z}\right| = \left|\frac{1}{z^2} + \sum_{a \in L \setminus \{0\}} \frac{z}{a^2(z-a)}\right| \leq P(q), \tag{1}$$

where $\zeta(z)$ is the Weierstrass ζ-function.

We restrict us to the case when the lattice has generators $2\omega_1 = 1$, $2\omega_2 = i$; the general case can be obtained similarly for

$$2\omega_1 = 1, \quad 2\omega_2 = \lambda e^{i\theta}, \quad \lambda > 0, \quad \theta \in (0, \pi/2]$$

and then we have to use the homogeneity property of $\sigma(z)$.

We fix $z = n + \varepsilon_1 + (m + \varepsilon_2)i$, $m, n \in \mathbb{Z}$, $|\varepsilon_1| \le 1/2$, $|\varepsilon_2| \le 1/2$. Without loss of generality we can set $m \ge 1$, $n \ge 1$, $|z| \ge 3$ and let $q > 0$ be so small that the disks in $G_q(\sigma)$ are pairwise disjoint.

Let us define

$$f(\tau) = f_z(\tau) = \frac{z}{\tau^2(\tau - z)}.$$

Then

$$\sum_{a \in L\setminus\{0\}} \frac{z}{a^2(z-a)} = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} f(a_{kj}) + \sum_{j=-\infty, j\ne 0}^{\infty} f(a_{k0}) + \sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} f(a_{kj})$$
$$+ \sum_{k=-\infty}^{\infty} f(a_{km}) + \sum_{j=m+1}^{\infty} \sum_{k=-\infty}^{\infty} f(a_{kj}) = S_1 + S_2 + S_3 + S_4 + S_5.$$

We shall estimate each sum using transition to the corresponding integral by the trapezoidal rule. For S_2 we consider the curve

$$C_2 = (-\infty; -1] \cup C_0 \cup [1; +\infty),$$

where $C_0 = \left\{\tau : \tau = \tau(t) = t + \frac{i}{2}(1 + \cos \pi t), t \in [-1; 1]\right\}$. Then

$$\int\limits_{C_0} f(\tau) d\tau = \int\limits_{-1}^{1} f(\tau(t))(1 - \frac{\pi i}{2} \sin \pi t) dt = \frac{f(-1)}{2} + \frac{f(1)}{2} + f(i) + R_0, \tag{2}$$

where using the trapezoidal rule the remainder term for a complex-valued function has an estimate ([8], p. 162):

$$|R_0| \le \frac{2 \cdot 2^3}{12 \cdot 2^2} \max_{t \in [-1;1]} |F''(t)|, \quad F(t) = f(\tau(t))\tau'(t).$$

Since $f(\tau) = \dfrac{1}{\tau^2} + \dfrac{1}{z\tau} + \dfrac{1}{z(z-\tau)}$ and $|z - \tau| \ge 1$, $|\tau| \ge 1/2$ for $\tau \in C_0$, we have

$$|R_0| \le \frac{1}{3}\left(\frac{\pi^3}{2} + 1\right) \max_{\tau \in C_0} \{|f''(\tau)| + 3|f'(\tau)| + |f(\tau)|\} \le K_1 + \frac{K_2}{|z|},$$

($K_1, K_2, \ldots$ are positive constants). Thus from (2) we get

$$\frac{f(-1)}{2} + \frac{f(1)}{2} = \int\limits_{C_0} f(\tau) d\tau + \frac{z}{z-i} - R_0 = \int\limits_{C_0} f(\tau) d\tau + O(1), \quad |z| \ge 3. \tag{3}$$

Similarly, for S_4 we consider the curve

$$C_4 = (-\infty + mi; n - 2 + mi] \cup \tilde{C} \cup [n + 2 + mi; +\infty + mi),$$

where $\tilde{C} = \left\{\tau : \tau = \tau(t) = t + n + \frac{i}{2}(1 + \cos\frac{\pi t}{2}) + mi, t \in [-2; 2]\right\}$. Then

$$\int_{\tilde{C}} f(\tau)d\tau = \int_{-2}^{2} f(\tau(t))(1 - \frac{\pi i}{4}\sin\frac{\pi t}{2})dt$$
$$= \frac{f(n-2+mi)}{2} + \frac{f(n+2+mi)}{2} + f(n-1+i(m+\frac{1}{2}))(1+\frac{\pi i}{4})$$
$$+f(n+i(m+1)) + f(n+1+i(m+\frac{1}{2}))(1-\frac{\pi i}{4}) + \tilde{R}, \tag{4}$$

where

$$(|\tau| \geq 1, \quad |z-\tau| \geq 1/5, \quad |z| \geq 3, \quad F(t) = f(\tau(t))\tau'(t))$$

$$|\tilde{R}| \leq \frac{2 \cdot 4^3}{12 \cdot 4^2} \max_{t\in[-2;2]} |F''(t)| = O(1).$$

Since

$$|f(\tau)| \leq \frac{|z|}{|\tau - z|(|z| - |\tau - z|)^2},$$

and therefore

$$\left|\left(1+\frac{\pi i}{4}\right) f\left(n-1+i\left(m+\frac{1}{2}\right)\right)\right| \leq \frac{2\sqrt{2}|z|}{(|z|-2)^2},$$
$$\left|\left(1-\frac{\pi i}{4}\right) f\left(n+1+i\left(m+\frac{1}{2}\right)\right)\right| \leq \frac{2\sqrt{2}|z|}{|z|^2},$$
$$|f(n+i(m+1))| \leq \frac{2|z|}{(|z|-2)^2},$$

then from (4) we deduce

$$\frac{f(n-2+mi)}{2} + \frac{f(n+2+mi)}{2} = \int_{\tilde{C}} f(\tau)d\tau + O(1), \quad |z| \geq 3. \tag{5}$$

Let us consider S_1. We shall use the trapezoidal rule to inner sum for each segment of the form $[k+ij; k+1+ij]$:

$$\sum_{k=-\infty}^{\infty} f(a_{kj}) = \sum_{k=-\infty}^{\infty} \left(\int\limits_{k+ij}^{k+1+ij} f(\tau)d\tau + R_k^{(j)} \right) = \int\limits_{-\infty+ij}^{\infty+ij} f(\tau)d\tau + \sum_{k=-\infty}^{\infty} R_k^{(j)}.$$

Since the function $f(\tau)$ has a third-order zero at infinity, then according to the residue theorem we have

$$\int\limits_{-\infty+ij}^{\infty+ij} f(\tau)d\tau = 2\pi i \left(\mathrm{res}_{\tau=0}\frac{\mathrm{z}}{\tau^2(\mathrm{z}-\tau)} + \mathrm{res}_{\tau=\mathrm{z}}\frac{\mathrm{z}}{\tau^2(\mathrm{z}-\tau)}\right) = 0.$$

Now we estimate remainder terms ($|\tau| \geq 1$).

$$\begin{aligned}
|R_k^{(j)}| \leq & \frac{2}{12} \max_{\tau\in[k+ij;k+1+ij]} \left| \frac{6}{\tau^4} + \frac{2}{z\tau^3} + \frac{2}{z(z-\tau)^3} \right| \\
\leq & \frac{7}{6} \max_{x\in[k;k+1]} \left(\frac{1}{(x^2+j^2)^{3/2}} + \frac{1}{((x-n-\varepsilon_1)^2+(j-m-\varepsilon_2)^2)^{3/2}} \right) \\
\leq & \frac{7}{6} \max_{x\in[k;k+1]} \left(\frac{1}{(x^2+1)^{3/4}|j|^{3/2}} + \frac{1}{((x-n-\varepsilon_1)^2+1)^{3/4}|j|^{3/2}} \right);
\end{aligned}$$

therefore

$$\sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} |R_k^{(j)}| \leq K_3 \left(\sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \right) \left(\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1/2)^{3/4}} \right) = O(1).$$

Hence

$$S_1 = \sum_{j=-\infty}^{-1} \left(\int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau + \sum_{k=-\infty}^{\infty} R_k^{(j)} \right) = O(1), \quad |z| \geq 3. \tag{6}$$

For S_2 with $a_{k0} \in C_2$, by analogy, in view of (5), we obtain

$$\begin{aligned}
S_2 = & \left(\sum_{k=-\infty}^{-1} f(a_{k0}) + \frac{f(-1)}{2} \right) + \left(\frac{f(-1)}{2} + \frac{f(1)}{2} \right) + \left(\frac{f(1)}{2} + \sum_{k=1}^{\infty} f(a_{k0}) \right) \\
= & \int_{-\infty}^{-1} f(\tau)d\tau + \sum_{k=-\infty}^{-2} R_k^{(0)} + \int_{C_0} f(\tau)d\tau + O(1) + \int_{1}^{+\infty} f(\tau)d\tau + \sum_{k=1}^{\infty} R_k^{(0)} \\
= & \int_{C_2} f(\tau)d\tau + \sum_{k=-\infty, k\neq -1,0}^{\infty} R_k^{(0)} + O(1).
\end{aligned}$$

Since $\tau = 0$ is located under the curve C_2, we have

$$\int_{C_2} f(\tau)d\tau = 2\pi i \operatorname{res}_{\tau=\mathrm{z}} \frac{\mathrm{z}}{\tau^2(\mathrm{z}-\tau)} = -\frac{2\pi \mathrm{i}}{\mathrm{z}}.$$

Farther ($k \neq -1, 0$, $|x| \geq 1$),

$$|R_k^{(0)}| \leq \frac{1}{6} \max_{x\in[k;k+1]} \left(\frac{7}{|x|^3} + \frac{2}{3|x-z|^3} \right), \qquad \sum_{k=-\infty, k\neq -1;0}^{\infty} R_k^{(0)} \leq K_4 \sum_{k=2}^{\infty} \frac{1}{k^3},$$

and

$$S_2 = -\frac{2\pi i}{z} + O(1) = O(1), \quad |z| \geq 3. \tag{7}$$

We estimate S_3 by analogy to S_1:

$$\sum_{k=-\infty}^{\infty} f(a_{kj}) = \int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau + \sum_{k=-\infty}^{\infty} R_k^{(j)},$$

$$\int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau = 2\pi i \mathrm{res}_{\tau=\mathrm{z}} \frac{\mathrm{z}}{\tau^2(\mathrm{z}-\tau)} = -\frac{2\pi \mathrm{i}}{\mathrm{z}},$$

$$|R_k^{(j)}| \le \frac{7}{6} \max_{x\in[k;k+1]} \left(\frac{1}{(x^2+j^2)^{3/2}} + \frac{1}{((x-n-\varepsilon_1)^2+(j-m-\varepsilon_2)^2)^{3/2}} \right)$$
$$\le \frac{7}{6} \max_{x\in[k;k+1]} \left(\frac{1}{(x^2+1)^{3/4}|j|^{3/2}} + \frac{1}{((x-n-\varepsilon_1)^2+1/2)^{3/4}(j-m-\varepsilon_2)^{3/2}} \right)$$

and

$$\sum_{j=1}^{m-1} \sum_{k=-\infty}^{\infty} |R_k^{(j)}| \le K_5 \left(\sum_{j=1}^{m-1} \frac{1}{j^{3/2}} \right) \left(\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1)^{3/4}} \right)$$
$$+K_6 \left(\sum_{j=1}^{m-1} \frac{1}{(k^2+1/2)^{3/4}} \right) \left(\sum_{d=1}^{m-1} \frac{1}{(d-1/2)^{3/2}} \right) = O(1);$$

hence for $|z| \ge 3$

$$S_3 = \sum_{j=1}^{m-1} \left(\int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau + \sum_{k=-\infty}^{\infty} R_k^{(j)} \right) = -\frac{(m-1)2\pi i}{z} + O(1) = O(1). \tag{8}$$

Let us consider S_4. Here $a_{km} \in C_4$ and C_4 contains a neighbourhood of the point z. By analogy to S_2, in view of (5), we get

$$S_4 = \left(\sum_{k=-\infty}^{n-3} f(a_{km}) + \frac{f(n-2+im)}{2} \right) + \left(\frac{f(n-2+mi)}{2} + \frac{f(n+2+mi)}{2} \right)$$
$$+ \sum_{k=n-1}^{n+1} f(k+im) + \left(\frac{f(n+2+im)}{2} + \sum_{k=n+3}^{\infty} f(a_{km}) \right)$$
$$= \int_{C_4} f(\tau)d\tau + \sum_{k=-\infty, k\ne n-2,n-1,n,n+1}^{\infty} R_k^{(m)} + O(1),$$

because

$$|z-\tau| \ge \frac{q}{|\tau|}$$

for $\tau = k+im \in C_4$ and therefore

$$\left| \sum_{k=n-1}^{n+1} f(k+im) \right| \le \sum_{k=n-1}^{n+1} \frac{|z|}{q|k+im|} \le \frac{K_7}{q}.$$

Since $\tau = z$ is located under the curve C_4, we have $\int_{C_4} f(\tau)d\tau = 0$. Farther ($k \neq n-2, n-1, n, n+1$):

$$|R_k^{(m)}| \leq \frac{1}{6} \max_{x\in[k;k+1]} \left(\frac{7}{(x^2+m^2)^{3/2}} + \frac{2}{3|x-n-\varepsilon_1|^3} \right),$$

$$\sum_{k=-\infty, k\neq n-2,n-1,n,n+1}^{\infty} R_k^{(m)} \leq K_8 \left(\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1)^{3/2}} + \sum_{d=2}^{\infty} \frac{1}{(d-1/2)^3} \right) = O(1)$$

and for $|z| \geq 3$, $z \notin G_q(\sigma)$,

$$S_4 = O(1). \tag{9}$$

Finally, we consider S_5. As for S_1, we have

$$\sum_{k=-\infty}^{\infty} f(a_{kj}) = \int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau + \sum_{k=-\infty}^{\infty} R_k^{(j)}, \quad \int_{-\infty+ij}^{\infty+ij} f(\tau)d\tau = 0,$$

$$|R_k^{(j)}| \leq \frac{7}{6} \max_{x\in[k;k+1]} \left(\frac{1}{(x^2+1)^{3/4} j^{3/2}} + \frac{1}{((x-n-\varepsilon_1)^2+1/2)^{3/4}(j-m-\varepsilon_2)^{3/2}} \right),$$

$$\sum_{j=m+1}^{\infty} \sum_{k=-\infty}^{\infty} |R_k^{(j)}| \leq K_9 \left(\left(\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1)^{3/4}} \right) \left(\sum_{j=m+1}^{\infty} \frac{1}{j^{3/2}} \right) + \left(\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1/2)^{3/4}} \right) \left(\sum_{d=1}^{\infty} \frac{1}{(d-1/2)^{3/2}} \right) \right) = O(1);$$

hence for $|z| \geq 3$

$$S_5 = O(1). \tag{10}$$

Combining all the estimates (6)–(10), we obtain (1). The theorem is proved.

References

[1] M. Sheremeta, *Analytic Functions of Bounded l-Index*, Monograph Series. 6, VNTL Publishers, 1999, 141 p.

[2] B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math. 2, Amer. Math. Soc., Providence, Rhode Island, 1968, 298–307.

[3] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 2., Fizmatgiz, Moskva 1963.

[4] S. M. Shah, *Entire function satisfying a linear differential equation*, J. Math. Mech. **18** (1968/69), 131–136.

[5] A. A. Gol'dberg, *An estimate of logarithmic derivative modulus of Mittag-Leffler function and its application*, Matem. Studii **5** (1996), 21–30 (in Ukrainian).

[6] M. T. Bordulyak, *Boundedness of value distribution of Mittag-Leffler function*, Matem. Studii **9**, no. 2 (1998), 177–186 (in Ukrainian).

[7] M. M. Sheremeta and M. T. Bordulyak, *Boundedness of l-index of Laguerra-Polya entire functions*, Ukr. Math. J. **55**, no. 1 (2003), 91–99 (in Ukrainian).

[8] G. M. Fichtenholz, *Differential and integral calculus*, vol. 2, Izd. Nauka, Moskva 1970 (in Russian).

Faculty of Mechanics and Mathematics,
Lviv Ivan Franko National University
Universytets'ka 1, UA-79000 Lviv
e-mail: mbordulyak@yahoo.com

Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 13, 2012

O OGRANICZONOŚCI l-INDEKSU FUNKCJI σ WEIERSTRASSA

S t r e s z c z e n i e

Wykazujemy, że jest spełniona hipoteza Szeremety o ograniczoności funkcji σ Weierstrassa dla $l(r) = r$, $r \geq 1$.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 57–63

In memory of
Professor Promarz M. Tamrazov

Andriy L. Targonskii

EXTREMAL PROBLEMS FOR PARTIALLY NONOVERLAPPING DOMAINS ON EQUIANGULAR SYSTEM OF POINTS

Summary

In this note derivation of the sharp estimate for inner radius of partially non-overlapping domains is given. The problems of such type arise for the first time in M. A. Lavrentiev's paper [1]. The result of this work was generalized and strengthened in [2–15]. In papers [7, 8, 10] general systems of points were introduced, called n-radial systems of points. In this note we generalize some results of [7].

Keywords and phrases: geometric function theory, equiangular system of points

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of possitive integers and real numbers, respectively, $\mathbb{C}$ – the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ – the Riemannian sphere, $\mathbb{R}_+ = (0, \infty)$.

For a fixed number $n \in \mathbb{N}$ a system points

$$A_n = \{a_k \in \mathbb{C} : k = \overline{1, n}\},$$

forms an *n-equiangular system of points* if for all $k = \overline{1, n}$ the following relation is satisfied:

$$\arg a_k = \frac{2\pi}{n}(k-1),\ k = \overline{1, n}. \tag{1}$$

Such a system will be considered an angular domain:

$$P_k = \left\{ w \in \mathbb{C} : \frac{2\pi}{n}(k-1) < \arg w < \frac{2\pi}{n}k \right\}, \quad k = \overline{1, n}.$$

Here we follow the notation of the work [7].

For arbitrary n-equiangular system of points *controlling functional* we consider the

$$\mu\left(A_n\right) := \prod_{k=1}^{n} \chi\left(\left|\frac{a_{k+1}}{a_k}\right|^{\frac{n}{4}}\right) \cdot |a_k|,$$

where

$$\chi(t) = \frac{1}{2}(t + \frac{1}{t}), \quad t \in \mathbb{R}_+.$$

Let D, $D \subset \overline{\mathbb{C}}$ be an arbitrary open set and $w = a \in D$. Let $D(a)$ be the connected component of D containing a. For arbitrary n-equiangular system of points

$$A_n = \{a_k\}_{k=1}^{n}$$

and open set D, $A_n \subset D$, we define $D_k(a_p)$ being the connected component of $D(a_p) \bigcap \overline{P_k}$, that contains the point

$$a_p, \quad k = \overline{1, n}, \quad p = k, k+1, \quad s = \overline{1, m}, \quad a_{n+1} := a_1.$$

Let $D_k(0)$ (resp. $D_k(\infty)$) denote the connected component $D(0) \bigcap \overline{P_k}$ (resp. $D(\infty) \bigcap \overline{P_k}$) that contains the point $w = 0$ (resp. $w = \infty$).

We say that the open set D, $\{0, \infty\} \cup A_n \subset D$ satisfies the conditions of non-overlapping relatively to the n-equiangular system of points A_n if it satisfies the condition

$$\begin{aligned} &\Big[D_k(a_k) \bigcap D_k(a_{k+1})\Big] \bigcup \Big[D_k(0) \bigcap D_k(a_k)\Big] \bigcup \Big[D_k(0) \bigcap D_k(\infty)\Big] \bigcup \\ &\bigcup \Big[D_k(\infty) \bigcap D_k(a_k)\Big] \bigcup \Big[D_k(\infty) \bigcap D_k(a_{k+1})\Big] \bigcup \Big[D_k(0) \bigcap D_k(a_{k+1})\Big] = \varnothing, \end{aligned} \tag{2}$$

$k = \overline{1, n}$, on all angular domains $\overline{P_k}$.

