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SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

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Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

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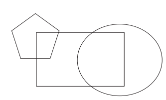

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References

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TABLE DES MATIÈRES

Professor
Claude Surry
* 2.12.1945 † 19.12.2013

Photo by Prof. Z.-D. Zhang

Chers amis Yolande, Fréderic et Gildas Surry,

C'est avec une immense tristesse que nous venons d'apprendre le décès de notre cher collègue et ami, Professeur Claude Surry. Il était non seulement un chercheur impliqué et professionnel, mais essentiellement, nous nous souvenons de lui en tant qu'un vrai ami – gentil, cordial, avec la riche personnalité pleine d'empathie.

Nous allons manquer de ses conseils amicaux, et de son sens de l'humour qui a toujours enrichi nos discussions autour d'une tasse de café ou d'un verre du vin.

Nous prions pour Claude et pour vous.

Nous espérons que vous recevrez la consolation chrétienne dans votre douleur et perte.

Dear Friends Yolande, Frederic and Gildas Surry,

It is with great pain and sorrow that we received your information about the death of our friend and colleague Professor Claude Surry.

He was not only an involved and highly professional researcher but, first of all, we remember Claude as a true friend – kind, warm and of rich and empathic personality.

We will miss his friendly advice and his rich sense of humour, always enriching our informal talks over a cup of coffee or a glass of wine.

We pray for Claude and for you.

We hope that you will receive Christian consolidation in your pain and loss.

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Dedicated to the memory of our Professors
on the occasion of the 60th anniversary
of appearance of their fundamental publications

Zbigniew Jerzy Jakubowski

SOME REMARKS ON THE ORIGINS OF COMPLEX ANALYSIS IN ŁÓDŹ

Summary

The article has a review and recollective character. Its guiding idea is a 60th anniversary of appearance of the work by Z. Charzyński entitled "Sur les fonctions univalentes bornées".

Keywords and phrases: complex analysis in Łódź, bounded univalent functions, Charzyński Z., Janowski W.

Professor Julian Ławrynowicz on June 18th, 2013, at the session of Mathematical Commission of the III Department of the Łódz Society of Mathematics, proposed a project to commemorate a 60th anniversary of Scientific School of Complex Analysis in Łódź, which was initiated by a fundamental Zygmunt Charzyński's work entitled "Sur les fonctions univalentes bornées", [1]. The project was accepted. It remained to set about to its realization. Obviously, three restrictions must have been taken into consideration: mathematicians researching in the field of complex analysis (which used to be called as analytic functions), a time of the formation of scientific institutions before and after the Second World War, the place Łódź. The mentioned criteria are not actually "sharp". For example, a given paper can concern couple branches of mathematics, can be written in the time when the author was working in Warsaw, and was published in the "Łódź time", has couple co-authors, or non-local author published a paper in a publisher in Łódź, etc. From the above remarks there arise an author's request of the present paper for understanding.

As it is well known, (e.g., [2]), in the time before the Second World War in Łódź there were running some institutions with educational and scientific character (including the department of Wolna Wszechnica Polska in Łódź). However, it is difficult to find among them significant scientific mathematical research, especially in the field of analytic functions. After the war, many mathematicians came to Łódź, where some of them for several years, only travelled to Łódź, a large group, however, remained permanently and of course the latter had a significant impact on the emergence and development of mathematics in Łódź. Among them, one should look for the pioneers of complex analysis.

Among the non-resident people it is worthy to mention Jerzy Popruženko, who worked in Łódź in years 1946–1952 and among his earlier works there are two: "Sur l'analycité des ensembles (A)", 1932, [3] and "Le principle de Dirichlet et les ensembles (A)", 1934, [4] concerning the theory of analytic functions. For couple years (1945–1954) dr. Hanna Szmuszkowicz worked in Educational College (Państwowa Wyższa Szkoła Pedagogiczna) and in University of Łódź (UŁ). In the sectional article by Z. Charzyński and L. Kaczmarek [5] there are mentioned her three papers [6,7] and [8] written together with S. Mazurkiewicz and her two own [9,10] concerning the theory of analytic functions, also published in the thirties. I also found the information that H. Szmuszkowicz gave lectures in UŁ on analytic functions. One of the mentioned work is entitled: "Sur les zéros des fonctions quasi-analytiques" (B), [8]. There was not found any information that J. Popruženko and H. Szmuszkowicz had influenced the development of analytic functions in Łódź, however the above mentioned facts are worthy to be recalled.

It is known (e.g., [2]) that a great significance on the development of mathematics in Technical University of Łódź had (despite the short stay) Professor Witold Pogorzelski. Among his doctoral students there were W. Krysicki (1950) and D. Sadowska (1956). In the literature there is relatively little of attention to the activity of W. Pogorzelski in the field of analytic functions. In the article by M. Biernacki and F. Leja (Biography of Mathematics in the ten year period of 1944–1954), Part VI. Analytic functions, [11]) there were mentioned four of his works. In volume XX.3 of Annales Polonici Mathematici, 1967–68, p. 352, among the mentioned twenty articles by W. Pogorzelski one can find references 17 and 19 [12, 13] concerning the classical theory of analytic functions.

It seems interesting that a resident of Łódź and professors's student, abovementioned Danuta Sadowska, has an article entitled: "Sur une probléme aux limites de la théorie des fonctions analytiques", 1960, [14]. W. Pogorzelski in 1954 at the conference in Łódź devoted to the theory of analytic functions [2], gave a talk on "Analytic functions and integral equations". He is also an author of "Mathematical analysis", which eighth part is devoted to "Functions with complex variable". (Warszawa, PWN, 1956, volume IV). It is also worth to pay attention on the article by J. Wolska-Bochenek on the occasion of 45th anniversary of scientific work of Professor Pogorzelski, [15].

Looking for the oldest traces of complex analysis in Łódź, one must not omit Zygmunt Zahorski. Z. Charzyński and L. Kaczmarek [5] listed five of his works, including: "On a problem of M. F. Leja", 1947 and "On zeros of quasi-analytic (B) functions", 1947 [16, 17]. In the aforementioned article [11] M. Biernacki and F. Leja write:

1) "The Leja's theory was completed in many details by Z. Zahorski".

2) "In a completely different approach Z. Zahorski dealt with singularities in his works on the borderland of analytic functions and real functions". In the sectional article by Z. Charzyński [18] there are also given the information about the Z. Zahorski's research in the complex domain. On the aforementioned conference (year 1954) Z. Zahorski gave a talk entitled "The application of resultanta to a certain extreme issue". We do not have the content of the paper, but the subject is close to the extreme issues of geometric function theory. More information about the Professor can be found in the J. S. Lipiński's article [19].

In the second half of the forties, Zygmunt Charzyński, Witold Janowski and also Romuald Zawadzki appear in Łódź (from Warsaw, Łowicz and Suwałki respectively). They are "rich" in experience after didactic and organisational practice and the first two are after work in higher education in Warsaw (until 1939). In Łódź at school Lucjan Siewierski worked (as it later turned out, he was a wonderful teacher). It turns out that all four had known each other from mathematical studies in the University of Warsaw. W. Janowski together with R. Zawadzki were in oflag in Woldenburg, where they organized the secret teaching of different branches of mathematics. Among the belongings that W. Janowski took after the liberation in 1945, there were his notes from the classes and the exams. This notes were useful later for a number of former officers. As one can observe, the "spark" was needed in order to form a positive activity. For example – a scientific one.

Z. Charzyński such a "spark" brought from Warsaw. In 1938 he started his work as a Professor W. Sierpiński's assistant in University of Warsaw. In the spring of 1939, he passed his master's examination. During the war, in addition to working as a clerk [20], he started the investigations concerning properties of holomorphic bounded and univalent functions. I do not remember if he explained why this issue was interesting for him, and possible to realize. He created his own variational method for the aforementioned functions, and in consequence, he obtained the differential-functional equation for the extreme functions with respect to the functionals dependent on the finite number of coefficients of the expansions in Taylor's series of such functions in the unit disk. He presented these results on Mathematical Congress in Wrocław (December 12–14, 1946), and the abstract of the article appeared in 1948, [21], whereas the full text of the work was published in the known series of Disertationes Mathematicae (Rozprawy Matematyczne), 1953, [1], therefore sixty years ago. This work played a crucial and inspiring role in the process of forming a strong group of mathematicians interested in the geometric theory of analytic functions and other problems of this theory.

In 1950, Z. Charzyński and W. Janowski (using the Z. Charzyński's variational method) transfer the Charzyński's result to the case of the functionals defined on the mentioned class of $S(M)$ functions of the form $f(z) = z + A_2z^2 + \cdots$ holomorphic univalent and bounded ($|f(z)| < M$ for $|z| < 1$; $M > 1$) dependent on the infinite number of coefficients, [22]. W. Janowski investigated a functional $\mathcal{F}(f) = \arg f(z)/z$, $f \in S(M)$ and using the primary theorem from the mentioned work [1] obtained the limits of the aforementioned functional for every $M > 1$ and arbitrarily fixed z, $0 < |z| < 1$, [23]. There are two co-authored articles from 1959 by Z. Charzyński and W. Janowski in the domain of values of coefficients A_2 and A_3 in the class $S(M)$ [24] and in the domain of values of functions in this class [25]. On their basis there are mentioned earlier the general results from the articles [1] and [22]. It is also worthy to mention the elaborated by W. Janowski method of solving the differential-functional equations in the paper [26] about the estimation of the functional $|f'(z)|$, $f \in S(M)$. Z. Charzyński believed that the presented here investigations were pioneer for these type of concrete issues.

It is worth noting that the mentioned general Z. Charzyński's and Z. Charzyński-W. Janowski's theorems were transferred to the case of subclass $S_r(M) \subset S(M)$ of the functions with all coefficients real. This was made by I. Dziubiński [27] and Z. Charzyński-H. Śmiałkówna [28]. Obviously, the Z. Charzyński's idea of variation had to be modified to the case of symmetric functions.

The mentioned four papers about the equations of "Charzyński type" had a lot of other applications, including the doctoral and habilitation thesis. One can list here the Charzyński-Tammi's hypothesis and its beautiful solution in L. Siewierski's work, 1960, [29]. The other was an interesting hypothesis antipodal to the previous, where there was a case of M sufficiently large and the coefficients with even indexes, [30]. L. Mikołajczyk determined the domain of values of coefficients A_2 and A_3 in the class of holomorphic bounded and univalent functions with all coefficients real, [31]. Z. Jakubowski investigated the different problems concerning the functional $|A_3 - \alpha A_2^2|$, $\alpha \in \mathbb{R}$, [32]. The above problem was inspired by the master's seminars, which in 1954–1955 were run by Z. Charzyński and W. Janowski (including the G. M. Gołuzin variational method), the Z. Charzyński's monographic lecture (the Löwner's method and its applications), and also the seminar for the (W. Janowski's) research workers, when as a master I had a talk about the paper [33] on the estimation of the coefficient A_3 in the class $S(M)$. In the latter work, as in many others, there was used again the Zygmunt Charzyński's theorem, [1].

In the discussed topic the famous Bieberbach conjecture is very important. In 1955 P. R. Garabedian and M. Schiffer, [34] determined the result in a case of coefficient A_4. However, the paper was very long (38 pages) and difficult to read. I remember when professor Charzyński talking about it, questioned whether the article is easy to be checked. During his stay in the United States, Z. Charzyński together with M. Schiffer published in 1960 two papers containing new proofs, the article [35] consists of five pages and can be presented during a master's seminar. These works have been accepted with universal acclaim and attracted renewed interest of the mentioned hypothesis in many centres around the world.

It is difficult in a short article to discuss all the research directions in the theory of complex analysis initiated by professor Charzyński, and also by professor Janowski and his students. For example one can list a paper [36] about the univalent algebraic and bounded functions – 1955 and its development in publications: [37] by L. Siewierski, [38], [39] co-authored with Janina Śladkowska, [40] by Romuald Zawadzki and in the Józef Janikowski's manuscript (about a uniformity of certain class of algebraic functions of third degree by a method with differential equations). Leon Mikołajczyk investigated for example the functions meromorphic univalent and bounded from below, [41], Izydor Dziubiński – the quasi-starlike functions, [42]. Obviously, a list of the doctoral dissertations is very long.

I hope that further recollective articles let us with satisfaction to recall the past sixty years of complex analysis in Łódź. It is also probably worth looking in the sectional articles, which were already published [20, 43–51], and one can think how much we owe our teachers. Obviously, a great significance for a development of complex analysis in Łódź had:

1) co-operated next conferences every four years after the mentioned conference on analytic functions in 1954,

2) a wide collaboration of "our seniors" with the centres of complex analysis in Cracow (F. Leja, ...) and Lublin (M. Biernacki, ...),

3) international cooperation (M. Schiffer, O. Tammi and many others).

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WSPOMNIENIA O POCZĄTKACH ANALIZY ZESPOLONEJ W ŁODZI

S t r e s z c z e n i e

Artykuł ma charakter przeglądowy i wspomnieniowy. Ideą przewodnią jest 60-ta rocznica ukazania się pracy Z. Charzyńskiego pt. "Sur les fonctions univalentes bornées".

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B U L L E T I N

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In memory of
Professor Luboš Valenta

Leszek Wojtczak and Štefan Zajac

ON THE VALENTA MODEL AND ITS ACTUALITY II

Summary

The model for ferromagnetic thin films originally introduced by Luboš Valenta at the level of molecular field approximation (MFA) and now modified in the case of layered nanoparticles described by means of the reactions field approach (RFA) is still of great interest for modern physics and technology.

In this context the present paper is a contribution to the previous one, devoted to the Valenta model and its actuality, I, published in *Bulletin de la Société des Scienices et des Lettres de Łódź; Série: Recherches sur les Déformations* (**25** (2005), 13, ŁTN, Łódź, 2005).

The recent results obtained within the modified version of the Valenta model were presented during the 15th Czech and Slovak Conference on Magnetism (Košice 2013) in poster form entitled "Topicality of the Valenta model for the magnetization in thin films and surfaces" (CSMAG'13 Abstracts, Košice, 2013, P4-01). For that reason we remember this fact in honor of P. J. Šafarik University in 50. anniversary of its foundation.

Keywords and phrases: ferromagnetic thin films, spin autocorrelation functions, Valenta model modified by Reaction Field Approach

1. Introduction

Fifty five years ago Luboš Valenta introduced the model for ferromagnetic thin films [1, 2–4, 5] which describes the spontaneous magnetization, its angular and spatial distributions leading to the construction of spin waves resonances, the calculations in terms of the order-disorder theory as well as the phase transitions including also the instability conditions.

The model was later extended to its more general form which is now known in literature as the Valenta model [1, 2–4, 6]. It is applicable not only to ferromagnetic thin films but it can also be applied to the description of the lattice thermodynamics as well as to electronic phenomena and order-disorder effects.

The Valenta model for magnetic films has been proceeded by the pioneering work [2] concerning the angular distribution of magnetization in one-dimensional toroid. The work has been founded as a good starting point for explanation of the surface deformed in rare-earth thin films with heliomagnetic structure. Mössbauer effect as well as neutron inelastic magnetic scattering certify non colinear distribution of magnetization in low dimensional solid magnetic systems.

Recently, the Valenta model is also applied to the description of nanoparticles when the extended form of the model is modified by RFA where the spin correlations are not neglected, in contrast to MFA when the spin correlations do not appear.

Thus, we introduce now the Valenta model modified by RFA [7–10] in order to test the fundamental parameters like the phase transition temperature $T^C_{\mathrm{RFA}}(n)$ and $T^C_{\mathrm{MFA}}(n)$ as well as the spin waves resonance linewidth $\Gamma^{\mathrm{RFA}}_{\tau h}$ and $\Gamma^{\mathrm{MFA}}_{\tau h}$ considered for each resonance peak $(\tau h)^*$ derived by means of a frequency spectrum which satisfies the difference equations of universal character [11, 12] for spin wave propagation (SWR) [13] or Green's function averages (GFA) [14] are related to the discussed coefficients in MFA [3] or RFA [9]. The symbol h denotes the wave vector in the plane perpendicular to the direction labelled by τ. In particular, the last method seems to us very convenient for nano-structures.

In order to introduce the characterization by means of parameter K we interpret it as the surface anisotropy of the model assumed that the Curie temperature in MFA is given by [3]

$$T^C_{\mathrm{MFA}}(n) = T^C_{\mathrm{MFA}}(\infty) \cdot \frac{s_{11} + s_{12}\cos\frac{\pi}{n+1}}{s_{11} + 2s_{12}} \tag{1}$$

where

$$T^C_{\mathrm{MFA}}(\infty) = \frac{zJ}{k_B}, \quad z = s_{11} + s_{12}, \tag{2}$$

and the boundary condition $(\alpha = \pi)$ should be valid.

The relation (2) shows that the Curie temperature is proportional to the exchange integral J which determines the interaction between two neighbouring spins $(s_{\nu\mu}; \nu, \mu = 1, 2)$ denotes the number of nearest neighbours in the plane μ when the central spin is localized in the plane ν.

Similar calculations lead to the Curie temperature in RFA, namely [9]

$$T^C_{\mathrm{RFA}}(n) = T^C_{\mathrm{MRA}}(n) \frac{1}{1+\frac{K}{J}} G^{-1}\left(1 + \frac{K}{J}\right) \tag{3}$$

where

$$G(s) = \frac{1}{nN} \sum_{\tau h} \frac{1}{\left[s - \frac{I(\tau,h)}{I(1,0)}\right]} \tag{4}$$

stands for the "lattice" Green function.

The aim of our presentation is to discuss the relation between MFA and RFA considerations taking into account the role of RFA in the calculations concerning the Curie temperature.

The next experimental verification concern the description of the spin waves resonances terms and the explanation of the spectral structure of excitations. The third domain of our interest in this work is the description of the topmost surface layer and its influence on the thermodynamical properties, first of all, the behaviour of the spontaneous magnetization leads to the picture of stochastic structures which decay spontaneously. These mentioned phenomena having the mutual behaviours are similar in their solutions.

The spontaneous magnetization considered in its local equilibrium can be obtained by means of the use of the MFA or RFA in their standard form of the quantum mechanics which gives

$$\langle S_R^z \rangle_{\mathrm{MFA}} = \frac{1}{2} \tanh\left(\frac{J}{2k_B T} \sum_g \langle S_{R+g}^z \rangle_{\mathrm{MFA}} \right) \tag{5}$$

in the case of $S = 1/2$ in MFA, while the result in RFA is of the form [9, 18]

$$\langle S_R^z \rangle_{\mathrm{RFA}} = \frac{1}{2} \tanh\left(J \sum_g \langle S_{R+g}^z \rangle + (K - \lambda) \langle S_R^z \rangle_{\mathrm{RFA}} \right) \tag{6}$$

where

$$\lambda = Jz \left(s - \frac{1}{G^1(s)} \right) \tag{7}$$

for $s = 1 + \frac{K}{J}$ with the summation over g which runs over the distance between two of neighbouring spins. K is the anisotropy parameter.

2. MFA and RFA

The Valenta model modified by RFA is discussed in the context of its discretization and its thermodynamics corresponding to the Valenta model which can be considered in two variants, (RFA) and (MFA).

Next, the kinetic equation is based on the Oguchi approach [1] for the damping term and the Néel [6] construction for the thermodynamics of inhomogeneous systems which are described by the equation of the diffusion type with the damping [8]. In this context the basic differential equation is calculated in MFA or RFA procedure applied to the Oguchi method.

The damping corfficient $\Gamma_{\tau h}$ and the diffusion constant $\Lambda_{\tau h}$ are connected by

$$\tau^*_{\tau h}\left(\Gamma_{\tau h} + \frac{z}{a^2}\Lambda_{\tau h}\right) = 1 \tag{8}$$

hence, the coefficients $\Gamma_{\tau h}$ and $\Lambda_{\tau h}$ can be calculated separately. In advance we obtain

$$\Gamma_{\tau h} = \frac{1}{\tau^*_{\tau h}}\left[1 - \frac{1}{2}\left(1 - 4\langle S^z_{\tau h}\rangle^2\right)\frac{T_C(\infty)}{T}\right] \tag{9}$$

and

$$\Lambda_{\tau h} = \frac{1}{\tau^*_{\tau h}}\cdot\frac{1}{2}\left[1 - 4\langle S^z_{\tau h}\rangle^2\frac{T_C(\infty)}{T}\right] \tag{10}$$

with $\tau^*_{\tau h}$ standing for the linewidth parameter temperature dependent.

We can see that the damping coefficient $\Gamma_{\tau h}$ as well as the diffusion constant $\Lambda_{\tau h}$ depend on the spontaneous magnetization $M_{\tau h} = g\mu_B\langle S^z_{\tau h}\rangle$, namely

$$\langle S^z_{\tau h}\rangle = \frac{1}{n}\sum_{\nu j} T_{\nu j \tau h}\langle S^z_{\nu j}\rangle \tag{11}$$

with the coefficients $T_{\nu j \tau h}$ satisfying the well known equation [11, 12] which together with the orthogonality conditions and the boundary conditions allow us to find the solutions interpreted as the third component of the propagation wave vector connected with the perpendicular wave amplitudes which describe only the properties of the boundary surfaces.

The inhomogeneities of the magnetization $\langle S^z_{\nu j}\rangle$ are connected with the creation $a^+_{\nu j}$ and annihilation $a^-_{\nu j}$ operators for magnons and they can determine the relation between the third component of magnetization and the number of magnons, namely

$$\langle S^z_{\nu j}\rangle = S - n_{\nu j}, \qquad n_{\nu j} = a^+_{\nu j}a^-_{\nu j}. \tag{12}$$

In this manner, taking into consideration, that

$$n_{\nu j} = \sum_{\tau h} T_{\nu j \tau h} n_{\tau h} \tag{13}$$

we present a scheme of mutual dependences leading to the main self consistent relations between the wave and particle quantum mechanics formulation.

First of all, the number of magnons $n_{\tau h}$ determined with respect to peaks of the energetic spectrum $E^s_{\tau h}$ for which, in consequence, we have

$$n_{\tau h} = \left(\frac{1}{e^{\beta E^s_{\tau h}} - 1} + \frac{(n+1)}{1 - e^{B(n+1)E^s_{\tau h}}}\right); \quad \beta = (k_B T)^{-1} \tag{14}$$

instead of that determined by the boson statistics when the second term is vanishing [cf. 17, 18].

The occupied number of quasi-particles is in fact determined by structural behavior for which the equation describing the linear transformation. The Curie temperature behavior is one of the most interesting results in connection with the construction of models presented.

The Curie point in RFA can be seen in contrast to the Curie point in MFA. That in the case of the isotropic interactions ($K = 0,\ s = 1,\ \Delta J = J^z - J^* = 0$) the T^c_{RFA} is leading to zero ($G^{-1}(1) = 0$) when it is compared with T^c_{MFA} which has the limited value. This result is in agreement with the rigorous theorems by Mermin and Wagner [12] for localized order or by Gosh in the case of band theory.

Thus, we can conclude that the Curie temperature behavior is one of the most interesting results in connection with the constructing of the model RFA in context to MFA.

The second effect observed in the case of the phase transition temperature and its behavior is the interval of spin autocorrelation function which can be considered in the conditions when the influence of the correlation symmetry is important not only for the scattering but also for the spin autocorrelation time. The recent one is evidently closer to the experimental data than to those obtained on the basis of the Ornstein-Zernike radial function.

3. Theory and experiments

The interplay between the theory and experiment is still important for the fundamental physics and modern technology. Moreover, recent achievements obtained in the case of local nanoparticles show that the progress of the surface physics seems to be expected.

In particular, the use of the Valenta model considered for the Curie temperature belonging to the interval from the RFA to MFA level can be treated as a new original methodology which allows us to interpret magnetic fluctuations. We introduce the Curie temperature dependence on temperature. From theoretical point of view we consider the relation between the Curie point and the anisotropy parameter. From experimental point of view analyze the phase transition in relation to the surface properties. In this manner the present paper is of an proper example of above mentioned relation which corresponds to the measurement of the autocorrelation time.

Fig. 1 presents a typical experimental device profiting from the spin wave resonances observed in the form of the peaks of the energetic spectrum $E_{\tau h}$,

$$(n_{\tau h}) = f(E_{\tau h}), \tag{15}$$

where $n_{\tau h}$ given by (14), is a number of quasi-particles which occupy the sublattices being in fact determined by structural behavior for which the determinant is defined by the linear transformation describing the transition from an arbitrary sublattice to diagonal one. f denotes the Fermi-Dirac distribution. Next, we can see that the RFA allows us to conclude that the character of the phase transition reflects the fluctuating character connected with methodology, influence of its nature on a system has the meaning of the second. Fig. 2 shows the geometry of the SWR experiment parameter describing the linewidth $\Gamma_{\tau h}$ as related to the linewidth $\Gamma_{\tau h}$. In the spin waves resonance (SWR) amplitude in the dependence on the Curie temperature by means of the experimental connected with the magnetic adsorption power measurements.

The alternative magnetic field h^* causes the precession around magnetization in the plane perpendicular to the surface. The change of the field $H_\perp$ leads to the change of the intensity $P^{0-1}_{\tau h=0}$ which is determined by the matrix elements corresponding to the inelastic magnetic scattering on the surfaces. The power intensity is found originally by the resonance conditions

$$P^{0-1}_{\tau h=0} \sim P_0 \left[\sum_\nu T_{\nu\tau} \langle S^z_\nu \rangle\right] \left[\sum_\nu T_{\nu\tau}\right]. \tag{16}$$

Another example to the presented here interpretations is connected with the relation between the Curie temperature and properties of a sample like the observation of elementary excitations via the adsorption power measurements (Fig. 2).

The effective parameter related to the linewidth can be observed in the spin waves resonance (SWR) experiments. The power intenstity is found originally by the resonance condition related to the environment can be reduced to two relations

$$T_{\tau h} \sim |\sum_{\nu j} T_{\nu j \tau h}| \neq 0 \tag{17}$$

and

$$T_{\tau h} \sim |\sum_{\nu j} T_{\nu j \tau h} \langle S^z_{\nu j} \rangle| \neq 0 \tag{18}$$

which are very well known in literature. The conditions (17) and (18) are satisfied when the surface anisotropy is taken into account. At the same time it is worthwhile to notice that in the case of regular homogeneous surfaces the conditions (17) and (18) are not fulfilled, so that the ideal samples cannot consider for discussed experiments.

Therefore, magnetic phenomena expected at the surfaces, interfaces, or first of all superfacial layers receive great attention. This arises from the fact that magnetic structures serve as almost ideal systems to explore basic ideas in physics [17–22]. However, for several decades the experiments and the theory were not developed at the same level of precision.

In order to consider an example of the interplay between theory and experiment we take into account ferromagnetic thin films with the hexagonal cobalt structure, we describe their properties. For this purpose we divide a cobalt sample into layers parallel to the plane $xy(0001)$ which remains determined by spins belonging to one of the sublattices A and B. In this manner we obtain n layers of type A and n layers of type B. The layers are directed in z-axis which is perpendicular to the plane xy whose position in two-dimensional space of spins is extended. The magnetization is assumed to be along the z-axis. Its characteristics is determined by the thermodynamic average and the distribution of the spin directions at every point of the discretized lattice.

Let us remark that the lattice of hexagonal cobalt sample is characteristic for the structure of 2D graphene [24] and the analogy between two sublattices which

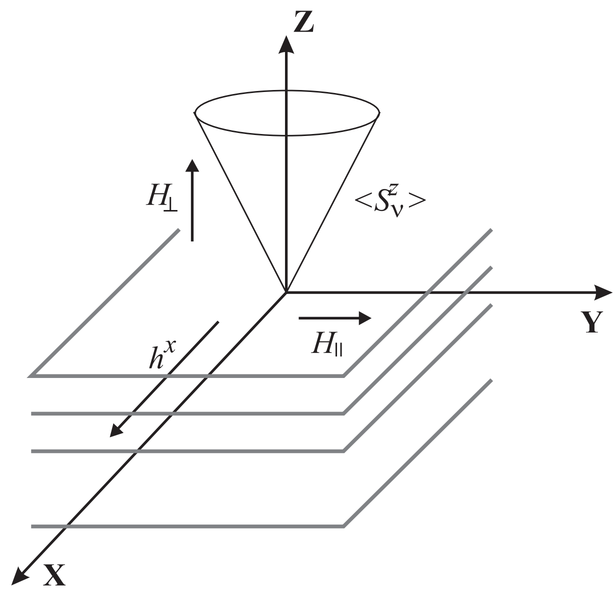


Fig. 1: The geometry of the spin waves resonance (SWR) experiment. The alternating magnetic field h^x causes the precession around magnetisation in the plane perpendicular to the surface. The intensity P_τ is determined by the matrix elements given by (15).

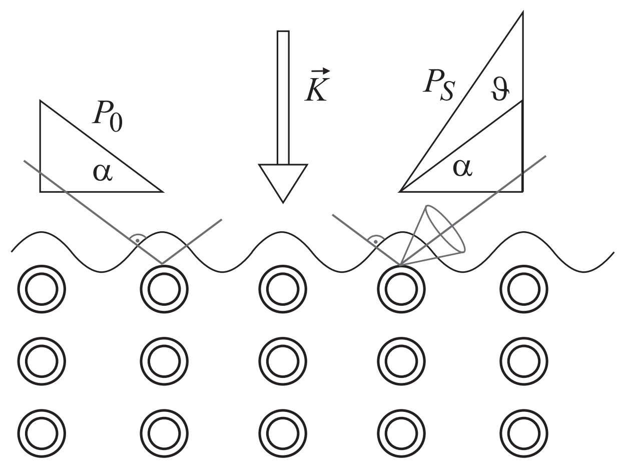


Fig. 2: Trajectories on the surfaces corresponding to the scattering phenomena which appear on the surface [cf. 15], or more precisely speaking on the atoms localized into lattice sites connected with magnons. The impulse momentum P_0 and P_S as well as the production of a quasi particle satisfy the momentum and the energy conversation law. The incident and scattered particles are a source of the energy of the motion in the plane parallel to the surface [cf. 16].

are properly chosen allows us to find the structure belonging to the regular lattice. For the investigations of surface magnetic structures we use ion-induced capture or emission of the spin polarized from magnetic surfaces which are powerful means for prolong various surface properties, in particular, surface magnetic properties. In order to continue various methods, we can profit from the theoretical description of the methods choosing experimental investigations which allow us to find the methods experimentally precise with a great level of applications.

Electron capture spectrometry (ECS) and spin-polarized electron emission spectroscopy (SPEES), electron spin polarization (ESP) existing at magnetic surface with extremely high surface sensivity.

Experimental details on ECS, SPEES and ESP are given in the papers published by C. Rau [15] ECS procedure allows us to study long-ranged and short-ranged ferromagnetic order at surfaces of magnetic materials. The physical process in ECS is the capture of one or two spin polarized electrons during grazing angle surface reflection of fast ions. In the case of deuterons with the energy 150 keV and the angle of incidence 0.2° the distance of closes approach to the reflecting surface amount to 0.1 mm (Fig. 3ab) and the ions probe spin polarized electron densities of state at the topmost surface layer. The long ranged ferromagnetic order is detected by exploiting one-electron capture processes $(\mathrm{D}^+ + e^- \to \mathrm{D}^0)$ [15]. The short ranged ferromagnetic order is detected by exploiting two electron capture processes $(\mathrm{D}^+ + 2e^- \to \mathrm{D}^0$ or $\mathrm{H}^+ = 2e^- \to \mathrm{H}^-)$.

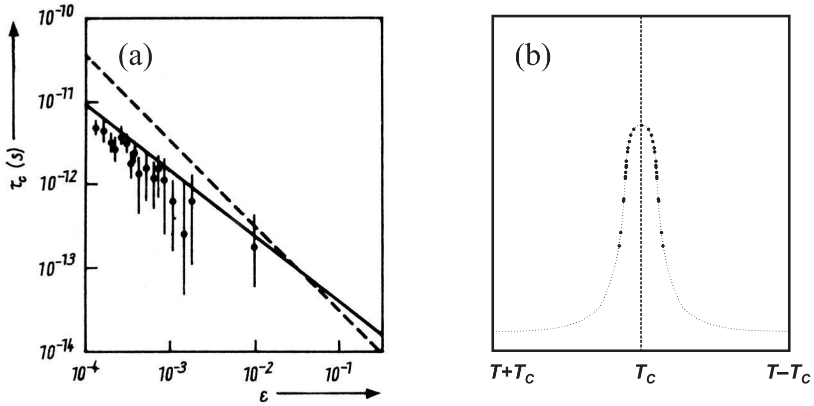
In angle and energy resolved SPEED, small angle surface scattering of energetic (5.150 keV) ions $(\mathrm{H}^+,\ \mathrm{He}^+$ or $\mathrm{Ne}^+)$ is utilized to study.

It is found that at $\mathrm{Ni}(hkl)$ surfaces the short ranged ferromagnetic order exists even at $2T_{\mathrm{Cs}}$ where T_{Cs} means the Curie temperature the emission of spin polarized secondary and Auger electrons as a measure of long ranged ferromagnetic order which can be interpreted as the spin correlations when the measurements are made above the Curie temperature of a sample.

Fig. 2 illustrates ion trajectories for scattering angles α varying this angle from 0.2° up to 45° allows to vary the probing depth from the topmost surface layer to interface and deeper layers, allowing us to perform magnetic depth profiling.

The ESP of secondary (or Auger) electrons emitted along the surface normal is detected by using Mott detectors. The EPS is defined by $P = (n^+ - n^-)/(n^+ + n^-)$ with n^+ and n^- being the numbers of majority and minority of spin electrons, respectively. The case $P > 0$ is related to a predominance of majority spin electrons parallel to the total magnetization while the case $P < 0$ refers to a predominance of minority spin electrons antiparallel to the total magnetization.

We can see that the new spin-sensitive spectroscopies discussed in the excellent review reported by Rau [15] (i.e. ECS, SPEES) as well as SIMPA (scaning ions microscopy with polarization analysis (SIMPA)), permit very selective investigations of surface magnetic structures and promise to reveal many new and fascinating phenomena in the future. In particular, SIMPA enables us to study and fabricate



$$\tau_0 = f\left(\frac{T-T_C}{T_C}\right)$$

Fig. 3: Temperature dependence ($\varepsilon = (T - T_C)/T_C$) of the spin autocorrelation time in a nickel sample (a). The solid curve is related to the correlation function with the lattice symmetry [21]. The dashed curve is related to the Ornstein-Zernike function. Experimental points are taken from the paper of Kobeissi et al. [23]. The temperature diffusion (b) on the linewidth as compared with experimental points.

in situ nano structured 3D, 2D and 1D magnetic elements. SIMPA allows us for detailed observations of the internal structure of magnetic domains and domain walls by providing high resolution and the surface electron spin polarization.

Fig. 3 shows that the spin auto-relation time is of stochastic nature. Transforming the picture presented in Fig. 3a to the coordinate system $(T_C, T_0(T_C))$ presented in Fig. 3b we can see that the τ_0 is of the Gaussian-like form experimentally confirmed. The magnetization is of the same properties due to the universal theory of homogeneous functions, namely we can write

$$\tau_0^* \sim \left[1 - \left(\frac{T_C - T}{T_C}\right)^{2\beta}\right]^{\beta+1}.$$

The Curie temperature depend on temperature via the autocorrelation time τ^* temperature dependent. The temperature dependence of the magnetisation is described by the critical exponent β.

The experiments presented in the paper of Rau [15] provide clear evidence that the described methods are powerful techniques to study a topmost surface and interface layer magnetic properties. The results collected by Rau concern the nickel, iron, hcp cobalt samples as well as several systems, magnetically exotic, like vanadium. Terbium films seem to us extremely interesting for considerations. In particular, the cobalt topmost planes are interesting because their band structure remains very similar to the planes of graphene whose properties are intensively studied [24].

4. Final conclusions

The main result of the present paper is to bring a comparison between the Valenta model originally applied in MFA approach and the model modified in terms of RFA, introduced to the theoretical construction, considered in the both cases; thin films as well nanoparticles structures.

The evident advantage in the case of RFA method is observed when a generalized susceptibility considerations are included to the sample energy minimization and lead to the conclusion that the convergence of a mean number of magnons is obtained even in thin films, and in contrast to the result of MFA calculations.

We consider the above problem as the explanation of the spontaneous magnetization in some isotropic layered system which gives the average magnetization vanishing at the temperature assumed to be different from zero. For this purpose, we remember that a thin film in the Valenta model is treated as a set of n monoatomic layers parallel with the film surfaces. The set of layers is equivalent in their interpretation to Néel sublattices [6] embedded in the limited space of the discrete geometry. Of course, the construction of the lattice for the structural form in the case of RFA is the same. In terms of thermodynamics we consider properties of a sample treated as the composition of layers which form homogeneous independent subsystems. Thus, the relation between the main values of spontaneous magnetization and the effective number of magnons is different when MFA or RFA are applied.

Concluding we can see that the mean number of particles vanishes when $T \neq 0$ and it takes the value different from zero when $T = 0$. The second conclusion which is important for the present paper and brings the interpretations of great interest for the physics methodology refers to the interplay between theory and experiment.

The theory and, first of all, its development from MFA to RFA shows the interpretation of the considered effects at the surface. At the same time, the theoretical description is an inspiration of new experimental techniques based on the investigated effects. This interdependence is seen particularly in the surface physics domain. The relation between theory and experiment is an leading factor in the progress of coherent and successive interpretations. The method applied to the long and short-ranged ferromagnetic order at the topmost surface layer as well as a layer in the middle of interface is an example of mutual considerations.

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O MODELU VALENTY I JEGO PRZYSTOSOWANIU DO RZECZYWISTOŚCI II

S t r e s z c z e n i e

Model opisujący cienkie warstwy ferromagnetyczne, wprowadzony przez profesora Luboša Valentę, na poziomie przbliżenia pola molekularnego (MFA), a obecnie zmodyfikowany w przypadku warstwowej struktury nanocząstek na poziomie przybliżenia pola reakcji (RFA) budzi wciąż duże zainteresowanie zastosowaniem metody we współczesnej fizyce i technologii.

W tym kontekście obecna praca jest przyczynkiem do pracy, poświęconej modelowi Valenty i jego przystosowaniu do rzeczywistości I, opublikowanej w *Bulletin de la Société des Scienices et des Lettres de Łódź; Série: Recherches sur les déformations* (**25** (2005), 13, ŁTN, Łódź, 2005).

Ostatnio otrzymane rezultaty w modelu Valenty zmodyfikowanym na poziomie RFA były prezentowane podczas 15-tej Czesko-Słowackiej Konferencji o Magnetyzmie (Košice 2013) w formie prezentacji posterowej.

Niniejszy artykuł stanowi okazję, aby przypomnieć o tym w 50. rocznicę utworzenia Uniwersytetu J. P. Šafarika w Koszycach.

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Dedicated to the memory of our Professors

Julian Ławrynowicz, Kiyoharu Nôno, Daiki Nagayama, and Osamu Suzuki

BINARY AND TERNARY CLIFFORD ANALYSIS ON NONION ALGEBRA AND su(3)

Summary

Concepts of binary and ternary extensions are considered and the extension theory is developed on nonion algebra and su(3). Concepts of binary and ternary Clifford algebras are studied by the Galois theory. The corresponding Dirac-like operators and Klein-Gordon-like operators are associated and quark models are constructed. As an example the Galois extension structures for su(3) are constructed and the quark model due to Gell-Mann is reconstructed.

Keywords and phrases: noncommutative Galois extension, ternary Clifford algebra, ternary Clifford analysis, quark model

Introduction

In [5, 7] Kerner has introduced a concept of ternary algebra and has given trials for the quark confinement by this concept. He has also introduced a concept of ternary Clifford analysis and its ternary Dirac-like operator and Klein-Gordon-like operator. In this paper we shall develop a concept of noncommutative Galois theory and discuss the binary/ternary Clifford analysis by use of the binary/ternary Galois extension. In some sense the paper summarizes our previous papers [8–13].

0.1. Binary Clifford analysis

We call in this paper the usual Clifford algebra *binary Clifford algebra*. For the case of binary Clifford algebra $Cl_2(n)$ with generators $(T_1, T_2, \dots, T_n)$ satisfying the commutation relations

$$T_bT_c + T_cT_b = \pm 2\delta^{bc} I_n \tag{1}$$

we have the generation scheme

$$Cl_2(2) \Rightarrow Cl_2(4) \Rightarrow Cl_2(8) \Rightarrow \dots \ . \tag{2}$$

We can show that this scheme can be described in terms of successive binary Galois extensions (Theorem 1).

0.2. Ternary Clifford analysis

The purpose of this paper is to analyze the ternary version of the observations. We develop the concept of ternary Clifford algebra and find standard ternary Clifford algebras, and give their generation scheme. We study the concept of ternary Clifford algebra.

Definition 1. We call an algebra with generators T_a, T_b, T_c *ternary Clifford algebra* when they satisfy two sets of commutation relations (1), *nondegenerate cyclic conditions*

$$\begin{aligned}
&T_aT_bT_c + T_bT_cT_a + T_cT_aT_b = \eta^{abc} I_3,\\
&\eta^{abc} = \eta^{bca} = \eta^{cab},\\
&\eta^{111} = \eta^{222} = \eta^{333} = 1, \quad \eta^{123} = \eta^{231} = \eta^{312} = \mathbf{j}^2, \quad \text{where} \quad \mathbf{j}^3 = 1,\\
&\eta^{321} = \eta^{213} = \eta^{123} = \mathbf{j},
\end{aligned} \tag{3}$$

and *degenerate cyclic conditions*

$$T_aT_bT_c + \mathbf{j}T_bT_cT_a + \mathbf{j}^2T_cT_aT_b = 0 \quad \text{or} \quad T_aT_bT_c + \mathbf{j}^2T_bT_cT_a + \mathbf{j}T_cT_aT_b = 0 \tag{4}$$

where two of them are identical.

We denote the algebra by $Cl_3(3)$. At first we consider ternary Clifford algebras on the nonion algebra. Then we proceed to the construction of generation scheme of ternary Clifford algebras. The heart of this paper is a study of noncommutative Galois extension for the construction of Clifford algebras. Then we can introduce the *ternary Clifford algebra* $Cl_3(n)$ with generators $T_1, \dots, T_n$, $n = 3^p$, and shall find a generation scheme

$$Cl_3(3) \Rightarrow Cl_3(9) \to Cl_3(27) \Rightarrow \dots \ . \tag{5}$$

In order to discuss physical applications we have to consider a successive noncommutative Galois extension of binary and ternary Clifford algebras. At first we consider the noncommutative Galois extension structure su(3). By this we can obtain the *algebras which admit binary and ternary Clifford structures at the same time*. Then we can expect to obtain the generation scheme

$$Cl_2(3) \Rightarrow Cl_3(3) \Rightarrow Cl_2(9) \Rightarrow Cl_3(9) \Rightarrow \dots \ . \tag{6}$$

This scheme is suggested only in the final section and with be discussed in a forthcoming paper.

More precisely, in Sect. 1 we introduce a concept of noncommutative Galois extension and give some basic facts on noncommutative Galois extensions [2, 12]. In particular, we notice that although the commutative Galois theory is developed quite well, the noncommutative Galois theory has been developed quite little [2, 12]. In Sect. 2 we are concerned with a relationship between binary Galois extensions and binary Clifford algebra. We can prove that any Clifford algebra with negative signature defines a noncommutative binary Galois extension (Theorem 1). In Sect. 3 we concentrate ourselves on the noncommutative Galois theory only for the nonion algebra (Theorem 2).

In Sect. 4 we consider the ternary Clifford analysis on the nonion algebra by the use of ternary Galois extensions. Then we can obtain a standard ternary Clifford algebra which is called *ternary algebra of nonion type* (Theorem 3). In Sect. 5 we can introduce a generation scheme of ternary Clifford algebras of nonion type. In Sect. 6 we construct noncommutative binary and ternary extension on su(3). Then we can prove that su(3) has a successive extension of binary and ternary extensions (Theorem 4). In the final Sect. 7 we propose a construction scheme of quark models and thus can obtain the so-called Gell-Mann model [3] by the use of noncommutative Galois extensions.

1. Binary and ternary noncommutative Galois extensions

In this section we develop concepts of noncommutative Galois extensions of binary type and ternary type.

1.1. Examples of noncommutative Galois extension

We adapt the notation and definitions of our previous paper [10] (the same journal and year, Sect. 1). We quote

Proposition 1. 1) *The relation*

$$A'_1(\tau)A'_1(\tau) = {A'}_2^{(\pm)}(\tau)$$

implies that

$${A'}_r^{(\pm)}(\tau)A'_1(\tau) = {A'}_{r+1}^{(\pm)}(\tau).$$

2) *When the condition*

$$x\tau = \tau x' \ (\exists x' \in A') \quad \textit{for} \quad \forall x \in A'$$

is satisfied, then the extensions (1), (2), (3) *are identical each other.*

We give some examples of binary and ternary extensions.

Example 1: **complex numbers**

$$\begin{aligned} R[\sqrt{-1}] &= \{\theta_1 1 + \theta_2\sqrt{-1} | \theta_1, \theta_2 \in \mathbb{R}\} \\ &= \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix} \middle| \ \theta_1, \theta_2 \in \mathbb{R} \right\}. \end{aligned}$$

Example 2: **quaternionic numbers**

They can be obtained by the *left-module* or the *right-module* noncommutative Galois extension of complex numbers:

$$C[\sqrt{-1}_2 = \{\theta_1 1 + \theta_2\sqrt{-1}_2 | \theta_1, \theta_2 \in \mathbb{C}\} = \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix} | \theta_1, \theta_2 \in \mathbb{C} \right\}$$
$$= \left\{ \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ -\theta_2 & \theta_1 & \theta_4 & -\theta_3 \\ -\theta_3 & -\theta_4 & \theta_1 & \theta_2 \\ -\theta_4 & \theta_3 & -\theta_2 & \theta_1 \end{pmatrix} \middle| \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \right\}.$$

Example 3: **cubic root numbers.** For a basic *ternary* Galois extension a natural example is provided by the cubic root numbers; cf. [10], Sect. 3 and [12], Sect. 2:

$$R\left[\sqrt[3]{1}\right] = \left\{\theta_1 1 + \theta_2 \mathbf{j} + \theta_3 \mathbf{j}^2 | \theta_1, \theta_2, \theta_3 \in \mathbb{R}\right\}$$
$$\cong \left\{ \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \\ \theta_3 & \theta_1 & \theta_2 \\ \theta_2 & \theta_3 & \theta_1 \end{pmatrix} \middle| \theta_1, \theta_2, \theta_3 \in \mathbb{R} \right\} / I_3.$$

In the next section we shall give bimodule ternary extensions in the nonion algebra.

1.2. Successive Galois extensions

We take an extension $A_2 = A_1[\tau_2](\tau_2^{k'} = 1)$ at first. Then we consider the extension $A_2 = A_1[\tau_2]$ $(\tau_2^{k'} = 1)$ which is called *successive extension of k-nary and k'-nary* extensions and is denoted by $A_0[\tau_1, \tau_2]$. As a special successive extension, we can make the tensor product extension of bimodule type:

$$A_2 = A_0[\tau_1 \otimes \tau_2], \quad A_2 = \left\{\sum x_{i,j}\tau_j^i \otimes \tau_2^i x'_{i,j} |\ x_{i,j}, x'_{i,j} \in A_0\right\}. \tag{7}$$

The successive extensions of the other types can be defined in a completely analogous manner. Example 2 gives the tensor product extension of the binary extensions. The basic notations on Galois extensions are listed as 1)–5) in [10], Sect. 1.2.

2. Binary Clifford algebras and noncommutative Galois extensions

In this section we discuss relationships between binary Clifford algebras and binary noncommutative Galois extensions, and prove that *a binary Clifford algebra introduces a binary Galois extension*. Here we are concerned with Clifford algebras of negative signature.

Let us proceed to a Clifford pair of noncommutative Galois extensions. We notice that Galois extensions do not necessarily have a structure of Clifford algebra.

Example 4 ([10], formula (26)):

$$R\left[\sqrt[2]{I_2},\sqrt[2]{I_2}\right]\cong\left\{\begin{array}{c}\mathbb{C}\times\mathbb{C},\\ \mathbb{H}.\end{array}\right.$$

We can find only one Clifford pair (e_1,e_4) for $\mathbb{C}\times\mathbb{C}$ as follows:

$$\begin{aligned}\mathbb{C}\times\mathbb{C}&=\{x_1e_1+x_2e_2+x_3e_3+x_4e_4|x_1,x_2,x_3,x_4\in\mathbb{R}\}\\ &=\left\{x_1\begin{pmatrix}\mathbf{I}&0\\0&\mathbf{I}\end{pmatrix}+x_2\begin{pmatrix}0&\mathbf{J}\\\mathbf{J}&0\end{pmatrix}+x_3\begin{pmatrix}\mathbf{J}&0\\0&\mathbf{J}\end{pmatrix}+\begin{pmatrix}0&\mathbf{I}\\-\mathbf{I}&0\end{pmatrix}\right\}\end{aligned}\tag{8}$$

where $\mathbf{I},\mathbf{I}\in M_2(\mathbb{R})$, is the unit matrix, and $\mathbf{J},\mathbf{J}\in M_2(\mathbb{R})$, is the complex structure. We can see that every pair (e_i,e_j), $i,j\neq 1$; $i\neq j$, for $\mathbb{H}$ is a Clifford pair:

$$\begin{aligned}\mathbb{H}&=\{x_1f_1+x_2f_2+x_3f_3+x_4f_4|\ x_1,x_2,x_3,x_4\in\mathbb{R}\}\\ &=\left\{x_1\begin{pmatrix}\mathbf{I}&0\\0&-\mathbf{I}\end{pmatrix}+x_2\begin{pmatrix}0&\mathbf{J}\\\mathbf{J}&0\end{pmatrix}+x_3\begin{pmatrix}\mathbf{J}&0\\0&-\mathbf{J}\end{pmatrix}+\begin{pmatrix}0&\mathbf{I}\\-\mathbf{I}&0\end{pmatrix}\right\}.\end{aligned}\tag{9}$$

We have

Theorem 1. *For a Clifford algebra $\mathcal{A}$ with generators $T_1,T_2,\dots,T_n$, there exists a sequence of noncommutative binary Galois extensions of $\mathbb{R}$ which realizes the given Clifford algebra $\mathcal{A}$. Namely, we have the following sequences of binary Galois extensions $\mathcal{A}_k|\ k=1,2,\dots,m$:*

$$\begin{aligned}T_iT_j+T_jT_i=-2\delta_{ij}I_n\Rightarrow\mathcal{A}_k&=\mathcal{A}_{k-1}\left[\sqrt[2]{-I_n}\right],\quad k=1,2,\dots,m;\\ \mathcal{A}_m&=\mathcal{A},A_0=\mathbb{R}.\end{aligned}\tag{10}$$

Proof. We prove the assertion by induction with respect to m. Complex numbers can be obtained by the commutative Galois extensions of real numbers (see Example 1). We can give a construction of the Clifford algebras. Let $T_1,T_2,\dots,T_m$ be a system of generators of a Clifford algebra $\mathcal{A}_m$. Setting

$$\hat{T}_i=\begin{pmatrix}T_i&0\\0&-T_i\end{pmatrix},\quad i=1,2,\dots,n;\quad \hat{H}_{n+1}=\begin{pmatrix}0&\mathbf{I}\\-\mathbf{I}&0\end{pmatrix},\tag{11}$$

we get Clifford algebra which is generated by $\{\hat{T}_1,\hat{T}_2,\dots,\hat{T}_n,\hat{H}_{n+1}\}$ on one hand, and the right (or left) module binary extension of $\mathcal{A}_n$ by $\hat{H}_{m+1}$ on the other hand. Hence we arrive at the desired assertion.