For a system of domains

$$\{B_k\}_{k=1}^{n}, \qquad k = \overline{1, n},$$

we define the *system of partially non-overlapping domains*, if

$$D := \bigcup_{k=1}^{n} B_k, \tag{3}$$

is an open set, by the condition (2).

Let $r(B; a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ [4–6, 14]. In this note we study the following

Problem. Let $n \in \mathbb{N}$, $n \geq 2$. We wish to find the maximum functional

$$\left(r\left(B_0\,;\,0\right) \cdot r\left(B_\infty\,;\,\infty\right)\right)^{\gamma} \cdot \prod_{k=1}^{n} r\left(B_k\,;\,a_k\right),$$

where $A_n = \{a_k\}_{k=1}^{n}$ is an arbitrary n-equiangular system of points realizing the relation (1), and $\{B_0, \{B_k\}_{k=1}^{n}, B_\infty\}$ – an arbitrary set of partially nonoverlapping domains satisfying the condition (3), $0 \in B_0$, $\infty \in B_\infty$, $a_k \in B_k$, and all a_k being extremal, $k = \overline{1,n}$.

This problem is solved in [7] for an open set. Our result reads:

Theorem 1. *Let $\gamma \in \mathbb{R}_+$, $n \in \mathbb{N}$, $n \geq 3$. Then for an arbitrary n-equiangular system of points* (1)*, the condition*

$$\mu\left(A_n\right) = 1$$

is satisfied and for an arbitrary set partially nonoverlapping domains $\{B_0, B_k, B_\infty\}$, the satisfied condition (3)*,*

$$a_k \in B_k \subset \overline{\mathbb{C}}, \quad k = \overline{1,n}, \quad 0 \in B_0 \subset \overline{\mathbb{C}}, \quad \infty \in B_\infty \subset \overline{\mathbb{C}},$$

$$\begin{aligned} &\left(r\left(B_0\,;\,0\right) \cdot r\left(B_\infty\,;\,\infty\right)\right)^\gamma \cdot \prod_{k=1}^{n} r\left(B_k\,;\,a_k\right) \\ &\leq \left(r\left(B_0^0\,;\,0\right) \cdot r\left(B_\infty^0\,;\,\infty\right)\right)^\gamma \cdot \prod_{k=1}^{n} r\left(B_k^0\,;\,a_k^0\right). \end{aligned} \tag{4}$$

The equality in (4) *is obtained when points $\{a_k^0\}$ and domains $\{B_0^0, B_k^0, B_\infty^0\}$, $k = \overline{1,n}$ are the poles and the circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + \left(n^2 - 2\gamma\right) w^n + \gamma}{w^2\left(w^n - 1\right)^2} dw^2, \tag{5}$$

respectively.

Corollary 1. *If the assumptions of the theorem are satisfied, then we have*

$$\begin{aligned} &\left(r\left(B_0\,;\,0\right) \cdot r\left(B_\infty\,;\,\infty\right)\right)^\gamma \cdot \prod_{k=1}^{n} r\left(B_k\,;\,a_k\right) \\ &\leq \left(\frac{4}{n}\right)^n \cdot \left(\frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{4\gamma}{n^2}}}{\left|\frac{2\sqrt{\gamma}}{n} - 1\right|^{\left(\frac{2\sqrt{\gamma}}{n}-1\right)^2} \cdot \left(\frac{2\sqrt{\gamma}}{n} + 1\right)^{\left(\frac{2\sqrt{\gamma}}{n}+1\right)^2}} \right)^{\frac{1}{2}n}. \end{aligned} \tag{6}$$

The equality in (6) *holds, when points $\{a_k\}$ and domains $\{B_0, B_k, B_\infty\}$, $k = \overline{1,n}$ are the poles and the circular domains of the quadratic differential* (5)*, respectively.*

Proof of the theorem. Take an open set D, define (3) and assume the condition (2).

Then,

$$B_0, B_k, B_\infty \subset D, \quad k = \overline{1, n}. \tag{7}$$

We take into account that

$$\begin{gathered} r\left(B_0, 0\right) \leq r\left(D, 0\right), \quad r\left(B_\infty, \infty\right) \leq r\left(D, \infty\right), \\ r\left(B_k, a_k\right) \leq r\left(D, a_k\right), \quad k = \overline{1, n}. \end{gathered} \tag{8}$$

Indeed, a theorem given in Hayman's monograph [14, p. 99] states that if

$$a \in B_1 \subset B_2 \subset \overline{\mathbb{C}},$$

then

$$r\left(B_1; a\right) \leq r\left(B_2; a\right).$$

On the other hand, Theorem 5.3.2 in the monograph [7, p. 264] staes that if $n \in \mathbb{N}$, $\gamma \in (0; \infty)$, then for arbitrary n-equiangular system of points $A_n = \{a_k\}_{k=1}^n$

$$\mu\left(A_n\right) = 1,$$

and for an arbitrary open set D, $\{0, \infty\} \cup A_n \subset D \subset \overline{\mathbb{C}}$ with (3), we have the inequality

$$\begin{gathered} \left(r\left(D\,;\,0\right) \cdot r\left(D\,;\,\infty\right)\right)^\gamma \cdot \prod_{k=1}^{n} r\left(D\,;\,a_k\right) \\ \leq \left(r\left(D_0^0\,;\,0\right) \cdot r\left(D_\infty^0\,;\,\infty\right)\right)^\gamma \cdot \prod_{k=1}^{n} r\left(D_k^0\,;\,a_k^0\right). \end{gathered}$$

The equality is attained in this inequality for

$$D^{(0)} = \bigcup_{k=0}^{n+1} D_k^{(0)} \quad \text{and} \quad A_n^{(0)} = \left\{a_k^{(0)}\right\}_{k=1}^{n},$$

where $\left\{D_k^{(0)}\right\}_{k=0}^{n+1}$ are circular domains, while $A_n^{(0)} \cup \{0; \infty\}$ are the poles of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + \left(n^2 - 2\gamma\right) w^n + \gamma}{w^2\left(w^n - 1\right)^2} dw^2,$$

and

$$0 \in D_0^{(0)}, \quad \infty \in D_{n+1}^{(0)}, \quad a_k^{(0)} \in D_k^{(0)}, \quad k = 1, 2, ..., n.$$

Besides, Corollary 4.1.3 in the same reference [7, p. 169] says that if

$$\alpha_1, \alpha_2, \rho, \rho_1, \rho_2 \in (0; \infty), \quad a_0 = 0, \quad a_\infty = \infty, \quad a_1 = i\rho_1, \quad a_2 = -i\rho_2.$$

then, for arbitrary nonoverlapping domains B_q, $a_q \in B_q \subset \overline{\mathbb{C}}$, $q \in \{0,1,2,\infty\}$, we have the inequality

$$(r\,(B_0\,;\,0)\cdot r\,(B_\infty\,;\,\infty))^{\alpha_1}\cdot\left(\frac{r\,(B_1\,;\,a_1)\cdot r\,(B_2\,;\,a_2)}{|a_1-a_2|^2}\right)^{\alpha_2}$$

$$\leq\left(r\left(B_0^{(0)}\,;\,0\right)\cdot r\left(B_\infty^{(0)}\,;\,\infty\right)\right)^{\alpha_1}\cdot\left(\frac{r\left(B_1^{(0)}\,;\,i\rho\right)\cdot r\left(B_2^{(0)}\,;\,-i\rho\right)}{4\rho^2}\right)^{\alpha_2},$$

$B_q^{(0)}$, $q \in \{0,1,2,\infty\}$ being in relation to the circular domains quadratic differential

$$Q(w)dw^2=-\frac{w^4+2\left(1-2\frac{\alpha_2}{\alpha_1}\right)\rho^2w^2+\rho^4}{w^2\left(w^2+\rho^2\right)^2}dw^2,$$

$$0\in B_0^{(0)},\quad \infty\in B_\infty^{(0)},\quad i\rho\in B_1^{(0)},\quad -i\rho\in B_2^{(0)}.$$

Finally, we take into account Lemma 4.1.2 in the same reference [7, p. 189]. In the Corollary 4.1.3 of [7] (already quoted) we have the equalities

$$r\left(B_0^{(0)}\,;\,0\right)=\rho\cdot\frac{\sqrt{\alpha_1}\left|\sqrt{\alpha_2}-\sqrt{\alpha_1}\right|^{\frac{1}{2}\left(\sqrt{\frac{\alpha_2}{\alpha_1}}-1\right)}}{\left(\sqrt{\alpha_2}+\sqrt{\alpha_1}\right)^{\frac{1}{2}\left(\sqrt{\frac{\alpha_2}{\alpha_1}}+1\right)}},$$

$$r\left(B_\infty^{(0)}\,;\,\infty\right)=\frac{1}{\rho}\cdot\frac{\sqrt{\alpha_1}\left|\sqrt{\alpha_2}-\sqrt{\alpha_1}\right|^{\frac{1}{2}\left(\sqrt{\frac{\alpha_2}{\alpha_1}}-1\right)}}{\left(\sqrt{\alpha_2}+\sqrt{\alpha_1}\right)^{\frac{1}{2}\left(\sqrt{\frac{\alpha_2}{\alpha_1}}+1\right)}},$$

$$r\left(B_k^{(0)}\,;\,(-1)^{k+1}i\rho\right)=2\rho\cdot\frac{\sqrt{\alpha_2}\left|\sqrt{\alpha_2}-\sqrt{\alpha_1}\right|^{\frac{1}{2}\left(\sqrt{\frac{\alpha_1}{\alpha_2}}-1\right)}}{\left(\sqrt{\alpha_2}+\sqrt{\alpha_1}\right)^{\frac{1}{2}\left(\sqrt{\frac{\alpha_1}{\alpha_2}}+1\right)}},\quad k=1,2,$$

$$\left(r\left(B_0^{(0)}\,;\,0\right)\cdot r\left(B_\infty^{(0)}\,;\,\infty\right)\right)^{\alpha_1}\cdot\left(\frac{r\left(B_1^{(0)}\,;\,i\rho\right)\cdot r\left(B_2^{(0)}\,;\,-i\rho\right)}{4\rho^2}\right)^{\alpha_2}$$

$$=\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{\left|\sqrt{\alpha_1}-\sqrt{\alpha_2}\right|^{(\sqrt{\alpha_1}-\sqrt{\alpha_2})^2}\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)^{(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2}}.$$

Altogether (8) gives

$$(r\,(B_0\,;\,0)\cdot r\,(B_\infty\,;\,\infty))^{\gamma}\cdot\prod_{k=1}^{n}r\,(B_k\,;\,a_k)$$

$$\leq(r\,(D\,;\,0)\cdot r\,(D\,;\,\infty))^{\gamma}\cdot\prod_{k=1}^{n}r\,(D\,;\,a_k)\,.$$

The use of Theorem 5.3.2 in [7], completes the proof of our theorem, while the use of Lemma 4.1.2 in [7] and Corollary 4.1.3 in [7] completes the proof of our corollary.

References

[1] M. A. Lavrentiev, *On the theory of conformal mappings*, Trudy Fiz.-Mat. Inst. AN USSR **5** (1934), 159–245 (in Russian).

[2] G. M. Goluzin, *Geometric Theory of Functions of a Complex Varieble,* Izd. Nauka, Moskva 1966, 628 pp. (in Russian).

[3] G. P. Bakhtina, *The variational methods and quadratic differential in the problems for nonoverlapping domains*, Abstract for the Kand. Diss., An USSR, Kiev 1975, 11 pp. (in Russian).

[4] V. N. Dubinin, *The partition transformation domains and the problems about extremal partition*, Notes Scientific Sem. Leningrad, Mat. Inst. AN USSR **168** (1988), 48–66 (in Russian).

[5] V. N. Dubinin, *The symmetrization method in geometric theory of functions od the complex varieble*, Success Mat. Sciences **49**, no. 1 (295) (1994), 3–76 (in Russian).

[6] V. N. Dubinin, *The asymptotics modulus degenerate capacitors and some their applications*, Notes Scientific Sem. POMI **237** (1997), 56–73 (in Russian).

[7] A. K. Bakhtin, G. P. Bakhtina, Yu. B. Zelinskii, *Topological-algebraical Structures and Geometrics Methods in Complex Analysis*, Works the Inst. of Math. NAS Ukr. 73, Kyiv 2008, 308 pp. (in Russian).

[8] A. K. Bakhtin, *The inequalities for inner radius nonoverlapping domains and open sets*, UMZ **61**, no. 5 (2009), 596–610 (in Ukrainian).

[9] V. N. Dubinin, *On quadratic forms, the generate Green's function and Robben function*, Mat. Coll. **200**, no. 10 (2009), 25–38 (in Russian).

[10] A. K. Bakhtin and A. L. Targonskii, *The extremal problems and quadratic differentials*, Nonlinear Vibrations **8**, no. 3 (2005), 298–303 (in Russian).

[11] A. L. Targonskii, *The extremal problems on partially nonoverlapping domains on the Riemannian sphere*, DANU **9** (2008), 31–36 (in Russian).

[12] G. V. Kuzmina, *The problems on extremal partitioning of the Riemannian sphere*, Notes Scientific Sem. POMI **276** (2001), 253–275 (in Russian).

[13] E. G. Emel'yanov, *The problem on a maximum product degree conformal radius of the nonoverlapping domains*, Notes Scientific Sem. POMI **286** (2002), 103–114 (in Russian).

[14] W. K. Hayman, *Multivalent Functions*, Cambridge Univ. Press, Cambridge 1958, 180 pp.

[15] J. A. Jenkins, *Univalent Functions and Conformal Mappings*, Springer, Berlin-Göttingen-Heidelberg 1958, 256 pp.

Zhytomyr State University
Velyka Berdychivska Str. 40
UA-10008 Zhytomyr
Ukraine
e-mail: targonsk@zu.edu.ua

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 6, 2012

ZAGADNIENIA EKSTREMALNE DLA CZĘŚCIOWO NIEZACHODZĄCYCH NA SIEBIE OBSZARÓW NA UKŁADZIE RÓWNOKĄTNYCH PUNKTÓW

S t r e s z c z e n i e

W niniejszej nocie podajemy ostre oszacowanie wewnętrznego promienia dla częściowo niezachodzących na siebie obszarów. Zagadnienia tego typu pojawiają się po raz pierwszy w pracy M. A. Ławrentiewa [1]. Następnie pojawiło się szereg uogólnień [2–15]. W pracach [7, 8, 10] wprowadzone są ogólne układy punktów, noszące nazwę układów n-radialnych punktów. W naszej nocie uogólniamy niektóre z tych rezultatów.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 65–77

In memory of
Professor Promarz M. Tamrazov

Bogumiła Kowalczyk and Adam Lecko

RADIUS PROBLEM IN CLASSES OF POLYNOMIAL CLOSE-TO-CONVEX FUNCTIONS I

Summary

In this paper we study some radius problem in the classes of polynomial close-to-convex functions, namely the radius of a reciprocal dependence of the classes concerned. We prove some basic theorems in this subject and discuss a method for calculation of radii for special subclasses of polynomial close-to-convex functions.

For $\delta \in [-\pi/2, \pi/2]$, $\mu_i \in \mathbb{N}$ and distinct points $\xi_i \in \overline{\mathbb{D}} \setminus \{0\}$, $i = 1, \ldots, j$, we study some radius problem in the classes of functions f analytic in the unit disk $\mathbb{D}$ standardly normalized, satisfying the condition

$$\operatorname{Re}\left\{e^{i\delta}\prod_{i=1}^{j}(1-\xi_i z)^{\mu_i} f'(z)\right\} \geq 0, \quad z \in \mathbb{D}.$$

Keywords and phrases: Polynomial close-to-convex functions, radius problem, functions of bounded turning, functions convex in the direction of the imaginary axis

1. Introduction

The famous criterium of univalence due to Noshiro [9] and Warschawski [12] (see also [1, p. 88]) states that if $\delta \in [-\pi/2, \pi/2]$ and $f \in \mathcal{A}$, where $\mathcal{A}$ stands for the class of all analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ standardly normalized by $f(0) = f'(0) - 1 = 0$, satisfies the inequality

$$\operatorname{Re}\left\{e^{i\delta} f'(z)\right\} \geq 0, \quad z \in \mathbb{D}, \tag{1}$$

then f is univalent in $\mathbb{D}$. Functions having such a property are called of *bounded turning with argument* δ and form the class denoted usually as $\mathcal{P}'(\delta)$. In 1936 Robertson [10] proposed the condition

$$\operatorname{Re}\left\{e^{i\delta}\left(1-e^{-i(\mu-\nu)}z\right)\left(1-e^{-i(\mu+\nu)}z\right)f'(z)\right\}\geq 0, \quad z\in\mathbb{D}, \tag{2}$$

where $\mu,\nu\in[0,\pi]$ and $\delta=\mu-\pi/2$, as the analytical characterization of the class $\mathcal{CV}(i)$ of univalent functions called *convex in the direction of the imaginary axis* introduced by himself and he proved the equivalence partially. Further studies in this subject were continued by Hengartner and Schober [2], Royster and Ziegler [11] (see also [1, p. 203]), and the others. To verify Robertson's conjecture Hengartner and Schober [2] distinguished in $\mathcal{CV}(i)$ the subclass of functions f such that

$$\operatorname{Re}\left\{\left(1-z^2\right)f'(z)\right\}\geq 0, \quad z\in\mathbb{D}, \tag{3}$$

and (here slightly modified) the subclass of f such that

$$\operatorname{Re}\left\{(1-z)^2 f'(z)\right\}\geq 0, \quad z\in\mathbb{D}. \tag{4}$$

Note that in Robertson's inequality (2) and, particularly, in (3) and (4), appear square trinomials with roots of modulus one. Generalizing this concept the authors in [3] proposed to use in Robertson's formula an arbitrary polynomial P_Λ with roots outside of the unit disk. Including the identity as a 0 degree polynomial, i.e., the class $\mathcal{P}'(\delta)$ given by (1), the classes $\mathcal{C}(\delta,\Lambda)$ of analytic functions f satisfying the inequality (5) were defined. The case of square trinomials P_Λ with roots outside of $\mathbb{D}$ were examined in [5] and [8].

As was proved in [3] some specific ranges of roots of P_Λ produce classes of univalent functions, so in this way (5) in some cases can be treated as a criterium of univalence.

In this paper and in its sequel [4], we develop the result of [3] that the classes of polynomial close-to-convex functions defined by polynomials having different systems of their roots, do not contain each other. This allows to formulate some radius problem which was remarked in [3] only. The main goal of this work is to present the method of calculations of radii of a reciprocal dependence of classes of polynomial close-to-convex functions in a special case when $\delta_1=\delta_2=0$ in Definition 3.

Basing on these considerations, partial solutions are demonstrated in a sequel [4]. One of the result presented there (Part 2 of Corollary 4.3) states that every function f in $\mathcal{P}'(0)$ satisfies the inequality (3) in the disk of radius $\sqrt{\sqrt{2}-1}$ and the radius is the largest one.