Next we construct the corresponding Dirac-like operators: field operators of the Clifford algebra defined by the Galois extension of binary type. Choosing $\hat{T}_1,\hat{T}_2,\dots,\hat{T}_m$ and $\hat{H}_{m+1}$ we can introduce the following three operators on the m-dimensional Euclidean space:

$$
(12) \qquad
\begin{aligned}
D &= \hat{T}_1 \frac{\partial}{\partial x_1} + \hat{T}_2 \frac{\partial}{\partial x_2} + \ldots + \hat{T}_m \frac{\partial}{\partial x_m}, \\
D^* &= \hat{H}_{m+1} \left(\hat{T}_1 \frac{\partial}{\partial x_1} + \hat{T}_2 \frac{\partial}{\partial x_2} + \ldots + \hat{T}_m \frac{\partial}{\partial x_m} \right).
\end{aligned}
$$

The operators are called *Dirac-like operator* and its *conjugate operator* for the extension, and they satisfy the condition

$$
(13) \qquad \Delta = D^* D = D D^*, \quad \text{where} \quad \Delta = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_m^2} \right) \otimes 1_m .
$$

We call Δ *binary Klein-Gordon-like operator* (or binary Laplace-like operator).

3. Ternary noncommutative Galois extensions for the nonion algebra

We recall the concept of nonion algebra [6, 10] and make several constructions of successive ternary extensions. We call the algebra which is generated by two of the following three elements *nonion algebra*:

$$
(14) \qquad \hat{Q}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{Q}_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{Q}_3 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}.
$$

We can see that the linear basis over the complex field can be given as follows:

$$
Q_1 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
$$

$$
(15) \qquad \bar{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j}^2 & 0 & 0 \\ 0 & \mathbf{j} & 0 \end{pmatrix}, \quad \bar{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j} & 0 & 0 \\ 0 & \mathbf{j}^2 & 0 \end{pmatrix}, \qquad \bar{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \end{pmatrix}.
$$

The construction is related with the concept of cubic algebra. Namely, setting

$$
(16) \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

we introduce a ternary extension $\mathbb{B} = R[T_2]$ over $\mathbb{R}$ which is called *cubic algebra*:

$$
(17) \qquad \mathbb{B} = \{ \theta_1 T_1 + \theta_2 T_2 + \theta_3 T_3 |\ \theta_1, \theta_2, \theta_3 \in \mathbb{R} \}.
$$

Then we have the commutative ternary Galois extension which is isomorphic to the cubic root numbers of Example 3:

$$\begin{aligned}\mathbb{B}/I &\simeq \left\{\theta_1 1 + \theta_2 \mathbf{j} + \theta_3 \mathbf{j}^2 \,|\, \theta_1, \theta_2, \theta_3 \in \mathbb{R}\right\} \\ &\simeq \{\theta_1 R_1 + \theta_2 R_2 + \theta_3 R_3 | \theta_1, \theta_3, \theta_3 \in \mathbb{R}\},\end{aligned} \tag{18}$$

where $I = \{\theta(T_1 + T_2 + T_3), \theta \in \mathbb{R}\}$. The above enables us describing the structure of Galois extension on the nonion algebra as follows:

Theorem 2. **(1)** *We introduce the following ternary Galois extensions of bimodule type over* $B[\mathbf{j}]$, $\mathbb{B} = R[T_2]$, *called basic extensions:*

$$\begin{aligned}&\Delta[R] = \{xR_1x' + yR_2y' + zR_2z' |\; x, y, z, x', y', z' \in B[\mathbf{j}]\}, \\ &A[Q_i] = \{xR_ix' + yQ_iy' + zQ_i^2z' |\; x, y, z, x', y', z' \in B[\mathbf{j}]\},\ i = 1, 2, 3, \\ &A[\bar{Q}_i = \{xR_ix' + y\bar{Q}_iy' + z\bar{Q}_i^2z' |\; x, y, z, x', y', z' \in B[\mathbf{j}]\},\ i = 1, 2, 3.\end{aligned} \tag{19}$$

Then we have

$$B[\mathbf{j}] = A[\mathbb{R}] \quad \textit{and} \quad N = A[Q_i] = A[\bar{Q}_i], \quad i = 1, 2, 3.$$

Thus the nonion algebra N has bimodule ternary Galois extensions over $B[\mathbf{j}]$.

(2) $R_i, Q_i, \bar{Q}_j$ $(i, j = 1, 2, 3)$ *generate a subgroup of the Galois group as multiplication operator. Namely, setting*

$$A_U[R] = \left\{xR_1x' + yUR_2y' + z\bar{U}R_3z' |\; x, x', y, y', z, z' \in R[\mathbf{j}]\right\}, \tag{20}$$

where $U = R_1, Q_j, \bar{Q}_k$ $(i, j, k = 1, 2, 3)$, *we obtain new Galois extensions, so U determines a subgroup of the Galois group of extension which is generated by multiplicative elements and which is isomorphic to the permutation group of degree* 3.

(3) *The adjoint operation gives a part of generators of the Galois group of* $B[\mathbf{j}]\,[\sqrt[3]{1}]$*:*

$$\begin{aligned}&\mathrm{Ad}_{Q_i} R_1 = R_1, \quad \mathrm{Ad}_{Q_i} R_2 = \mathbf{j}R_2, \quad \mathrm{Ad}_{Q_i} R_3 = \mathbf{j}^2 R_3; \quad j = 1, 2, 3, \\ &\mathrm{Ad}_{Q_i} Q_1 = Q_1, \quad \mathrm{Ad}_{Q_i} Q_2 = \mathbf{j}Q_2, \quad \mathrm{Ad}_{Q_i} Q_3 = \mathbf{j}^2 Q_3; \quad i = 1, 2, 3, \\ &\mathrm{Ad}_{Q_i} \bar{Q}_1 = Q_1, \quad \mathrm{Ad}_{Q_i} \bar{Q}_2 = \mathbf{j}^2 \bar{Q}_2, \quad \mathrm{Ad}_{Q_i} \bar{Q}_3 = \mathbf{j}\bar{Q}_3; \quad i = 1, 2, 3.\end{aligned} \tag{21}$$

Proof of **(1)**. Clearly, $B[\mathbf{j}] = A[\mathbb{B}]$. We prove that $N = A[Q_1]$. The remaining assertions can be proved in the completely analogous manner. The statements on $R_i, Q_j, \bar{Q}_k (i, j, k = 1, 2, 3)$ can be obtained by the extension with the use of the $(Q, \bar{Q}, R)$-matrices product table: see [10], p. 104. From the definition we can see that $R_1, R_2, R_3, Q_1, Q_1^2 \in A[Q_1]$. Next we show that $\bar{Q}_i \in A[Q_1], i = 1, 2, 3$. From

$Q_1^2 = \bar{Q}_1$ we deduce that $\bar{Q}_1 R_3 = \mathbf{j}\bar{Q}_2$ and $\bar{Q}_1 = \mathbf{j}^2\bar{Q}_3$. Hence the assertion of **(1)** follows.

Proof of **(2)** is based upon the following obvious lemma:

Lemma 1. *With the preceding notation and assumptions we have*

$$
\begin{aligned}
&A_{Q_1}[\mathbb{R}] = A[Q_2],\ A_{Q_1}[Q_1] = A[Q_1],\ A_{Q_1}[Q_2] = A[Q_3],\ A_{Q_1}[Q_3] = A[Q_2],\\
&A_{Q_2}[\mathbb{R}] = A[Q_3],\ A_{Q_2}[Q_1] = A[Q_3],\ A_{Q_2}[Q_2] = A[Q_2],\ A_{Q_2}[Q_3] = A[Q_2],\\
&A_{Q_3}[\mathbb{R}] = A[Q_1],\ A_{Q_2}[Q_1] = A[Q_2],\ A_{Q_2}[Q_2] = A[Q_1],\ A_{Q_3}[Q_3] = A[Q_1],\\
&A_{R_2}[\mathbb{R}] = A[\mathbb{R}],\ A_{R_2}[Q_1] = A[Q_2],\ A_{R_2}[Q_2] = A[Q_3],\ A_{R_2}[Q_3] = A[Q_1],\\
&A_{R_3}[\mathbb{R}] = A[\mathbb{R}],\ A_{R_2}[Q_1] = A[Q_3],\ A_{R_3}[Q_2] = A[Q_1],\ A_{R_3}[Q_3] = A[Q_2].
\end{aligned}
\tag{22}
$$

By use of the lemma we can see that the group in question is isomorphic to the symmetry group of degree 3 when we pay attention to the actions Q_i on $A[Q_j]$, $i, j = 1, 2, 3$.

Proof of **(3)** is easy and may be omitted.

Remark 1. We may give some demonstration of the generation scheme of the total nonion algebra by elements of the Galois group in Fig. 1.

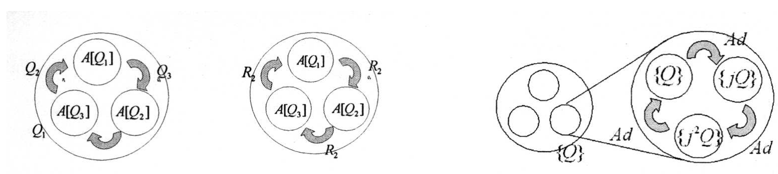

Fig. 1: The generation scheme of the total nonion algebra in relation with (21) and (22).

We may expext that the total Galois group of the extension $N = R[\sqrt[3]{1}, \sqrt[3]{1}]$ can be generated by the elements in **(2)** and **(3)** of Theorem 2.

4. Ternary Clifford algebra for the nonion algebra

We are going to investigate the structure of a ternary Clifford algebra in the case of the nonion algebra. We start with recalling Definition 1 of a ternary Clifford algebra. Next we construct the corresponding Dirac-like operators. Choosing $\{T_a, T_b, T_c\}$ we can introduce the following four operators on the 3-dimensional Euclidean space:

$$
\begin{aligned}
D &= T_a \frac{\partial}{\partial x_a} + T_b \frac{\partial}{\partial x_b} + T_c \frac{\partial}{\partial x_c},\\
D^* &= T_a \frac{\partial}{\partial x_a} + \mathbf{j}^2 T_b \frac{\partial}{\partial x_b} + \mathbf{j} T_c \frac{\partial}{\partial x_c},\\
D^{**} &= T_a \frac{\partial}{\partial x_a} + \mathbf{j} T_b \frac{\partial}{\partial x_b} + \mathbf{j}^2 T_c \frac{\partial}{\partial x_c}.
\end{aligned}
\tag{23}
$$

The operators are called *Dirac-like operators* and its *first* (resp. *second*) *conjugate operator* for the extension, and they satisfy condition

$$\Delta = DD^*D^{**}, \quad \text{where} \quad \Delta = \left(\frac{\partial^2}{\partial x_{a^2}} + \frac{\partial^2}{\partial x_{b^2}} + \frac{\partial^2}{\partial x_{c^2}} - 3\frac{\partial^2}{\partial x_a \partial x_b \partial x_c} \right) \otimes 1_3. \tag{24}$$

We call Δ *ternary Klein-Gordon-like operator.*

By a direct calculation we can arrive at

Proposition 2. *A triple $\{T_a, T_b, T_c\}$ determines a ternary Clifford algebra if and only if it determines the Dirac-like operators* (23)*.*

Now we extend Definition 1 as follows:

Definition 2. Consider an algebra with finite generators $T_1, T_2, \ldots, T_n$. Choose three of them, T_a, T_b, T_c, say. The triple chosen is called *Clifford triple* when it generates a ternary Clifford algebra.

Hence we can define the ternary Dirac-like operators and ternary Klein-Gordon-like operator.

Example 4. The generators T_1, T_2, T_3 of the cubic algebra defines a ternary Clifford algebra; see (16). Hence it is a Clifford triple.

We can determine ternary Clifford triples for the nonion algebra. By direct calculations we can prove the following

Theorem 3. *Let $\{Q_i, \bar{Q}_j, R_k\}$, $i, j, k = 1, 2, 3$, be the system of linear basis of the nonion algebra; see* (15)*. Then the ternary Clifford triples which are obtained from the system of generators can be listed as follows:*

Type I (*cubic extension type*)

$$\begin{gathered} \{Q_1, Q_1, Q_1\},\ \{Q_2, Q_2, Q_2\},\ \{Q_3, Q_3, Q_3\}; \\ \{\bar{Q}_1, \bar{Q}_1, \bar{Q}_1\},\ \{\bar{Q}_2, \bar{Q}_2, \bar{Q}_2\},\ \{\bar{Q}_3, \bar{Q}_3, \bar{Q}_3\}; \\ \{R_1, R_1, R_1\},\ \{R_2, R_2, R_2\},\ \{\bar{R}_2, \bar{R}_2, \bar{R}_2\}. \end{gathered} \tag{25}$$

Type II (*basic extension type*)

$$\{R_1, Q_1, \bar{Q}_1\},\ \{R_1, Q_2, \bar{Q}_2\},\ \{R_1, Q_3, \bar{Q}_3\}. \tag{26}$$

Type III (*general type*)

$$\{Q_1, Q_2, Q_3\},\ \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_3\},\ \{R_1, R_2, \bar{R}_2\}. \tag{27}$$

Hence we can introduce Dirac-like operators for the triples.

5. Nonion algebra construction of ternary Galois extensions

In Sect. 2 we have given generators of a binary Clifford algebra in terms of binary Galois extensions (Theorem 1). Now we consider the analogy of this fact for a ternary

Clifford algebra. In this section we introduce a concept of the nonion algebra construction of ternary Galois extensions and a standard construction of successive Galois extensions.

5.1. Basic construction by the cubic algebra/nonion algebra

Consider a sequence of successive ternary extensions over real numbers (of $\mathbb{R}$):

$$A_k = \sqrt[3]{I_n}[A_{k-1}], \quad k = 1, 2, \ldots, m; \quad A = A_m, \quad A_0 = \mathbb{R}. \tag{28}$$

We consider the case where $m = 2$. At first we notice that the successive extension is not unique. In fact, we can obtain a commutative extension by the successive extension of cubic extensions and a noncommutative extension by the nonion algebra (cf. [10], p. 103):

$$R\left(\sqrt[3]{I_3}, \sqrt[3]{I_9}\right] = \left\{ \begin{array}{c} \mathbb{B} \otimes \mathbb{B} \\ \mathbb{V}. \end{array} \right) \tag{29}$$

We may understand that the *cubic and nonion algebras are ternary counterparts of complex numbers and quaternions, respectively.*

1° We begin with the extension by the cubic algebra. Let A_1 be an algebra with a system of generators $\{S_1, S_2, \ldots, S_n\}$. Then we can obtain *cubic algebra extension construction* $A_1[\sqrt[3]{1}]$ by the tensor product extension $A_1 \otimes \mathbb{B}$, where $\mathbb{B}$ is the cubic algebra; cf. (16). Choosing a linear basic of $\mathbb{B}$, we define

$$S_i^{(\alpha)} = S_i \otimes T_\alpha, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, 3. \tag{30}$$

2° Next we proceed to the ternary extension

$$A_3 = A_1[\sqrt[3]{1}, \sqrt[3]{1}]$$

by the nonion algebra using the tensor product $A_1 \otimes \mathbb{V}$. We choose 9 generators (15) which are now denoted by $V_1, V_2, \ldots, V_9$. We introduce the elements

$$S_i^{(\alpha)} = S_i \otimes V_\alpha, \quad i = 1, 2, \ldots, n; \quad \alpha = 1, 2, \ldots, 9. \tag{31}$$

Repeating the process, we get a sequence of ternary extensions which we call *nonion algebra extension construction.*

5.2. Totally ternary Dirac-like operator

Finally we introduce the totally ternary Dirac-like operator. As we have seen, we can obtain the ternary Clifford algebra not for all triples involved, but for Clifford triples only. We notice the following obvious

Proposition 3. *Consider a nonion algebra extension construction. Let* $\{S_a, S_b, S_c\}$ *be a ternary Clifford triple of the generators; see* (30)*. Then we have the following ternary Clifford triples:*

$$\left\{S_a^{(1)}, S_b^{(2)}, S_c^{(3)}\right\}, \ \left\{S_a^{(i)}, S_b^{(i)}, S_c^{(i)}\right\}, \quad i = 1, 2, 3. \tag{32}$$

Hence, choosing the total set of Clifford triples, we can introduce the following Dirac-type operator which we call total Dirac-type operator:

$$D = \sum D^{(j)}, \ D^* = \sum D^{(j)*}, \ D^{**} = \sum D^{(j)**}, \tag{33}$$

where

$$D^{(i)}, D^{(i)*}, D^{(i)**}, \quad i = 1, 2, \dots, \tilde{m},$$

are the Dirac-like operator and its conjugate operators of each triple, $\tilde{m}$ being the number of Clifford triples. Introducing the product by taking the usual product only for the ternary Clifford triples and defining other products to be zero, we can introduce the following total Klein-Gordon-like operator:

$$D \circ D^* \circ D^{**} = \sum \Delta^{(j)},$$

where $\Delta^{(j)}$ is the Klein-Gordon-type operator for the j-th triple.

6. The Galois extension of su(3) and its Clifford analysis

In this section we discuss the structure the Galois extension for su(3). At first we recall some basic facts on su(3).

Example 5. We give a basis of the algebra due to Gell-Mann [3]:

$$f_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \tag{34}$$

$$f_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

Example 6. In connection with the preceding example we consider the linear space L_1 generated by 3 elements:

$$L_1 : e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

We also introduce two linear spaces L_2 and L_3:

$$L_2 : e_1' = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad e_2' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3' = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix},$$

(36)

$$L_3 : e_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad e_2'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_3'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Remark 2. Observe that $f_8 = (1/\sqrt{3})(e_3' + e_3'')$. Therefore, omitting one of e_3, e_3', e_3'', we obtain a system of basis of su(3).

With the help $L_i, i = 1, 2, 3$, we can formulate important properties of the binary and ternary Galois extension structures.

Theorem 4. 1). *We have the following adjoint representation on* $L_i, i = 1, 2, 3$*:*

$$\begin{gathered} He_1H^{-1} = -e_2, \quad He_2H^{-1} = e_1, \quad He_3H^{-1} = e_3, \\ H'e_1'{H'}^{-1} = -e_2', \quad H'e_2'{H'}^{-1} = -e_1', \quad H'e_3{H'}^{-1} = e_3', \\ H'e_1''{H'}^{-1} = -e_2'', \quad H'e_2''{H'}^{-1} = e_1'', \quad H'e_3''{H'}^{-1} = e_3'', \end{gathered} \tag{37}$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \tag{38}$$

Then, following the scheme $\mathrm{Ad}_g : A'[\tau] \to A'[\tau']$ *with*

$$\mathrm{Ad}_g\xi = g\xi g^{-1}, \quad g \in A'[\tau], \quad xg = gx \quad \textit{for} \quad x \in A'$$

we can define the ternary adjoint extension.

2). *The triples* $\{e_i, e_i', e_i''\}$, $i = 1, 2, 3$, *satisfy the conditions*

$$\begin{gathered} Ge_kG^{-1} = e'', \quad k = 1, 2, 3; \quad Ge_k'G^{-1} = e_k, \quad k = 1, 2, 3; \\ \textit{where} \quad G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ Ge_k''G^{-1} = e_k', \quad k = 1, 2; \quad Ge_3''G^{-1} = -e'. \end{gathered} \tag{39}$$

Proof of Assertion 1. Setting $V_0 = \{e_0, e_3\}$, $V_1 = \{e_0, e_1, e_3\}$ and $V_2 = \{e_0, e_2, e_3\}$, we have

$$V_2/V_0 = \mathrm{Ad}_H(V_1/V_0) \quad \textit{and} \quad \mathrm{su}(2)/V_0 \simeq V_1/V_0 + V_2/V_0;$$

cf. [10], p. 97, formula (5). Hence we can see that su(2) has the adjoint commutative Galois extension of bimodule type.

Proof of Assertion 2. Choosing $\tau = G$ and setting $V_i = \{e_i, e_i', e_i''\}$, $e = 1, 2, 3$, we have

$$V_{i+1} = \mathrm{Ad}_{G_i} V_i, \quad i = 1, 2, \quad \text{and} \quad \mathrm{su}(3) = (V_1 \otimes V_2 \otimes V_3)/\Gamma, \ \Gamma = \{e_3\};$$

cf. [10], p. 97, formula (5), and Remark 2. Hence the desired extension follows.

Remark 3. As a *summary of noncommutative Galois extension on* su(3) we may give the schematic comparison of that extension in terms of H_1 and G, as shown in [10], p. 106, Fig. 3.

As far as the corresponding *binary and ternary Dirac-like operators* are concerned, it is important to take into account the commutation relations

$$\begin{aligned} &e_1^2 = e_2^2 = e_3^2 = -1, \\ &e_1e_2 = -e_2e_1 = e_3, \ e_2e_3 = -e_3e_2 = e_1, \ e_3e_1 = -e_1e_3 = e_2. \end{aligned} \tag{40}$$

After the extension by $e_0 = \mathrm{diag}[1, 1, 0]$ we have the Clifford algebra which is isomorphic to the quaternion algebra. For e_i' and e_i'', $i = 1, 2, 3$, the above assertions are still valid. We are led to the following *binary Dirac-like operators* for $\{e_0, e_1, e_2, e_3\}$:

$$\begin{aligned} D &= e_0 \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}, \\ \bar{D} &= \bar{e}_0 \frac{\partial}{\partial x_0} + \bar{e}_1 \frac{\partial}{\partial x_1} + \bar{e}_2 \frac{\partial}{\partial x_2} + \bar{e}_3 \frac{\partial}{\partial x_3}. \end{aligned} \tag{41}$$

The Dirac-like operators for $\{e_0', e_1', e_2', e_3'\}$ and $\{e_0'', e_1'', e_2'', e_3''\}$ can be obtained in a similar manner. We can also introduce *ternary Dirac-like operators* for the above three quadruples which may be interpreted as operators leading to the ternary equations of quarks; cf. [10], Section 6 and 7, pp. 52–61, and Section 7 below:

$$\begin{aligned} D &= e_i \frac{\partial}{\partial \theta_1} + e_i' \frac{\partial}{\partial \theta_2} + e_i'' \frac{\partial}{\partial \theta_3}, \\ D^* &= e_i \frac{\partial}{\partial \theta_1} + \mathbf{j}^2 e_i' \frac{\partial}{\partial \theta_2} + \mathbf{j} e_i'' \frac{\partial}{\partial \theta_3}, \\ D^{**} &= e_i \frac{\partial}{\partial \theta_1} + \mathbf{j} e_i' \frac{\partial}{\partial \theta_2} + \mathbf{j}^2 e_i'' \frac{\partial}{\partial \theta_3}, \quad i = 1, 2, 3. \end{aligned} \tag{42}$$

We notice a duality structure between binary and ternary Galois extension as well as between binary particles and ternary particles, as shown in Fig. 2 (Theorem 5 in [11], p. 83).

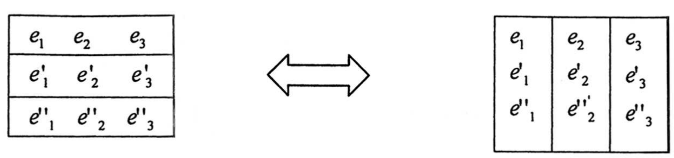

Fig. 2: Duality between the collections of binary and ternary Dirac-like operators.

7. Application to the theory of elementary particles

The present paper gives us new elements for conclusions and discussion at the end of [11], in particular in the context of Sections 5–8 of that paper.

7.1. The duality of mesons and baryons; cf. Sections 5 and 9 of [11]

A natural model and application of the preceding research is the structure of elementary particles [1, 2]. We know that mesons and baryons constitute with quarks. Moreover, we know the facts that each meson constitutes with a quark and an anti-quark, and that each baryon constitutes with only three quarks or anti-quarks. Till now we have no understanding of this fact. We can give an understanding of it using the duality structure of binary and ternary Galois extensions. In [11] and here we propose a model of generations of particles in terms of Galois extensions. We assume that quarks and anti-quarks are generated by binary extensions.

Hence we can see that the *binary extension generate mesons. Then we have a ternary Galois extension and baryons are created.* From the duality structure between these extensions, as we have seen in the case of su(3) (see Fig. 2), we conclude that baryons constitute only particles or anti-particles. The duality structure involved is precised in [11], p. 82, in Theorem 4 and illustrated by [11], Fig. 4.

7.2. Construction of quark models

The second application is the construction of quark models. We can realize the Gell-Mann model [3] by using the Galois extension structure on su(3).

At first we notice that we can introduce three kinds of quarks by the binary Galois extension on su(3). We may identify a quark as an up-quark, down-quark or strange quark, by

$$\{e_0, e_1, e_2, e_3\} \Rightarrow u, \quad \{e_0', e_1', e_2', e_3'\} \Rightarrow d, \quad \text{or} \quad \{e_0'', e_1'', e_2'', e_3''\} \Rightarrow s, \tag{43}$$

respectively. The construction given in [11], Sec. 6.4, pp. 84–85, leads to 10 baryons.

7.3. Introduction of colours

If we wish to introduce a concept of colours, we may consider the successive ternary extensions

$$C\left[\sqrt[3]{1}, \sqrt[3]{1_3}, \sqrt[3]{1_9}\right] \simeq \mathrm{su}[\sqrt[3]{1_9}]. \tag{44}$$

Transformations G_1, G_2, G_3 governing the nonion extension of su(3) and the idea of introducing colours of elementary particles in connection with the transformations G_1, G_2, G_3 governing the nonion extension of su(3) are given in [11], pp. 91–93 (in particular, see Fig. 14 and Fig. 16 therein), where the corresponding matrices G_1, G_2, G_3 are given by

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j}\ , \\ 1 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}. \tag{45}$$

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ANALIZA BINARNA I TERNARNA NA ALGEBRZE NONIONÓW A ALGEBRA su(3)

S t r e s z c z e n i e

Rozpatrywane są idee rozszerzeń binarnych i ternarnych oraz teoria rozszerzenia jest rozwijana na algebrez nonionów i na algebrze su(3). Idee binarnych i ternarnych algebr Clifforda są badane przy użyciu teorii Galois. Skonstruowanym obiektom przyporządkowane są stosowne operatory typu Diraca i operatory typu Kleina-Gordona oraz jest zrekonstruowany model kwarkowy Gell-Manna.