2. Preliminaries

For $r>0$ let

$$\mathbb{D}_r:=\{z\in\mathbb{C}:|z|<r\},\quad \overline{\mathbb{D}}_r:=\{z\in\mathbb{C}:|z|\leq r\},\quad \overline{\mathbb{D}}_r^0:=\overline{\mathbb{D}}_r\setminus\{0\}$$

and $\mathbb{T}_r:=\{z\in\mathbb{C}:|z|=r\}$. Let $\overline{\mathbb{D}}^0:=\overline{\mathbb{D}}\setminus\{0\}$ and $\mathbb{T}:=\mathbb{T}_1$.

Throughout the whole paper we assume that $\operatorname{Arg} z \in [-\pi, \pi)$ for all $z \neq 0$.

By $\mathcal{P}$ will be denoted the class of all analytic functions p in $\mathbb{D}$ with $p(0) = 1$ having a positive real part in $\mathbb{D}$. It is known that $\mathcal{P}$ forms a convex compact family in the class of all analytic functions in $\mathbb{D}$ with a standard topology.

For each $x \in \mathbb{T}$ and $\delta \in (-\pi/2, \pi, 2)$ let

$$p_{x,\delta}(z) := \frac{e^{i\delta} + e^{-i\delta} xz}{1 - xz}, \quad z \in \mathbb{C} \setminus \{1/x\}.$$

Let $p_x := p_{x,0}$. Clearly $p_x \in \mathcal{P}$ for every $x \in \mathbb{T}$. Let

$$\mathcal{P}_0 := \{p_x : z \in \mathbb{T}\}.$$

The set $\mathcal{P}_0$ contains all extreme points of $\mathcal{P}$.

For $k \in \mathbb{N}$ and $1 \leq j \leq k$ let

$$\Lambda_k^j := \{\{(\mu_i, \xi_i) : i = 1, \ldots, j\} :$$
$$\left. 1 \leq \mu_i \leq k,\ \sum_{i=1}^{j} \mu_i = k,\ \xi_i \in \overline{\mathbb{D}}^0,\ \xi_{i_1} \neq \xi_{i_2},\ i_1 \neq i_2,\ i_1, i_2 = 1, \ldots, j \right\}.$$

Particularly,

$$\Lambda_k^1 = \left\{\{(k, \xi)\} : \xi \in \overline{\mathbb{D}}^0\right\}$$

and

$$\Lambda_k^k = \left\{\{(1, \xi_i) : i = 1, \ldots, k\} : \xi_i \in \overline{\mathbb{D}}^0,\ \xi_{i_1} \neq \xi_{i_2},\ i_1 \neq i_2,\ i_1, i_2 = 1, \ldots, j\right\}.$$

For $k \in \mathbb{N}$ let

$$\Lambda_k := \bigcup_{j=1}^{k} \Lambda_k^j \qquad \text{and let} \qquad \Lambda_0 := \{\{(0, 0)\}\}.$$

Let

$$\Lambda := \bigcup_{k \in \mathbb{N} \cup \{0\}} \Lambda_k.$$

For $\mathit{\Lambda} = \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \Lambda \setminus \Lambda_0$ let

$$P_{\mathit{\Lambda}}(z) := \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i}, \quad z \in \mathbb{D}.$$

For $\mathit{\Lambda} \in \Lambda_0$ let

$$P_{\mathit{\Lambda}}(z) := 1, \quad z \in \mathbb{D}.$$

For fixed $k \in \mathbb{N}_0$ each polynomial $P_{\mathit{\Lambda}}$ with $\mathit{\Lambda} \in \Lambda_k$ is of a degree k.

The classes $\mathcal{C}(\delta; \mathit{\Lambda})$ defined below were introduced in [3].

Definition 1. Let $\delta \in [-\pi/2, \pi/2]$ and $\mathit{\Lambda} \in \Lambda$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{C}(\delta; \mathit{\Lambda})$ if and only if

$$\operatorname{Re}\left\{e^{i\delta} P_{\mathit{\Lambda}}(z) f'(z)\right\} \geq 0, \quad z \in \mathbb{D}. \tag{5}$$

Such functions are called *polynomial close-to-convex with argument δ*.

Remark 1. 1. When $\varLambda = \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \Lambda \setminus \Lambda_0$, then (5) is of the form

$$\operatorname{Re}\left\{ e^{i\delta} \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i} f'(z) \right\} \geq 0, \quad z \in \mathbb{D}. \tag{6}$$

2. For $\varLambda \in \Lambda_0$ we get the class $\mathcal{P}'(\delta)$ of functions of bounded turning with argument δ, called shortly of *bounded turning* when $\delta = 0$ (see [1, vol. I, p. 101; vol. II, p. 11]). Write $\mathcal{P}'$ instead of $\mathcal{P}'(0)$ (see [1, vol. I, p. 101]).

As was noted in [3] the following observation is true.

Observation 2.1. *The strict inequality in* (5) *holds if and only if* $\delta \in (-\pi/2, \pi/2)$.

Moreover directly from Definition 2 we have

Theorem 2.2. *Let* $\delta \in [-\pi/2, \pi/2]$, $\varLambda \in \Lambda$ *and* $f \in \mathcal{A}$. *Then* $f \in \mathcal{C}(\delta; \varLambda)$ *if and only if*

$$e^{i\delta} P_\varLambda(z) f'(z) = p(z) \cos\delta + i \sin\delta, \quad z \in \mathbb{D}, \tag{7}$$

for some $p \in \mathcal{P}$.

3. Radius problem

The inclusion relation between classes $\mathcal{C}(\delta; \varLambda)$ is a natural problem for the study. The theorem below was proved in [3]. To be self-contained we recall its proof.

Theorem 3.1. *Let* $\delta_1, \delta_2 \in (-\pi/2, \pi/2)$ *and* $\varLambda_1, \varLambda_2 \in \Lambda$ *be such that* $(\delta_1, \varLambda_1) \neq (\delta_2, \varLambda_2)$. *Then*

$$\mathcal{C}(\delta_1; \varLambda_1) \not\subset \mathcal{C}(\delta_2; \varLambda_2).$$

Proof. Let

$$\delta_1, \delta_2 \in (-\pi/2, \pi/2)$$

and

$$\varLambda_1 = \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \Lambda, \quad \varLambda_2 = \{(\nu_l, \zeta_l) : l = 1, \ldots, m\} \in \Lambda$$

be such that $(\delta_1, \varLambda_1) \neq (\delta_2, \varLambda_2)$. For $x \in \mathbb{T}$ let f_x be an analytic function defined by the equation

$$f_x'(z) = e^{-i\delta_1} \frac{1}{P_{\varLambda_1}(z)} p_{x,\delta_1}(z) \tag{8}$$

for $z \in \mathbb{C} \setminus \{1/x, -e^{2i\delta_1}/x, 1/\xi_1, \ldots, 1/\xi_j\}$. Since

$$e^{i\delta_1} P_{\varLambda_1}(z) f_x'(z) = p_{x,\delta_1}(z) = p_x(z) \cos\delta_1 + i \sin\delta_1, \quad z \in \mathbb{D},$$

so, in view of Theorem 2.2, $f_x \in \mathcal{C}(\delta_1; \varLambda_1)$ for each $x \in \mathbb{T}$. Showing that $f_x \notin \mathcal{C}(\delta_2; \varLambda_2)$ for some $x \in \mathbb{T}$ we prove the theorem.

In view of (8), for $z \in \mathbb{C} \setminus \{1/x, -e^{2i\delta_1}/x, 1/\xi_1, \ldots, 1/\xi_j\}$ we have

$$e^{i\delta_2} P_{\Lambda_2}(z) f_x'(z) = e^{i(\delta_2 - \delta_1)} \frac{P_{\Lambda_2}(z)}{P_{\Lambda_1}(z)} p_{x,\delta_1}(z).$$

Hence for $z \in \mathbb{C} \setminus \{1/x, -e^{2i\delta_1}/x, 1/\xi_1, \ldots, 1/\xi_j, 1/\zeta_1, \ldots, 1/\zeta_m\}$ we have

$$\arg\left\{e^{i\delta_2} P_{\Lambda_2}(z) f_x'(z)\right\} = \delta_2 - \delta_1 + \arg \frac{P_{\Lambda_2}(z)}{P_{\Lambda_1}(z)} + \arg p_{x,\delta_1}(z). \tag{9}$$

Note first that for each $x \in \mathbb{T}$ the circle $\mathbb{T}$ is the union of two complementary arcs, say $I_+(x)$ and $I_-(x)$, with endpoints at $1/x$ and $-e^{2i\delta_1}/x$ such that

$$\arg p_{x,\delta_1}(z) = \frac{\pi}{2}, \quad z \in I_+(x), \tag{10}$$

and

$$\arg p_{x,\delta_1}(z) = -\frac{\pi}{2}, \quad z \in I_-(x). \tag{11}$$

From the assumption that $(\delta_1, \Lambda_1) \neq (\delta_2, \Lambda_2)$ it follows that $\delta_1 \neq \delta_2$ and $\Lambda_1 = \Lambda_2$ or $\delta_1 = \delta_2$ and $\Lambda_1 \neq \Lambda_2$, or $\delta_1 \neq \delta_2$ and $\Lambda_1 \neq \Lambda_2$.

I. If $\delta_1 \neq \delta_2$ and $\Lambda_1 = \Lambda_2$, then (9), with $\arg 1 = 0$, equals

$$\arg\left\{e^{i\delta_2} P_{\Lambda_2}(z) f_x'(z)\right\} = \delta_2 - \delta_1 + \arg p_{x,\delta_1}(z), \quad z \in \mathbb{C} \setminus \{1/x\}.$$

When $\delta_2 - \delta_1 \in (0, \pi)$, then taking an arbitrary $x \in \mathbb{T}$ and any $z_0 \in I_+(x)$ the above and (10) yield

$$\arg\left\{e^{i\delta_2} P_{\Lambda_2}(z_0) f_x'(z_0)\right\} = \delta_2 - \delta_1 + \frac{\pi}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Since the above holds for $z \in \mathbb{D}$ near z_0, it follows that $f_x \notin \mathcal{C}(\delta_2; \Lambda_2)$.

Analogously, using the arc $I_-(x)$, we prove the case $\delta_2 - \delta_1 \in (-\pi, 0)$.

II. Let $\Lambda_1 \neq \Lambda_2$. Without loss of generality we can assume that the rational function $P_{\Lambda_2}/P_{\Lambda_1}$ is of the simplest form, i.e. after reducing common factors. Then the function

$$z \mapsto g(z) := \frac{P_{\Lambda_2}(z)}{P_{\Lambda_1}(z)} \tag{12}$$

is analytic in $\mathbb{C} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}$. Since $g(0) = 1$ and g is non-vanishing in $\mathbb{D}$, so

$$\partial g(\mathbb{D}) \cap [0, 1) \neq \emptyset.$$

Thus there exists $z_0 \in \mathbb{T} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}$ such that

$$\arg g(z_0) \in (0, 2\pi).$$

From now on we consider all cases where $g(z_0)$ with respect to its argument can lie in the plane. We use also the observation that for a given $z_0 \in \mathbb{T}$ we can find $x_0 \in \mathbb{T}$ in such a way that either $z_0 \in I_+(x_0)$ or $z_0 \in I_-(x_0)$. Then either (10) or (11) with $x = x_0$ and $z = z_0$ hold and having such choice we prove that

$$\begin{aligned} &\arg\left\{e^{i\delta_2} P_{\Lambda_2}(z_0) f_{x_0}'(z_0)\right\} \\ &= \delta_2 - \delta_1 + \arg g(z_0) + \arg p_{x_0,\delta_1}(z_0) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{aligned}$$

If the above holds for $z_0 \in \mathbb{T}$, then it holds for $z \in \mathbb{D}$ near z_0, also. Consequently $f_{x_0} \notin \mathcal{C}(\delta_2; \Lambda_2)$.

Let us start noting that $\delta_2 - \delta_1 \in (-\pi, \pi)$.

1. Let $\delta_2 - \delta_1 \in [0, \pi/2]$.

If $\arg\, g(z_0) \in (0, \pi/2)$, then we take $x_0 \in \mathbb{T}$ such that $z_0 \in I_+(x_0)$.

If $\delta_2 = \delta_1$ and $\arg\, g(z_0) = \pi/2$, then we take $x_0 \in \mathbb{T}$ as above.

If $\arg\, g(z_0) \in (\pi/2, \pi)$, then we consider two subcases: if

$$\delta_2 - \delta_1 + \arg\, g(z_0) \in \left(\frac{\pi}{2}, \pi\right),$$

then we take $x_0 \in \mathbb{T}$ such that $z_0 \in I_+(x_0)$; if

$$\delta_2 - \delta_1 + \arg\, g(z_0) \in \left(\pi, \frac{3\pi}{2}\right),$$

then we take $x_0 \in \mathbb{T}$ such that $z_0 \in I_-(x_0)$.

If $\delta_2 - \delta_1 = \pi/2$ and $\arg\, g(z_0) = \pi$, then we take $x_0 \in \mathbb{T}$ such that $z_0 \in I_-(x_0)$.

Except from two cases considered below further cases for δ_1, δ_2 and $\arg\, g(z_0)$ we prove in the same manner.

2. Now we consider two cases:

$$\delta_2 = \delta_1, \quad \operatorname{Im}\, g(z) \equiv 0, \tag{13}$$

and

$$\delta_2 - \delta_1 = \frac{\pi}{2}, \quad \operatorname{Re}\, g(z) \equiv 0, \tag{14}$$

for $z \in \mathbb{T} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}$.

Suppose that (13) holds. Since $g(0) = 1$ and $0 \notin g(\mathbb{D})$, we see that $\partial g(\mathbb{D})$ is a half-line with $0 \in \partial g(\mathbb{D})$ going to ∞. Thus P_{Λ_1} and P_{Λ_2} vanish at some points on $\mathbb{T}$, say at ξ_1 and ζ_1. Set $x := \xi_1$. Since

$$g(z) = \frac{P_{\Lambda_2}(z)}{P_{\Lambda_1}(z)} = \frac{q(z)}{(1-\xi_1 z)^{\mu_1}},$$

where

$$q(z) = \frac{\prod_{l=1}^{m}(1-\zeta_l z)^{\nu_l}}{\prod_{i=2}^{j}(1-\xi_i z)^{\mu_i}}, \quad z \in \mathbb{C} \setminus \{1/\xi_1, \ldots, 1/\xi_j\},$$

so

$$g(z) p_{\xi_1, \delta_1}(z) = \frac{q_1(z)}{(1-\xi_1 z)^{\mu_1+1}}, \tag{15}$$

where

$$q_1(z) := \left(e^{i\delta_1} + e^{-i\delta_1}\xi_1 z\right) q(z), \quad z \in \mathbb{C} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}.$$

As $\delta_1 \neq \pm\pi/2$, and $\xi_1 \notin \{\zeta_1, \ldots \zeta_m\}$, so

$$q_1(1/\xi_1) = \left(e^{i\delta_1} + e^{-i\delta_1}\right) q(1/\xi_1) \neq 0. \tag{16}$$

Thus the rational function

$$z \mapsto \frac{q_1(z)}{(1-\xi_1 z)^{\mu_1+1}}, \quad z \in \mathbb{C} \setminus \{1/\xi_1, \ldots, 1/\xi_j\},$$

has a pole of order $\mu_1 + 1 \geq 2$ at $1/\xi_1$. Observe additionally that the function

$$\mathbb{D} \ni z \mapsto \frac{1}{1 - \xi_1 z}$$

maps univalently the unit disk $\mathbb{D}$ onto the half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > 1/2\}$. Thus for $z \in \mathbb{D}$ we have

$$-(\mu_1 + 1)\frac{\pi}{2} < \arg \frac{1}{(1 - \xi_1 z)^{\mu_1 + 1}} < (\mu_1 + 1)\frac{\pi}{2}.$$

Since $\mu_1 + 1 \geq 2$ and $q_1(1/\xi_1)$ is a finite nonzero number as was noted in (16), we see that we can find points in $\mathbb{D}$ near a pole $1/\xi_1$ such that

$$\arg \frac{q_1(z)}{(1 - \xi_1 z)^{\mu_1 + 1}} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Thus, in view of (15), $f_{\xi_1} \notin \mathcal{C}(\delta_2; \varLambda_2)$.

Suppose that (14) holds. Then $\partial g(\mathbb{D})$ is a straight line or a half-line with $0 \in \partial g(\mathbb{D})$ going to ∞. As in the above we take $x := \xi_1$, where $P_{\varLambda_1}(\xi_1) = 0$. Now

$$e^{i(\delta_2 - \delta_1)} g(z) p_{\xi_1, \delta_1}(z) = g_1(z) p_{\xi_1, \delta_1}(z),$$

where $g_1 = ig$. Further argumentation is the same as in the previous case.

3. When $\delta_2 - \delta_1 \in (\pi/2, \pi)$ and further when $\delta_2 - \delta_1 \in (-\pi, 0]$ we argue in the same manner as in Parts 1 and 2.

The definition below is a consequence of the last theorem.

Definition 2. Let $\delta_1, \delta_2 \in (-\pi/2, \pi/2)$ and $\varLambda_1, \varLambda_2 \in \Lambda$ be such that $(\delta_1, \varLambda_1) \neq (\delta_2, \varLambda_2)$. By $R(\delta_1, \delta_2; \varLambda_1, \varLambda_2)$ we denote the largest radius $r \in (0, 1)$ of the disk $\mathbb{D}_r$ such that

$$\operatorname{Re}\left\{e^{i\delta_2} P_{\varLambda_2}(z) f'(z)\right\} > 0, \quad z \in \mathbb{D}_r, \tag{17}$$

for all $f \in \mathcal{C}(\delta_1; \varLambda_1)$.

From now on, throughout the whole paper we consider the case when $\delta_1 = \delta_2 = 0$ only. For short we will write

$$\mathcal{C}(\varLambda) := \mathcal{C}(0; \varLambda), \quad R(\varLambda_1, \varLambda_2) := R(0, 0; \varLambda_1, \varLambda_2).$$

For $\varLambda_1, \varLambda_2 \in \Lambda$ let

$$Q_{\varLambda_1, \varLambda_2}(z) := \frac{P_{\varLambda_2}(z)}{P_{\varLambda_1}(z)}, \quad z \in \mathbb{D}.$$

If $\varLambda_1 \in \Lambda$, then $\varLambda_1 \in \Lambda_{k_1}$ for some $k_1 \in \mathbb{N}_0$. Thus either

$$\varLambda_1 = \varLambda_1^j := \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \Lambda_{k_1}^j$$

or

$$\varLambda_1 = \varLambda_1^0 := \{(\mu_0, \xi_0)\},$$

where $\mu_0 = \xi_0 = 0$. Analogously, if $\varLambda_2 \in \Lambda$, then $\varLambda_2 \in \Lambda_{k_2}$ for some $k_2 \in \mathbb{N}_0$. Thus either

$$\varLambda_2 = \varLambda_2^m := \{(\nu_l, \zeta_l) : l = 1, \ldots, m\} \in \Lambda_{k_2}^m$$

or

$$\Lambda_2 = \Lambda_2^0 := \{(\nu_0, \zeta_0)\},$$

where $\nu_0 = \zeta_0 = 0$.