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Bogumiła Kowalczyk and Adam Lecko

POLYNOMIAL CLOSE-TO-CONVEX FUNCTIONS I

PRELIMINARIES AND THE UNIVALENCE PROBLEM

Summary

For $\delta \in [-\pi/2, \pi/2]$, $\mu_i \in \mathbf{N}$ and distinct points $\xi_i \in \mathbf{C}$, $0 < |\xi_i| \le 1$, $i = 1, \ldots, j$, we introduce the classes of analytic functions f in the unit disk $\mathbf{D}$ standardly normalized, satisfying the condition

$$\operatorname{Re}\left\{ \mathrm{e}^{\mathrm{i}\delta} \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i} f'(z) \right\} \ge 0, \quad z \in \mathbf{D},$$

called *polynomial close-to-convex functions*. Basic properties of considered classes are discussed.

Keywords and phrases: univalent function, close-to-convex function, polynomial close-to-convex function, function convex in one direction

1. Introduction

In this paper we introduce the classes of analytic functions standardly normalized defined by the condition (2.1). The inequality (2.1) generalizes the well known Robertson's characterization [21] of functions convex in one direction, further studied by many authors (see e.g. [9], [22], [8, pp. 193-206]). Note that in Robertson's formula recalled as (2.17), appears a quadratic trinomial with roots on the unit circle. What is interesting, the geometrical property of functions defined by (2.17) when roots of the quadratic trinomial are distinct, is essentially different when they are identical. After suitably boundary normalization, the first case leads to the inequality (2.18);

the second one to the inequalities (2.19) and (2.20) which correspond to the classes of functions called convex in the positive (negative) direction of the imaginary (real) axis (for references see e.g. [3], [15], [16, Chapter VI] and [5]).

In the sequence of papers [12–14] of the second author and in [18] with Yaguchi, Robertson's formula was generalized by considering polynomials of a degree at most two with roots outside of the unit disk. Such defined classes form subclasses of close-to-convex functions and have some geometrical properties (see [14]).

In this paper we replace a quadratic trinomial in Robertson's formula by an arbitrary polynomial P_Λ with roots outside of the unit disk. Such generalization is quite natural and seems to be worth of study. One of the question is to characterize these P_Λ for which functions f satisfying (2.1) are univalent. Then (2.1) can be treated as a criterium of univalence. This problem is studying in Section 3. Further discussion on the class of polynomial close-to-convex functions is continued in [11].

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Its subclass of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{S}^*$ denote the class of *starlike* functions, i.e., $f \in \mathcal{S}^*$ iff $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbf{D} \setminus f^{-1}(0).$$

Let $\mathcal{S}^c$ denote the class of *convex* functions, i.e., $f \in \mathcal{S}^c$ iff $f \in \mathcal{A}$ and

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathbf{D} \setminus (f')^{-1}(0).$$

For each $\delta \in (-\pi/2, \pi/2)$, let $\mathcal{K}_\delta$ denote the class of *close-to-convex functions with argument* δ, i.e., $f \in \mathcal{K}_\delta$ iff $f \in \mathcal{A}$ and there exists $g \in \mathcal{S}^c$ such that

$$\operatorname{Re} \left\{ \mathrm{e}^{\mathrm{i}\delta} \frac{f'(z)}{g'(z)} \right\} > 0, \quad z \in \mathbf{D}. \tag{1.1}$$

Let

$$\mathcal{K} := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{K}_\delta$$

denote the class of *close-to-convex functions*. Recall that

$$\mathcal{S}^c \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

Given $\delta \in (-\pi/2, \pi/2)$, let $\mathcal{P}(\delta)$ denote the class of all analytic functions p in $\mathbf{D}$ with $p(0) = \mathrm{e}^{\mathrm{i}\delta}$, having a positive real part in $\mathbf{D}$. Let $\mathcal{P} := \mathcal{P}(0)$. The following observation is well known.

Observation 1.1. *Let $\delta \in (-\pi/2, \pi/2)$. Then the following three statements are equivalent:*

(a) $p \in \mathcal{P}(\delta)$;

(b) $q = (p - i \sin \delta)/\cos \delta \in \mathcal{P}$;

(c)

$$p(z) = \frac{\mathrm{e}^{\mathrm{i}\delta} - \mathrm{e}^{-\mathrm{i}\delta}\omega(z)}{1+\omega(z)}, \quad z \in \mathbf{D}, \tag{1.2}$$

for some Schwarz function ω, *i.e., an analytic self-mapping* ω *of* $\mathbf{D}$ *keeping the origin fixed.*

For $\delta \in (-\pi/2, \pi/2)$ and $x \in \mathbf{T}$ let

$$\widetilde{L}_{\delta,x}(z) := \frac{\mathrm{e}^{\mathrm{i}\delta} + \mathrm{e}^{-\mathrm{i}\delta}xz}{1-xz}, \quad z \in \mathbf{C} \setminus \{1/x\},$$

$L_{\delta,x} := (\widetilde{L}_{\delta,x})_{|\mathbf{D}}$ and $L_x := L_{0,x}$. Clearly, by (1.2), $L_{\delta,x} \in \mathcal{P}(\delta)$. Let

$$\mathcal{P}_0 := \{L_x : x \in \mathbf{T}\}\,.$$

2. Preliminaries

Let $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $\overline{\mathbf{D}} := \{z \in \mathbf{C} : |z| \le 1\}$ and $\overline{\mathbf{D}}^0 := \overline{\mathbf{D}} \setminus \{0\}$. For $k \in \mathbf{N}$ and $1 \le j \le k$ let

$$\begin{aligned}\mathbf{\Lambda}_k^j := \Big\{\{(\mu_i, \xi_i) : i = 1, \ldots, j\} :\\ \mu_i \in \mathbf{N},\ \sum_{i=1}^{j} \mu_i = k,\ \xi_i \in \overline{\mathbf{D}}^0,\ \xi_{i_1} \ne \xi_{i_2},\ i_1 \ne i_2,\ i_1,\, i_2 = 1, \ldots, j\Big\}.\end{aligned}$$

Particularly,

$$\mathbf{\Lambda}_k^1 = \Big\{\{(k, \xi)\} : \xi \in \overline{\mathbf{D}}^0\Big\}$$

and

$$\begin{aligned}\mathbf{\Lambda}_k^k = \Big\{\{(1, \xi_i) : i = 1, \ldots, k\} :\\ \xi_i \in \overline{\mathbf{D}}^0,\ \xi_{i_1} \ne \xi_{i_2},\ i_1 \ne i_2,\ i_1,\, i_2 = 1, \ldots, k\Big\}.\end{aligned}$$

For $k \in \mathbf{N}$ let

$$\mathbf{\Lambda}_k := \bigcup_{j=1}^{k} \mathbf{\Lambda}_k^j$$

and for $j \in \mathbf{N}$ let

$$\mathbf{\Lambda}^j := \bigcup_{k=1}^{\infty} \mathbf{\Lambda}_k^j.$$

Let

$$\mathbf{\Lambda}_0 = \mathbf{\Lambda}^0 := \{\{(0,0)\}\}.$$

At the end, let

$$\mathbf{\Lambda} := \bigcup_{k \in \mathbf{N}_0} \mathbf{\Lambda}_k.$$

For $\Lambda := \{(\mu_i, \xi_i) : i = 1, \dots, j\} \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$ let

$$P_\Lambda(z) := \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i}, \quad z \in \mathbf{C}.$$

For $\Lambda \in \mathbf{\Lambda}_0$ let

$$P_\Lambda(z) := 1, \quad z \in \mathbf{C}.$$

Given $k \in \mathbf{N}_0$, for each $\Lambda \in \mathbf{\Lambda}_k$ a polynomial P_Λ is of a degree k.

Now we introduce the classes $\mathcal{C}(\delta; \Lambda)$ being the subject of our studies.

Definition 2.1. Let $\delta \in [-\pi/2, \pi/2]$ and $\Lambda \in \mathbf{\Lambda}$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{C}(\delta; \Lambda)$ if and only if

$$\operatorname{Re}\left\{e^{i\delta} P_\Lambda(z) f'(z)\right\} \geq 0, \quad z \in \mathbf{D}. \tag{2.1}$$

Let us call functions in $\mathcal{C}(\delta; \Lambda)$ as *polynomial close-to-convex with respect to* Λ *with argument* δ. Given $\Lambda \in \mathbf{\Lambda}$, let

$$\mathcal{C}(\Lambda) := \bigcup_{\delta \in [-\pi/2, \pi/2]} \mathcal{C}(\delta; \Lambda)$$

be the class of functions called *polynomial close-to-convex with respect to* Λ. Given $k \in \mathbf{N}_0$ and $\delta \in [-\pi/2, \pi/2]$, let

$$\mathcal{C}_\delta^{(k)} := \bigcup_{\Lambda \in \mathbf{\Lambda}_k} \mathcal{C}(\delta; \Lambda)$$

be the class of functions called *polynomial close-to-convex with argument* δ *of a degree* k. Given $\delta \in [-\pi/2, \pi/2]$, let

$$\mathcal{C}_\delta := \bigcup_{k \in \mathbf{N}_0} \mathcal{C}_\delta^{(k)}$$

be the class of functions called *polynomial close-to-convex with argument* δ, and given $k \in \mathbf{N}_0$, let

$$\mathcal{C}^{(k)} := \bigcup_{\Lambda \in \mathbf{\Lambda}_k} \mathcal{C}(\Lambda)$$

be the class of functions called *polynomial close-to-convex a degree* k. At the end, let

$$\mathcal{C} := \bigcup_{k \in \mathbf{N}_0} \mathcal{C}^{(k)}$$

be the class of functions called *polynomial close-to-convex*.

Remark 2.2. (a) When $\Lambda := \{(\mu_i, \xi_i) : i = 1, \dots, j\} \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$, then (2.1) is of the form

$$\operatorname{Re}\left\{e^{i\delta} \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i} f'(z)\right\} \geq 0, \quad z \in \mathbf{D}. \tag{2.2}$$

(b) For $\Lambda \in \mathbf{\Lambda}_0$ the get the class of functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{e^{i\delta} f'(z)\right\} \geq 0, \quad z \in \mathbf{D}. \tag{2.3}$$

For short, denote it as $\mathcal{C}(\delta;0)$.

For $\mathit{\Lambda} := \{(\mu_i,\xi_i) : i = 1,\dots,j\} \in \mathbf{\Lambda}\setminus\mathbf{\Lambda}_0$ let

$$h_{\mathit{\Lambda}}(z) := \frac{z}{P_{\mathit{\Lambda}}(z)} = \frac{z}{\prod_{i=1}^{j}(1-\xi_i z)^{\mu_i}}, \quad z \in \mathbf{D}, \tag{2.4}$$

and

$$g_{\mathit{\Lambda}}(z) := \int_0^z \frac{h_{\mathit{\Lambda}}(t)}{t}dt = \int_0^z \frac{dt}{\prod_{i=1}^{j}(1-\xi_i t)^{\mu_i}}, \quad z \in \mathbf{D}.$$

Clearly,

$$zg'_{\mathit{\Lambda}}(z) = h_{\mathit{\Lambda}}(z), \quad z \in \mathbf{D}. \tag{2.5}$$

For $\mathit{\Lambda} \in \mathbf{\Lambda}_0$ let

$$h_{\mathit{\Lambda}}(z) = g_{\mathit{\Lambda}}(z) := z, \quad z \in \mathbf{D}. \tag{2.6}$$

Observe now that (2.1) can be rewritten as

$$\begin{aligned}\operatorname{Re}\left\{e^{i\delta}P_{\mathit{\Lambda}}(z)f'(z)\right\} &= \operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{h_{\mathit{\Lambda}}(z)}\right\}\\ &= \operatorname{Re}\left\{e^{i\delta}\frac{f'(z)}{g'_{\mathit{\Lambda}}(z)}\right\} \geq 0, \quad z \in \mathbf{D}.\end{aligned} \tag{2.7}$$

Observation 2.1. *The strict inequality in* (2.1) *holds if and only if* $\delta \in (-\pi/2,\pi/2)$.

Proof. ($\Rightarrow$) Let $\delta \in [-\pi/2,\pi/2]$, $\mathit{\Lambda} \in \mathbf{\Lambda}$, $f \in \mathcal{C}(\delta;\mathit{\Lambda})$ and the strict inequality in (2.1) holds. Since it holds at $z = 0$,

$$\operatorname{Re}\left\{e^{i\delta}P_{\mathit{\Lambda}}(0)f'(0)\right\} = \operatorname{Re}\left(e^{i\delta}\right) = \cos\delta > 0,$$

so $\delta \in (-\pi/2,\pi/2)$.

($\Leftarrow$) Let $\delta \in (-\pi/2,\pi/2)$. Suppose, on the contrary, that there exist $\mathit{\Lambda}_0 \in \mathbf{\Lambda}$, $f_0 \in \mathcal{C}(\delta;\mathit{\Lambda}_0)$ and $z_0 \in \mathbf{D}$ such that (2.1) with $\mathit{\Lambda} := \mathit{\Lambda}_0$ and $f := f_0$ holds, i.e.,

$$\operatorname{Re}\left\{e^{i\delta}P_{\mathit{\Lambda}_0}(z)f_0'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.8}$$

and the left-hand side of (2.8) equals zero at z_0. Then, by the minimum principle for harmonic functions, the left-hand side of (2.8) vanishes identically in $\mathbf{D}$. Thus

$$e^{i\delta}P_{\mathit{\Lambda}_0}(z)f_0'(z) = ai, \quad z \in \mathbf{D},$$

for some $a \in \mathbf{R}$. Particularly, it holds for $z = 0$, so

$$e^{i\delta}P_{\mathit{\Lambda}_0}(0)f_0'(0) = e^{i\delta} = ai.$$

Thus either $\delta = -\pi/2$ and $a = -1$, or $\delta = \pi/2$ and $a = 1$, which contradicts the assumption.

Theorem 2.2. (i) *Let $\delta \in (-\pi/2, \pi/2)$, $\Lambda \in \mathbf{\Lambda}$ and $f \in \mathcal{A}$. Then $f \in \mathcal{C}(\delta;\Lambda)$ if and only if the function*

$$p(z) := \mathrm{e}^{\mathrm{i}\delta} P_\Lambda(z) f'(z), \quad z \in \mathbf{D}, \tag{2.9}$$

belongs to the class $\mathcal{P}(\delta)$.

(ii)

$$\mathcal{C}(-\pi/2;\Lambda) = \mathcal{C}(\pi/2;\Lambda) = \{g_\Lambda\}. \tag{2.10}$$

Proof. (i) Let $\delta \in (-\pi/2, \pi/2)$, $\Lambda \in \mathbf{\Lambda}$ and $f \in \mathcal{A}$. Define the function p by (2.9). Then p is analytic in $\mathbf{D}$ and

$$p(0) = \mathrm{e}^{\mathrm{i}\delta} P_\Lambda(0) f'(0) = \mathrm{e}^{\mathrm{i}\delta}. \tag{2.11}$$

Assume that $f \in \mathcal{C}(\delta;\Lambda)$. Then the inequality (2.1) holds and, by Observation 2.1, this inequality is strict. Thus by (2.9) we have

$$\operatorname{Re} p(z) = \operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta} P_\Lambda(z) f'(z)\right\} > 0, \quad z \in \mathbf{D}.$$

This with (2.11) yields that $p \in \mathcal{P}(\delta)$.

Vice versa, assume that p given by (2.9) belongs to $\mathcal{P}(\delta)$. It follows that the inequality (2.1) holds, which means that $f \in \mathcal{C}(\delta;\Lambda)$.

(ii) Let $\delta := -\pi/2$ and $\Lambda \in \mathbf{\Lambda}$. Suppose that $f \in \mathcal{C}(-\pi/2;\Lambda)$. Then by (2.1) we have

$$\operatorname{Re}\{-\mathrm{i} P_\Lambda(z) f'(z)\} = \operatorname{Im}\{P_\Lambda(z) f'(z)\} \geq 0, \quad z \in \mathbf{D}. \tag{2.12}$$

Since

$$\operatorname{Im}\{P_\Lambda(0) f'(0)\} = 0,$$

by the minimum principle for harmonic functions, the left-hand side of (2.12) vanishes identically in $\mathbf{D}$. Thus

$$P_\Lambda(z) f'(z) = a, \quad z \in \mathbf{D},$$

for some $a \in \mathbf{R}$. Particularly, it holds for $z = 0$, so

$$1 = P_\Lambda(0) f'(0) = a.$$

Thus

$$P_\Lambda(z) f'(z) = 1, \quad z \in \mathbf{D}. \tag{2.13}$$

Hence and in view of (2.4) we have

$$z f'(z) = \frac{z}{P_\Lambda(z)} = h_\Lambda(z), \quad z \in \mathbf{D}.$$

Consequently, by (2.5), $f = g_\Lambda$. Analogously, we prove the case $\delta := \pi/2$.

Let $\delta \in (-\pi/2, \pi/2)$ and $\Lambda \in \mathbf{\Lambda}$. Take $p \equiv \mathrm{e}^{\mathrm{i}\delta}$ in $\mathbf{D}$. Clearly, $p \in \mathcal{P}(\delta)$. Setting p into (2.9) we get (2.13), and hence $f = g_\Lambda$. Thus $g_\Lambda \in \mathcal{C}(\delta;\Lambda)$, so taking into account (2.10) we have

Corollary 2.3.

$$\bigcap_{\delta\in[-\pi/2,\pi/2]} \mathcal{C}(\delta;\Lambda) = \{g_\Lambda\}. \tag{2.14}$$

Historical background

1. The class $\mathcal{C}(\delta;0)$, usually denoted as $\mathcal{P}'(\delta)$, is well known. It contains functions called *of the boundary rotation with argument* δ. On the other hand, the inequality (2.3) is the famous criterium of univalence due to Noshiro [19] and Warschawski [23] (see also [8, p. 88]).

2. For $\Lambda := \{(1,1)\}$, $\Lambda := \{(1,1),(1,-1)\}$ and $\Lambda := \{(2,1)\}$ the inequality (2.1) was mentioned by Ozaki [20, p. 186] as a criterium of univalence in each case.

3. The classes $\mathcal{C}(\delta;\Lambda)$ with $\Lambda := \{(1,\xi_1)\} \in \mathbf{\Lambda}$, $\Lambda := \{(1,\xi_1),(1,\xi_2)\} \in \mathbf{\Lambda}$ and $\Lambda := \{(2,\xi_1)\} \in \mathbf{\Lambda}$, were introduced by the second author in [14]. Assuming for convenience that $\xi_1, \xi_2 \in \overline{\mathbf{D}}$, write (2.1) as

$$\operatorname{Re}\left\{e^{i\delta}(1-\xi_1 z)(1-\xi_2 z)f'(z)\right\} \geq 0, \quad z \in \mathbf{D}. \tag{2.15}$$

For specific ξ_1, ξ_2 these classes were examined in [12], [13], [14] and [18].

4. Setting

$$\xi_1 := \alpha e^{-i(\mu+\nu)},\ \xi_2 := \beta e^{-i(\mu-\nu)}, \quad \alpha,\beta \in [0,1],\ \mu,\nu \in [0,\pi],$$

rewrite (2.15) as

$$\operatorname{Re}\left\{e^{i\delta}\left(1-\left(\alpha e^{-i\nu}+\beta e^{i\nu}\right)e^{-i\mu}z+\alpha\beta e^{-2i\mu}z^2\right)f'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.16}$$

and denote the class of such functions f by $\widetilde{\mathcal{C}}(\delta;\alpha,\beta,\mu,\nu)$.

5. When $\beta := \alpha \in [0,1]$ the inequality (2.16) defines the class $\widetilde{\mathcal{C}}(\delta;\alpha,\alpha,\mu,\nu)$ of functions f satisfying the inequality

$$\operatorname{Re}\left\{e^{i\delta}\left(1-2\alpha e^{-i\mu}\cos(\nu)z+e^{-2i\mu}\alpha^2 z^2\right)f'(z)\right\} \geq 0, \quad z \in \mathbf{D},$$

investigated in [18].

6. For $\alpha := 1$ and $\delta := \mu - \pi/2$ the last inequality is of the form

$$\operatorname{Re}\left\{-ie^{i\mu}\left(1-2e^{-i\mu}\cos(\nu)z+e^{-2i\mu}z^2\right)f'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.17}$$

and defines the class $\widetilde{\mathcal{C}}(\mu-\pi/2;1,1,\mu,\nu)$. The inequality (2.17) was proposed by Robertson [21] to characterize analytically the class $\mathcal{CV}(i)$ of functions *convex in one direction* introduced by himself. Partially proof given by Roberston was completed by Hengartner and Schober [9]. A supplement of their proof was done by Royster and Ziegler [22] (cf. [8, pp. 193-206]). To complete the proof, Hengartner and Schober [8] distinguished in the class $\mathcal{CV}(i)$ three subclasses, namely, $\widetilde{\mathcal{C}}(0;1,1,\pi/2,\pi/2)$, $\widetilde{\mathcal{C}}(\mu-\pi/2;1,1,\mu,0)$ and $\widetilde{\mathcal{C}}(\mu-\pi/2;1,1,\mu,\pi)$ defined, respectively, as

$$\operatorname{Re}\left\{(1-z^2)f'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.18}$$

$$\operatorname{Re}\left\{-ie^{i\mu}\left(1-e^{-i\mu}z\right)^2 f'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.19}$$

$$\operatorname{Re}\left\{-\mathrm{i}e^{i\mu}\left(1+\mathrm{e}^{-\mathrm{i}\mu}z\right)^2 f'(z)\right\} \geq 0, \quad z \in \mathbf{D}. \tag{2.20}$$

7. Setting $\xi_1 = \xi_2 := e^{-i\delta}$ in (2.15) we get the inequality

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\left(1-\mathrm{e}^{-\mathrm{i}\delta}z\right)^2 f'(z)\right\} \geq 0, \quad z \in \mathbf{D}, \tag{2.21}$$

which defines the class of functions studied by Ciozda [2], [3] and [4], called there as *convex in the negative direction of the real axis*. By the analogy to this notion, functions satisfying (2.19) ((2.20)) were recalled in [15] as *convex in positive* (*negative*) *direction of the imaginary axis*. Classes defined by (2.19)-(2.21) were recently studied in [16, Chapter VI], [6], [17], [1], [5] and [7].

3. Univalence problem

In this section we present some univalence problem of functions in considered classes.

Theorem 3.1. *Let $j \in \mathbf{N}_0$. If*

$$\Lambda := \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \mathbf{\Lambda}^j \tag{3.1}$$

and

$$\sum_{i=1}^{j} \mu_i \frac{|\xi_i|}{1+|\xi_i|} \leq 1, \tag{3.2}$$

then $h_\Lambda \in \mathcal{S}^*$.

Proof. The case $\Lambda \in \mathbf{\Lambda}^0$ is obvious, since then by (2.6), $h_\Lambda(z) = z,\ z \in \mathbf{D}$. Let $j \in \mathbf{N}$ and $\Lambda \in \mathbf{\Lambda}^j$ be as in (3.1). From (2.4) we have

$$\begin{aligned}\operatorname{Re}\frac{zh'_\Lambda(z)}{h_\Lambda(z)} &= \operatorname{Re}\left\{1 - \frac{zP'_\Lambda(z)}{P_\Lambda(z)}\right\} \\ &= \operatorname{Re}\left\{1 + \sum_{i=1}^{j}\mu_i \frac{\xi_i z}{1-\xi_i z}\right\} = 1 + \sum_{i=1}^{j}\mu_i \operatorname{Re}\frac{\xi_i z}{1-\xi_i z}, \quad z \in \mathbf{D}.\end{aligned} \tag{3.3}$$

Note that for $\xi \in \overline{\mathbf{D}}^0$,

$$\begin{aligned}\operatorname{Re}\frac{\xi z}{1-\xi z} &= \frac{1}{2}\left(\frac{\xi z}{1-\xi z} + \frac{\overline{\xi z}}{1-\overline{\xi z}}\right) \\ &= \frac{\operatorname{Re}(\xi z) - |\xi|^2|z|^2}{|1-\xi z|^2} \geq \frac{-|\xi||z| - |\xi|^2|z|^2}{(1+|\xi||z|)^2} \\ &> \frac{-|\xi| - |\xi|^2}{(1+|\xi|)^2} = -\frac{|\xi|}{1+|\xi|}, \quad z \in \mathbf{D}.\end{aligned}$$

Hence, from (3.3) and (3.2) we get

$$\operatorname{Re}\frac{zh'_\Lambda(z)}{h_\Lambda(z)} > 1 - \sum_{i=1}^{j}\mu_i \frac{|\xi_i|}{1+|\xi_i|} \geq 0, \quad z \in \mathbf{D}.$$

Thus $h_\Lambda \in \mathcal{S}^*$.

Observe that for arbitrary $\xi_1,\ \xi_2 \in \overline{\mathbf{D}}$,

$$\frac{|\xi_1|}{1+|\xi_1|} \le \frac{1}{2}, \quad \frac{|\xi_1|}{1+|\xi_1|} + \frac{|\xi_2|}{1+|\xi_2|} \le 1. \tag{3.4}$$

This is easily seen by noting that the function

$$[0,1] \ni r \mapsto \frac{r}{1+r} \tag{3.5}$$

is strictly increasing.

Since $\Lambda \in \mathbf{\Lambda}_1$ if and only if $\Lambda = \{(1,\xi_1)\},\ \xi_1 \in \overline{\mathbf{D}}_0$, and $\Lambda \in \mathbf{\Lambda}_2$ if and only if $\Lambda = \{(1,\xi_1),(1,\xi_2)\}$ or $\Lambda = \{(2,\xi_1)\}$, $\xi_1,\ \xi_2 \in \overline{\mathbf{D}}_0$, so in view of (3.4) and Theorem 3.1 we have

Corollary 3.2. *If $\Lambda \in \mathbf{\Lambda}_k,\ k = 0,1,2$, then $h_\Lambda \in \mathcal{S}^*$.*

Corollary 3.3. *Let $k \ge 2$. If $\Lambda := \{(\mu_i,\xi_i) : i = 1,\dots,j\} \in \mathbf{\Lambda}_k$ and*

$$|\xi_i| \le \frac{1}{k-1}, \quad i = 1,\dots,j, \tag{3.6}$$

then $h_\Lambda \in \mathcal{S}^*$.