Let now $f \in \mathcal{C}(\Lambda_1)$. Then by Theorem 2.2 we have

$$P_{\Lambda_1}(z) f'(z) = p(z), \quad z \in \mathbb{D},$$

for some $p \in \mathcal{P}$. Hence (17) has the form

$$\begin{aligned} \text{(18)} \qquad \operatorname{Re}\{Q_{\Lambda_1,\Lambda_2}(z)p(z)\} &= \operatorname{Re}\left\{Q_{\Lambda_1^j,\Lambda_2^m}(z)p(z)\right\} \\ &= \operatorname{Re}\left\{\frac{\prod_{l=0}^m (1-\zeta_l z)^{\nu_l}}{\prod_{i=0}^j (1-\xi_i z)^{\mu_i}} p(z)\right\} > 0, \quad z \in \mathbb{D}_r. \end{aligned}$$

In this way, in order to calculate $R(\Lambda_1, \Lambda_2) = R\left(\Lambda_1^j, \Lambda_2^m\right)$ it is enough to calculate the largest $r \in (0,1)$ such that (18) is satisfied in the disk $\mathbb{D}_r$ for all functions $p \in \mathcal{P}$.

For each $z \in \mathbb{D}$ let $J_z^{\Lambda_1;\Lambda_2} : \mathcal{P} \to \mathbb{C}$ be a functional defined as

$$J_z^{\Lambda_1;\Lambda_2}(p) := Q_{\Lambda_1,\Lambda_2}(z)p(z), \quad p \in \mathcal{P}.$$

Clearly,

$$J_0^{\Lambda_1;\Lambda_2} \equiv 1.$$

Consequently, (18) is of the form

$$\text{(19)} \qquad \operatorname{Re} J_z^{\Lambda_1;\Lambda_2}(p) > 0, \quad z \in \mathbb{D}_r.$$

For short we will write J_z instead of $J_z^{\Lambda_1;\Lambda_2}$. For further discussion we assume without loss of generality that the ratio

$$Q_{\Lambda_1,\Lambda_2} = P_{\Lambda_2}/P_{\Lambda_1}$$

is after reducing all common factors.

The method of proof is based on the following observation. Note that for each $z \in \mathbb{D} \setminus \{0\}$, $J_z : \mathcal{P} \to \mathbb{C}$ is a continuous linear functional over the class $\mathcal{P}$. From the fact that $\mathcal{P}$ is a convex compact family in the class of all analytic functions in $\mathbb{D}$ it follows that in order to calculate the radius $R(\Lambda_1, \Lambda_2)$ it suffices to consider (18), so (19), over the set $\mathcal{P}_0$ of all extreme points of $\mathcal{P}$. Thus the problem reduces to find the largest $r \in (0,1)$ such that

$$\operatorname{Re} J_z(p_x) > 0, \quad z \in \mathbb{D}_r,$$

for all functions $p_x \in \mathcal{P}_0$, i.e., to find the largest $r \in (0,1)$ such that

$$|\operatorname{Arg} J_z(p_x)| < \frac{\pi}{2}, \quad z \in \mathbb{D}_r,$$

or equivalently

$$-\frac{\pi}{2} < \operatorname{Arg}\{Q_{\Lambda_1,\Lambda_2}(z)p_x(z)\} < \frac{\pi}{2}, \quad z \in \mathbb{D}_r,$$

for all $p_x \in \mathcal{P}_0$. Hence and from the maximum principle for harmonic functions we see that

$$R(\varLambda_1, \varLambda_2) = \min\left\{R^+(\varLambda_1, \varLambda_2), R^-(\varLambda_1, \varLambda_2)\right\},$$

where

$$R^+(\varLambda_1, \varLambda_2) := r \quad \text{and} \quad R^-(\varLambda_1, \varLambda_2) := r$$

are the smallest roots r in $(0,1)$ of the following equations:

$$\begin{aligned} (20) \qquad & \max_{z\in\overline{\mathbb{D}}_r}\max_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\varLambda_1,\varLambda_2}(z)p_x(z)\right\} \\ &= \max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\varLambda_1,\varLambda_2}(z)p_x(z)\right\} = \frac{\pi}{2} \end{aligned}$$

and

$$\begin{aligned} (21) \qquad & \min_{z\in\overline{\mathbb{D}}_r}\min_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\varLambda_1,\varLambda_2}(z)p_x(z)\right\} \\ &= \min_{z\in\mathbb{T}_r}\min_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\varLambda_1,\varLambda_2}(z)p_x(z)\right\} = -\frac{\pi}{2}, \end{aligned}$$

respectively. Note that for each $r \in (0,1)$,

$$(22) \qquad \max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \operatorname{Arg} p_x(z) = \arcsin\frac{2r}{1+r^2}$$

and

$$(23) \qquad \min_{z\in\mathbb{T}_r}\min_{x\in\mathbb{T}} \operatorname{Arg} p_x(z) = -\arcsin\frac{2r}{1+r^2}.$$

Since the function $Q_{\varLambda_1,\varLambda_2}$ is nonvanishing and analytic in $\mathbb{D}$ with $Q_{\varLambda_1,\varLambda_2}(0) = 1$, so $Q_{\varLambda_1,\varLambda_2}(\mathbb{T}_r)$, for each $r \in (0,1)$, is a closed analytic curve which surrounds 1. Thus, let $r^+(\varLambda_1, \varLambda_2) := r$ be the smallest root r in $(0,1)$ of the equation

$$(24) \qquad \max_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\varLambda_1,\varLambda_2}(z) = \frac{\pi}{2}$$

if it is exists or set $r^+(\varLambda_1, \varLambda_2) := 1$ otherwise. Similarly, let $r^-(\varLambda_1, \varLambda_2) := r$ be the smallest root r in $(0,1)$ of the equation

$$\min_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\varLambda_1,\varLambda_2}(z) = -\frac{\pi}{2}$$

if it is exists or set $r^-(\varLambda_1, \varLambda_2) := 1$ otherwise.

Fix $r \in (0, r^+(\varLambda_1, \varLambda_2))$. Let $\theta_0 \in \mathbb{R}$ be such that for $z_0 := re^{i\theta_0}$,

$$(25) \qquad \max_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\varLambda_1,\varLambda_2}(z) = \operatorname{Arg} Q_{\varLambda_1,\varLambda_2}(z_0).$$

Since $r \in (0, r^+(\varLambda_1, \varLambda_2))$, so from (24) it follows that

$$(26) \qquad \max_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\varLambda_1,\varLambda_2}(z) \in (0, \pi/2).$$

Further, observe that taking $x_0 := ie^{-i\theta_0} \in \mathbb{T}$ and using (22) we have

$$\begin{aligned} \operatorname{Arg} p_{x_0}(z_0) &= \operatorname{Arg}\frac{1+x_0z_0}{1-x_0z_0} = \operatorname{Arg}\frac{1+ir}{1-ir} \\ &= \arcsin\frac{2r}{1+r^2} = \max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \operatorname{Arg} p_x(z). \end{aligned}$$

This, (25) and (26) yield

$$\begin{aligned}
&\operatorname{Arg}\left\{Q_{\Lambda_1,\Lambda_2}(z_0)p_{x_0}(z_0)\right\}\\
&\le \max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\Lambda_1,\Lambda_2}(z)p_x(z)\right\}\\
&= \max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \left\{\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\operatorname{Arg} p_x(z)\right\}\\
&\le \max_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}}\operatorname{Arg} p_x(z)\\
&= \operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z_0)+\operatorname{Arg} p_{x_0}(z_0)\\
&= \operatorname{Arg}\left\{Q_{\Lambda_1,\Lambda_2}(z_0)p_{x_0}(z_0)\right\}.
\end{aligned}$$

Thus for $r\in(0,r^+(\Lambda_1,\Lambda_2))$ we have

$$\begin{aligned}
&\max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\Lambda_1,\Lambda_2}(z)p_x(z)\right\}\\
&= \max_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\max_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}}\operatorname{Arg} p_x(z)\\
&= \max_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\arcsin\frac{2r}{1+r^2}.
\end{aligned}\tag{27}$$

In a similar way we prove that for $r\in(0,r^-(\Lambda_1,\Lambda_2))$ holds

$$\begin{aligned}
&\min_{z\in\mathbb{T}_r}\min_{x\in\mathbb{T}} \operatorname{Arg}\left\{Q_{\Lambda_1,\Lambda_2}(z)p_x(z)\right\}\\
&= \min_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\min_{z\in\mathbb{T}_r}\max_{x\in\mathbb{T}}\operatorname{Arg} p_x(z)\\
&= \min_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)-\arcsin\frac{2r}{1+r^2}.
\end{aligned}\tag{28}$$

Consequently, by (27), for $r\in(0,r^+(\Lambda_1,\Lambda_2))$ the equation (20) is of the form

$$\max_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\arcsin\frac{2r}{1+r^2}=\frac{\pi}{2},\tag{29}$$

and, by (28), for $r\in(0,r^-(\Lambda_1,\Lambda_2))$ the equation (21) is of the form

$$\min_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)-\arcsin\frac{2r}{1+r^2}=-\frac{\pi}{2},$$

i.e.,

$$-\min_{z\in\mathbb{T}_r}\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z)+\arcsin\frac{2r}{1+r^2}=\frac{\pi}{2}.\tag{30}$$

Thus $R^+(\Lambda_1,\Lambda_2)$ is the smallest root in $(0,r^+(\Lambda_1,\Lambda_2))$ of the equation (29) and $R^-(\Lambda_1,\Lambda_2)$ is the smallest root in $(0,r^-(\Lambda_1,\Lambda_2))$ of the equation (30).

Note finally that both solutions exist. To see this, consider the equation (29). Taking into account (26) and the strict monotonicity of the function

$$[0,1]\ni r\mapsto \arcsin\frac{2r}{1+r^2}$$

from 0 at 0 to $\pi/2$ at 1, we see at once that $R^+(\Lambda_1,\Lambda_2)$ exists. Analogously we prove that the equation (30) has a solution $R^-(\Lambda_1,\Lambda_2)$.

For each $r \in (0, r^+(\Lambda_1, \Lambda_2))$ let

$$\varphi(r; \Lambda_1, \Lambda_2) := \max_{z \in \mathbb{T}_r} \operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z). \tag{31}$$

By (26) we see that

$$\varphi(r; \Lambda_1, \Lambda_2) \in (0, \pi/2). \tag{32}$$

Then (29) is of the form

$$\varphi(r; \Lambda_1, \Lambda_2) + \arcsin \frac{2r}{1+r^2} = \frac{\pi}{2}.$$

Hence

$$\sin\varphi(r; \Lambda_1, \Lambda_2) \frac{1-r^2}{1+r^2} + \sqrt{1 - \sin^2\varphi(r; \Lambda_1, \Lambda_2)} \frac{2r}{1+r^2} = 1.$$

Consequently

$$4r^2 \left(1 - \sin^2\varphi(r; \Lambda_1, \Lambda_2)\right) = \left((1+r^2) - (1-r^2)\sin\varphi(r; \Lambda_1, \Lambda_2)\right)^2.$$

Equivalently,

$$\left((1+r^2)\sin\varphi(r; \Lambda_1, \Lambda_2) + r^2 - 1\right)^2 = 0.$$

In this way to calculate $R^+(\Lambda_1, \Lambda_2)$ it is enough to find the smallest root in $(0, r^+(\Lambda_1, \Lambda_2))$ of the equation

$$\sin\varphi(r; \Lambda_1, \Lambda_2) = \frac{1-r^2}{1+r^2} \tag{33}$$

or, taking into account (32), the smallest root in $(0, r^+(\Lambda_1, \Lambda_2))$ of the equation

$$\sqrt{1 - \sin^2\varphi(r; \Lambda_1, \Lambda_2)} = \cos\varphi(r; \Lambda_1, \Lambda_2) = \frac{2r}{1+r^2}. \tag{34}$$

Analogously, if for each $r \in (0, r^-(\Lambda_1, \Lambda_2))$ we set

$$\psi(r; \Lambda_1, \Lambda_2) := -\min_{z \in \mathbb{T}_r} \operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z), \tag{35}$$

then we write (30) as

$$\psi(r; \Lambda_1, \Lambda_2) + \arcsin \frac{2r}{1+r^2} = \frac{\pi}{2}.$$

Arguing as for φ, to calculate $R^-(\Lambda_1, \Lambda_2)$ it is enough to find the smallest root in $(0, r^-(\Lambda_1, \Lambda_2))$ of the equation

$$\sin\psi(r; \Lambda_1, \Lambda_2) = \frac{1-r^2}{1+r^2}, \tag{36}$$

or of the equation

$$\sqrt{1 - \sin^2\psi(r; \Lambda_1, \Lambda_2)} = \cos\psi(r; \Lambda_1, \Lambda_2) = \frac{2r}{1+r^2}. \tag{37}$$

Summarizing, we proved

Theorem 3.2.

$$R(\Lambda_1, \Lambda_2) = \min\left\{R^+(\Lambda_1, \Lambda_2), R^-(\Lambda_1, \Lambda_2)\right\},$$

where $R^+(\Lambda_1, \Lambda_2)$ *is the smallest root in* $(0, r^+(\Lambda_1, \Lambda_2))$ *of the equation* (33) ((34)) *and* $R^-(\Lambda_1, \Lambda_2)$ *is the smallest root in* $(0, r^-(\Lambda_1, \Lambda_2))$ *of the equation* (36) ((37)).

Part (i) of the next theorem is an immediate consequence of the equations (33) and (36). Part (ii) follows from the fact that

$$\operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z) = -\operatorname{Arg} Q_{\Lambda_2,\Lambda_1}(z).$$

Theorem 3.3. *Assume that* $r^+(\Lambda_1, \Lambda_2) = r^-(\Lambda_1, \Lambda_2)$ *and for* $r \in (0, r^+(\Lambda_1, \Lambda_2))$ *holds*

$$\max_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z) = -\min_{z\in\mathbb{T}_r} \operatorname{Arg} Q_{\Lambda_1,\Lambda_2}(z).$$

(i) *Then* $R(\Lambda_1, \Lambda_2)$ *is the smallest root in* $(0, r^+(\Lambda_1, \Lambda_2))$ *of the equation* (33) (34).

(ii)

$$R(\Lambda_1, \Lambda_2) = R(\Lambda_2, \Lambda_1).$$

References

[1] A. W. Goodman , *Univalent Functions*, Mariner, Tampa, Florida, 1983.

[2] W. Hengartner and G. Schober, *On schlicht mappings to domains convex in one direction*, Comm. Math. Helv. **45** (1970), 303–314.

[3] B. Kowalczyk and A. Lecko, *Polynomial close-to-convexity*, Submited.

[4] B. Kowalczyk and A. Lecko, *Radius problem in classes of polynomial close-to-convex functions, II. Partial solutions*, Bull. Soc. Sci. Lettres Łódź, Sér. Rech. Déform. **63**, no. 2 (2013), in print.

[5] A. Lecko, *Some subclasses of close-to-convex functions*, Ann. Polon. Math. **58**, no. 1, (1993), 53–64

[6] A. Lecko, *On reciprocal dependence of some classes of regular functions*, Folia Sci. Univ. Tech. Resov., Matematyka **15**, no. 127 (1994), 35–53.

[7] A. Lecko, *On a radius problem in some subclasses of univalent functions, I*, Folia Sci. Univ. Tech. Resov., Matematyka **19**, no, 147 (1996), 35–54.

[8] A. Lecko, *A generalization of analytic condition for convexity in one direction*, Demonstr. Math. **30**, no. 1 (1997), 155–170.

[9] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ. Jap. **2**, (1934–35), 129–155.

[10] M. S. Robertson, *Analytic functions star-like in one direction*, Amer. J. Math. **58**, (1936), 465–472.

[11] W. C. Royster and M. Ziegler, *Univalent functions convex in one direction*, Publ. Math. Debrecen **23**, no. 3–4 (1976), 339–345.

[12] S. E. Warschawski , *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc. **38**, no. 2 (1935), 310–340.

Department of Applied Mathematics
University of Warmia and Mazury
Słoneczna 54, 10-710 Olsztyn
Poland

Department of Analysis and Differential Equations
University of Warmia and Mazury
Słoneczna 54, 10-710 Olsztyn
Poland
e-mail: alecko@matman.uwm.edu.pl

Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on February 7, 2013

PROBLEM PROMIENI W KLASACH FUNKCJI WIELOMIANOWO PRAWIE WYPUKŁYCH I

S t r e s z c z e n i e

W pracy tej rozważany jest pewien problem związany ze wzajemną zależnością klas funkcji wielomianowo prawie wypukłych, a mianowicie promień wzajemnej zależności rozważanych klas. Udowodniono podstawowe twierdzenia związane z badanym zagadnieniem oraz omówiona została metoda wyznaczania szukanych promieni dla pewnych szczególnych podklas funkcji prawie wypukłych.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 79–83

In memory of
Professor Promarz M. Tamrazov

Dominika Klimek-Smęt and Andrzej Michalski

JACOBIAN ESTIMATES FOR HARMONIC MAPPINGS GENERATED BY CONVEX CONFORMAL MAPPINGS

Summary

Clunie and Sheil-Small introduced in [1] the class S_H of all normalized univalent and sense-preserving harmonic functions in the unit disc Δ. It is well-known that every $f \in S_H$ has the canonical representation $f = h + \overline{g}$, where h and g are analytic in Δ. In [8] we have defined the class $\tilde{S}_H$ consisting of all $f \in S_H$ such that $h + \varepsilon g$ is a convex conformal mapping for some $\varepsilon \in \overline{\Delta}$. The main results of this paper are the Jacobian estimates for the class $\tilde{S}_H$ and its certain subclasses.

Keywords and phrases: univalent harmonic mappings, conformal mappings onto regular polygons

1. Introduction

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\Delta} := \{z \in \mathbb{C} : |z| \le 1\}$ be the open and closed unit disc, respectively. A complex-valued harmonic function $f : \Delta \to \mathbb{C}$ can be uniquely represented as

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic in Δ with $g(0) = 0$. Consequently, such a function f can be uniquely determined by coefficients of the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \tag{2}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$.

A harmonic function f, not identically constant, satisfying (1) is said to be sense-preserving in Δ iff it satisfies the condition $g' = \omega h'$,where $\omega : \Delta \to \Delta$ is an analytic function. The sense-preservity of harmonic function f is closely related to its Jacobian J_f, which can be expressed in terms of h and g as follows

$$\mathrm{J}_f(z) = |h'(z)|^2 - |g'(z)|^2, \quad z \in \Delta. \tag{3}$$

Recall that the necessary and sufficient condition for f to be locally univalent and sense-preserving in Δ is $\mathrm{J}_f(z) > 0$, $z \in \Delta$.

Clunie and Sheil-Small introduced in [1] the family S_H of all univalent and sense-preserving harmonic functions f satisfying (1) in the unit disc Δ, such that $h(0) = 0$ and $h'(0) = 1$. In [8] we have defined the class $\tilde{S}_H$ of all $f \in S_H$ such that $h + \varepsilon g$ is a convex conformal mapping for some $\varepsilon \in \overline{\Delta}$. In particular, the classical coefficient problem in the class $\tilde{S}_H$ has been solved there. In [4] and [5] we have constructed a lot of various examples of functions of this class (graphical images shown).

The main aim of this paper is to give the Jacobian estimate for $\tilde{S}_H$. Additionally, we provide the Jacobian estimate for two interesting subclasses of $\tilde{S}_H$, i.e. $\widehat{S}_H$ composed of all $f \in S_H$ such that h is a convex function and $\overline{S}_H$ of all $f \in S_H$ with $h(z) = z$, $z \in \Delta$. The classes $\widehat{S}_H$ and $\overline{S}_H$ first have been studied in [7] and [6], respectively.

2. Main results

First we establish a genaral Jacobian estimate of $f \in S_H$ in terms of the distortion of its analytic part h, where h is taken from the representation (1) of f.