Proof. Since, by the monotonicity of the function (3.5) and by (3.6),

$$\frac{|\xi_i|}{1+|\xi_i|} \le \frac{\dfrac{1}{k-1}}{1+\dfrac{1}{k-1}} = \frac{1}{k},$$

in view of the fact that $\sum_{i=1}^{j} \mu_i = k$, we obtain

$$\sum_{i=1}^{j} \mu_i \frac{|\xi_i|}{1+|\xi_i|} \le \frac{1}{k} \sum_{i=1}^{j} \mu_i = 1.$$

Applying now Theorem 3.1, we complete the proof. Note that the case $k = 2$ follows from Corollary 3.2, also.

Particularly, we have

Corollary 3.4. *Let $j,\ l \in \mathbf{N}$ be such that $jl \ge 2$, and let $\kappa_1,\dots,\kappa_j \in \mathbf{T}$ be distinct points. If*

$$\Lambda := \{(l,\kappa_1/(jl-1)),\dots,(l,\kappa_j/(jl-1))\}, \tag{3.7}$$

then $h_\Lambda \in \mathcal{S}^*$.

For $j = 1$ and $l := k$, from the above we have

Corollary 3.5. *Let $k \ge 2$ and $\kappa \in \mathbf{T}$. If $\Lambda := \{(k,\kappa/(k-1))\}$, then $h_\Lambda \in \mathcal{S}^*$.*

Observe that for Λ given by (3.7), we have

$$\lim_{l\to+\infty} h_\Lambda(z) = \lim_{l\to+\infty} \frac{z}{\left(1-\dfrac{\kappa_1}{jl-1}z\right)^l \ldots \left(1-\dfrac{\kappa_j}{jl-1}z\right)^l}$$
$$= z\exp(\kappa z) =: \varphi_\kappa(z), \quad z \in \mathbf{D},$$

where

$$\kappa := \frac{1}{j}\sum_{i=1}^{j} \kappa_i. \tag{3.8}$$

Clearly, $\varphi_\kappa \in \mathcal{S}^*$.

Corollary 3.6. *Let $j \in \mathbf{N}$. If $\Lambda := \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \mathbf{\Lambda}^j \setminus \mathbf{\Lambda}_1$ and*

$$|\xi_i| \le \frac{1}{\mu_i j - 1}, \quad i = 1, \ldots, j, \tag{3.9}$$

then $h_\Lambda \in \mathcal{S}^*$.

Proof. Since, from (3.9) and from the monotonicity of the function (3.5), we have

$$\frac{|\xi_i|}{1+|\xi_i|} \le \frac{\dfrac{1}{\mu_i j - 1}}{1 + \dfrac{1}{\mu_i j - 1}} = \frac{1}{\mu_i j},$$

so

$$\sum_{i=1}^{j} \mu_i \frac{|\xi_i|}{1+|\xi_i|} \le \sum_{i=1}^{j} \mu_i \frac{1}{\mu_i j} = \sum_{i=1}^{j} \frac{1}{j} = 1,$$

and Theorem 3.1 completes the proof.

Theorem 3.7. *Let $\delta \in (-\pi/2, \pi/2)$ and $j \in \mathbf{N}_0$. If (3.1) and (3.2) hold, then*

$$\mathcal{C}(\delta; \Lambda) \subset \mathcal{K}_\delta. \tag{3.10}$$

Moreover

$$\mathcal{C}(\Lambda) \subset \mathcal{K}. \tag{3.11}$$

Proof. Let $\delta \in (-\pi/2, \pi/2)$ and $j \in \mathbf{N}_0$. Let (3.1) and (3.2) hold. Let $f \in \mathcal{C}(\delta; \Lambda)$. By Observation 2.1, the strict inequality in (2.1), so in (2.7) holds, i.e.,

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{h_\Lambda(z)}\right\} = \operatorname{Re}\left\{e^{i\delta}\frac{f'(z)}{g'_\Lambda(z)}\right\} > 0, \quad z \in \mathbf{D}. \tag{3.12}$$

Since by Theorem 3.1, $h_\Lambda \in \mathcal{S}^*$, so in view of the Alexander relation (2.5), $g_\Lambda \in \mathcal{S}^c$. Thus by (3.12) and (1.1), $f \in \mathcal{K}_\delta$.

Note that by (2.14), $g_\Lambda \in \mathcal{C}(\delta;\Lambda)$, so by (3.10), $g_\Lambda \in \mathcal{K}_\delta$ for every $\delta \in (-\pi/2, \pi/2)$, and consequently, $g_\Lambda \in \mathcal{K}$. Hence, by (2.10) and by (3.10), we get the inclusion (3.11).

Particularly, from the above by using Corollary 3.2, we have

Theorem 3.8. *Let* $\delta \in (-\pi/2, \pi/2)$. *If* $\Lambda \in \mathbf{\Lambda}_k,\ k = 0, 1, 2,$ *then*

$$\mathcal{C}(\delta;\Lambda) \subset \mathcal{K}_\delta.$$

Moreover, (3.11) *holds.*

Remark 3.9. By (2.10) the classes $\mathcal{C}(-\pi/2;\Lambda)$ and $\mathcal{C}(\pi/2;\Lambda)$ with

(i) $\Lambda := \{(1,\xi)\} \in \mathbf{\Lambda}$ contain only the convex function

$$g_\Lambda(z) = \int_0^z \frac{dt}{1-\xi t} = -\frac{1}{\xi}\log(1-\xi z), \quad \log 1 = 0,\ z \in \mathbf{D}.$$

(ii) $\Lambda := \{(1,\xi_1),(1,\xi_2)\} \in \mathbf{\Lambda}$ contain only the convex function

$$g_\Lambda(z) = \int_0^z \frac{dt}{(1-\xi_1 t)(1-\xi_2 t)}$$
$$= \frac{1}{\xi_2-\xi_1}\log\frac{1-\xi_1 z}{1-\xi_2 z}, \quad \log 1 = 0,\ z \in \mathbf{D}.$$

(iii) $\Lambda := \{(2,\xi)\} \in \mathbf{\Lambda}$ contain only the convex function

$$g_\Lambda(z) = \int_0^z \frac{dt}{(1-\xi t)^2} = \frac{z}{1-\xi z}, \quad z \in \mathbf{D}.$$

Remark 3.10. Under the assumption of Corollary 3.4, consider the class $\mathcal{C}(\delta;\Lambda)$ with Λ given by (3.7), i.e., the class of functions $f \in \mathcal{A}$ such that

$$\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\prod_{i=1}^{j}\left(1-\frac{\kappa_i}{jl-1}z\right)^l f'(z)\right\} \ge 0, \quad z \in \mathbf{D}. \tag{3.13}$$

By Theorem 3.7, such functions are univalent.

(a) When $j = 1$ and $l := k$, (3.13) with $\kappa_1 := \kappa$ is of the form

$$\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\left(1-\frac{\kappa}{k-1}z\right)^k f'(z)\right\} \ge 0, \quad z \in \mathbf{D}.$$

(b) When $l \to +\infty$ in (3.13), we get the inequality

$$\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\exp(-\kappa z)f'(z)\right\} = \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\delta}\frac{zf'(z)}{\varphi_\kappa(z)}\right\} \ge 0, \quad z \in \mathbf{D},$$

with κ given by (3.8). The class of such functions $f \in \mathcal{A}$ is the subject of studies in the forthcoming paper [10].

A natural question which we can consider, is to describe sets of Λ for which $\mathcal{C}(\delta;\Lambda) \subset \mathcal{S}$.

Definition 3.11. For $\delta \in [-\pi/2, \pi/2]$ and $k \in \mathbf{N}_0$ let

$$U_\delta^{(k)} := \{\varLambda \in \mathbf{\Lambda}_k : \mathcal{C}(\delta; \varLambda) \subset \mathcal{S}\}, \quad U_\delta := \bigcup_{k \in \mathbf{N}_0} U_\delta^{(k)},$$

$$U^{(k)} := \bigcup_{\delta \in [-\pi/2, \pi/2]} U_\delta^{(k)}, \quad U := \bigcup_{\delta \in [-\pi/2, \pi/2]} \bigcup_{k \in \mathbf{N}_0} U_\delta^{(k)}.$$

From Theorem 3.8 we have

Corollary 3.9.

$$U^{(0)} = \mathbf{\Lambda}_0, \quad U^{(1)} = \mathbf{\Lambda}_1, \quad U^{(2)} = \mathbf{\Lambda}_2.$$

Theorem 3.7 and Corollary 3.3 yield

Observation 3.10.

$$\left\{ \{(\mu_i, \xi_i) : j = i, \ldots, j\} \in \mathbf{\Lambda} : \sum_{i=1}^{j} \mu_i \frac{|\xi_i|}{1 + |\xi_i|} \leq 1 \right\} \subset U.$$

For $k \geq 2$,

$$\left\{ \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \mathbf{\Lambda}_k : |\xi_i| \leq \frac{1}{k-1}, \ i = 1, \ldots, j \right\} \subset U^{(k)}.$$

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FUNKCJE WIELOMIANOWO PRAWIE WYPUKŁE I
WPROWADZENIE I ZAGADNIENIE JEDNOLISTNOŚCI

S t r e s z c z e n i e

W pracy tej zdefiniowane są klasy funkcji analitycznych unormowanych w kole jednostkowym nazwane funkcjami *wielomianowo prawie-wypukłymi*. Badane są podstawowe własności takich funkcji, między innymi rozważane jest zagadnienie jednolistności.

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In memory of
Academician Ljubomir Iliev (1913–2000)

Bogumiła Kowalczyk and Adam Lecko

POLYNOMIAL CLOSE-TO-CONVEX FUNCTIONS II
INCLUSION RELATION AND COEFFICIENT FORMULAE

Summary

We continue the research of [5] by studing the inclusion relation and coefficient formula for the classes of polynomial close-to-convex functions.

Keywords and phrases: univalent function, close-to-convex function, polynomial close-to-convex function, function convex in one direction

4. Inclusion relation

In this section we deal with the problem of inclusion relation between classes $\mathcal{C}(\delta;\Lambda)$. Theorem 4.2 presented below was shown in [6] by a different method of proof than that used here. For $z_0 \in \mathbf{C}$ and $r > 0$ let $\mathbf{D}(z_0, r) := \{z \in \mathbf{C} : |z - z_0| < r\}$.

Lemma 4.1. *Let $\delta \in (-\pi/2, \pi/2)$. If $p \in \mathcal{P}(\delta)$ is analytic at $z_0 \in \mathbf{T}$ and $p(z_0) = 0$, then $p'(z_0) \neq 0$, i.e., z_0 is the zero of p of the order* 1.

Proof. Let $\delta \in (-\pi/2, \pi/2)$ and $p \in \mathcal{P}(\delta)$ satisfies the assumption. Note that a function $\mathbf{D} \ni z \to p(z_0 z)$ is in $\mathcal{P}(\delta)$, so without loss of generality, consider $p \in \mathcal{P}(\delta)$ which is analytic at 1 and $p(1) = 0$. By (1.3) of [5],

$$\psi(z) := \frac{1 - \mathrm{e}^{-\mathrm{i}\delta} p(z)}{1 + \mathrm{e}^{\mathrm{i}\delta} p(z)}, \quad z \in \mathbf{D}, \tag{4.1}$$

is a Schwarz function. Moreover, ψ is analytic at 1 with

$$\psi(1) = \frac{1 - \mathrm{e}^{-\mathrm{i}\delta} p(1)}{1 + \mathrm{e}^{\mathrm{i}\delta} p(1)} = 1.$$

Hence and from the well known Julia-Wolff-Carathéodory Theorem (see e.g. [8, p. 82]) it follows that $\psi'(1) > 0$. But in view of (4.1) we have

$$p'(1) = -\frac{1}{2}e^{i\delta}\psi'(1),$$

so $p'(1) \neq 0$. This ends the proof of the lemma. □

Theorem 4.2. *Let $\delta_i \in (-\pi/2, \pi/2)$ and $\Lambda_i \in \mathbf{\Lambda}$ for $i = 1, 2$, be such that $(\delta_1, \Lambda_1) \neq (\delta_2, \Lambda_2)$. Then*

$$\mathcal{C}(\delta_1; \Lambda_1) \not\subset \mathcal{C}(\delta_2; \Lambda_2).$$

Proof. Let $\delta_1, \delta_2 \in (-\pi/2, \pi/2)$ and

$$\Lambda_1 := \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \mathbf{\Lambda},$$
$$\Lambda_2 := \{(\nu_l, \zeta_l) : l = 1, \ldots, m\} \in \mathbf{\Lambda}$$

be such that $(\delta_1, \Lambda_1) \neq (\delta_2, \Lambda_2)$. Given $x \in \mathbf{T}$, let

$$\widetilde{f}_x(z) := e^{-i\delta_1}\int_0^z \frac{\widetilde{L}_{\delta_1,x}(t)}{P_{\Lambda_1}(t)}dt, \quad z \in \mathbf{C} \setminus \{1/x, 1/\xi_1, \ldots, 1/\xi_j\}. \tag{4.2}$$

Clearly, $\widetilde{f}_x$ is analytic and

$$e^{i\delta_1}P_{\Lambda_1}(z)\widetilde{f}'_x(z) = \widetilde{L}_{\delta_1,x}(z), \quad z \in \mathbf{C} \setminus \{1/x, 1/\xi_1, \ldots, 1/\xi_j\}. \tag{4.3}$$

Since P_{Λ_1} does not vanish in $\mathbf{D}$, so by (4.2), $f_x := (\widetilde{f}_x)_{|\mathbf{D}}$ is analytic in $\mathbf{D}$. Moreover, as $L_{\delta_1,x} := \left(\widetilde{L}_{\delta_1,x}\right)_{|\mathbf{D}} \in \mathcal{P}(\delta_1)$, from (4.3) and Theorem 2.4 of [5] it follows that $f_x \in \mathcal{C}(\delta_1; \Lambda_1)$. Thus

$$\mathcal{F} := \{f_x : x \in \mathbf{T}\} \subset \mathcal{C}(\delta_1; \Lambda_1).$$

Showing that $f_{x_0} \notin \mathcal{C}(\delta_2; \Lambda_2)$ for some $f_{x_0} \in \mathcal{F}$, we prove the theorem.

Let for $x \in \mathbf{T}$,

$$\widetilde{p}_{\delta_2,x}(z) := e^{i\delta_2}P_{\Lambda_2}(z)\widetilde{f}'_x(z), \quad z \in \mathbf{C} \setminus \{1/x, 1/\xi_1, \ldots, 1/\xi_j\}.$$

Thus by (4.3) we have

$$\widetilde{p}_{\delta_2,x}(z) = e^{i(\delta_2-\delta_1)}Q(z)\widetilde{L}_{\delta_1,x}(z), \quad z \in \mathbf{C} \setminus \{1/x, 1/\xi_1, \ldots, 1/\xi_j\} \tag{4.4}$$

where

$$Q(z) := \frac{P_{\Lambda_2}(z)}{P_{\Lambda_1}(z)}, \quad z \in \mathbf{C} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}. \tag{4.5}$$

We will show now that

$$(\widetilde{p}_{\delta_2,x_0})_{|\mathbf{D}} \notin \mathcal{P}(\delta_2)$$

for some $x_0 \in \mathbf{T}$. Consequently, Theorem 2.4 of [5] yields

$$f_{x_0} := (\widetilde{f}_{x_0})_{|\mathbf{D}} \notin \mathcal{C}(\delta_2; \Lambda_2).$$

Note that for $z \in \mathbf{T} \setminus \{1/x\}$ we have

$$\operatorname{Re} \widetilde{L}_{\delta_1,x}(z) = \operatorname{Re} \frac{e^{i\delta_1} + e^{-i\delta_1} xz}{1 - xz} \tag{4.6}$$

$$= \operatorname{Re} \left\{ \cos(\delta_1) \frac{1 + xz}{1 - xz} + i \sin(\delta_1) \right\} = \cos(\delta_1) \operatorname{Re} \frac{1 + xz}{1 - xz}$$

$$= \cos(\delta_1) \frac{1 - |x|^| z|}{|1 - xz|^2} = 0.$$

Hence and from (4.4), for $z \in \mathbf{T} \setminus \{1/x, 1/\xi_1, \ldots, 1/\xi_j\}$, we have

$$\operatorname{Re} \widetilde{p}_{\delta_2,x}(z) = \operatorname{Re} \left\{ \mathrm{e}^{\mathrm{i}(\delta_2 - \delta_1)} Q(z) \widetilde{L}_{\delta_1,x}(z) \right\} \tag{4.7}$$

$$= -\left(\sin(\delta_2 - \delta_1) \operatorname{Re} Q(z) + \cos(\delta_2 - \delta_1) \operatorname{Im} Q(z)\right) \operatorname{Im} \widetilde{L}_{\delta_1,x}(z).$$

Since $|\delta_1| < \pi/2$, so $1/x \neq -\mathrm{e}^{2\mathrm{i}\delta_1}/x$. Hence, by (4.6), by the fact that

$$\widetilde{L}_{\delta_1,x}\left(-\mathrm{e}^{2\mathrm{i}\delta_1}/x\right) = 0$$

and by the injectivity of $\widetilde{L}_{\delta_1,x}$, it follows that $1/x$ and $-\mathrm{e}^{2\mathrm{i}\delta_1}/x$ are the end points of the two disjoint open arcs of $\mathbf{T}$, say $I_+(x)$ and $I_-(x)$, such that

$$I^+ = \widetilde{L}_{\delta_1,x}^{-1}\left(\{iy : y > 0\}\right), \quad I^- = \widetilde{L}_{\delta_1,x}^{-1}\left(\{iy : y < 0\}\right).$$

Thus

$$\operatorname{Im} \widetilde{L}_{\delta_1,x}(z) > 0, \quad z \in I_+(x), \tag{4.8}$$

and

$$\operatorname{Im} \widetilde{L}_{\delta_1,x}(z) < 0, \quad z \in I_-(x). \tag{4.9}$$

Since $(\delta_1, \Lambda_1) \neq (\delta_2, \Lambda_2)$, so $\delta_1 \neq \delta_2$ and $\Lambda_1 = \Lambda_2$, or $\delta_1 = \delta_2$ and $\Lambda_1 \neq \Lambda_2$, or $\delta_1 \neq \delta_2$ and $\Lambda_1 \neq \Lambda_2$.

(I) Let $\delta_1 \neq \delta_2$ and $\Lambda_1 = \Lambda_2$. Then $Q \equiv 1$ and (4.7) reduces to

$$\operatorname{Re} \widetilde{p}_{\delta_2,x}(z) = -\sin(\delta_2 - \delta_1) \operatorname{Im} \widetilde{L}_{\delta_1,x}(z), \quad z \in \mathbf{T} \setminus \{1/x\}. \tag{4.10}$$

Take any $x_0 \in \mathbf{T}$. Note that $\delta_2 - \delta_1 \in (-\pi, \pi)$. When $\delta_2 - \delta_1 \in (0, \pi)$, then take any $z_0 \in I_+(x_0)$. Then, in view of (4.8), we have

$$\sin(\delta_2 - \delta_1) \operatorname{Im} \widetilde{L}_{\delta_1,x_0}(z_0) > 0. \tag{4.11}$$

When $\delta_2 - \delta_1 \in (-\pi, 0)$, then take any $z_0 \in I_-(x_0)$. Then, in view of (4.9), we have again (4.11). Thus (4.10) is negative for $x := x_0$ and $z := z_0$. By the continuity of $\widetilde{p}_{\delta_2,x_0}$ at z_0 it follows that (4.10) is negative for $z \in \mathbf{D}$ near z_0, so $(\widetilde{p}_{\delta_2,x_0})_{|\mathbf{D}} \notin \mathcal{P}(\delta_2)$. Consequently, by Theorem 2.4 of [5], $f_{x_0} \notin \mathcal{C}(\delta_2; \Lambda_2)$.

(II) Let $\Lambda_1 \neq \Lambda_2$. Then $Q \not\equiv 1$. Without loss of generality we can assume that the rational function $Q = P_{\Lambda_2}/P_{\Lambda_1}$ is of the simplest form, i.e., after reducing common factors. We consider two cases.

(1) Suppose that there exists $z_0 \in \mathbf{T} \setminus \{1/\xi_1, \ldots, 1/\xi_j\}$ such that

$$\sin(\delta_2 - \delta_1) \operatorname{Re} Q(z_0) + \cos(\delta_2 - \delta_1) \operatorname{Im} Q(z_0) \neq 0. \tag{4.12}$$

If (4.12) is positive, then we take any $x_0 \in \mathbf{T}$ in order to $z_0 \in I_+(x_0)$, otherwise, we take any $x_0 \in \mathbf{T}$ in order to $z_0 \in I_-(x_0)$. In both cases, by using (4.8) and (4.9), respectively, we get

$$(\sin(\delta_2-\delta_1)\operatorname{Re} Q(z_0)+\cos(\delta_2-\delta_1)\operatorname{Im} Q(z_0))\operatorname{Im}\widetilde{L}_{\delta_1,x_0}(z)>0.$$

Hence we see that (4.7) is negative for $z := z_0$ and $x := x_0$. As in Part (I), by continuity of $\widetilde{p}_{\delta_2,x_0}$ at z_0, we deduce that $f_{x_0} \notin \mathcal{C}(\delta_2;\Lambda_2)$.

(2) Suppose that for every $z \in \mathbf{T}\setminus\{1/\xi_1,\ldots,1/\xi_j\}$,

$$\sin(\delta_2-\delta_1)\operatorname{Re} Q(z)+\cos(\delta_2-\delta_1)\operatorname{Im} Q(z)=0. \tag{4.13}$$

Let

$$L_{\delta_1,\delta_2} := \{w \in \mathbf{C} : \sin(\delta_2-\delta_1)\operatorname{Re} w+\cos(\delta_2-\delta_1)\operatorname{Im} w = 0\}.$$

The set L_{δ_1,δ_2} is a straight line going through the origin. By (4.13) we have

$$\{Q(z) : z \in \mathbf{T}\setminus\{1/\xi_1,\ldots,1/\xi_j\}\} \subset L_{\delta_1,\delta_2}.$$

Hence and by the fact that $0 \in L_{\delta_1,\delta_2}$, either

$$0 \in \{Q(z) : z \in \mathbf{T}\setminus\{1/\xi_1,\ldots,1/\xi_j\}\}, \tag{4.14}$$

or $0 \in Q(\mathbf{D})$. But by (4.5), $Q \neq 0$ in $\mathbf{D}$, so (4.14) holds. Since $\{1/\zeta_1,\ldots,1/\zeta_m\}$ is the set of all zeros of Q, from (4.14) it follows that $1/\zeta_l \in \mathbf{T}\setminus\{1/\xi_1,\ldots,1/\xi_j\}$ for some $l \in \{1,\ldots,m\}$; say $1/\zeta_1 \in \mathbf{T}\setminus\{1/\xi_1,\ldots,1/\xi_j\}$.

Set $x_0 := -\mathrm{e}^{2\mathrm{i}\delta_1}\zeta_1$. For

$$z \in \mathbf{C}\setminus\{-\mathrm{e}^{-2\mathrm{i}\delta_1}/\zeta_1, 1/\xi_1,\ldots,1/\xi_j\}$$

we have

$$\begin{aligned}\widetilde{p}_{\delta_2,x_0}(z) &= \mathrm{e}^{\mathrm{i}(\delta_2-\delta_1)}Q(z)\widetilde{L}_{\delta_1,x_0}(z)\\ &= \mathrm{e}^{\mathrm{i}(\delta_2-\delta_1)}\frac{\prod_{l=1}^m(1-\zeta_l z)^{\nu_l}}{\prod_{i=1}^j(1-\xi_i z)^{\mu_i}}\cdot\frac{\mathrm{e}^{\mathrm{i}\delta_1}(1-\zeta_1 z)}{1+\mathrm{e}^{2\mathrm{i}\delta_1}\zeta_1 z} = (1-\zeta_1 z)^{\nu_1+1}q(z),\end{aligned} \tag{4.15}$$

where

$$q(z) := \frac{\prod_{l=2}^m(1-\zeta_l z)^{\nu_l}}{\prod_{i=1}^j(1-\xi_i z)^{\mu_i}}\cdot\frac{\mathrm{e}^{\mathrm{i}\delta_2}}{1+\mathrm{e}^{2\mathrm{i}\delta_1}\zeta_1 z}.$$

As $\delta_1 \neq \pm\pi/2$, and $\zeta_1 \notin \{\xi_1,\ldots\xi_j,\zeta_2,\ldots,\zeta_m\}$, so $q(1/\zeta_1) \in \mathbf{C}\setminus\{0\}$. Thus there exists $\varepsilon > 0$ such that $\widetilde{p}_{\delta_2,x_0}$ is analytic in $\mathbf{D}(1/\zeta_1,\varepsilon)$, so analytic in $\mathbf{D}\cup\mathbf{D}(1/\zeta_1,\varepsilon)$, having the zero of order $\nu_1+1 \geq 2$ at $1/\zeta_1$ and nonvanishing in $\mathbf{D}(1/\zeta_1,\varepsilon)\setminus\{1/\zeta_1\}$. Moreover $\widetilde{p}_{\delta_2,x_0}(0) = \mathrm{e}^{\mathrm{i}\delta_2}$. Since $1/\zeta_1$ is the zero of $\widetilde{p}_{\delta_2,x_0}$ of order at least 2, by applying Lemma 4.1 with $\delta := \delta_2$ and $z_0 := 1/\zeta_1$, we conclude that $(\widetilde{p}_{\delta_2,x_0})_{|\mathbf{D}} \notin \mathcal{P}(\delta_2)$. Consequently, by Theorem 2.4 of [5], $f_{x_0} \notin \mathcal{C}(\delta_2;\Lambda_2)$, which ends the proof of the theorem. □

Definition 4.3. Let $\delta_i \in (-\pi/2,\pi/2)$ and $\Lambda_i \in \mathbf{\Lambda}$ for $i = 1,2$. By $R(\delta_1,\delta_2;\Lambda_1,\Lambda_2)$ we denote the largest radius in $(0,1]$ such that

$$\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\delta_2}P_{\Lambda_2}(z)f'(z)\right\} > 0, \quad z \in \mathbf{D}_{R(\delta_1,\delta_2;\Lambda_1,\Lambda_2)},$$

for all $f \in \mathcal{C}(\delta_1;\Lambda_1)$.

By Theorem 4.2 we have

Corollary 4.3. *Let $\delta_i \in (-\pi/2, \pi/2)$ and $\Lambda_i \in \mathbf{\Lambda}$ for $i = 1, 2$, be such that $(\delta_1, \Lambda_1) \neq (\delta_2, \Lambda_2)$. Then*

$$R\,(\delta_1, \delta_2; \Lambda_1, \Lambda_2) < 1.$$

Some selected radii $R\,(\delta_1, \delta_2; \Lambda_1, \Lambda_2)$ was calculated in [6] and [7].

5. Coefficients formulas

In this section we present some relations on coefficients for functions in $\mathcal{C}(\delta; \Lambda)$.

For $t \in \mathbf{R} \setminus \{0\}$ the following formula holds (e.g. [9, str. 47])

$$\frac{1}{(1-z)^t} = \sum_{m=0}^{\infty} \binom{t+m-1}{m} z^m, \quad z \in \mathbf{D}. \tag{5.1}$$

Let $\delta \in (-\pi/2, \pi/2)$ and

$$\Lambda := \{(\mu_i, \xi_i) : i = 1, \ldots, j\} \in \mathbf{\Lambda}. \tag{5.2}$$

Let $f \in \mathcal{C}(\delta; \Lambda)$ be of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbf{D}. \tag{5.3}$$

By Theorem 2.4 of [5], the function

$$p(z) := \mathrm{e}^{\mathrm{i}\delta} P_\Lambda(z) f'(z), \quad z \in \mathbf{D}, \tag{5.4}$$

belongs to $\mathcal{P}(\delta)$ so, by Observation 1.1 of [5], the function

$$q(z) := \frac{1}{\cos\delta}(p(z) - i\sin\delta), \quad z \in \mathbf{D}, \tag{5.5}$$

belongs to $\mathcal{P}$ and is of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbf{D}. \tag{5.6}$$

From (5.4) and (5.5) it follows that

$$\mathrm{e}^{\mathrm{i}\delta} \prod_{i=1}^{j} (1 - \xi_i z)^{\mu_i} f'(z) = q(z)\cos\delta + i\sin\delta, \quad z \in \mathbf{D}. \tag{5.7}$$

Note that when $\Lambda \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$, then $\xi_i \neq 0,\ i = 1, \ldots, j$, in (5.7); and when $\Lambda \in \mathbf{\Lambda}_0$, then $\xi_1 = \xi_2 = \ldots = \xi_j = 0$ in (5.7).

Let us set $a_1 := 1$ and $a_{-n} := 0$ for $n \in \mathbf{N}_0$. Under this notation we have

Theorem 5.1. *Let $\delta \in (-\pi/2, \pi/2)$, $\Lambda \in \mathbf{\Lambda}$ be of the form (5.2) and let*

$$k := \sum_{i=1}^{j} \mu_i.$$

Let for $f \in \mathcal{C}(\delta; \Lambda)$ of the form (5.3) and $q \in \mathcal{P}$ of the form (5.6) the equality (5.7) hold. Then

(i)

$$\sum_{m=0}^{k} (-1)^m (n+1-m) a_{n+1-m} A_m = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta, \quad n \in \mathbf{N}, \tag{5.8}$$

where $A_0 := 1$ and

$$A_m := \sum_{\substack{\lambda_i \in \mathbf{N}_0 \\ \lambda_i \le \mu_i,\ i=1,\dots,j \\ \lambda_1+\dots+\lambda_j=m}} \binom{\mu_1}{\lambda_1} \dots \binom{\mu_j}{\lambda_j} \xi_1^{\lambda_1} \dots \xi_j^{\lambda_j}, \quad 1 \le m \le k. \tag{5.9}$$

(ii)

$$a_n = \frac{1}{n} \left(B_{n-1} + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) \sum_{i=1}^{n-1} B_{n-1-i} c_i \right), \quad n \ge 2, \tag{5.10}$$

where $B_0 := 1$ and

$$\begin{aligned} B_m :={}& \\ ={}& \sum_{\substack{\lambda_i \in \mathbf{N}_0,\ i=1,\dots,j \\ \lambda_1+\dots+\lambda_j=m}} \binom{\mu_1+\lambda_1-1}{\lambda_1} \dots \binom{\mu_j+\lambda_j-1}{\lambda_j} \xi_1^{\lambda_1} \dots \xi_j^{\lambda_j}. \end{aligned} \tag{5.11}$$

Proof. (i) Let $\Lambda \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$. Then

$$\begin{aligned} &\prod_{i=1}^{j} (1-\xi_i z)^{\mu_i} = (1-\xi_1 z)^{\mu_1} \dots (1-\xi_j z)^{\mu_j} \\ &= \left(1 - \binom{\mu_1}{1}\xi_1 z + \binom{\mu_1}{2}\xi_1^2 z^2 - \dots \pm \binom{\mu_1}{\mu_1}\xi_1^{\mu_1} z^{\mu_1}\right) \\ &\times \dots \times \left(1 - \binom{\mu_j}{1}\xi_j z + \binom{\mu_j}{2}\xi_j^2 z^2 - \dots \pm \binom{\mu_j}{\mu_j}\xi_j^{\mu_j} z^{\mu_j}\right) \\ &= 1 - \left[\binom{\mu_1}{1}\xi_1 + \binom{\mu_2}{1}\xi_2 + \dots + \binom{\mu_j}{1}\xi_j\right] z \\ &+ \left[\binom{\mu_1}{2}\xi_1^2 + \binom{\mu_2}{2}\xi_2^2 + \dots + \binom{\mu_j}{2}\xi_j^2 \right. \\ &+ \binom{\mu_1}{1}\binom{\mu_2}{1}\xi_1\xi_2 + \binom{\mu_1}{1}\binom{\mu_3}{1}\xi_1\xi_3 + \dots + \binom{\mu_1}{1}\binom{\mu_j}{1}\xi_1\xi_j \end{aligned}$$

$$+\binom{\mu_2}{1}\binom{\mu_3}{1}\xi_2\xi_3+\binom{\mu_2}{1}\binom{\mu_4}{1}\xi_2\xi_4+\dots+\binom{\mu_2}{1}\binom{\mu_j}{1}\xi_2\xi_j$$

$$+\dots+\binom{\mu_{j-1}}{1}\binom{\mu_j}{1}\xi_{j-1}\xi_j\Bigg]z^2+\dots$$

$$+\left[(-1)^k\binom{\mu_1}{\mu_1}\binom{\mu_2}{\mu_2}\dots\binom{\mu_k}{\mu_k}\xi_1^{\mu_1}\xi_2^{\mu_2}\dots\xi_j^{\mu_j}\right]z^k$$

$$=\sum_{m=0}^{k}(-1)^m\left(\sum_{\substack{\lambda_i\in\mathbf{N}_0\\ \lambda_i\le\mu_i,\ i=1,\dots,j\\ \lambda_1+\dots+\lambda_j=m}}\binom{\mu_1}{\lambda_1}\dots\binom{\mu_j}{\lambda_j}\xi_1^{\lambda_1}\dots\xi_j^{\lambda_j}\right)z^m$$

$$=\sum_{m=0}^{k}(-1)^m A_m z^m,$$

where $A_0=1$ and A_m is given by (5.9). Hence, from (5.7), (5.3) and (5.6) we obtain

$$\mathrm{e}^{\mathrm{i}\delta}\left(1-A_1z+A_2z^2-\dots+(-1)^kA_kz^k\right) \tag{5.12}$$

$$\times(1+2a_2z+3a_3z^2+\dots+(n+1)a_{n+1}z^n+\dots)$$

$$=\mathrm{e}^{\mathrm{i}\delta}+c_1\cos(\delta)z+c_2\cos(\delta)z^2+\dots+c_n\cos(\delta)z^n+\dots,\quad z\in\mathbf{D}.$$

Thus

$$\mathrm{e}^{\mathrm{i}\delta}+\mathrm{e}^{\mathrm{i}\delta}(2a_2-A_1)z+\mathrm{e}^{\mathrm{i}\delta}(3a_3-2a_2A_1+A_2)z^2+\dots+$$

$$+\mathrm{e}^{\mathrm{i}\delta}\left((n+1)a_{n+1}-na_nA_1+\dots+(-1)^k(n+1-k)a_{n+1-k}A_k\right)z^n+\dots$$

$$=\mathrm{e}^{\mathrm{i}\delta}+c_1\cos(\delta)z+c_2\cos(\delta)z^2+\dots+c_n\cos(\delta)z^n+\dots,\quad z\in\mathbf{D}.$$

Hence for $1\le n<k$,

$$(n+1)a_{n+1}-na_nA_1+(n-1)a_{n-1}A_2-\dots+(-1)^nA_n=c_n\mathrm{e}^{-\mathrm{i}\delta}\cos\delta,$$

and for $n\ge k$,

$$(n+1)a_{n+1}-na_nA_1+(n-1)a_{n-1}A_2-\dots+(-1)^k(n+1-k)a_{n+1-k}A_k$$
$$=c_n\mathrm{e}^{-\mathrm{i}\delta}\cos\delta.$$

The above two formulas can be written as (5.8).

Let now $\Lambda\in\mathbf{\Lambda}_0$. Then $k=0$ and the equality (5.12) holds with

$$A_1=A_2=\dots=A_k=0.$$

Thus we get the equality

$$(n+1)a_{n+1}=c_n\mathrm{e}^{-\mathrm{i}\delta}\cos\delta, \tag{5.13}$$

being the special case of (5.8) for $k=0$.

(ii) Let $\Lambda\in\mathbf{\Lambda}\setminus\mathbf{\Lambda}_0$. From (5.7) we get

$$f'(z)=\mathrm{e}^{-\mathrm{i}\delta}\frac{1}{\prod_{i=1}^{j}(1-\xi_iz)^{\mu_i}}\left(q(z)\cos\delta+i\sin\delta\right),\quad z\in\mathbf{D}. \tag{5.14}$$

Using (5.1), for $i = 1, \dots, j$, we have

$$\frac{1}{(1-\xi_i z)^{\mu_i}} = \sum_{m_i=0}^{\infty} \binom{\mu_i + m_i - 1}{m_i} \xi_i^{m_i} z^{m_i}, \quad z \in \mathbf{D}_{1/|\xi_i|}.$$

Hence and by the fact that $1/|\xi_i| \geq 1$ for $i = 1, \dots, j$, we obtain

$$\begin{aligned}
\frac{1}{\prod_{i=1}^{j}(1-\xi_i z)^{\mu_i}} &= \frac{1}{(1-\xi_1 z)^{\mu_1}} \cdots \frac{1}{(1-\xi_j z)^{\mu_j}} \\
&= \left(\sum_{m_1=0}^{\infty} \binom{\mu_1 + m_1 - 1}{m_1} \xi_1^{m_1} z^{m_1} \right) \cdots \left(\sum_{m_j=0}^{\infty} \binom{\mu_j + m_j - 1}{m_j} \xi_j^{m_j} z^{m_j} \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{\substack{\lambda_i \in \mathbf{N}_0,\ i=1,\dots,j \\ \lambda_1 + \dots + \lambda_j = m}} \binom{\mu_1 + \lambda_1 - 1}{\lambda_1} \cdots \binom{\mu_j + \lambda_j - 1}{\lambda_j} \xi_1^{\lambda_1} \cdots \xi_j^{\lambda_j} \right) z^m \\
&= \sum_{m=0}^{\infty} B_m z^m, \quad z \in \mathbf{D},
\end{aligned}$$

where $B_0 = 1$ and B_m is given by (5.11). Hence, from (5.14), (5.3) and (5.6), for $z \in \mathbf{D}$ we obtain

$$\begin{aligned}
&z + 2a_2 z^2 + \dots + n a_n z^n + \dots \\
&= z\mathrm{e}^{-\mathrm{i}\delta} \left(1 + B_1 z + B_2 z^2 + \dots + B_n z^n + \dots\right) \\
&\times \left(\mathrm{e}^{\mathrm{i}\delta} + c_1 \cos(\delta) z + c_2 \cos(\delta) z^2 + \dots + c_n \cos(\delta) z^n + \dots\right) \\
&= \left(z + B_1 z^2 + B_2 z^3 + \dots + B_{n-1} z^n + \dots\right) \\
&\times \left(1 + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) c_1 z + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) c_2 z^2 + \dots + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) c_n z^n + \dots\right) \\
&= z + \left(B_1 + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) c_1\right) z^2 + \dots + \left(B_{n-1} + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) \sum_{i=1}^{n-1} B_{n-1-i} c_i\right) z^n + \dots .
\end{aligned}$$

Comparing the coefficients, for $n \geq 2$ we get

$$n a_n = B_{n-1} + \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) \sum_{i=1}^{n-1} B_{n-1-i} c_i, \tag{5.15}$$

which yields (5.10).

Let now $\Lambda \in \mathbf{\Lambda}_0$. Then $k = 0$ and the equality (5.15) holds with $B_i = 0,\ i \in \mathbf{N}$, i.e.,

$$n a_n = \mathrm{e}^{-\mathrm{i}\delta} \cos(\delta) B_0 c_{n-1} = c_{n-1} \mathrm{e}^{-\mathrm{i}\delta} \cos\delta \tag{5.16}$$

being the special case of (5.10) for $\xi_1 = \xi_2 = \dots = \xi_j = 0$. Note that (5.16) coincides with (5.13). □

The corollary below is a consequence of the relation (5.8).

Corollary 5.2. *Let $\delta \in (-\pi/2, \pi/2)$, $\Lambda \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$ and $f \in \mathcal{C}(\delta; \Lambda)$ be of the form* (5.3). *Then for $n \in \mathbf{N}$ holds:*

(1) *If $\Lambda := \{(1, \xi)\}$, then*

$$(n+1)a_{n+1} - na_n\xi = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

Particularly, when $\Lambda = \{(1,1)\}$, then

$$(n+1)a_{n+1} - na_n = c_n e^{-i\delta} \cos\delta.$$

(2) *If $\Lambda := \{(2, \xi)\}$, then*

$$(n+1)a_{n+1} - 2na_n\xi + (n-1)a_{n-1}\xi^2 = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

Particularly, when $\Lambda = \{(2,1)\}$, then

$$(n+1)a_{n+1} - 2na_n + (n-1)a_{n-1} = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

(3) *If $\Lambda := \{(3, \xi)$, then*

$$(n+1)a_{n+1} - 3na_n\xi + 3(n-1)a_{n-1}\xi^2 - (n-2)a_{n-2}\xi^3 = \\ = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

Particularly, when $\Lambda = \{(3,1)\}$, then

$$(n+1)a_{n+1} - 3na_n + 3(n-1)a_{n-1} - (n-2)a_{n-2} = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

(4) *If $\Lambda := \{(1, \xi_1), (1, \xi_2)\}$, then*

$$(n+1)a_{n+1} - na_n(\xi_1 + \xi_2) + (n-1)a_{n-1}\xi_1\xi_2 = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

Particularly, when $\Lambda = \{(1,-1), (1,1)\}$, then

$$(n+1)a_{n+1} - (n-1)a_{n-1} = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

(5) *If $\Lambda := \{(1, \xi_1), (1, \xi_2), (1, \xi_3)$, then*

$$(n+1)a_{n+1} - na_n(\xi_1 + \xi_2 + \xi_3) + (n-1)a_{n-1}(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) + \\ -(n-2)a_{n-2}\xi_1\xi_2\xi_3 = c_n \mathrm{e}^{-\mathrm{i}\delta} \cos\delta.$$

The corollary below is a consequence of the relation (5.10).

Corollary 5.3. *Let $\delta \in (-\pi/2, \pi/2)$, $\Lambda \in \mathbf{\Lambda} \setminus \mathbf{\Lambda}_0$ and $f \in \mathcal{C}(\delta; \Lambda)$ be of the form* (5.3). *Then for $n \in \mathbf{N}$ holds:*

(1) *If $\Lambda := \{(k, \xi)$, $k \in \mathbf{N}$, then*

$$a_n = \frac{1}{n}\left[\binom{k+n-2}{n-1}\xi^{n-1} + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\binom{k+n-2-i}{n-1-i}\xi^{n-1-i}c_i\right].$$

Particularly, when $\Lambda = \{(k,1)\}$, then

$$a_n = \frac{1}{n}\left[\binom{k+n-2}{n-1} + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\binom{k+n-2-i}{n-1-i}c_i\right].$$

(a) *If* $\Lambda := \{(1,\xi)\}$, *then*

$$a_n = \frac{1}{n}\left(\xi^{n-1} + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\xi^{n-1-i}c_i\right).$$

Particularly, when $\Lambda = \{(1,1)\}$, *then*

$$a_n = \frac{1}{n}\left(1 + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}c_i\right).$$

(b) *If* $\Lambda := \{(2,\xi)\}$, *then*

$$a_n = \xi^{n-1} + \frac{1}{n}\mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}(n-i)\xi^{n-1-i}c_i.$$

Particularly, when $\Lambda = \{(2,1)\}$, *then*

$$a_n = 1 + \frac{1}{n}\mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}(n-i)c_i$$

(see [10], [1], [2]).

(c) *If* $\Lambda := \{(3,\xi)$, *then*

$$a_n = \frac{n+1}{2}\xi^{n-1} + \frac{1}{n}\mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\frac{(n-i)(n+1-i)}{2}\xi^{n-1-i}c_i.$$

Particularly, when $\Lambda = \{(3,1)\}$, *then*

$$a_n = \frac{n+1}{2} + \frac{1}{n}\mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\frac{(n-i)(n+1-i)}{2}c_i.$$

(2) *If* $\Lambda := \{(1,\xi_1),(1,\xi_2)\}$, *then*

$$a_n = \frac{1}{n}\left[\sum_{\lambda=0}^{n-1}\xi_1^{n-1-\lambda}\xi_2^{\lambda} + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\left(\sum_{\lambda=0}^{n-1-i}\xi_1^{n-1-i-\lambda}\xi_2^{\lambda}\right)c_i\right].$$

Particularly, when $\Lambda = \{(1,-1),(1,1)\}$, *then*

$$a_n = \frac{1}{n}\left[\sum_{\lambda=0}^{n-1}(-1)^{n-1-\lambda} + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\left(\sum_{\lambda=0}^{n-1-i}(-1)^{n-1-i-\lambda}\right)c_i\right],$$

i.e.,

$$a_{2n} = \frac{1}{2n}\mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n}c_{2i-1},$$

$$a_{2n+1} = \frac{1}{2n+1}\left(1 + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n}c_{2i}\right)$$

(see [10], [11]).

(4) *If* $\Lambda := \{(1,\xi_1),(1,\xi_2),(1,\xi_3)\}$, *then*

$$a_n = \frac{1}{n}\Bigg[\sum_{\lambda_1+\lambda_2+\lambda_3=n-1} \xi_1^{\lambda_1}\xi_2^{\lambda_2}\xi_3^{\lambda_3} + \\ + \mathrm{e}^{-\mathrm{i}\delta}\cos(\delta)\sum_{i=1}^{n-1}\Bigg(\sum_{\lambda_1+\lambda_2+\lambda_3=n-1-i} \xi_1^{\lambda_1}\xi_2^{\lambda_2}\xi_3^{\lambda_3}\Bigg) c_i\Bigg].$$

Since $|c_n| \le 2,\ n \in \mathbf{N}$, (see e.g. [3, Vol. I, p. 80]), from (5.8) we get

Corollary 5.4.

$$\left|\sum_{m=0}^{k}(-1)^m(n+1-m)a_{n+1-m}A_m\right| \le 2\cos\delta. \tag{5.17}$$

Remark 5.5. (a) Note that from (5.10) we have

$$a_2 = \frac{1}{2}\left(B_1 + c_1\mathrm{e}^{\mathrm{i}\delta}\cos\delta\right), \tag{5.18}$$

where by (5.11),

$$B_1 = \sum_{i=1}^{j}\mu_i\xi_i.$$

Thus

$$|a_2| \le \frac{1}{2}|B_1| + \cos\delta.$$

Particularly, when $B_1 = 0$, then

$$|a_2| \le \cos\delta;$$

when $|B_1| \le 2$ and $\delta = 0$, then

$$|a_2| \le 2.$$

(b) From Corollary 5.3(2) for the class $\mathcal{C}(0;\Lambda),\ \Lambda := \{(1,-1),(1,1)\}$, it follows that

$$|a_n| \le 1, \quad n \ge 2,$$

(see [4], [3, p. 201]).

Observe that for $\Lambda := \{(k,1)\},\ k \in \mathbf{N}$, we have

$$B_1 = \sum_{i=1}^{j}\mu_i = k,$$

and then for the class $\mathcal{C}(0;\Lambda)$ by (5.18) we get

$$a_2 = \frac{1}{2}(k + c_1). \tag{5.19}$$

Setting $c_1 = 2$, i.e., taking $p := L_1 \in \mathcal{P}$, from(5.19) we have

$$a_2 = 1 + \frac{k}{2}$$

for the corresponding function $f \in \mathcal{C}(0; \Lambda)$. Since for $k \geq 3$ we have $a_2 > 2$, the following result follows.

Theorem 5.5. *For every $k \geq 3$ and $\Lambda := \{(k, 1)\}$,*

$$\mathcal{C}(0; \Lambda) \not\subset \mathcal{S}.$$

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FUNKCJE WIELOMIANOWO PRAWIE WYPUKŁE II
RELACJA ZAWIERANIA SIĘ I WZORY NA WSPÓŁCZYNNIKI

S t r e s z c z e n i e

W pracy tej będącej kontynuacją pracy *Funkcje wielomianowo prawie-wypukłe I* z tegoż tomu *Bulletin de la Société des Sciences et des Lettres de Łódź, Série: Recherches sur les Déformations*, badany jest problem zawierania się klas funkcji wielomianowo prawie-wypukłych oraz problem współczynników funkcji z tych klas.

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In memory of
Academician Ljubomir Iliev (1913–2000)

Lilia N. Apostolova

SQUARE ROOTS OF BICOMPLEX NUMBERS

Summary

The square roots of the bicomplex number $A = a + ib + jc + ijd$, where a, b, c, d are real numbers and i, j, ij are the bicomplex units, are found. The solutions of the quadratic equation $X^2 + pX + q = 0$ of the bicomplex variable X and bicomplex parameters p, q, are given.

Keywords and phrases: bicomplex number, square root, quadratic equation

The bicomplex numbers are introduced by C. Segre in [5]. Algebraic investigations of these numbers and of the hyperbolic numbers, which form their subalgebra, are made in [3], [4], [7]. Functions of bicomplex variable are study in [2], [6].

Let us recall the definition of the algebra of bicomplex numbers $\mathbf{C}(j)$. It is defined as follows

$$\mathbf{C}(j) = \{x + iy + ju + ijv : \; i^2 = j^2 = -1, \; ij = ji, \; x, y, u, v \in \mathbf{R}\}.$$

The addition and the multiplication by real scalar are defined componentwise, and the multiplication of elements of the algebra is defined by opening the brackets and using the identities of the units i and j. The algebra $\mathbf{C}(j)$ is an associative, commutative algebra with divisors of zero. So are for example the numbers $X(ij-1)$, where X is an arbitrary bicomplex number. Actually, the product of this number with $ij + 1$ is equal to zero.

In the article [3] is proved the following result:

Let

$$p_n(w) = a_n w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0$$

be a polynomial in $\mathbf{C}(j)$. Then $w = w_+ + w_-$, $a_i = a_i^+ + a_i^-$, where $w_+, a_i^+ \in I(e_+)$ and $w_-, a_i^- \in I(e_-)$ are elements of the idempotents $I(e_+)$ and $I(e_-)$,

$$e_+ = \frac{1+ij}{2}, \qquad e_- = \frac{1-ij}{2}.$$

Then the equation

$$p_n(w) = 0$$

is reduced to the system

$$a_n^+ w_+^n + a_{n-1}^+ w_+^{n-1} + \ldots + a_1^+ w_+ + a_0^+ = 0,$$
$$a_n^- w_-^n + a_{n-1}^- w_-^{n-1} + \ldots + a_1^- w_- + a_0^- = 0.$$

The set of zeros of the polynomial $p_n(w) = 0$ coinsides with the couples (z_1, z_2) of the bicomplex solutions of two polynomials with complex coefficients of order $k \leq n$ and $l \leq n$, respectively. A consequence is that the number of the zeros of the polynomial $p_n(w)$ when it is a finite number is no more then the number $kl \leq n^2$.