Theorem 1. *If $f \in S_H$ satisfies* (1) *and $\alpha := |g'(0)|$ then*

$$\frac{(1-\alpha^2)(1-|z|^2)|h'(z)|^2}{(1+\alpha|z|)^2} \le \mathrm{J}_f(z) \le \begin{cases} \frac{(1-\alpha^2)(1-|z|^2)|h'(z)|^2}{(1-\alpha|z|)^2}, & |z| < \alpha, \\ |h'(z)|^2, & |z| \ge \alpha, \end{cases} \tag{4}$$

where $z \in \Delta$.

Proof. Observe, that if $f \in S_H$ then h' does not vanish in Δ and the dilatation of f can be expressed as $\omega = g'/h'$. So we can write the Jacobian of f given by (3) in the form

$$\mathrm{J}_f(z) = |h'(z)|^2 \left(1 - |\omega(z)|^2\right), \quad z \in \Delta. \tag{5}$$

Let $g'(0) = \alpha \mathrm{e}^{\mathrm{i}\varphi}$ and consider the function

$$\Omega(z) = \frac{\mathrm{e}^{-\mathrm{i}\varphi}\omega(z) - \alpha}{1 - \alpha \mathrm{e}^{-\mathrm{i}\varphi}\omega(z)}, \quad z \in \Delta.$$

Clearly Ω satisfies the assumptins of the Schwarz lemma, which gives

$$\left|\mathrm{e}^{-\mathrm{i}\varphi}\omega(z) - \alpha\right| \le |z| \left|1 - \alpha \mathrm{e}^{-\mathrm{i}\varphi}\omega(z)\right|, \quad z \in \Delta$$

or equivalently

$$\left| e^{-i\varphi}\omega(z) - \frac{\alpha(1-|z|^2)}{1-\alpha^2|z|^2} \right| \le \frac{(1-\alpha^2)|z|}{1-\alpha^2|z|^2}, \quad z \in \Delta.$$

Hence, by use of triangle inequality, we obtain

$$|\omega(z)| \le \frac{\alpha+|z|}{1+\alpha|z|},\ z \in \Delta \quad \text{and} \quad |\omega(z)| \ge \frac{\alpha-|z|}{1-\alpha|z|},\ |z| < \alpha. \tag{6}$$

Finally, by applying (6) to (5) we have (4) and the proof is completed. ∎

This theorem is the main tool to prove the Jacobian estimates for the classes $\overline{S}_H$ and $\widehat{S}_H$.

Corollary 2. *If* $f \in \overline{S}_H$ *satisfies* (1) *and* $\alpha := |g'(0)|$ *then*

$$\frac{(1-\alpha^2)(1-|z|^2)}{(1+\alpha|z|)^2} \le \mathrm{J}_f(z) \le \begin{cases} \frac{(1-\alpha^2)(1-|z|^2)}{(1-\alpha|z|)^2}, & |z| < \alpha, \\ 1, & |z| \ge \alpha, \end{cases} \tag{7}$$

where $z \in \Delta$. *The estimate cannot be improved.*

Proof. Let $f \in \overline{S}_H$. By definition of the class $\overline{S}_H$ we have $h(z) = z$ and so $h'(z) = 1$, $z \in \Delta$. Using Theorem 1, i.e. putting $|h(z)| = 1$ into (4) we obtain (7) and the proof is completed.

The sign of equality in the inequalities (7) holds for functions of the form:

$$\Delta \ni z \mapsto z + \overline{\int_0^z \frac{\zeta-\alpha}{1-\alpha\zeta}\frac{\zeta-c}{1-c\zeta}\mathrm{d}\zeta}$$

with suitably chosen $c \in (\alpha, 1]$ ∎

Corollary 3. *If* $f \in \widehat{S}_H$ *satisfies* (1) *and* $\alpha := |g'(0)|$ *then*

$$\frac{(1-\alpha^2)(1-|z|)}{(1+|z|)^3(1+\alpha|z|)^2} \le \mathrm{J}_f(z) \le \begin{cases} \frac{(1-\alpha^2)(1+|z|)}{(1-|z|)^3(1-\alpha|z|)^2}, & |z| < \alpha, \\ \frac{1}{(1-|z|)^4}, & |z| \ge \alpha, \end{cases} \tag{8}$$

where $z \in \Delta$. *The estimate cannot be improved.*

Proof. Let $f \in \widehat{S}_H$. By definition of the class $\widehat{S}_H$ the function h is convex, analytic and suitably normalized (properties of such functions can be found in [3]). So we can use the following well-known estimate

$$\frac{1}{(1+|z|)^2} \le |h'(z)| \le \frac{1}{(1-|z|)^2}, \quad z \in \Delta.$$

Combining this with the inequality (4) of Theorem 1 we obtain (8) and the proof is completed.

The sign of equality in the inequalities (8) holds for functions of the form:

$$\Delta \ni z \mapsto \frac{z}{1-z} + \overline{\int_0^z \frac{1}{(1-z)^2}\frac{\zeta-\alpha}{1-\alpha\zeta}\frac{\zeta-c}{1-c\zeta}\mathrm{d}\zeta}$$

with suitably chosen $c \in (\alpha, 1]$ ■

As we can see the clasical methods of complex analysis and known distortion estimate for convex conformal mappings lead to the estimates for the classes $\overline{S}_H$ and $\widehat{S}_H$. To obtain the Jacobian estimate for $f \in \tilde{S}_H$, we use a general estimate given by Schaubroeck in [6] and later improved by Sobczak-Kneć, Starkov and Szynal in [7].

Theorem 4. *If* $f \in \tilde{S}_H$ *satisfies* (1) *and* $\alpha := |g'(0)|$ *then*

$$\frac{(1-\alpha^2)(1-|z|)^2}{(1+|z|)^6} \le \mathrm{J}_f(z) \le \frac{(1-\alpha^2)(1+|z|)^2}{(1-|z|)^6}, \quad z \in \Delta. \tag{9}$$

Proof. It is quiet obvious (see [8, Proposition 2.1]) that $\tilde{S}_H$ is affine invariant family, i.e. for all $f \in \tilde{S}_H$ satisfying (1) and $\delta \in \Delta$ the function

$$f_\delta(z) := \frac{f(z) + \delta \bar{f}(z)}{1 + \delta g'(0)}, \quad z \in \Delta$$

belongs to the class $\tilde{S}_H$. Similarly, it is not hard to prove (see [8, Proposition 2.2]) that $\tilde{S}_H$ is linear invariant family, i.e. for all $f \in \tilde{S}_H$ and all conformal disc automorphisms φ the function

$$f_\varphi(z) := \frac{f(\varphi(z)) - f(\varphi(z))}{\varphi'(0)h'(\varphi(0))}, \quad z \in \Delta$$

belongs to the class $\tilde{S}_H$.

These two invariance properties allow us to apply the Jacobian estimate given in [10, Theorem 2.1] to the class $\tilde{S}_H$. Thus, for $f \in \tilde{S}_H$ satisfying (1) we obtain

$$\frac{(1-\alpha^2)(1-|z|)^{2\beta-2}}{(1+|z|)^{2\beta+2}} \le \mathrm{J}_f(z) \le \frac{(1-\alpha^2)(1+|z|)^{2\beta-2}}{(1-|z|)^{2\beta+2}}, \quad z \in \Delta, \tag{10}$$

where $\alpha := |g'(0)|$ and $\beta := \sup\{|a_2|\}$ taken over all functions of the class $\tilde{S}_H$ satisfying (1) with the expansion (2).

Finally, we need to refer to [8, Corollary 3.2], which says that if $f \in \tilde{S}_H$ satisfies (1) with the expansion (2), then $|a_n| < n$, $n = 2, 3, 4, \ldots$ and the bound is sharp (although not attained). In particular, this implies that $\beta = 2$, which combined with the inequality (10) gives (9) and the proof is completed. ■

The inequalities (9) of Theorem 4 do not seem to be best possible.

References

[1] J. G. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 3–25.

[2] P. L. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Math. 156, Cambridge Univ. Press, Cambridge 2004.

[3] P. L. Duren, *Univalent functions*, Grundlehren Math. Wiss. 259, Springer-Verlag, Berlin-New York 1983.

[4] D. Klimek-Smęt and A. Michalski, *Univalent harmonic functions generated by conformal mappings onto regular polygons*, Bull. Soc. Sci. Lettres Łódź **59**, Ser. Rech. Deform. **58** (2009), 33–44.

[5] D. Klimek and A. Michalski, *Univalent harmonic mappings with dilatation onto regular polygons by hypergeometric functions*, Sci. Bull. Chełm **2** (2006), 83–90.

[6] D. Klimek and A. Michalski, *Univalent anti-analytic perturbation of the identity in the unit disc*, Sci. Bull. Chełm **1** (2006), 67–76.

[7] D. Klimek-Smęt and A. Michalski, *Univalent anti-analytic perturbation of convex conformal mapping in the unit disc*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A **61** (2007), 39–49.

[8] D. Klimek-Smęt and A. Michalski, *Harmonic mappings generated by convex conformal mappings*, preprint.

[9] L. E. Schaubroeck, *Subordination of planar harmonic functions*, Complex Variables Theory Appl. **41** (2000), 163-178.

[10] M. Sobczak-Kneć, V. V. Starkov and J. Szynal, *Old and new order of linear invariant family of harmonic mappings and the bound for Jacobian*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A **65(2)** (2011), 191–202.

Department of Applied Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 5
PL-20-031 Lublin, Poland
Department of Complex Analysis

Faculty of Mathematics
and Natural Sciences
The John Paul II Catholic University
of Lublin, Konstantynów 1H
PL-20-950 Lublin, Poland
e-mail: amichal@kul.lublin.pl

Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 29, 2012

OSZACOWANIA JAKOBIANU DLA ODWZOROWAŃ HARMONICZNYCH GENEROWANYCH PRZEZ ODWZOROWANIA KONFOREMNE WYPUKŁE

S t r e s z c z e n i e

Clunie i Sheil-Small wprowadzili w pracy [1] klasę S_H wszystkich unormowanych funkcji harmonicznych, jednolistnych i zachowujących orientację w kole jednostkowym Δ. Wiadomo, że każda funkcja $f \in S_H$ ma kanoniczną reprezentację $f = h + \bar{g}$, gdzie h i g są funkcjami analitycznymi w kole Δ. W pracy [8] zdefiniowaliśmy klasę $\tilde{S}_H$ złożoną z wszystkich takich funkcji $f \in S_H$, że $h + \epsilon g$ jest odwzorowaniem konforemnym pracy są oszacowania jakobinianu dla klasy $\tilde{S}_H$ i pewnych jej podklas.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 85–88

Zhidong Zhang and Norman H. March

CONFORMAL INVARIANCE IN THE THREE-DIMENSIONAL (3D) ISING MODEL AND QUATERNIONIC GEOMETRIC PHASE IN QUATERNIONIC HILBERT SPACE III

REJOINDER TO THE COMMENT TO THE RESPONSE TO "ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES" BY J. H. H. PERK

Summary

We show that singularities at $\beta = 0$ are different for the hard core model and the Ising model and that there is no upper bound at $\beta = 0$ for series of the Ising model. The high-temperature series cannot serve as a standard for judging a putative exact solution of the 3D Ising model. Furthermore, the free energy per site f and the reduced free energy βf lose their definitions at $\beta = 0$, and thus either of them could have two different forms for high-temperature series expansions at/below infinite temperature. Three independent Virasoro algebras for 3D conformal field theory can be written within the $3+1$-dimensional space (i.e., 3-sphere) with weight factors.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1.

This is a Rejoinder to Professor J.H.H. Perk's recent Comment [1] to our Response [2] to his paper entitled "Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices" [3].

We emphasized that all the debates [1–9], up to date, have focused only on whether the high-temperature series can serve as a standard for judging a putative exact solution of the 3D Ising model, not on the validity of the topological approach developed in [10] and the correctness of its consequence (i.e., the conjectured exact solution).

2.

As already pointed out in our previous responses [2,5,8], all the well-known theorems (including Perk's new approach in [3]) for the convergence of the high-temperature series of the Ising model are rigorously proved only for β $(= 1/k_BT) > 0$, not for infinite temperature ($\beta = 0$). Lebowitz and Penrose indicated clearly in the abstract of [11] that their proof for the analyticity of the free energy per site and the distribution function of the Ising model is for $\beta > 0$. If the Ising model were a special case of a hard core on the lattice, Lebowitz and Penrose would prove its analyticity for $\beta = 0$ immediately after their proof for a hard core potential. One sentence would be enough for such proof. But, certainly, it is not the case. Their proof for the Ising model is related with the Yang-Lee Theorems [12, 13] for $\beta > 0$, and for the analyticity of the function βp. Here, we have to inspect the definition of the hard-core model and the Ising model to discern the difference between them, and also to clarify the conditions of their proof for the hard-core model. The hard-core potential is defined by

$$\varphi(r) = +\infty \quad \text{for} \quad r \leq a, \quad \text{and} \quad \varphi(r) < \infty \quad \text{for} \quad r > a,$$

where a is a positive constant ($a > 0$) [11]. The Ising ferromagnet is isomorphic to a lattice gas with an attractive interaction potential with

$$\varphi(0) = +\infty, \quad \text{and} \quad \varphi(r) \leq 0 \quad \text{for} \quad r \neq 0;$$

cf. [11]. So, $a = 0$ for the Ising lattice (see also the definition (9) in page 411 of [13]). The key distinction between the two models is whether a is zero or a positive constant. Though Lebowitz and Penrose claimed that the hard-core systems are analytic in β at $\beta = 0$, actually, their proof concerns βp (the series (4) of [11]), and not p itself. (For equivalence between βp and p, setting $\beta = 1$ equalizes to $T = 1/k_B \neq \infty$.) Regardless of this flaw in their proof, on the other hand, we can show that their claim for the hard-core systems is not appropriate for the Ising model. Equation (20) of [11] is claimed to define an upper bound on the positive values $|z|$ for hard-core potentials. However, setting $a = 0$ for the Ising model, one would have $|z| < \infty$ since the volume $K_v(0) = 0$. Clearly, behaviours at $\beta = 0$ are different for the hard-core model ($a > 0$) and the Ising model ($a = 0$). There is no upper bound at $\beta = 0$ for series (even for the function βp) of the Ising model, which means that the high-temperature series of the Ising model could be divergent at $\beta = 0$.

In their proofs for the Ising model, Lebowitz and Penrose also relied on (or referred to) Gallavotti et al.'s proofs [14–16]. But, it is evident that none of Gallavotti et al.'s proofs for the Ising model touches $\beta = 0$ [14–16]. On the second page (p. 275) of [14] for a detailed proof, they put, for convenience, $\beta = 1$. When they defined $Z_\Lambda(\Phi)$ in eq. (5) of [16], they also set $\beta = 1$. This condition of $\beta = 1$ is contradictory with $\beta = 0$, as mentioned above (see [2] for detailed discussion). The inequality

$$\sum_{T \cap X - \phi, T \neq \phi} | K_{\beta\phi'}(X, T) | \leq [\exp(e^{\beta\|\phi'\|} - 1) - 1]$$

of [15] (or Proposition 1 (i.e., eq. (17)) of [14]) is invalid for $\beta = 0$. All the facts indicate that $\beta = 0$ cannot be included within the radius of convergence, since it makes the condition for such proof invalid.

3.

The Yang-Lee theory [12,13] is only focused on the zeros of the grand partition function Z, since their interest is the phase transition at finite temperature, not infinite temperature. We have shown in [2] that for behaviours at infinite temperature, one has to study both the zeros and the poles of the grand partition function (i.e., $Z = 0$ and $Z^{-1} = 0$). Both the free energy per site f and the reduced free energy βf lose their definitions at $\beta = 0$, and thus either of them could have two different forms at/below infinite temperature for high-temperature series expansions, as revealed in [10].

For the singularities at/near infinite temperature, we do not mean that there is a phase transition like what we have at the critical point. There is a change of topological phase factors corresponding to the change of topological structures of the 3D Ising system as interactions are turned on. All the facts above indicate that the high-temperature series cannot serve as a standard for judging a putative exact solution of the 3D Ising model.

For the procedure developed in [17] for extending 2D conformal field theory to be appropriate for three dimensions, three independent Virasoro algebras with weight factors (Re $| e^{i\phi_i} |$ could be zero or non-zero values) guarantee that three independent Virasoro algebras can be written within the $3+1$-dimensional space (i.e., 3-sphere), and one does not need to introduce a 6-dimensional $(2+2+2)$-space for it.

We do not want to repeat the criticisms in [10] on disadvantages of numerical techniques (such as Monte Carlo methods), and the criticisms in [2] on the failure of Perk's theorem in [3]. We do not want to repeat the comparison in [10, 18, 19] of the conjectured exact solution with experimental data.

Acknowledgment

ZDZ appreciates the National Natural Science Foundation of China (under grant number 50831006). NHM thanks Professors D. Lamoen, and C. Van Alsenoy for making possible his continuing affiliation with the University of Antwerp.

References

[1] J. H. H. Perk, Bull. Soc. Sci. Lettres Łodź Sér. Rech. Déform. **62** (2012), 71–74; arXiv:1209.0731v2.

[2] Z. D. Zhang and N. H. March, Bull. Soc. Sci. Lettres Łodź Sér. Rech. Déform. **62** (2012), 61–69, as part II; arXiv:1209.3247.

[3] J. H. H. Perk, Bull. Soc. Sci. Lettres Łodź Sér. Rech. Déform. **62** (2012), 45–59; arXiv:1209.0731v1.

[4] F. Y. Wu, B. M. McCoy, M. E. Fisher, and L. Chayes, Phil. Mag. **88** (2008), 3093.

[5] Z. D. Zhang, Phil. Mag. **88** (2008), 3097.
[6] F. Y. Wu, B. M. McCoy, M. E. Fisher, and L. Chayes, Phil. Mag. **88** (2008), 3103.
[7] J. H. H. Perk, Phil. Mag. **89** (2009), 761.
[8] Z. D. Zhang, Phil. Mag. **89** (2009), 765.
[9] J. H. H. Perk, Phil. Mag. **89** (2009), 769.
[10] Z. D. Zhang, Phil. Mag. **87** (2007), 5309.
[11] J. L. Lebowitz and O. Penrose, Commun. Math. Phys. **11** (1968), 99.
[12] C. N. Yang and T. D. Lee, Phys. Rev. **87** (1952), 404.
[13] T. D. Lee and C. N. Yang, Phys. Rev. **87** (1952), 410.
[14] G. Gallavotti and S. Miracle-Sol, Commun. Math. Phys. **7** (1968), 274.
[15] G. Gallavotti, S. Miracle-Sol, and D. W. Robinson, Phys. Lett. **25A** (1967), 493.
[16] G. Gallavotti and S. Miracle-Sol, Commun. Math. Phys. **5** (1967), 317.
[17] Z. D. Zhang and N. H. March, Bull. Soc. Sci. Lettres Łodź Sér. Rech. Déform. **62** (2012), pp. 35–44, as part I; arXiv:1110.5527.
[18] Z. D. Zhang and N. H. March, Phase Transitions **84** (2011), 299.
[19] N. H. March and Z. D. Zhang, Phys. Lett. A **373** (2009), 2075.

Shenyang National Laboratory for Materials Science
Institute of Metal Research
and International Centre for Materials Physics
Chinese Academy of Sciences
72 Wenhua Road, Shenyang 110016
People's Republic of China
e-mail: zdzhang@imr.ac.cn

Department of Physics
University of Antwerp
Antwerp, Belgium
and
Oxford University
Oxford, England
e-mail: iris.howard@telenet.be

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 6, 2012

NIZMIENNICZOŚĆ KONFOREMNA TRÓJWYMIAROWEGO (3D) MODELU ISINGA A KWATERNIONOWA FAZA GEOMETRYCZNA W KWATERNIONOWEJ PRZESTRZENI HILBERTA III

ODNIESIENIE SIĘ DO KOMENTARZA DO ODPOWIEDZI NA “BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA” J. H. H. PERKA

S t r e s z c z e n i e

Wykazujemy, że osobliwości dla $\beta = 0$ są różne dla modelu twardego rdzenia i modelu Isinga oraz że dla $\beta = 0$ dla szeregu z modelu Isinga nie istnieje kres górny. Zatem wysokotemperaturowy szereg nie może służyć do oceny domniemanego dokładnego rozwiązania dla modelu Isinga 3D. Co więcej, energia swobodna odniesiona do miejsca f i zredukowana energia swobodna βf nie są zdefiniowane dla $\beta = 0$, a więc każda z nich może mieć dwie różne postaci dla rozwinięcia w szereg wydoko-temperaturowy dla/poniżej nieskończonej temperatury. W $3 + 1$-wymiarowej przestrzeni (tj. dla 3-sfery) możemy zapisać trzy niezależne algebry Virasoro dla konforemnej teorii pola 3D.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 89–93

Jacques H.H. Perk

ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES III

REJOINDER TO THE 2nd COMMENT (REJOINDER) TO THE COMMENT TO THE RESPONSE TO "ERRONEOUS SOLUTION OF THREE-DIMENSIONAL (3D) SIMPLE ORTHORHOMBIC ISING LATTICES" BY Z.-D. ZHANG

Summary

The responses by Zhang and March to recent comments on several of their papers are questionable, misleading and outright wrong.

Keywords and phrases: Ising lattice, quaternionic phase, quaternionic Hilbert space, conformal invariance in 3 dimensions

1. The series test is all-deciding

In 2007 Zhang claimed to have an exact solution of the three-dimensional Ising model [1]. However, Zhang's "putative" solution is wrong failing the series test [2] and this has been backed up recently by a detailed elementary proof [3]. It is, therefore, surprising that Zhang and March [4–6] still defend the refuted claims of [1], even after a subsequent comment [7]. No further comments should be necessary, but being accused of perpetrating fraud at the end of section 2 of [4] and in section 4.4 of [6] I feel compelled to respond one more time with further details.

The proof given in [3] shows that βf, with f the free energy per site and $\beta = 1/k_{\mathrm{B}}T$ inversely proportional to the absolute temperature T, is analytic in β with a finite radius of convergence r, i.e.

$$\beta f = \sum_{i=0}^{\infty} a_i \beta^i, \quad |\beta| < r. \tag{1}$$

In Appendix A of [1] Zhang gave two different series expansions for $\log \lambda = -\beta f$ in powers of $\tanh K$ with $K = \beta J$ and from this it was already manifest that the "putative" solution has a series expansion different from the well-known high-temperature expansion, see also section 3.1 of [3] for more discussion.

To counter the series test failure, Zhang and March [5,6] claim that tests should be applied to f, which equals $-\infty$ at $\beta = 0$. They claim that this singularity invalidates the application of the series test at $\beta = 0$. However, this argument is misleading as

$$f = \frac{a_0}{\beta} + \sum_{i=1}^{\infty} a_i \beta^{i-1} \tag{2}$$

is a convergent Laurent series with the same coefficients, so that it has to obey the same series test. The simple pole at $\beta = 0$ is of no consequence, as in statistical mechanics the partition function per site z, $\log z = -\beta f$, is more fundamental. It relates to the normalization $Z = z^N$ of the Boltzmann–Gibbs canonical ensemble distribution of the Ising model with N sites, in the large-N limit.

The failure of the series test raises the question if one of the conjectures in [1] fails and it is now easy to show that this is the case.

2. Conjecture 1 is manifestly wrong

The original paper [1] has an error in the application of the Jordan–Wigner transformation pointed out in [8]. This error has only been corrected explicitly in a very recent paper [6], which makes it easy to pinpoint the error with Conjecture 1 [1, 6].

It is well-known that in the spinor representation of the orthogonal groups each element g can be written as a "fermionic Gaussian" of the form

$$g = \exp\biggl(\sum_i \sum_j A_{ij}\Gamma_i\Gamma_j\biggr), \quad A_{ji} = -A_{ij}, \tag{3}$$

with Clifford algebra elements satisfying $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\delta_{ij}$ and complex coefficients A_{ij}. In [1, 9] the Γ_i's are also written as P_i's and Q_i's and we note that the Γ_i's can be expressed as linear combinations of fermion creation and annihilation operators $c_i^\dagger$ and c_i. The spinor representation has been used in the Ising context by Kaufman [9] in 1949. The closure property of Lie groups says that any product or inverse of elements of the form (3) is again of the same fermionic Gaussian form.

The factors in (15) of [6] commute, compare (7a) in [1], so that $\boldsymbol{V}_3$ can be rewritten as

$$\boldsymbol{V}_3 = \exp\biggl(-\mathrm{i}K'' \sum_{j=1}^{nl} \Gamma_{2j}\biggl[\prod_{k=j+1}^{j+n-1} \mathrm{i}\Gamma_{2k-1}\Gamma_{2k}\biggr]\Gamma_{2j+2n-1}\biggr), \tag{4}$$

which is clearly not of the form (3) and, therefore, not an element of the group. Multiplying $\boldsymbol{V}_3$ with group elements $\boldsymbol{V}_2$, $\boldsymbol{V}_1$ and $\boldsymbol{V}_4'$, as given in (16), (17) and (18) in [6], cannot give a group element as implicitly claimed in [6]. Zhang and March violate a fundamental property of Lie groups. Hence, Conjecture 1 as stated in [6] fails.

3. Zeros of Z^{-1} are irrelevant

Zhang and March repeatedly [4–6] claim the importance of the zeros of Z^{-1}, with $Z = z^N$ the total partition function. However, unlike the complex Yang–Lee zeros of Z [10, 11], the zeros of Z^{-1} are irrelevant. For a finite number N of sites, Z is a finite Laurent polynomial in e^K and only can become infinite when $\mathrm{Re}K = \pm\infty$, i.e. zero-temperature type limits.

For the infinite system, $N \to \infty$, and finite K, the infinity of Z should be seen as just a manifestation of the thermodynamic limit, in which $z = Z^{1/N}$ remains finite. One can easily see that $\beta f < 0$ for K real, so that $Z = \mathrm{e}^{-N\beta f} = \infty$ for all real K when $N = \infty$. There is nothing special about the zeros of Z^{-1}. If Zhang and March want to claim differently they face inconsistencies even in the one-dimensional Ising model.

4. Other issues

Bringing up [12, 13] at great length is only a smoke screen, as [3] made no use of these references and provides an independent derivation. Zhang and March also misrepresent statements in section 3 of [7]: Setting $\beta = 1$ in [12, 13] is no loss of generality, as βf is only a function of $K = \beta J$. Having $J \equiv K$ and choosing a fixed $\bar{J}$ and a new $\beta = J/\bar{J} \not\equiv 1$, we can write $J = K = \beta\bar{J}$. Thus we recover the general case with both a fully variable β (including $\beta = 0$) and a new J (omitting the bar on $\bar{J}$). This is said in another equivalent way in section 3 of [7].

Next, Zhang and March fail to realize that $K_{\beta\phi'}(X, T)$ vanishes for $\beta = 0$, so that the inequality in [13] does not fail for $\beta = 0$, contrary to what is said in [4–6]. Also, this inequality plays no role in the proof of [3], so that bringing it up can only be seen as a diversion.

Finally, only in two dimensions is the conformal group infinite-dimensional, so that there is inconsistency in [6, 14] beyond the fact that these papers build on an erroneous solution of the 3D Ising model. That Zhang and March write $\mathrm{Re}|\mathrm{e}^{\mathrm{i}\phi_i}|$, the real part of a positive real number, is objectionable too.

5. Conclusion

As should already have been clear from [2, 8], Zhang's very long paper [1] and all the works building on it are in error. Some further errors have been shown explicitly above, including why Conjecture 1 does not hold.

Acknowledgments

The research was supported in part by NSF grant PHY 07-58139.

References

[1] Z.-D. Zhang, *Conjectures on the exact solution of three-dimensional (3D) simple orthorhombic Ising lattices*, Philos. Mag. **87** (2007), 5309–5419 [arXiv:cond-mat/0705.1045 (pp. 1–176)].

[2] F. Y. Wu, B. M. McCoy, M. E. Fisher, and L. Chayes, *Comment on a recent conjectured solution of the three-dimensional Ising model*, Philos. Mag. **88** (2008), 3093–3095; **89** (2009), 195 [arXiv:0811.3876].

[3] J. H. H. Perk, *Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 3 (2012), 45–59 [arXiv:1209.0731].

[4] Z.-D. Zhang and N. H. March, *Conformal invariance in the three dimensional (3D) Ising model and quaternionic geometric phase in quaternionic Hilbert space II. Response to "Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices" by J. H. H. Perk*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 3 (2012), 61–69 [arXiv:1209.3247].

[5] Z.-D. Zhang and N. H. March, *Conformal invariance in the three dimensional (3D) Ising model and quaternionic geometric phase in quaternionic Hilbert space III. Rejoinder to the Comment to the Response to "Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices" by J. H. H. Perk*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **63**, no. 1 (2013), xx–xx.

[6] Z.-D. Zhang and N. H. March, *Mathematical structure of the three-dimensional (3D) Ising model*, Chin. Phys. B **22** (2013), 030513 (15 pp.).

[7] J. H. H. Perk, *Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices II. Comment to the Response to "Erroneous solution of three-dimensional (3D) simple orthorhombic Ising lattices" by Z.-D. Zhang*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 3 (2012), 71–74 [arXiv:1209.0731].

[8] J. H. H. Perk, *Comment on 'Conjectures on exact solution of three-dimensional (3D) simple orthorhombic Ising lattices'*, Philos. Mag. **89** (2009), 761–764 [arXiv:0811.1802].

[9] B. Kaufman, *Crystal statistics. II. Partition functions evaluated by spinor analysis*, Phys. Rev. **76** (1949), 1232–1243.

[10] C. N. Yang and T. D. Lee, *Statistical theory of equations of state and phase transitions. I. Theory of condensation*, Phys. Rev. **87** (1952), 404–409.

[11] T. D. Lee and C. N. Yang, *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model*, Phys. Rev. **87** (1952), 410–419.

[12] J. L. Lebowitz and O. Penrose, *Analytic and clustering properties of thermodynamic functions and distribution functions for classical lattice and continuum systems*, Commun. Math. Phys. **11** (1968), 99–124.

[13] G. Gallavotti, S. Miracle-Solé, and D. W. Robinson, *Analyticity properties of a lattice gas*, Phys. Lett. A **25** (1967), 493–494.

[14] Z.-D. Zhang and N. H. March, *Conformal invariance in the three dimensional (3D) Ising model and quaternionic geometric phase in quaternionic Hilbert space*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 3 (2012), 35–44 [arXiv:1110.5527].

Department of Physics
Oklahoma State University
145 Physical Sciences
Stillwater, OK 74078-3072, USA
e-mail: perk@okstate.edu
jhhp@jperk.phy.okstate.edu

Presented by Leszek Wojtczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 6, 2012

BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA III

KOMENTARZ DO ODPOWIEDZI NA "BŁĘDNE ROZWIĄZANIE TRÓJWYMIAROWYCH (3D) PROSTOPADŁOŚCIENNYCH SIATEK ISINGA" Z.-D. ZHANGA

S t r e s z c z e n i e

Odpowiedź Zhanga i Marcha na niedawne uwagi do szeregu ich prac są wątpliwe, zwodnicze i zupełnie błędne.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2013 Vol. LXIII

Recherches sur les déformations no. 1

pp. 95–112

In memory of
Professor Promarz M. Tamrazov

Julian Ławrynowicz, Kiyoharu Nôno, Daiki Nagayama, and Osamu Suzuki

A METHOD OF NONCOMMUTATIVE GALOIS THEORY FOR CONSTRUCTION OF QUARK MODELS (KOBAYASHI-MASUKAWA MODEL) I

SUCCESSIVE GALOIS EXTENSIONS

Summary

Concepts of binary and ternary Galois extension are introduced and the gauge theory with the symmetry group of the Galois groups is developed. Concepts of binary and ternary Clifford algebras are developed and the corresponding Dirac operators and Klein-Gordon operators are associated. In the second part of the paper, by use of the Galois extension structure of su(3), the quark models of Gell'Man model will be constructed. By use of the binary extension of su(3), the Kobayashi-Masukawa model will be constructed.

Keywords and phrases: noncommutative Galois extension, ternary Clifford algebra, ternary Clifford analysis, quark model

Introduction

We know that we have anti-particles for particles as for elementary particles. Hence we have the binary duality structure for particles (Fig. 1). Next we proceed to quarks. We know that we have two kinds of quarks; (1) Binary particles (Mesons) and (2) ternary particles (Baryons) (Fig. 2). Hence we have both of binary and ternary structures for elementary particles. In this paper we introduce a concept of noncommutative Galois theory and give an understandings for these structures.

We summarize some basic facts on elementary particles. It is well known [4] that the quark model due to Gell'Mann can be understood by use of the representation theory of SU(3). The representation $3\otimes\bar{3}$ introduces the baryons which are generated by three flavours u, d and s quarks. The representation $3 \otimes 3 \otimes 3$ introduces the concept of colours. The complete description of the baryons is given by six flavours and their corresponding three colors (Fig. 10). In [5, 6, 7], Kerner has introduced a concept of ternary Clifford algebra and gave trials for the quark confinement by this method. He has introduced a concept of ternary Pauli-exclusion principle and tried to determine the possible quarks. He has also introduced a concept of ternary Clifford analysis and the corresponding ternary Dirac and Klein-Gordon operators.

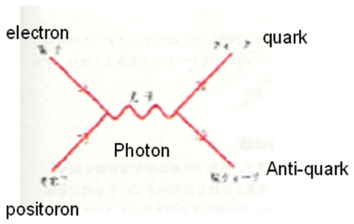

Fig. 1: Examples of binary duality structure for particles.

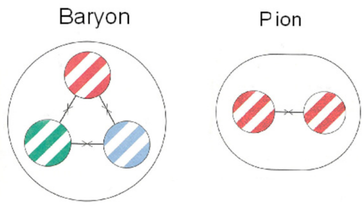

Fig. 2: Examples of binary and ternary particles.

In this paper, we develop a concept of noncommutative Galois extension theory and show that the noncommutative Galois theory can describe the basic facts on elementary particles. We introduce two kinds of Galois extensions. The first one is binary extension and the other is ternary extension. At first we analyze the non-commutative Galois extension structure on su(3) and find that it has the successive extension of binary and ternary extension: $R[\sqrt[2]{-1}, \sqrt[2]{-1}, \sqrt[3]{1}]$ (Theorem 1). The first extension describes the binary Clifford algebras and their binary Dirac operators (Theorem 2). The other can describe ternary Clifford algebra and the ternary Dirac operators quite well which are introduced by Kerner [see Definition 6, (50), (50'), (51)]. Next we proceed to quark models. (Part II of the paper; this journal, next issue).

1. Binary and ternary noncommutative Galois extensions

In this section we develop concepts of noncommutative Galois extensions of binary type and ternary type. As for another approach to Galois extensions, see [2, 10].

1.1. Definitions of non commutative Galois extensions

We develop a concept of noncommutative Galois extension. Let A be an algebra and A' be a subalgebra of A. We choose an element $\tau \in A$ with $\tau^k = 1$ and prepare the following elements:

(1) $$A'_1(\tau) = \{x\tau x' : x,\ x' \in A'\}, \quad A'_r(\tau) = A'_{r-1}(\tau)A'_1(\tau) \quad (r = 2, 3, \dots).$$

Introducing the elements

$$A_r(\tau) = \sum_{m=0}^{r} A'_r(\tau) \quad (A'_0(\tau) = A'), \tag{2}$$

we make the following

Definition 1. **(1)** We take an element $\tau \in A$ with the condition $\tau^k = 1$. We call $A_k(\tau)$ *k-nary* noncommutative Galois extension, when

$$A_r(\tau) \neq A_{r+1}(\tau) \quad (r = 0, 1, 2, \dots,\ k-2) \quad \text{and} \quad A_{k-1}(\tau) = A_k(\tau).$$

The extension is called *proper* when $\tau^\rho \notin A'$ $(\rho = 1, \dots,\ k-1)$. In the case where $k = 2$ (resp. $k = 3$), it is called *binary* (resp. *ternary*) *extension*, respectively.

(2) When $A'_1(\tau)A'_1(\tau)\dots A'_1(\tau) = {A'}^{(\pm)}_r(\tau)$ (the r-times product) and ${A'}^{(\pm)}_r(\tau) = \{x\tau^r x' : x,\ x' \in A'\}$, we consider

$${A'}^{(\pm)}_r(\tau) = \sum_{i=0}^{r} {A'}^{(\pm)}_i(\tau). \tag{3}$$

We can define the *bimodule k-nary extension* $A^{(\pm)}_k(\tau)$ replacing $A_r(\tau)$ with $A^{(\pm)}_r(\tau)$. The following extension is called *right* (resp. *left*) *module noncommutative Galois extension of k-nary type* respectively:

$$\begin{aligned} {A'}^{(+)}_k(\tau) &= \left\{ \sum_{\rho=0}^{k-1} \tau^\rho \zeta_\rho | \zeta_\rho \in A' \right\}, \\ {A'}^{(-)}_k(\tau) &= \left\{ \sum_{\rho=0}^{k-1} \zeta'_\rho \tau^\rho | \zeta'_\rho \in A' \right\}. \end{aligned} \tag{4}$$

(3) We assume that we have an element $g \in A$ with $g^k = 1$. Then we can define the commutative Galois extension which is called *adjoint Galois extension* when we have linear subspaces V_i $(i = 1, 2, \dots, k)$ of A satisfying the condition

$$A = V_1 \oplus \dots \oplus V_k, \quad V_i = \mathrm{Ad}_{g^i} V_1. \tag{5}$$

In this paper we denote one of k-nary noncommutative Galois extensions in (2) and (3) (in Definition 1) by $A'[\tau]$ simply. Moreover we denote it by $A'[\sqrt[k]{1}]$ when the extension τ with $\tau^k = 1$ is given.

Obviously, we have

Proposition 1. 1) *The relation*

$$A'_1(\tau)A'_1(\tau) = {A'}^{(\pm)}_2(\tau)$$

implies that

$${A'}^{(\pm)}_r(\tau)A'_1(\tau) = {A'}^{(\pm)}_{r+1}(\tau).$$

2) *When the condition:*

$$x\tau = \tau x' \ (\exists x' \in A') \quad \textit{for} \quad \forall x \in A'$$

is satisfied, then the extensions (1), (2), (3) *are identical each other.*

1.2. Basic notations on Galois extensions

We prepare some basic notations on Galois extensions.

1) We take two extensions $A'[\tau]$ $(\tau^k = 1)$ and $A'[\tau']$ $({\tau'}^k = 1)$. They are equivalent each other when there exists an isomorphism

$$\theta : A'[\tau] \to A'[\tau'] \quad \text{satisfying} \quad \theta_{|A'} = 1_\theta. \tag{6}$$

2) When the isomorphism $\theta : A'[\tau] \to A'[\tau']$ is given by a multiplication operator:

$$\theta^j \tau^j = {\tau'}^j (\theta^j \in A'[\tau]) \quad \text{and} \quad \theta^j x = x \ (x \in A') \quad (j = 1, 2, \ldots, k-1), \tag{6$'$}$$

it is called θ-*equivalent*.