In this article we would like to find the square roots of bicomplex number

$$A = a + ib + jc + ijd,$$

given in real representation, i.e. to solve the equation

$$X^2 = A, \tag{1}$$

where $X = x + iy + ju + ijv$ and $a, b, c, d,\ x, y, u$ and v are real numbers. We obtain

$$(x + iy + ju + ijv)^2 = a + ib + jc + ijd \tag{2}$$

and it is true that

$$\begin{aligned} (x + iy + ju + ijv)^2 &= x^2 + ixy + jxu + ijxv + iyx \\ -y^2 + ijyu - jyv + jux + jiuy - u^2 &- iuv + ijvx - jvy - ivu + v^2 \\ = x^2 - y^2 - u^2 + v^2 + 2ixy + 2jxu &+ 2ijxv + 2ijyu - 2jyv - 2iuv \\ &= a + ib + jc + ijd. \end{aligned}$$

So the following system of four quadratic equations with four real variables x, y, u, v and four real parameters a, b, c, d arises

$$x^2 - y^2 - u^2 + v^2 = a, \tag{3}$$

$$2xy - 2uv = b, \tag{4}$$

$$2xu - 2yv = c \tag{5}$$

$$2xv + 2yu = d. \tag{6}$$

The system of two equations (3) and (6) is equivalent to the following system of two equations

$$(x + v)^2 - (y - u)^2 = a + d, \tag{i}$$

$$(x - v)^2 - (y + u)^2 = a - d. \tag{ii}$$

The system of equations (4) and (5) is equivalent to the following system of two equations

$$2(y+u)x - 2(y+u)v = b+c \iff 2(y+u)(x-v) = b+c, \qquad \text{iii)}$$

$$2(y-u)x + 2(y-u)v = b-c \iff 2(u-y)(x+v) = b-c. \qquad \text{iv)}$$

Now, the system i), ii), iii), iv) can be written as two systems of two equations. The equations i) and iv) gives the system of two equations for the unknown $x+v$ and $y-u$ as follows

$$(x+v)^2 - (y-u)^2 = a+d, \qquad \text{a)}$$

$$2(x+v)(y-u) = b-c. \qquad \text{b)}$$

The equations ii) and iii) gives the system of two equations for the unknown $x-v$ and $y+u$ as follows

$$(x-v)^2 - (y+u)^2 = a-d, \qquad \text{c)}$$

$$2(x-v)(y+u) = b+c. \qquad \text{d)}$$

1. Square roots of bicomplex number

1.1. Square roots of bicomplex number $a+ib+jc+ijd$, when $b \neq \pm c$

Theorem 1. *The bicomplex number* $a+ib+jc+ijd$, *where* a, b, c, d *are real numbers and* i, j *are the imaginary units of the algebra of bicomplex numbers* $\mathbf{C}(j)$, *in the case* $b \neq \pm c$ *has* 4 *square roots given by the formula*

$$X(\varepsilon_1, \varepsilon_2) = \varepsilon_1 \frac{1+ij}{2\sqrt{2}} \sqrt{a+d+\sqrt{(a+d)^2+(b-c)^2}}$$
$$+ i\varepsilon_2 \; sign \; (b+c) \frac{1+ij}{2\sqrt{2}} \sqrt{-(a-d)+\sqrt{(a-d)^2+(b+c)^2}}$$
$$+ i\varepsilon_1 \; sign \; (b-c) \frac{1-ij}{2\sqrt{2}} \sqrt{-(a+d)+\sqrt{(a+d)^2+(b-c)^2}}$$
$$+ \varepsilon_2 \frac{1-ij}{2\sqrt{2}} \sqrt{a-d+\sqrt{(a-d)^2+(b+c)^2}},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$ *and* $sign \; (b \pm c)$ *are the signs of the nonzero numbers* $b \pm c$, *respectively.*

Proof. As $b \neq c$, from the equations a) and b) follows the equation

$$(x+v)^2 - \frac{(b-c)^2}{4(x+v)^2} = a+d \qquad (7)$$

and as $b \neq -c$ from the equations c) and d) follows

$$(x-v)^2 - \frac{(b+c)^2}{4(x-v)^2} = a-d. \qquad (8)$$

From the equation (7) we obtain the biquadratic equation

$$4(x+v)^4 - 4(a+d)(x+v)^2 = (b-c)^2,$$

which is equivalent to the equation

$$(2(x+v)^2 - a - d)^2 = (b-c)^2 + (a+d)^2.$$

As we ask the real solutions of this equation, we obtain the solution, satisfying the inequality

$$2(x+v)^2 = a + d + \sqrt{(a+d)^2 + (b-c)^2} \geq 0,$$

namely,

$$x + v = \varepsilon_1 \sqrt{\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}}, \tag{9}$$

where $\varepsilon_1 = \pm 1$.

From the equation (8) we obtain the biquadratic equation

$$4(x-v)^4 - 4(a-d)(x-v)^2 = (b+c)^2,$$

which is equivalent to the equation

$$(2(x-v)^2 - a + d)^2 = (b+c)^2 + (a-d)^2.$$

As we ask the real solutions of this equation, we consider the solution, for which

$$2(x-v)^2 = a - d + \sqrt{(a-d)^2 + (b+c)^2} \geq 0$$

and finally

$$x - v = \varepsilon_2 \sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}, \tag{10}$$

where $\varepsilon_2 = \pm 1$.

From the equations (9) and (10) we obtain the following numbers for x and v in the case $b \neq \pm c$

$$x = \frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} + \frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}$$

and

$$v = \frac{\varepsilon_1}{2}\sqrt{\frac{a+d}{2} + \frac{1}{2}\sqrt{(a+d)^2 + (b-c)^2}} - \frac{\varepsilon_2}{2}\sqrt{\frac{a-d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + (b+c)^2}}.$$

To find the real numbers y and u in the considered case we work as follows. As $b \neq c$ from the equations a) and b) follows the equation

$$\frac{(b-c)^2}{4(y-u)^2} - (y-u)^2 = a + d, \tag{11}$$

and as $b \neq -c$ from the equations c) and d) follows

$$\frac{(b+c)^2}{4(y+u)^2} - (y+u)^2 = a - d. \tag{12}$$

From the equation (11) we obtain the biquadratic equation

$$4(y-u)^4 + 4(a+d)(y-u)^2 = (b-c)^2,$$

which is equivalent to the equation

$$(2(y-u)^2 + a + d)^2 = (a+d)^2 + (b+c)^2.$$

As we ask the real solutions of this equation, we obtain

$$2(y-u)^2 = -a - d + \sqrt{(a+d)^2 + (b-c)^2} \geq 0$$

and finally using the conditions in the equations (9) and equation a) we obtain

$$y - u = \frac{\varepsilon_1}{\sqrt{2}} \text{ sign(b-c)} \sqrt{-(a+d) + \sqrt{(a+d)^2 + (b-c)^2}}, \tag{13}$$

where sing $(b-c)$ is equal to 1 when the real number $b-c$ is positive and to -1, when this number is negative.

We obtain the biquadratic equation from the equation (12)

$$4(y+u)^4 + 4(a-d)(y+u)^2 = (b+c)^2,$$

which is equivalent to the equation

$$(2(y+u)^2 + a - d)^2 = (a-d)^2 + (b+c)^2.$$

As we ask the real solutions of this equation, we obtain

$$2(y+u)^2 = -a + d + \sqrt{(a-d)^2 + (b+c)^2} > 0$$

and finally

$$y + u = \frac{\varepsilon_2}{2} \text{ sign } (b+c) \sqrt{-(a-d) + \sqrt{(a-d)^2 + (b+c)^2}}, \tag{14}$$

where sign $(b+c)$ is equal to the sign of the real nonzero number $b+c$.

We obtain the following real numbers y and u from the equations (13) and (14)

$$y = \frac{\varepsilon_1}{2\sqrt{2}} \text{ sign } (b-c) \sqrt{-(a+d) + \sqrt{(a+d)^2 + (b-c)^2}}$$

$$+ \frac{\varepsilon_2}{2\sqrt{2}} \text{ sign } (b+c) \sqrt{-(a-d) + \sqrt{(a-d)^2 + (b+c)^2}}$$

and

$$u = \frac{\varepsilon_1}{2\sqrt{2}} \text{ sign } (b-c) \sqrt{-(a+d) + \sqrt{(a+d)^2 + (b-c)^2}}$$

$$- \frac{\varepsilon_2}{2\sqrt{2}} \text{ sign } (b+c) \sqrt{-(a-d) + \sqrt{(a-d)^2 + (b+c)^2}},$$

where sign $(b-c)$ is equal to the sign of the real nonzero number $b-c$ and sign $(b+c)$ is equal to the sign of the real nonzero number $b+c$.

Then we obtain solution $x+iy+ju+ijv$ in the considered case. The square roots are the following bicomplex numbers

$$\begin{aligned}\sqrt{A}=\sqrt{a+ib+jc+ijd}=x+iy+ju+ijv=X(\varepsilon_1,\varepsilon_2)\\ =\varepsilon_1\frac{1+ij}{2\sqrt{2}}\sqrt{a+d+\sqrt{(a+d)^2+(b-c)^2}}\\ +i\varepsilon_2 \text{ sign } (b+c)\frac{1+ij}{2\sqrt{2}}\sqrt{-(a-d)+\sqrt{(a-d)^2+(b+c)^2}}\\ +i\varepsilon_1 \text{ sign } (b-c)\frac{1-ij}{2\sqrt{2}}\sqrt{-(a+d)+\sqrt{(a+d)^2+(b-c)^2}}\\ +\varepsilon_2\frac{1-ij}{2\sqrt{2}}\sqrt{a-d+\sqrt{(a-d)^2+(b+c)^2}},\end{aligned}\tag{15}$$

where $\varepsilon_1,\varepsilon_2=\pm1$, sign $(b+c)$ are the signs of the nonzero numbers $b+c$ and sign $(b-c)$ are the signs of the nonzero numbers $b-c$.

1.2. Square roots of bicomplex number $a+ib+jc+ijd$, when $b=c\neq0$ and $b=-c\neq0$

Theorem 2. *The bicomplex number $a+ib+jc+ijd$, where a,b,c,d are real numbers and i, j are the imaginary units of the algebra of bicomplex numbers $\mathbf{C}(j)$, in the case $b=c\neq0$ has the following square roots*

– in the case $a+d<0$ there exist 4 square roots, given by the formula

$$X(\varepsilon_1,\varepsilon_2)=\sqrt{a+(i+j)b+ijd}$$
$$=\varepsilon_1\frac{(1-ij)}{2\sqrt{2}}\sqrt{a-d+\sqrt{4b^2+(a-d)^2}}+i\varepsilon_2\frac{(1+ij)}{2}\sqrt{-a-d}$$
$$+i\varepsilon_1\frac{(1-ij)}{2\sqrt{2}}\quad sign\ \ b\ \sqrt{-(a-d)+\sqrt{4b^2+(a-d)^2}},$$

where $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$ when $a+d<0$.

– in the case $a+d>0$ there exist 4 square roots, given by the formula

$$X(\varepsilon_1,\varepsilon_2)=\sqrt{a+(i+j)b+ijd}$$
$$=\frac{1+ij}{2}\varepsilon_1\sqrt{a+d}+\frac{1-ij}{2\sqrt{2}}\varepsilon_2\sqrt{a-d+\sqrt{4b^2+(a-d)^2}}$$
$$+\frac{i(1-ij)\ sign\ b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a-d)+\sqrt{4b^2+(a-d)^2}},$$

where $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$.

– in the case $a=d$ there exist 2 square roots, given by the formula

$$X(\varepsilon_1)=\sqrt{a(1+ij)+b(i+j)}$$
$$=\frac{1-ij}{2}\varepsilon_1\sqrt{a+\sqrt{a^2+b^2}}+\frac{i(1-ij)\ sign\ b}{8}\varepsilon_1\sqrt{-a+\sqrt{a^2+b^2}},$$

where $\varepsilon_1=\pm1$.

The proof of the theorem is similar to the proof of the Theorem 1.

Theorem 3. *The bicomplex number* $a+ib+jc+ijd$*, where* a,b,c,d *are real numbers and* i*,* j *are the imaginary units of the algebra of bicomplex numbers* $\mathbf{C}(j)$*, in the case* $b=-c\neq 0$ *has square roots as follows*

– in the case $a-d<0$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{a+(i-j)b+ijd}=x+iy+ju+ijv=X(\varepsilon_1,\varepsilon_2)$$
$$=\frac{(1+ij)}{\sqrt{2}}\varepsilon_1\sqrt{a+d+\sqrt{4b^2+(a+d)^2}}+\frac{i(1-ij)}{2}\varepsilon_1\sqrt{-a+d}+$$
$$+\frac{i(1+ij)\ sign\ b}{2\sqrt{2}}\varepsilon_2\sqrt{-a-d+\sqrt{4b^2+(a+d)^2}},$$

where $\varepsilon_1=\pm1$*,* $\varepsilon_2=\pm1$*,* $a<d$*.*

– in the case $a-d>0$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{A}=\sqrt{a+(i-j)b+ijd}=x+iy+ju+ijvX(\varepsilon_1,\varepsilon_2)$$
$$=\varepsilon_1\frac{(1-ij)}{2}\sqrt{a-d}+\frac{1+ij}{2\sqrt{2}}\varepsilon_2\sqrt{a+d+\sqrt{(a+d)^2+4b^2}}$$
$$+\frac{i(1+ij)\ sign\ b}{2\sqrt{2}}\varepsilon_2\sqrt{-(a+d)+\sqrt{(a+d)^2+4b^2}},$$

where $\varepsilon_1=\pm1$*,* $\varepsilon_2=\pm1$*,* $a>d$*.*

– in the case $a=d$ *there exist* 2 *square roots, given by the formula*

$$X(\varepsilon_1)=\sqrt{a(1+ij)+b(i-j)}$$
$$=\frac{1+ij}{2}\varepsilon_1\sqrt{a+\sqrt{a^2+b^2}}+\frac{i(1+ij)\ sign\ b}{8}\varepsilon_1\sqrt{-a+\sqrt{a^2+b^2}},$$

where $\varepsilon_1=\pm1$*.*

The proof of the theorem is similar to the proof of the Theorem 1.

1.3. Square roots of bicomplex number $a+ib+jc+ijd$, when $b=c=0$

Theorem 4. *The bicomplex number* $a+ib+jc+ijd$*, where* a,b,c,d *are real numbers and* i*,* j *are the imaginary units of the algebra of bicomplex numbers* $\mathbf{C}(j)$ *in the case* $b=c=0$ *has square roots as follows*

– in the case $a<-|d|$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{A}=\sqrt{a+ijd}=iy+ju=X(\varepsilon_1,\varepsilon_2)$$
$$=\frac{i(1-ij)\varepsilon_1}{2}\sqrt{-a-d}+\frac{i(1+ij)\varepsilon_2}{2}\sqrt{-a+d},$$

where $\varepsilon_1=\pm1$*,* $\varepsilon_2=\pm1$*;*

– in the case $a>|d|$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{A} = \sqrt{a + ijd} = x + ijv = X(\varepsilon_1, \varepsilon_2)$$
$$= \frac{(1+ij)\varepsilon_1}{2}\sqrt{a+d} + \frac{(1-ij)\varepsilon_2}{2}\sqrt{a-d},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$*;*

– *in the case* $d > |a|$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{A} = \sqrt{a + ijd} = x(1+ij) + iy(1-ij) = X_1$$
$$= \frac{(1+ij)}{2}\varepsilon_1\sqrt{a+d} + i\frac{(1-ij)}{2}\varepsilon_2\sqrt{-a+d},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$*;*

– *in the case* $|a| > d$ *there exist* 4 *square roots, given by the formula*

$$\sqrt{A} = \sqrt{a(1-ij)} = x(1-ij) + iy(1+ij) = X(\varepsilon_1, \varepsilon_2)$$
$$= \frac{(1-ij)}{2}\varepsilon_1\sqrt{a-d} + i\frac{(1+ij)}{2}\varepsilon_2\sqrt{-a-d},$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$*;*

– *in the case* $a = d \neq 0$ *we shall find the square roots of the element* $a(1+ij)$ *which belongs to the idempotent* $I\left(\frac{1+ij}{2}\right)$*. In this case there exist* 2 *square roots, given by the formula*

$$\sqrt{A} = \sqrt{a(1+ij)} = X(\varepsilon_1)$$
$$= \begin{cases} \varepsilon_1 \frac{1+ij}{2}\sqrt{a} & \text{if } \ a > 0, \\ i\varepsilon_1 \frac{1+ij}{2}\sqrt{-a} & \text{if } \ a < 0, \end{cases}$$

where $\varepsilon_1 = \pm 1$*;*

– *in the case* $-a = d \neq 0$ *we shall find the square roots of the element* $a(1-ij)$ *which belongs to the idempotent* $I\left(\frac{1-ij}{2}\right)$*, in this case there exist* 2 *square roots, given by the formula*

$$\sqrt{A} = \sqrt{a(1-ij)} = X(\varepsilon_1)$$
$$= \begin{cases} \varepsilon_1 \frac{1-ij}{2}\sqrt{a} & \text{if } \ a > 0, \\ i\varepsilon_1 \frac{1-ij}{2}\sqrt{-a} & \text{if } \ a < 0, \end{cases}$$

where $\varepsilon_1 = \pm 1$.

Theorem 5. *The algebra of bicomplex numbers* $\mathbf{C}(j)$ *does not admit solutions of the equation* (1) *for the bicomplex number* $A = a + ib + jc + ijd$ *in the case* $a = b = c = d = 0$*, i.e. the equation* $X^2 = 0$ *does not admit nonzero solutions.*

Proof. In this case the system of equations (3), (4), (5), (6) seems as follows

$$x^2 - y^2 - u^2 + v^2 = 0, \ 2xy - 2uv = 0, \ 2xu - 2yv = 0, \ 2xv + 2yu = 0.$$

We obtain the equivalent system of equations

$(x+v)^2 = (y-u)^2, \quad (x-v)^2 = (y+u)^2, \quad (x+v)(y-u) = 0, \quad (x-v)(y+u) = 0,$

which is equivalent to the following system of four linear equations of first order

$$x+v+y-u=0, \quad x-v+y+u=0,$$
$$x+v-y+u=0, \quad x-v-y-u=0.$$

But this system of linear equations has only the zero solutions.

2. Quadratic equation

We shall write the solutions of the quadratic equation in the algebra $\mathbf{C}(j)$ of the bicomplex numbers, using the find above square roots of bicomplex number.

Theorem 6. *The quadratic equation*

$$x^2 + px + q = 0$$

with bicomplex coefficients

$p,\ q \in \mathbf{C}(j), \quad p = p_0 + ip_1 + jp_2 + ijp_3, \quad q = q_0 + iq_1 + jq_2 + ijq_3, \quad p_k, q_k \in \mathbf{R}$

for $k = 0, 1, 2, 3$ *has the following solutions*

$$x_+ = -\frac{p}{2} + X \quad \text{and} \quad x_- = -\frac{p}{2} - X,$$

where

$$X = \sqrt{\frac{p^2}{4} - q}$$

is one of the given in section 1 *bicomplex square roots of the bicomplex number* $\frac{1}{4}p^2 - q$.

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PIERWIASTKI KWADRATOWE Z LICZB BI-ZESPOLONYCH

S t r e s z c z e n i e

Wyznaczono pierwiastki kwadratowe z liczby bi-zespolonej $A = a + ib + jc + ijd$, gdzie a, b, s, d są liczbami rzeczywistymi, i, j, ij zaś – jednostkami bi-zespolonymi. Podajemy rozwiązania równania kwadratowego $X^2 + pX + q = 0$ z niewiadomą bi-zespoloną X i bi-zespolonymi parametrami p, q.

Appendix. Erratum to the paper:

L. N. Apostolova, S. Dimiev, P. Stoev, *Hyperbolic hypercomplex $\bar{\partial}$-operators, hyperbolic CR-equations, and harmonicity II, Fundamental solutions for hyperholomorphic operators and hyperbolic 4-real geometry*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **60** (2010), 61–72.

The first six lines on the page 65 have to be modified as follows

It is fulfilled $dx_0dx_1dx_2dx_3 = rsdrdsdt_1dt_2$ and

$$\tilde{\alpha}^{-1} = \left(\tilde{\zeta} + \tilde{\eta}j_2\right)^{-1} = \left(r\left(\cosh t_1 + \sinh t_1 j_1\right) + s\left(\cosh t_2 + \sinh t_2 j_1\right) j_2\right)^{-1} =$$

$$= \frac{x_0(x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_1x_2x_3}{(x_0 + x_1 + x_2 + x_3)(x_0 - x_1 + x_2 - x_3)(x_0 + x_1 - x_2 - x_3)(x_0 - x_1 - x_2 + x_3)} +$$

$$+ j_1 \frac{x_1(-x_0^2 + x_1^2 - x_2^2 - x_3^2) + 2x_0x_2x_3}{(x_0 + x_1 + x_2 + x_3)(x_0 - x_1 + x_2 - x_3)(x_0 + x_1 - x_2 - x_3)(x_0 - x_1 - x_2 + x_3)} +$$

$$+ j_2 \frac{x_2(-x_0^2 - x_1^2 + x_2^2 - x_3^2) + 2x_0x_1x_3}{(x_0 + x_1 + x_2 + x_3)(x_0 - x_1 + x_2 - x_3)(x_0 + x_1 - x_2 - x_3)(x_0 - x_1 - x_2 + x_3)} +$$

$$+ j_1j_2 \frac{x_3(-x_0^2 - x_1^2 - x_2^2 + x_3^2) + 2x_0x_1x_2}{(x_0 + x_1 + x_2 + x_3)(x_0 - x_1 + x_2 - x_3)(x_0 + x_1 - x_2 - x_3)(x_0 - x_1 - x_2 + x_3)}.$$

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In memory of
Academician Ljubomir Iliev (1913–2000)

Mariya I. Mitreva

ON THE INVERSION THEOREM OF WEI-LIANG CHOW

Summary

According to the famous theorem of Wei-Liang Chow (1949) each analytic subset of $\mathbb{P}^m$ is a projective algebraic set. In the case of an analytic subset of $\mathbb{C}^m$ there are different criteria in order for it to be algebraic. The present author proves three criteria in the opposite direction.

Keywords and phrases: compact complex analytic variety, complex projective space, projective algebraic set

1.

In connection with his famous theorem that each analytic subset of $\mathbb{P}^m$ is a projective algebraic set, Wei-Liang Chow [6] (see also [2–5, 7, 8] had proved seven different criteria, in terms of the behaviour at infinity, in order for it to be algebraic. Roughly speaking, every subvariety of a projective space is a projective variety. The present author proves three criteria in the opposite direction.

2.

Let $z \in \mathbb{C}$, $\mathrm{w} = (w_0, w_1, \dots, w_n) \in \mathbb{C}^n$. Consider the space $\mathbb{P}^m$ of all classes of equivalence in terms of the relation

$$(w) = (w_0, w_1, \dots, w_m) = \left(\frac{w}{w_0}\right) = \left(\frac{w_1}{w_0}, \dots, \frac{w_m}{w_0}\right),$$

for $w_0 \neq 0$. $\mathbb{P}^m$ is the complex projective space. We denote the elements of this space $[w]$ $(w \neq 0)$ and the mapping, which translates a point w into $\mathbb{P}^m$ denote

$$\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{P}^m;$$

π is the mapping of projectivity [1].

Let also $\tilde{w} = (w_1, w_2, \dots, w_m)$, and φ be the mapping of transformation to local coordinates of the rule

$$\varphi : (z, [w]) \to (z, \tilde{w}),$$

which acts as the same $\varphi : \mathbb{C}^n \times \mathbb{P}^m \to \mathbb{C}^n \times \mathbb{C}^m$. Let also

$$H_0 = \{[w],\ w_0 \neq 0\}, \quad \text{and} \quad H^0 = \{[w],\ w_0 = 0\}.$$

Now we consider a pseudopolynomial of m variables, with coefficients $a_i(z)$, $i = 1, 2, \dots, n$, which are analytic functions, depending of n variables:

$$P_q(z, \tilde{w}) = P_q(z_1, \dots, z_n, w_1, \dots, w_m) = a_0(z)\tilde{w}^q + a_1(z)\tilde{w}^{q-1} + \dots + a_{q-1}(z)\tilde{w} + a_q(z)$$

and we put

$$P_q^*(z, w) = P_q(z, \frac{\tilde{w}}{w_0}) w_0^q.$$

Correspondingly, let A be the set in $\mathbb{C}^n \times \mathbb{C}^m$, given by the equation $P_q(z, \tilde{w}) = 0$ and A^* is the set in $\mathbb{C}^n \times \mathbb{P}^m$ with the condition $P_q^*(z, w) = 0$, i.e.

$$\begin{aligned} A &= \{(z, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C}^m, P_q(z, \tilde{w}) = 0\}, \\ A^* &= \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^{m+1}, P_q^*(z, w) = 0\} \\ &= \{(z, (w_0, \tilde{w})) \in \mathbb{C}^n \times \mathbb{C}^{m+1}, P_q(z, \frac{\tilde{w}}{w_0}) w_0^q = 0\}. \end{aligned}$$

3.

We give here the next conclusion about the analytical sets A and A^*

Proposition 1. *The set $\varphi^{-1}A$ is analytical with respect to w in the set $\mathbb{C}^n \times H_0$.*

Remark 1. Here

$$\mathbb{C}^n \times H_0 = \mathbb{C}^n \times \mathbb{P}^m \setminus H^0,$$

means that

$$H_0 = \{[w] \in \mathbb{P}^m, w_0 \neq 0\},$$

and

$$H^0 = \{[w] \in \mathbb{P}^m, w_0 = 0\}.$$

Therefore we consider only such points, for which $w_0 \neq 0$.

Remark 2. The set A contains all the rays in $\mathbb{C}^n \times \mathbb{C}^m$ for which $w_0 \neq 0$ and

$$\frac{w_1}{w_0} = \frac{w_2}{w_0} = \dots = \frac{w_n}{w_0},$$

and the mapping φ acts from

$$\mathbb{C}^n \times \mathbb{C}^{m+1} \setminus \{w_0 = 0\} \quad \text{to} \quad \mathbb{C}^n \times \mathbb{P}^m \setminus H^0$$

(one ray by φ goes to one point).

Hence the map $\varphi^{-1}A$ is a multivalued map and acts from $\mathbb{P}^m \setminus H^0$.

Remark 3. The definition of the mapping φ is in the sense of local coordinates.

Proof of Proposition 1. Because φ is the map of kind to passing to local coordinates of $\varphi^{-1}A$, it is an analytic set on the manifold. According to the definition of the analytic set on the manifold, the set $\varphi^{-1}A$ is analytic:

$$A = \{(z, \tilde{w}), P_q(z, \tilde{w}) = 0\}.$$

4.

Moreover, we have

Proposition 2. *The same set $\varphi^{-1}A$ is analytic in $\mathbb{C}^n \times \mathbb{P}^m$.*

Proof. We can represent $\varphi^{-1}A$ as given:

$$\varphi^{-1}A = A^* \cap \mathbb{C}^n \times H_0 = A^* \setminus (\mathbb{C}^n \times H^0).$$

Moreover, A^* is an analytic set in $\mathbb{C}^n \times \mathbb{P}^m$, the set $\mathbb{C}^n \times H^0$ is also analytic, and according to the properties of these sets the closure of the difference of two analytic sets is analytic set too [3, p. 46]

Proposition 3. *The projection $\tilde{\pi} : (z, [w]) \to z$ of every analytic set $\mathcal{L}$ in $\mathbb{C}^n \times \mathbb{P}^m$ is analytic again set in $\mathbb{C}^n$.*

Proof. It is evident that the mapping $\tilde{\pi}^*$ is homeomorphic with the required property (as the space $\mathbb{P}^m$). Then, using the Remmert's theorem about the property mappings [4] we conclude that $\tilde{\pi} * \mathcal{L}$ is compact. The proof is complete.