3) When the automorphism is given by the adjoint operator: $\mathrm{Ad}_g : A'[\tau] \to A'[\tau']$

$$\mathrm{Ad}_g \xi = g \xi g^{-1} \ (g \in A'[\tau]) \quad (xg = gx \quad \text{for} \quad x \in A'), \tag{7}$$

it is called Ad-*equivalent*.

4) When $A'[\tau] = A'[\tau']$ holds, they are called identical. We notice that we have the identical extension: $A'[\tau_1] = A'[\tau_1^2]$ for an arbitrary ternary extension.

5) We introduce a concept of Galois group for an extension $A = A'[\tau]$. The automorphism group of A whose elements act as identity on A' is called *Galois group of the extension*. The automorphism $\theta : A'[\tau] \to A'[\tau]$ arising from multiplication $\theta(\xi) = \lambda\xi$ $(\lambda \in A)$ or $\theta(\xi) = \mathrm{Ad}_g \xi$ gives an example of element of the Galois group; (6). The Galois group of $A = A'[\tau]$ is denoted by $G = G(A'[\tau])$. We can introduce a concept of hierarchy structure of the extension. For a subalgebra A' with $A_0 \subseteq A' \subseteq A_1$, we consider a subgroup $G' \subset G_1$ which preserves the identity operation on A' which is called the Galois subgroup of A' and denoted by $G' = G(A')$. Then for an increasing sequence of subalgebras: $A_0 \subseteq A_1' \subseteq A_2' \subset \ldots \subset A_1$, we can find decreasing sequence $G \supseteq G_1' \supseteq G_2' \supseteq \ldots \supseteq G_1$ $(= e)$. We can also find the duality structure:

$$\begin{gathered} A_0 \subseteq A_1' \subseteq A_2' \subset \ldots \subset A_1 \\ \Updownarrow \\ G \supseteq G_1' \supseteq G_2' \supseteq \ldots \supseteq G_1 \ (= e). \end{gathered} \tag{8}$$

We shall describe the hierarchy structure for special ternary extensions.

2. The gauge theory of elementary particles with noncommutative Galois group

Choosing a Galois group as the gauge group, we make a gauge theory. We formulate the gauge theory on a space-time manifold M and later restrict our consideration to a single local coordinate. The gauge theory on a general manifold and its geometry will be discussed in a forthcoming paper. We choose the manifold M with a covering $\{U_\lambda\}$. We make a fibre bundle with Galois extension algebras as fibre and with gauge group of the Galois groups. Namely, we define the fibre space $\pi : F \to M$ by $\pi^{-1}(U_\lambda = U_\lambda \times A$, where A is a Galois extension algebra. The identification rules between local charts on the intersections are given as follows:

$$\pi^{-1}(U_\lambda) = U_\lambda \times A \qquad \Leftrightarrow \qquad \pi^{-1}(U_\mu) = U_\mu \times A,$$

$$(x_\lambda, \xi_\lambda) = (x_\mu, \xi_\mu) \tag{9}$$

if and only if

$$\xi_\lambda = g_{\lambda\mu}\xi_\mu g_{\lambda\mu}^{-1}, \quad x_\lambda = x_{\lambda\mu}(x_\mu) \tag{10}$$

where $g_{\lambda\mu}$ is a Galois group element on $U_\lambda \cap U_\mu$. In the following we restrict ourselves to binary and ternary extensions algebras separately.

2.1. Gauge theory of binary Galois extensions

At first we restrict ourselves to the gauge theory on a single coordinate. Hence we describe them without suffixes. We take a binary Galois extension $A = A_0[\sqrt{-1}]$ over an algebra A_0. We denote the generator H of the Galois group G. Then we have the following decomposition $A = A_0' \oplus A_0''$:

$$A_0' = K[e_1, e_2, \ldots, e_m] \ (\cong A_0), \qquad A_0'' = K[e_1', e_2', \ldots, e_m'], \tag{11}$$

where $(e_1, e_2, \ldots, e_m)$ is a system of generators of A_0 and $e_i' = He_iH^{-1}$. We assume that we have a function space L_0 on the space-time where the algebra acts: $A : L_0 \to L_0$. Then we have the extension of the function spaces L $(= L_0' + L_0'')$ with the gauge transform: $\tau : H \times L \to L$:

$$\tau(H, \psi) \to H\psi. \tag{12}$$

We notice that $\tau(L_0') = L_0''$ and $\tau(L_0'') = L_0'$. We consider the following linear spaces which are called basic Galois elements:

$$\begin{cases} A_0'(\theta_1, \theta_2, \ldots, \theta_m) = \{\theta_1 e_1 + \theta_2 e_2 + \ldots + \theta_m e_m | \theta_1, \theta_2, \ldots, \theta_m \in K\}, \\ \\ A_0''(\theta_1, \theta_2, \ldots, \theta_m) = \{\theta_1 e_1' + \theta_2 e_2' + \ldots + \theta_m e_m' | \theta_1, \theta_2, \ldots, \theta_m \in K\}. \end{cases} \tag{13}$$

Then we can obtain the gauge invariant spaces which may supply the observables in physics: As examples, we may take

$$\left\{\begin{aligned} \tilde{A}_0' &= A_0'(\theta_1, \theta_2, \dots, \theta_m) + A_0''(\theta_1, \theta_2, \dots, \theta_m), \\ \tilde{A}_0'' &= A_0'(\theta_1, \theta_2, \dots, \theta_m) A_0''(\theta_1, \theta_2, \dots, \theta_m). \end{aligned}\right. \tag{14}$$

By use of local data $\{A_{0,\lambda}', A_{0,\lambda}'', H_\lambda : g_{\lambda\omega}^{(2)}\}$, we formulate the gauge theory of binary extensions. We shall describe mesons in quark models in this framework.

2.2. Gauge theory of ternary Galois extensions

In an analogous manner, we can formulate a gauge theory of ternary extensions. We take a ternary Galois extension over an algebra A_0: $A = A_0[\sqrt[3]{1}]$. We denote the generator of the Galois group by G . Then we have the following decomposition:

$$A = A_0' \oplus A_0'' \oplus A_0'''. \tag{15}$$

We denote the generators of A by $e_1, e_2, \dots, e_m$. Then we have

$$\begin{aligned} A_0' &= K[e_1, e_2, \dots, e_m] \quad (\cong A_0), \\ A_0'' &= K[e_1', e_2', \dots, e_m'] \quad (\cong A_0), \\ A_0''' &= K[e_1'', e_2'', \dots, e_m''] \quad (\cong A_0), \end{aligned} \tag{16}$$

where $e_i' = Ge_iG^{-1}$, $e_i'' = Ge_i'G^{-1}$. We assume that we have the operant space L_0 where A_0 acts as $A : L_0 \to L_0$ with (15). Then we have the following operant spaces over the extension:

$$L = L_0' \oplus L_0'' \oplus L_0'''.$$

Thus we have the gauge transform: $\mu : \mathbf{G} \times L_0 \to L_0$:

$$G \times \psi \to G\psi. \tag{17}$$

We can see that

$$\mu(L_0') = L_0'', \quad \mu(L_0'') = L_0''', \quad \mu(L_0''') = L_0.$$

We can obtain the following linear elements:

$$A_0'(\theta_1, \theta_2, \dots, \theta_m), \quad A_0''(\theta_1, \theta_2, \dots, \theta_m), \quad A_0'''(\theta_1, \theta_2, \dots, \theta_m) \tag{18}$$

and their gauge invariant spaces:

$$\left\{\begin{aligned} &A_0'(\theta_1, \theta_2, \dots, \theta_m) \oplus A_0''(\theta_1, \theta_2, \dots, \theta_m) \oplus A_0'''(\theta_1, \theta_2, \dots, \theta_m) \\ &= \sum_{(i,j)=(',\,'',\,''')} A_0^i(\theta_1, \theta_2, \dots, \theta_m) A_0^j(\theta_1, \theta_2, \dots, \theta_m) \\ &\times\ A_0'(\theta_1, \theta_2, \dots, \theta_m) A_0''(\theta_1, \theta_2, \dots, \theta_m) A_0'''(\theta_1, \theta_2, \dots, \theta_m). \end{aligned}\right. \tag{19}$$

By use of local data $\{A_{0,\lambda}', A_{0,\lambda}'', A_{0,\lambda}''', H_\lambda : g_{\lambda\omega}^{(3)}\}$, we formulate the gauge theory of ternary extensions. This will be discussed for baryons in quark models.

2.3. Galois extension of binary and ternary Clifford type

Next we proceed to special realizations of gauge spaces. Setting

$$\text{(20)}\qquad \begin{cases} D_0'(\theta_1, \theta_2, \dots, \theta_m) = e_1\theta_1 + \theta_2 e_2 + \dots + \theta_m e_m, \\ \\ D_0''(\theta_1, \theta_2, \dots, \theta_m) = \theta_1 e_1' + \theta_2 e_2' + \dots + \theta_m e_m', \end{cases}$$

we introduce the concept of binary extension of Clifford type, when the spaces satisfy the condition

$$\text{(21)}\qquad D_0' D_0''(\theta_1, \theta_2, \dots, \theta_m) = ((\theta_1^2, \theta_2^2, \dots, \theta_m^2) \otimes 1.$$

Setting

$$\text{(22)}\qquad \begin{cases} D_0'(\theta_1, \theta_2, \dots, \theta_m) = (\theta_1 1 + \theta_2 G + .\theta_3 G^2), \\ \\ D_0'(\theta_1, \theta_2, \dots, \theta_m) = (\theta_1 1 + \theta_2 j^2 G + \theta_3 j G^2), \\ \\ D_0'(\theta_1, \theta_2, \dots, \theta_m) = (\theta_1 1 + \theta_2 j G + .\theta_3 j^2 G^2), \end{cases}$$

we introduce the ternary extension of ternary Clifford type (of the first kind):

$$\text{(23)}\qquad D_0' D_0'' D_0'''(\theta_1, \theta_2, \theta_3) = \theta_1^2 + \theta_2^2 + \theta_3^2 - 3\theta_1\theta_2\theta_3) \otimes 1.$$

2.4. Statements of main results of the paper

We state main results of this paper:

(0) We formulate the physics of quark theory as the invariant spaces with respect to the Galois group.

(1) The binary and ternary Clifford algebras are introduced and the Dirac operators are associated. We can find a successive extension of binary and ternary extensions on su(3) satisfying required properties (Theorem 1).

(2) For a binary Clifford algebra there exists a sequence of noncommutative binary Galois extensions of the real field $\mathbb{R}$ which realize the given Clifford algebra (Theorem 2).

(3) For the case of a successive extension of binary and ternary extensions, we can find a duality structure between binary and ternary Galois extensions. By this we can show the structures of mesons and baryons (Theorem 3).

(4) By the following sequences of extensions, we can derive the so-called Gell Mann model and Kobayashi-Masukawa model in a unified manner (Theorem 4):

$$\begin{aligned} \mathbb{R} \quad &\Rightarrow \quad R[\sqrt{1}] \Rightarrow R\left[\sqrt[2]{-1}, \sqrt[2]{-1}\right] \Rightarrow R\left[\sqrt[2]{-1}, \sqrt[2]{-1}, \sqrt[3]{1}\right] \\ &\Rightarrow \quad R\left[\sqrt[2]{-1}, \sqrt[2]{-1}, \sqrt[3]{1}, \sqrt[2]{-1}\right] \Rightarrow R\left[\sqrt[2]{-1}, \sqrt[2]{-1}, \sqrt[3]{1}, \sqrt[2]{-1}, \sqrt[3]{1}\right]. \end{aligned}$$

3. Standard construction of successive Galois extensions

In this section we introduce a concept of successive extensions of binary and ternary extensions and discuss their standard constructions.

3.1. Successive extensions

We consider successive Galois extensions. We take an extension:

$$A_1 = A_0[\tau_1](\tau_1^k = 1)$$

at first. Then we consider the extension

$$A_2 = A_1[\tau_2](\tau_2^{k'} = 1)$$

which is called *successive extension of k-nary and k'-nary extensions* and is denoted by $A_0[\tau_1, \tau_2]$. As a special successive extension, we can make the tensor product extension of bimodule type:

$$A_2 = A_0[\tau_1 \otimes \tau_2], \quad A_2 = \left\{ \sum x_{i,j} \tau_1^i \otimes \tau_2^j x'_{i,j} | x_{i,j}, x'_{i,j} \in A_0 \right\}. \tag{24}$$

The successive extensions of general type can be defined in a completely analogous manner. The quaternion algebra is the tensor product extension (or cross product) of the binary extensions; see (28).

3.2. Binary/ternary successive extensions

We introduce standard constructions of successive binary and ternary Galois extensions. We treat the following three kinds of successive extensions:

$$\begin{aligned} &(\mathbf{5}) \quad R[\sqrt[2]{I}, \sqrt[2]{I}, \\ &(\mathbf{6}) \quad R[\sqrt[2]{I}, \sqrt[3]{I}, \\ &(\mathbf{7}) \quad R[\sqrt[3]{I}, \sqrt[3]{I}. \end{aligned} \tag{25}$$

3.2.1. The basic construction by complex numbers algebra/quaternion algebra

At first we notice that a binary successive extension is not unique. In fact, we can obtain a commutative extension by the successive extension by complex numbers and a noncommutative extensions by quaternion algebra:

$$R[\sqrt[2]{-I}, \sqrt[2]{-I}] \cong \left\{ \begin{array}{c} \mathbb{C} \otimes \mathbb{C}, \\ \mathbb{H}. \end{array} \right. \tag{26}$$

The first one is the complex number $R[\sqrt{-1}]$:

$$R[\sqrt{-1}] = \left\{ \theta_1 1 + \theta_2 \sqrt{-1} | \theta_1, \theta_2 \in R \right\} = \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix} | \theta_1, \theta_2 \in R \right\}. \tag{27}$$

The quaternion numbers are obtained by the left module, or right module noncommutative Galois extension of the complex numbers:

$$
(28)\quad \begin{aligned} C[\sqrt{-1_2} &= \left\{\theta_1 1 + \theta_2\sqrt{-1_2} | \theta_1, \theta_2 \in C\right\} = \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix} | \theta_1, \theta_2 \in C \right\} \\ &= \left\{ \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ -\theta_2 & \theta_1 & \theta_4 & -\theta_3 \\ -\theta_3 & -\theta_4 & \theta_1 & \theta_2 \\ -\theta_4 & \theta_3 & -\theta_2 & \theta_1 \end{pmatrix} | \theta_1, \theta_2, \theta_3, \theta_4 \in R \right\}. \end{aligned}
$$

3.2.2. The basic construction by cubic algebra/nonion algebra

Next we proceed to the successive extension of cubic extensions and a noncommutative extensions by nonion algebra [6, 7]:

$$
(29)\quad R[\sqrt[3]{I_3}, \sqrt[3]{I_9} \cong \left\{ \begin{array}{c} \mathbb{B} \otimes \mathbb{B}, \\ \mathbb{V}. \end{array} \right.
$$

We may understand that the cubic algebra and nonion algebra are the ternary counterparts complex and quaternion, respectively. We begin with a basic ternary Galois extension: Setting

$$
(30)\quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1_3 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

we introduce a ternary extension $\mathbb{B}$ ($= R[T_3]$) over $\mathbb{R}$ which is called *cubic algebra*:

$$
(31)\quad \mathbb{B} = \{\theta_1 T_1 + \theta_2 T_2 + \theta_3 T_3 | \theta_1, \theta_2, \theta_3 \in \mathbb{R}\},
$$

Then we have the commutative ternary Galois extension which is isomorphic to the cubic root numbers:

$$
(32)\quad \mathbb{B}/I \cong \left\{\theta_1 1 + \theta_2 \mathbf{j} + \theta_3 \mathbf{j}^2 | \theta_1, \theta_2, \theta_3 \in \mathbb{R}\right\},
$$

where

$$
\mathbf{j}^3 = 1 \quad \text{and} \quad I = \{\theta(T_1 + T_2 + T_3), \theta \in \mathbb{R}\}.
$$

We call the algebra which is generated by two of the following three elements *nonion algebra* [6]:

$$
(33)\quad Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix},
$$

We can see that the linear basis over the complex field can be given as follows:

$$Q_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\bar{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j}^2 & 0 & 0 \\ 0 & \mathbf{j} & 0 \end{pmatrix}, \quad \bar{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j} & 0 & 0 \\ 0 & \mathbf{j}^2 & 0 \end{pmatrix}, \quad \bar{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{34}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \end{pmatrix}.$$

The following extension of bimodule type over $B[j]$, where $\mathbb{B} = R[T_2]$, are obtained, which are called basic extensions:

$$\begin{cases} A[R] \;= \{xR_1x' + yR_2y' + zR_3z' | x, y, z, x', y', z' \in B[\mathbf{j}]\}, \\[2ex] A[Q_i] = \{xR_1x' + yQ_iy' + zQ_i^2z' | x, y, z, x', y', z' \in B[\mathbf{j}]\} \ (i = 1, 2, 3), \\[2ex] A[\bar{Q}_i] = \{xR_1x' + y\bar{Q}_iy' + z\bar{Q}_i^2z' | x, y, z, x', y', z' \in B[\mathbf{j}]\} \ (i = 1, 2, 3). \end{cases} \tag{35}$$

Proposition 2. *We have*

$$B[\mathbf{j}] = A[R] \quad \textit{and} \quad N = A[Q_i] = A[\bar{Q}_i] \ (i = 1, 2, 3).$$

Hence the nonion algebra N has bimodule ternary Galois extensions over $B[\mathbf{j}]$.

We can prove the proposition by use of the product tables of nonion algebra and cubic algebra.

Tab. 1. The $(Q, \bar{Q}, R)$-matrices product table.

	Q_1	Q_2	Q_3	$\overline{Q}_1$	$\overline{Q}_2$	$\overline{Q}_3$	R_1	R_2	R_3
Q_1	$\overline{Q}_1$	$\mathbf{j}^2\overline{Q}_3$	$\mathbf{j}\overline{Q}_2$	R_1	$\mathbf{j}^2R_3$	$\mathbf{j}R_2$	Q_1	Q_2	Q_3
Q_2	$\mathbf{j}\overline{Q}_3$	$\overline{Q}_2$	$\mathbf{j}^2\overline{Q}_1$	$\mathbf{j}R_2$	R_1	$\mathbf{j}^2R_3$	Q_2	Q_3	Q_1
Q_3	$\mathbf{j}^2\overline{Q}_2$	$\mathbf{j}\overline{Q}_1$	$\overline{Q}_3$	$\mathbf{j}^2R_3$	$\mathbf{j}R_2$	R_1	Q_3	Q_1	Q_2
$\overline{Q}_1$	R_1	R_2	R_3	Q_1	$\mathbf{j}^2Q_3$	$\mathbf{j}Q_2$	$\overline{Q}_1$	$\mathbf{j}^2\overline{Q}_3$	$\mathbf{j}\overline{Q}_2$
$\overline{Q}_2$	R_3	R_1	R_2	$\mathbf{j}Q_3$	Q_2	$\mathbf{j}^2Q_1$	$\overline{Q}_2$	$\mathbf{j}^2\overline{Q}_1$	$\mathbf{j}\overline{Q}_3$
$\overline{Q}_3$	R_2	R_3	R_1	$\mathbf{j}^2Q_2$	$\mathbf{j}Q_1$	Q_3	$\overline{Q}_3$	$\mathbf{j}^2\overline{Q}_2$	$\mathbf{j}\overline{Q}_1$
R_1	Q_1	Q_2	Q_3	$\overline{Q}_1$	$\overline{Q}_2$	$\overline{Q}_3$	R_1	R_2	R_3
R_2	$\mathbf{j}^2Q_2$	$\mathbf{j}^2Q_3$	$\mathbf{j}^2Q_1$	$\overline{Q}_3$	$\overline{Q}_1$	$\overline{Q}_2$	R_2	R_3	R_1
R_3	$\mathbf{j}Q_3$	$\mathbf{j}Q_1$	$\mathbf{j}Q_2$	$\overline{Q}_2$	$\overline{Q}_3$	$\overline{Q}_1$	R_3	R_1	R_2

Tab. 2. The T)-matrices product table.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	T_1	T_2	T_3	T_4	T_5	T_6
T_2	T_2	T_3	T_1	T_5	T_6	T_4
T_3	T_3	T_1	T_2	T_6	T_4	T_5
T_4	T_4	T_6	T_5	T_1	T_3	T_2
T_5	T_5	T_4	T_6	T_2	T_1	T_3
T_6	T_6	T_5	T_4	T_3	T_2	T_1

3.2.3. The basic construction by su(3)

In turn we treat a successive extension of binary and ternary Galois extensions. We notice that the successive extension is not unique. In fact, we can obtain a commutative extension by the successive extension of complex number extensions and a commutative cubic extensions:

$$R\left[\sqrt[2]{I_3}, \sqrt[3]{I_9}\right] \approx \begin{cases} \mathbb{C}\otimes\mathbb{B}, \\ \mathrm{su}(3). \end{cases} \tag{36}$$

Next we discuss the structure of the Galois extension for su(3). At first we recall the basis of the algebra which are called Gell-Mann basis [4].

$$f_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(37)

$$f_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad f_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

We also consider the linear space L_1 generated by three elements

(38)

$$L_1: \quad e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and introduce the following two linear spaces L_2 and L_3:

$$(38)' \quad L_2: \quad e_1' = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad e_2' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3' = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix},$$

$$(38)'' \quad L_3: \quad e_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad e_2'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_3'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},$$

Remark 1. We notice the following relation:

$$f_8 = \frac{1}{\sqrt{3}}(e_3' + e_3'').$$

Hence omitting one of e_3, e_3', e_3'' we can obtain a system of basis of su(3).

Then we can prove the following theorem which plays the fundamental role in this paper:

Theorem 1. *We can find a successive extension of binary and ternary extensions on* su(3) *with the following properties:*

(**8**) su(3) *has the adjoint noncommutative Galois extension of bimodule type over the subalgebra A' generated by $\{e_1, e_2, e_3\}$.*

(**9**) *After the central extension, it is isomorphic to the quaternion algebra.*

(**10**) su(3) *is a binary Galois extension over $\mathbb{B}$, $\mathbb{B} = R[e_3]$.*

(**11**) su(3) *is a ternary Galois extension over* su(2)*.*

Remark 2. We can describe the structures in the following Fig. 3.

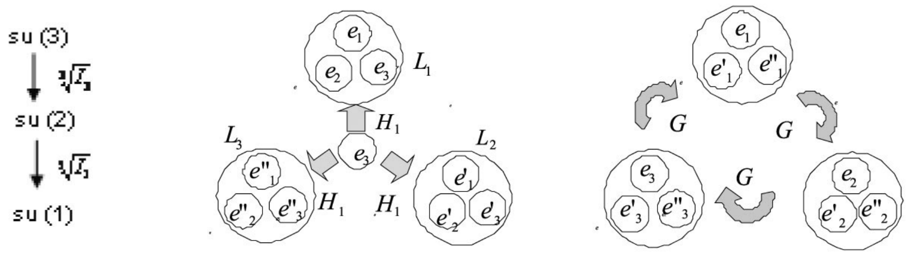

Fig. 3: Schematic comparison of the noncommutative Galois extension structures on su(3).

Proof. **Ad (8)**. In order to prove the first assertion it is enough to prove

Proposition 3. *We have the following adjoint representation on L_i $(i = 1, 2, 3)$; see* (**5**)*:*

$$(39) \qquad \begin{cases} He_1H^{-1} = -e_2, \quad He_2H^{-1} = e_1, \quad He_3H^{-1} = e_3, \\ H'e_1'H'^{-1} = -e_2', \quad H'e_2'H'^{-1} = -e_1', \quad H'e_3H'^{-1} = e_3', \\ H'e_1''H'^{-1} = e_2'', \quad H'e_2''H'^{-1} = e_1'', \quad H'e_3''H'^{-1} = e_3'', \end{cases}$$

where

$$(40) \qquad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad H' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Ad (9). We notice the following commutation relations:

$$(41)\qquad \begin{cases} e_1^2 = e_2^2 = e_3^2 = -1, \\ e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \end{cases}$$

After the extension by e_0 (= diag $[1,1,0]$), we have the Clifford algebra which is isomorphic to the quaternion algebra.

Ad (10). The assertion follows from the construction of quaternion, see (28).

Ad (11). In order to prove the assertion, we verify the following

Proposition 4. $\{e_i, e_i', e_i''\}$ $(i = 1,2,3)$ *satisfies the condition*

$$(42)\qquad \begin{cases} G e_k G^{-1} = e_k'' \quad (k = 1,2,3), \quad G e_k' G^{-1} = e_k \quad (k = 1,2,3), \\ \\ G e_k'' G^{-1} = e_k' \quad (k = 1,2), \quad G_1 e_3'' G^{-1} = -e_3', \end{cases}$$

where

$$(42)'\qquad G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence we can define the ternary adjoint extension.

4. Binary and ternary Dirac operators of noncommutative Galois extensions

In this section we discuss relationships between binary/ternary Clifford algebra and binary/ternary noncommutative Galois extension.

4.1. Noncommutative Galois theory of binary type and binary Clifford algebra

We show that a binary Clifford algebra introduces a binary Galois extension. Here we are concerned with Clifford algebras $Cl_p(\mathbb{C})$ with negative signature. We begin with the following definition:

Definition 2. (1) We take an algebra with generators: $T_1, T_2, \ldots, T_n$. A pair (T_a, T_b) $(a \neq b)$ is called (*binary*) *Clifford pair* when it satisfies the condition

$$(43)\qquad T_a T_b + T_b T_a = -2\delta^{ab} I_n.$$

(2) An algebra with generators $T_1, T_2, \ldots, T_n$ is called *binary Clifford algebra* when every pair is a binary Clifford pair.

Remark 3. We notice that Galois extensions do not necessarily have a structure of Clifford algebra. We recall the following two kinds of successive extensions of real numbers: One is commutative and the other is noncommutative; see (28):

$$R\left[\sqrt[2]{I_2},\sqrt[2]{I_2}\right] \approxeq \left\{ \begin{array}{l} \mathbb{C}\times\mathbb{C}, \\ \mathbb{H}. \end{array}\right. \tag{44}$$

We can find only one Clifford pair $\{e_1, e_4\}$ for $\mathbb{C}\times\mathbb{C}$ as follows:

$$\begin{aligned} \mathbb{C}\times\mathbb{C} &= \{x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 | x_1, x_2, \ldots, x_4 \in \mathbb{R}\} \\ &= \left\{ x_1\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + x_2\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} + x_3\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} + x_4\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\}, \end{aligned} \tag{45}$$

where I $(\in M_2(\mathbb{R}))$ is the identity matrix and J $(\in M_2(\mathbb{R}))$ is the complex structure. We see that every pair $\{e_i, e_j\}$ $(i, j \neq 1, i \neq j)$ for $\mathbb{H}$ is a Clifford pair:

$$\begin{aligned} \mathbb{H} &= \{x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 | x_1, \ldots, x_4 \in \mathbb{R}\} \\ &= \left\{ x_1\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + x_2\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} + x_3\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} + x_4\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\}. \end{aligned} \tag{46}$$

We can prove the following theorem:

Theorem 2. *For a binary Clifford algebra A with generators $T_1, T_2, \ldots, T_n$, there exists a sequence of noncommutative binary Galois extensions of the real field $\mathbb{R}$ which realize the given Clifford algebra A. Namely we have the following sequences of binary Galois extensions $\{A_k | k = 1, 2, \ldots, m\}$:*

$$\begin{aligned} T_iT_j + T_jT_i = -2\delta_{ij}I_n \Rightarrow A_k = A_{k-1}\left[\sqrt[2]{-I}\right] \\ (k = 1, 2, \ldots, m),\ (A_m = A,\ A_0 = \mathbb{R}). \end{aligned} \tag{47}$$

Proof. We prove the assertion by the induction on m. The complex numbers can be obtained by the commutative Galois extensions of real numbers; see (27). We can give a construction of the Clifford algebras. Let $(T_1, T_2, \ldots, T_m)$ be a system of generators of a Clifford algebra A_m. Setting

$$\hat{T}_i = \begin{pmatrix} T_i & 0 \\ 0 & -T_i \end{pmatrix}\ (i = 1, 2, \ldots, n), \quad \hat{H}_{n+1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{48}$$

we have a Clifford algebra which is generated by $(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_n, \hat{H}_{n+1})$ on the one hand and we have the right (or left) module binary extension of A_m by $\hat{H}_{m+1}$ on the other hand. Hence we can obtain the desired assertion.

4.2. Dirac operators for noncommutative Galois extensions

Next we are concerned with the field operators of the Clifford algebra defined by Galois extension of binary type. Choosing $(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_m)$ and $\hat{H}_{m+1}$, we can introduce the following three operators on the m-dimensional Euclidean space:

(49)
$$
\begin{aligned}
D &= \hat{T}_1\frac{\partial}{\partial x_1} + \hat{T}_2\frac{\partial}{\partial x_2} + \ldots + \hat{T}_m\frac{\partial}{\partial x_m},\\
D^* &= \hat{H}_{m+1}\left(\hat{T}_1\frac{\partial}{\partial x_1} + \hat{T}_2\frac{\partial}{\partial x_2} + \ldots + \hat{T}_m\frac{\partial}{\partial x_m}\right).
\end{aligned}
$$

The operators are called *Dirac operator* and its *conjugate operator* for the extension and they satisfy the condition

$$\Delta D^* D = D D^* \Delta = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_m^2}\right) \otimes 1_m.$$

The operator is called the *binary Klein-Gordon operator* [Laplace operator].

4.3. Noncommutative Galois theory of ternary type and ternary Clifford algebra

In turn we shall be concerned with the structure of ternary Clifford algebra. We put the following definition:

Definition 3. If (T_a, T_b, T_c) satisfies the following conditions, we say that it generates a *ternary Clifford algebra*:

(50)
$$
\begin{aligned}
&T_aT_bT_c + T_bT_cT_a + T_cT_aT_b = \eta^{abc}1,\\
&\eta^{abc} = \eta^{bca} = \eta^{cab},\\
&\eta^{111} = \eta^{222} = \eta^{333} = 1, \quad \eta^{123} = \eta^{231} = \eta^{312} = \mathbf{j}^2,\\
&\eta^{321} = \eta^{213} = \eta^{123} = \mathbf{j},
\end{aligned}
$$

(50)′ $\quad T_aT_bT_c + \mathbf{j}T_bT_cT_a + \mathbf{j}^2T_cT_aT_b = 0$ (or $T_aT_bT_c + \mathbf{j}^2T_bT_cT_a + \mathbf{j}T_cT_aT_b = 0$),

where two of them are identical.

Choosing (T_a, T_b, T_c), we introduce the following three operators on the 3-dimensional Euclidean space:

(51)
$$
\begin{aligned}
D &= T_a\frac{\partial}{\partial x_a} + T_b\frac{\partial}{\partial x_b} + T_c\frac{\partial}{\partial x_c},\\
D^* &= T_a\frac{\partial}{\partial x_a} + \mathbf{j}^2T_b\frac{\partial}{\partial x_b} + \mathbf{j}T_c\frac{\partial}{\partial x_c},\\
D^{**} &= T_1\frac{\partial}{\partial x_a} + \mathbf{j}T_2\frac{\partial}{\partial x_b} + \mathbf{j}^2T_3\frac{\partial}{\partial x_c}.
\end{aligned}
$$

The operators are called *Dirac operator* and its *conjugate operators of the Galois type* when they satisfy the following condition:

$$\begin{aligned} &\Delta = DD^*D^{**}, \\ &\Delta = \left(\frac{\partial^3}{\partial x_a^3} + \frac{\partial^3}{\partial x_b^3} + \frac{\partial^3}{\partial x_c^3} - 3\frac{\partial^3}{\partial x_a \partial x_b \partial x_c}\right) \otimes 1. \end{aligned} \tag{52}$$

The operator Δ is called the *ternary Klein-Gordon operator*. We can prove the following

Proposition 5. *A triple* $\{T_a, T_b, T_c\}$ *determines a ternary Clifford algebra if and only if it determines ternary Dirac operators.*

The proof is a direct calculation and may be omitted. We make the following

Definition 4. We take an algebra with finite generators: $(T_1, T_2, \dots, T_n)$. A triple (T_a, T_b, T_c) is called *Clifford triple* when it generates a ternary Clifford algebra.

Hence we can define the ternary Dirac operators and the corresponding Klein-Gordon operator.

Example. The generators T_1, T_2, T_3 of the cubic algebra determine a ternary Clifford algebra; see (30). Hence it is a Clifford triple.

4.4. Totally ternary Dirac operator

Finally we introduce the totally ternary Dirac operator. We can see that not every triple is a ternary Clifford triple. Hence, choosing the total set of Clifford triples $\{S_a^{(j)}, S_b^{(j)}, S_c^{(j)} : j = 1, 2, \dots, M\}$, we can introduce the following Dirac operator which is called *total Dirac operator*:

$$D = \sum D^{(j)}, \quad D^* = \sum D^{(j)*}, \quad D^{**} = \sum D^{(j)**}, \tag{53}$$

where

$$D^{(j)}, D^{(j)*}, D^{(j)**} \quad (j = 1, 2, \dots, M)$$

are the Dirac operator and its conjugate operators of the ternary triples, and M is the number of Clifford triples. Introducing the product by taking the usual product only for ternary Clifford triples and defining other products to be zero, we can introduce the following *total Klein-Gordon operator*:

$$D \circ D^* \circ D^{**} = \sum \Delta^{(j)}, \tag{54}$$

where $\Delta^{(j)}$ is the Klein-Gordon operator for each triple.

References

[1] V. Abramov, R. Kerner, and B. Le Roy, *A $\mathbb{Z}_3$-graded generalization of supersymmetry*, J. Math. Phys. **38** (1997), 1650–1669.

[2] S. Eidelman et al., *Particle data group: Review of particle Physics*, Physics Letters B **592**, nos. 1–4 (2004), 1–5.

[3] F. de Meyer and E. Ingram, *Separable Algebras over Commutative Rings* (Springer Lecture Notes in Math.), Springer, Berlin–Göttingen 1971, 181 pp.

[4] M. Gell'Mann and Y. Ne'emann, *The Eight-fold Way*, W. A. Benjamin, New York–Amsterdam 1964.

[5] R. Kerner, *$\mathbb{Z}_3$-graded algebras and the cubic root of supersymmetry translations*, J. Math. Phys. **33** (1997), 403–411.

[6] R. Kerner, *The cubic chessboard*, Classical and Quantum Gravity **14** (1997), A203–A225.

[7] R. Kerner and O. Suzuki, *Internal symmetric groups of cubic algebra*, Internat. J. of Geom. Methods in Modern Phys. **8**, no. 7 (2011), 1487–1506.

[8] M. Kobayashi and T. Masukawa, *CP-violation in the renormalizable theory of weak interaction*, Prog. Th. Physics, **49**, no. 2 (1973), 652–657,

[9] J. Ławrynowicz, K. Nôno, D. Nagayama, and O. Suzuki, *Non-commutative Galois theory on Nonion algebra and* su(3) *and its application to constructions of quark models*, Soryuusironnkennkyuu, Yukawa Institute, Kyoto 2012, pp. 145–157.

[10] T. Nakayam and G. Azumaya, *Algebra (II)*, [in Japanese], Iwanami Publisher, Tokyo 1954.

Department of Solid State Physics
University of Łódź
Pomorska 153/156, PL-90-236 Łódź
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, P.O. Box 21
PL-00-956 Warszawa
Poland
e-mail: jlawryno @uni.lodz.pl

Department of Mathematics
Fukuoka University of Education
Munakata-shi, Fukuoka
Japan
e-mail:nouno@fukuoka-edu.ac.jp

Department of Computer
and System Analysis
College of Humanities and Sciences
Nihon University, Sakurajosui 3-25-40
156-8550 Setagaya-ku, Tokyo
Japan
e-mail: osuzuki@chs.nihon-u.ac.jp

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 29, 2012

METODA NIEPRZEMIENNEJ TEORII GALOIS DLA KONSTRUKCJI MODELI KWARKÓW (MODEL KOBAYASHIEGO-MASUKAWY) I KOLEJNE ROZSZERZENIA GALOIS

S t r e s z c z e n i e

Idea binarnego i ternarnego rozszerzenia Galois jest rozważana w kontekście teorii cechowania z grupą symetrii z grup Galois. Również pojęcia binarnej i ternarnej algebry Clifforda są rozważane i użyte do określenia stowarzyszonych operatorów Diraca i Kleina-Gordona. W drugiej części pracy, przez zastosowanie struktury rozszerzenia Galois algebry su(3), skonstruujemy modele kwarkowe modelu Gell-Manna. Z kolei, przez zastosowanie rozszerzenia binarnego algebry su(3), skonstruujemy model Kobayashiego-Masukawy.

Rapporteurs – Referees

Richard A. Carhart (Chicago)
Fray de Landa Castillo Alvarado (México, D.F.)
Stancho Dimiev (Sofia)
Paweł Domański (Poznań)
Mohamed Saladin El Nashie (London)
Janusz Garecki (Szczecin)
Jerzy Grzybowski (Poznań)
Ryszard Jajte (Łódź)
Zbigniew Jakubowski (Łódź)
Tomasz Kapitaniak (Łódź)
Grzegorz Karwasz (Toruń)
Leopold Koczan (Lublin)
Dominique Lambert (Namur)
Andrzej Łuczak (Łódź)
Cecylia Malinowska-Adamska (Łódź)
Stefano Marchiafava (Roma)
Leon Mikołajczyk (Łódź)
Marek Moneta (Łódź)
Yuval Ne'eman (Haifa)
Adam Paszkiewicz (Łódź)
Krzysztof Podlaski (Łódź)
Yaroslav G. Prytula (Kyiv)
Henryk Puszkarski (Poznań)
Jakub Rembieliński (Łódź)
Carlos Rentería Marcos (México, D.F.)
Lino F. Reséndis Ocampo (México, D.F.)
Stanisław Romanowski (Łódź)
Monica Roşiu (Craiova)
Jerzy Rutkowski (Łódź)
Ken-Ichi Sakan (Osaka)
Hideo Shimada (Sapporo)
Józef Siciak (Kraków)
Józef Szudy (Toruń)
Luis Manuel Tovar Sánchez (México, D.F.)
Francesco Succi (Roma)
Massimo Vaccaro (Salerno)
Anna Urbaniak-Kucharczyk (Łódź)
Włodzimierz Waliszewski (Łódź)
Grzegorz Wiatrowski (Łódź)
Władysław Wilczyński (Łódź)
Hassan Zahouani (Font Romeu)
Lawrence Zalcman (Ramat-Gan)
Natalia Zoriĭ (Kyiv)

CONTENU DU VOLUME LXIII, no. 2