Remark 4.The map $f : Y \to B$ is proper if for every compact $K \subset B$ the set $f^{-1}(K)$ is in corresponding connected component of Y.

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O ODWROTNYM TWIERDZENIU WEI-LIANG CHOWA

S t r e s z c z e n i e

Zgodnie ze sławnym twierdzeniem Wei-Liang Chowa (1960) każdy podzbiór analityczny zespolonej przestrzeni rzutowej $\mathbb{P}^m$ jest rzutowym zbiorem algebraicznym. W przypadku analitycznego podzbioru przestrzeni $\mathbb{C}^m$ istnieją rozmaite kryteria na to, by był on algebraiczny. Obecna autorka dowodzi trzech kryteriów w odwrotnym kierunku.

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Gertruda Ivanova

REMARKS ON SOME MODIFICATION OF THE DARBOUX PROPERTY

Summary

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has $\mathcal{I}$-ap-Darboux property ($f \in \mathcal{D}_{\mathcal{I}-ap}$) if for each interval $(a, b) \subset \mathbb{R}$ and for each λ between $f(a)$ and $f(b)$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is $\mathcal{I}$-approximately continuous at x_0. Obviously, the family $\mathcal{D}_{\mathcal{I}-ap}$ is situated between the class $\mathcal{D}$ of Darboux functions and the class $\mathcal{D}_s$ of functions with strong Świątkowski property. We prove that our family is essentially different from both these families and from the family introduced by Grande in [2], i.e. from the family of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each interval $(a, b) \subset \mathbb{R}$ and for each λ between $f(a)$ and $f(b)$ there exists a point $x_0 \in (a, b)$ for which $f(x_0) = \lambda$ and f is approximately continuous at x_0.

Keywords and phrases: Darboux property, strong Świątkowski property, Baire property, $\mathcal{I}$-approximate continuity

Let $\mathcal{D}$ denote the class of Darboux functions. Put

$$< a, b >= (\min\{a, b\}, \max\{a, b\}) .$$

A function $f : \mathbb{R} \to \mathbb{R}$ has the strong Świątkowski property [3] if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in < f(a), f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is continuous at x_0. We will use the symbol $\mathcal{D}_s$ to denote the class of functions with strong Świątkowski property.

Z. Grande in 2009 [2] considered some modification of strong Świątkowski property changing the continuity with approximate continuity, i.e. a function $f : \mathbb{R} \to \mathbb{R}$ has ap-Darboux property if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in < f(a), f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is approximately continuous at x_0. A family of all functions with ap-Darboux property we will denote by $\mathcal{D}_{ap}$.

Let $\mathcal{I}$ be a σ-ideal of sets of the first category. We introduce the analogous modification of strong Świątkowski property changing the continuity with $\mathcal{I}$-approximate continuity, it means with the continuity with respect to the $\mathcal{I}$-density topology in the domain (see [1, 4–7]).

Definition 1. We will say that a function $f : \mathbb{R} \to \mathbb{R}$ has $\mathcal{I}$-ap-Darboux property if for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in < f(a), f(b) >$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is $\mathcal{I}$-approximately continuous at x_0.

We will denote by $\mathcal{D}_{\mathcal{I}-ap}$ a family of all functions with $\mathcal{I}$-ap-Darboux property.

Obviously if f has strong Świątkowski property then f has ap-Darboux property and $\mathcal{I}$-ap-Darboux property. It is easily seen that

$$D_s \subset \mathcal{D}_{ap} \cap \mathcal{D}_{\mathcal{I}-ap} \subset \mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{I}-ap} \subset \mathcal{D}.$$

We will prove that all these inclusions are proper. For this purpose we need some auxiliary lemmas.

If $A \subset \mathbb{R}$ then A' denote the complement of A. We will say that the sets of the form

$$\bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{or} \quad \bigcup_{n=1}^{\infty} [a_n, b_n]$$

are right interval sets at zero if $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$.

Lemma 1. *Let $A \subset \mathbb{R}$. If for each $n \in \mathbb{N}$ and for each interval $(a, b) \subset [0, 1]$ with length equal to $\frac{1}{n}$ there exists an open interval contained in $nA' \cap [0, 1]$, then 0 is not a right-hand $\mathcal{I}$-density point of A.*

Proof. Let $\{n_m\}_{m\in\mathbb{N}}$ be an arbitrary increasing sequence of natural numbers. Since for each $m \in \mathbb{N}$ and for each interval $(a, b) \subset [0, 1]$ with length equal to $\frac{1}{n_m}$ the set $(n_m \cdot A') \cap [0, 1]$ contains some open interval, so for each k the set

$$(0, 1) \cap \bigcup_{m=k}^{\infty} (n_m \cdot A')$$

contains some open set which is dense in $[0, 1]$. Therefore

$$(0, 1) \cap \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} (n_m \cdot A') = (0, 1) \cap \limsup_{m\to\infty} (n_m \cdot A')$$

contains a set which is residual in $[0, 1]$. Hence

$$(0, 1) \setminus \limsup_{m\to\infty} (n_m \cdot A') = \liminf_{m\to\infty} (n_m \cdot A) \cap (0, 1)$$

is a set of the first category. Using lemma 2.1.1. in [1] and from the arbitrariness of the sequence $\{n_m\}_{m\in\mathbb{N}}$ we obtain that 0 is not a right-hand $\mathcal{I}$-density point of A. □

Lemma 2. *There exists a right interval set A at zero such that 0 is a right-hand density point of A, and 0 is not a right-hand $\mathcal{I}$-density point of A.*

Proof. Let

$$a_n = \frac{1}{n+2}, \quad b_n = \frac{1}{n+1} - \frac{1}{(n+1)^2(n+2)}$$

for $n \in \mathbb{N}$. Put

$$A = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Observe that 0 is a right-hand density point of A. We have

$$d_+(A,0) = \liminf_{h\to 0} \frac{m(A\cap[0,h])}{h} = \liminf_{n\to\infty} \frac{m\left(A\cap\left[0,\frac{1}{n}\right]\right)}{\frac{1}{n}} \geq$$

$$\geq \liminf_{n\to\infty} \frac{\left(1-\frac{1}{n}\right) m\left(\left[\frac{1}{n+1},\frac{1}{n}\right]\right) + \left(1-\frac{1}{n+1}\right) m\left(\left[\frac{1}{n+2},\frac{1}{n+1}\right]\right) + \ldots}{\frac{1}{n}} \geq$$

$$\geq \liminf_{n\to\infty} \frac{\left(1-\frac{1}{n}\right) m\left(\left[0,\frac{1}{n}\right]\right)}{\frac{1}{n}} = 1.$$

Now we will prove that 0 is not a right-hand $\mathcal{I}$-density point of A. Observe that the length of the longest interval contained in $(n+1)\cdot A\cap[0,1]$ is equal to $\frac{n}{(n+1)(n+2)}$, so for each interval $(a,b)\subset[0,1]$ with length equal to $\frac{1}{n+1}$ there exists an open interval contained in $(n+1)\cdot A'\cap[0,1]$. Consequently, by Lemma 1, zeri is not a right-hand $\mathcal{I}$-density point of A. □

Theorem 3. *There exists a function $f:\mathbb{R}\to\mathbb{R}$ such that f has ap-Darboux property and f has not $\mathcal{I}$-ap-Darboux property, i.e. $f\in\mathcal{D}_{ap}\backslash\mathcal{D}_{\mathcal{I}-ap}$.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be the sequences defined in the previous lemma. Put

$$A_0 = (-\infty,0]\cup\bigcup_{n=1}^{\infty}(a_n,b_n)\cup[b_1,\infty).$$

Clearly, $d(A_0,0)=1$. Let

$$f(x) = \begin{cases} 1-x & \text{for } x\leq 0,\\ 1-\frac{1}{n} & \text{for } x\in[a_n,b_n],\ n\in\mathbb{N},\\ 0 & \text{for } x=\dfrac{a_n+b_{n+1}}{2},\ n\in\mathbb{N} \text{ and for } x\in[b_1,\infty),\\ \text{linear} & \text{on the intervals } \left[b_{n+1},\dfrac{a_n+b_{n+1}}{2}\right],\ \left[\dfrac{a_n+b_{n+1}}{2},a_n\right],\ n\in\mathbb{N}. \end{cases}$$

Then

(i) f is continuous at each point $x \in \mathbb{R}$, $x \neq 0$;

(ii) f is approximately continuous at 0, because $f \mid A_0$ is continuous at 0 and $d(A_0, 0) = 1$;

(iii) f has ap-Darboux property: for $y \in [0, \infty) \setminus \{1\}$ function f has the strong Świątkowski property and for $y = 1$ there exists a point $x_0 = 0$ such that $f(x_0) = f(0) = 1$ and f is approximately continuous at 0;

(iv) f has not $\mathcal{I}$-ap-Darboux property, $f \notin \mathcal{D}_{\mathcal{I}-ap}$, because f assumes value 1 only at the point 0 and f is not $\mathcal{I}$-approximately continuous at 0.

In order to prove the last property put $C = \left\{x \in \mathbb{R} : f(x) > \frac{1}{4}\right\}$. Obviously, for each $n \in \mathbb{N}$ and for each interval $(a, b) \subset [0, 1]$ with length equal to $\frac{1}{n}$ there exists an open interval contained in $n \cdot C' \cap [0, 1]$. By Lemma 1 zero is not the $\mathcal{I}$-density point of C. Simultaneously

$$f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) \subset C,$$

so f is not $\mathcal{I}$-approximately continuous at 0, because 0 is not the $\mathcal{I}$-density point of $f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$. Consequently, $f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ is not open in the $\mathcal{I}$-density topology. □

Lemma 4. *There exists a right interval set B at zero such that 0 is a right-hand $\mathcal{I}$-density point of B and 0 is not a right-hand density point of B.*

Proof. In the construction we will use a symmetric Cantor-type set C contained in $[0, 1]$ with positive measure $\alpha \in (0, 1)$, such that $\inf C = 0$ and $\sup C = 1$.

Let G_1 be a component interval of $[0, 1] \setminus C$, concentric with $[0, 1]$, which was removed in the first step of the construction. Let G_2 denote the union of the component intervals of $[0, 1] \setminus C$, which were removed in the first and second steps. Let G_n be a union of the component intervals of $[0, 1] \setminus C$, which were removed in the steps $1, \ldots, n$, $n \in \mathbb{N}$.

Put

$$B = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} G_n + \frac{1}{2^n}\right).$$

Obviously B is a right interval set at zero.

Let us show that 0 is not a right-hand density point of B. For this purpose it suffices to show that $d^+(B', 0) > 0$. Indeed, for each $k \in \mathbb{N}$ we have

$$m(G'_k \cap [0, 1]) > \alpha,$$

so

$$m\left(\frac{1}{2^k}(G'_k \cap [0, 1])\right) > \frac{1}{2^k} \cdot \alpha.$$

Hence

$$m\left(B' \cap \left[0, \frac{1}{2^n}\right]\right) = \sum_{k=n+1}^{\infty} m\left(\frac{1}{2^k}(G'_k \cap [0, 1]) + \frac{1}{2^k}\right) > \sum_{k=n+1}^{\infty} \frac{1}{2^k} \cdot \alpha = \frac{\alpha}{2^n},$$

it means $d^+(B', 0) \geq \alpha > 0$.

Now we will prove that 0 is a right-hand $\mathcal{I}$-density point of B. From Theorem 2.2.2, (iii) in [1] it is sufficient to show that for each increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers tending to infinity there exists a subsequence $\left\{t_{k_p}\right\}_{p\in\mathbb{N}}$ such that the set

$$\liminf_{p\to\infty}\left(t_{k_p}\cdot B\right)\cap(0,1)=\bigcup_{m=1}^{\infty}\bigcap_{p=m}^{\infty}\left(t_{k_p}\cdot B\right)\cap(0,1)$$

is residual in $[0,1]$.

So let $\{t_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers tending to infinity. We can assume that $t_1>1$.

For $k\in\mathbb{N}$ put

$$h_k=\frac{1}{t_k}. \tag{1}$$

Hence for each $k\in\mathbb{N}$ there exists exactly one number $n_k\in\mathbb{N}$ such that

$$h_k\in\left[\frac{1}{2^{n_k+1}},\frac{1}{2^{n_k}}\right).$$

So for $k\in\mathbb{N}$ we have

$$1\le h_k\cdot 2^{n_k+1}<2.$$

Let $\left\{h_{k_p}\cdot 2^{n_{k_p}+1}\right\}_{p\in\mathbb{N}}$ be a convergent subsequence,

$$h_{k_p}\cdot 2^{n_{k_p}+1}\xrightarrow{p\to\infty} g.$$

Obviously $g\in[1,2]$,

$$\frac{1}{h_{k_p}\cdot 2^{n_{k_p}}}\xrightarrow{p\to\infty}\frac{2}{g} \tag{2}$$

and $\frac{2}{g}\in[1,2]$.

Put

$$G=\bigcup_{n=1}^{\infty}G_n$$

and

$$B_0=\bigcup_{n=1}^{\infty}\left(\frac{1}{2^n}G+\frac{1}{2^n}\right).$$

Evidently G is open and dense in $[0,1]$, so B_0 is open and dense in $[0,1]$, too.

Now we will prove that

$$\liminf_{p\to\infty}\left(t_{k_p}\cdot B\right)\cap(0,1)=\bigcup_{m=1}^{\infty}\bigcap_{p=m}^{\infty}\left(t_{k_p}\cdot B\right)\cap(0,1)=$$

$$\bigcup_{m=1}^{\infty}\bigcap_{p=m}^{\infty}\left(\frac{1}{h_{k_p}}\cdot B\right)\cap(0,1)\supset\left(\frac{2}{g}\cdot B_0\right)\cap(0,1). \tag{3}$$

Let $x\in\left(\frac{2}{g}\cdot B_0\right)\cap(0,1)$. Since B_0 is open, so x belongs to some component of the set $\left(\frac{2}{g}\cdot B_0\right)\cap(0,1)$.

We have

$$B_0 = \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^n} \cdot \bigcup_{k=1}^{\infty} G_k \right) + \frac{1}{2^n} \right),$$

Therefore there exists a natural number n_0 such that

$$x \in \frac{2}{g} \left(\frac{1}{2^{n_0}} \cdot G + \frac{1}{2^{n_0}} \right),$$

so there exist two natural numbers n_0 and m_0 such that

$$x \in \frac{2}{g} \left(\frac{1}{2^{n_0}} \cdot G_{m_0} + \frac{1}{2^{n_0}} \right). \tag{4}$$

Let $n_{k_p} \geq m_0$. Then for $m = n_{k_p} + n_0$ we have

$$\begin{aligned}
\left(\frac{1}{h_{k_p}} \cdot B \right) \cap (0,1) &= \left(\frac{2^{n_{k_p}}}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot B \right) \cap (0,1) \\
&= \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \bigcup_{m=n_{k_p}+1}^{\infty} \left(\frac{2^{n_{k_p}}}{2^m} G_m + \frac{2^{n_{k_p}}}{2^m} \right) \\
&= \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \bigcup_{m=n_{k_p}+1}^{\infty} \left(\frac{1}{2^{m-n_{k_p}}} G_m + \frac{1}{2^{m-n_{k_p}}} \right) \\
&\supset \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right).
\end{aligned}$$

Using (2) we obtain

$$\frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right) \xrightarrow{p \to \infty} \frac{2}{g} \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right).$$

Hence and from (1) and (4) it follows that for each sufficiently big p

$$x \in \left(t_{k_p} \cdot B \right) \cap (0,1).$$

Therefore

$$x \in \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} \left(t_{k_p} \cdot B \right) \cap (0,1),$$

which gives (3). As B_0 is open and dense in $[0,1]$ and $\frac{2}{g} \geq 1$, so $\left(\frac{2}{g} \cdot B_0 \right) \cap (0,1)$ is also open and dense in $[0,1]$, so it is residual in $[0,1]$. Using (3) we obtain that $\liminf\limits_{p \to \infty} \left(t_{k_p} \cdot B \right) \cap (0,1)$ is residual in the interval $[0,1]$. Consequently, 0 is a right-hand $\mathcal{I}$-density point of B. □

Theorem 5. *There exists a function $g : \mathbb{R} \to \mathbb{R}$ such that g has $\mathcal{I}$-ap-Darboux property and g has not ap-Darboux property, i.e. $g \in \mathcal{D}_{\mathcal{I}-ap} \setminus \mathcal{D}_{ap}$.*

Proof. Analogously as in Theorem 3 we can construct a function g using the set B from Lemma 4. For this purpose consider a set

$$B_1 = (-\infty, 0] \cup B \cup [b_1, \infty) = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (a_n, b_n) \cup [b_1, \infty),$$

where a_n and b_n are the ends of the component intervals of B (defined in the previous lemma) such that $0 < ... < a_n < b_n < a_{n-1} < b_{n-1} < ... < a_1 < b_1$.

Put

$$g(x) = \begin{cases} 1 - x & \text{for } x \leq 0, \\ 1 - \dfrac{1}{n} & \text{for } x \in (a_n, b_n),\ n \in \mathbb{N}, \\ 0 & \text{for } x = \dfrac{a_n + b_{n+1}}{2},\ n \in \mathbb{N} \text{ and for } x \in [b_1, \infty), \\ \text{linear} & \text{on the intervals } \left[b_{n+1}, \dfrac{a_n + b_{n+1}}{2}\right],\ \left[\dfrac{a_n + b_{n+1}}{2}, a_n\right],\ n \in \mathbb{N}. \end{cases}$$

Then

(i) g is continuous at each point $x \in \mathbb{R}$, $x \neq 0$;

(ii) g is $\mathcal{I}$-approximately continuous at 0, because $g \mid B_1$ is continuous at 0 and 0 is the $\mathcal{I}$-density point of B_1;

(iii) g has $\mathcal{I}$-ap-Darboux property: for $y \in [0, \infty) \setminus \{1\}$ function g has the strong Świątkowski property and for $y = 1$ we have a point $x_0 = 0$ such that $g(x_0) = g(0) = 1$ and g is $\mathcal{I}$-approximately continuous at x_0.

(iv) g has not ap-Darboux property, because g assumes value 1 only at the point 0 and g is not approximately continuous at zero.

To prove the last property put $C = \left\{x \in \mathbb{R} : f(x) < \frac{3}{4}\right\}$. Obviously,

$$C \supset \bigcup_{n=1}^{\infty} \left(b_{n+1} + \frac{a_n - b_{n+1}}{4}, a_n - \frac{a_n - b_{n+1}}{4}\right),$$

and

$$B' \cap (0, b_1) = \bigcup_{n=1}^{\infty} [b_{n+1}, a_n].$$

So

$$m\left(C \cap \left[0, \frac{1}{2^n}\right]\right) \geq \frac{1}{2} m\left(B' \cap \left[0, \frac{1}{2^n}\right]\right) > \frac{1}{2} \cdot \frac{\alpha}{2^n}.$$

Hence $d^+(C, 0) \geq \frac{\alpha}{2} > 0$.

Consequently, 0 is not the density point of the set $g^{-1}\left(\left(\frac{3}{4}, \frac{5}{4}\right)\right)$ and g is not approximately continuous at 0. □

Lemma 6. *There exists a right interval set E_0 such that* 0 *is a right-hand dispersion and $\mathcal{I}$-dispersion point of E_0.*

Proof. For $n \in \mathbb{N}$ put

$$c_n = \frac{n+1}{(n+2)!}, d_n = \frac{1}{(n+1)!} \tag{5}$$

and let

$$E_0 = \bigcup_{n=1}^{\infty} [c_n, d_n].$$

Then

$$\lim_{n\to\infty} \frac{d_n - c_n}{c_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!} - \frac{n+1}{(n+2)!}}{\frac{n+1}{(n+2)!}} = \lim_{n\to\infty} \frac{\frac{1}{(n+2)!}}{\frac{n+1}{(n+2)!}} = 0$$

and

$$\limsup_{n\to\infty} \frac{d_{n+1}}{c_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+2)!}}{\frac{n+1}{(n+2)!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \in [0,1).$$

By Lemma 2.1.4 and Theorem 2.2.2, (iii) in [1] it follows that 0 is a right-hand $\mathcal{I}$-dispersion of E_0.

Simultaneously,

$$\begin{aligned}
\limsup_{h\to 0} \frac{m(E_0 \cap [0,h])}{h} &= \limsup_{n\to\infty} \frac{m(E_0 \cap [0,d_n])}{d_n} \\
&\le \lim_{n\to\infty} \frac{(d_n - c_n) + d_{n+1}}{d_n} \\
&= \lim_{n\to\infty} \frac{\frac{2}{(n+2)!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{2}{n+2} = 0,
\end{aligned}$$

so $d^+(E_0, 0) = 0$. □

Theorem 7. *There exists a function $h : \mathbb{R} \to \mathbb{R}$ such that h has ap-Darboux and $\mathcal{I}$-ap-Darboux properties, but it has not the strong Świątkowski property, i.e.*

$$h \in (\mathcal{D}_{ap} \cap \mathcal{D}_{\mathcal{I}-ap}) \setminus \mathcal{D}_s.$$

Proof. For our purpose we will use the set E_0 from the previous lemma. Let

$$a_n = \frac{1}{(n+2)!}, b_n = \frac{n+1}{(n+2)!} \tag{6}$$

for $n \in \mathbb{N}$ and

$$E = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (a_n, b_n) \cup [b_1, \infty).$$

Clearly,

$$\bigcup_{n=1}^{\infty} (a_n, b_n) = \left(0, \frac{1}{2}\right) \setminus E_0,$$

where E_0 is a set from the last lemma, so 0 is the density and $\mathcal{I}$-density point of E.

Put

$$h(x) = \begin{cases} 1-x & \text{for } x \le 0, \\[2ex] 1-\dfrac{1}{n} & \text{for } x \in (a_n, b_n),\ n \in \mathbb{N}, \\[2ex] 0 & \text{for } x = \dfrac{a_n + b_{n+1}}{2},\ n \in \mathbb{N}, \text{ and for } x \in [b_1, \infty), \\[2ex] \text{linear} & \text{on the intervals } \left[b_{n+1}, \dfrac{a_n + b_{n+1}}{2}\right], \left[\dfrac{a_n + b_{n+1}}{2}, a_n\right],\ n \in \mathbb{N}. \end{cases}$$

Then

(i) h is continuous at each point $x \in \mathbb{R}$, $x \neq 0$;

(ii) h is not continuous at 0 and assumes value 1 only at this point, hence h has not the strong Świątkowski property;

(iii) h is approximately continuous at 0, because $h \mid E$ is continuous at 0 and $d(E, 0) = 1$;

(iv) h has ap-Darboux property: for $y \in [0, \infty) \setminus \{1\}$ function h has the strong Świątkowski property and for $y = 1$ there exists a point $x_0 = 0$ such that $h(x_0) = h(0) = 1$ and h is approximately continuous at x_0;

(v) h is $\mathcal{I}$-approximately continuous at 0, because $h \mid E$ is continuous at 0 and 0 is the $\mathcal{I}$-density point of E;

(vi) h has $\mathcal{I}$-ap-Darboux property (the proof in analogous as in 4). □

Theorem 8. *There exists a Darboux function $s : \mathbb{R} \to \mathbb{R}$ such that s has neither ap-Darboux property nor $\mathcal{I}$-ap-Darboux property, i.e. $s \in \mathcal{D} \setminus (\mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{I}-ap})$.*

Proof. We can construct a function using the set E described in the previous theorem. Put

$$s(x) = \begin{cases} 1-x & \text{for } x \le 0, \\[2ex] 0 & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N},\ n \in \mathbb{N} \text{ and for } x \in [b_1, +\infty), \\[2ex] 1-\dfrac{1}{n} & \text{for } x = \dfrac{a_n + b_{n+1}}{2}, \\[2ex] \text{linear} & \text{on the intervals } \left[b_{n+1}, \dfrac{a_n + b_{n+1}}{2}\right], \left[\dfrac{a_n + b_{n+1}}{2}, a_n\right],\ n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are the sequences defined in (6).

Then

(i) s is continuous at each point $x \in \mathbb{R}$, $x \neq 0$;

(ii) s has Darboux property;

(iii) s is not approximately continuous at zero, because $d^{+}\left(\bigcup_{n=1}^{\infty}(a_n,b_n),0\right)=1$;

(iv) s has not ap-Darboux property, because it assumes value 1 only at the point 0 and s is not approximately continuous at 0;

(v) s is not $\mathcal{I}$-approximately continuous at 0, because 0 is the right-hand $\mathcal{I}$-density point of the set $\bigcup_{n=1}^{\infty}(a_n,b_n)$;

(vi) s has not $\mathcal{I}$-ap-Darboux property (analogously as in 4). □

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UWAGI O PEWNEJ MODYFIKACJI WŁASNÓSCI DARBOUX

S t r e s z c z e n i e

Mówimy, że funkcja $f:\mathbb{R}\to\mathbb{R}$ ma $\mathcal{I}$-własność Darboux ($f\in\mathcal{D}_{\mathcal{I}-ap}$), jeśli dla każdego przedziału $(a,b)\subset\mathbb{R}$ i każdej liczby $\lambda\in< f(a),f(b) >$ istnieje punkt $x_0\in(a,b)$ taki, że $f(x_0)=\lambda$ i f jest $\mathcal{I}$-aproksymatywnie ciągła w punkcie x_0. Oczywiście, rodzina $\mathcal{D}_{\mathcal{I}-ap}$ znajduje się pomiędzy rodziną $\mathcal{D}$ funkcji o własności Darboux i rodziną $\mathcal{D}_s$ funkcji o silnej własności Świątkowskiego. Pokazaliśmy, że rodzina $\mathcal{D}_{\mathcal{I}-ap}$ istotnie różnie się od rodziny wprowadzonej przez Grande w [2], tzn. od rodziny wszystkich funkcji $f:\mathbb{R}\to\mathbb{R}$ takich, że dla każdego przedziału $(a,b)\subset\mathbb{R}$ i każdej liczby $\lambda\in< f(a),f(b) >$ istnieje punkt $x_0\in(a,b)$ taki, że $f(x_0)=\lambda$ i f jest $\mathcal{I}$-aproksymatywnie ciągła w punkcie x_0.

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Recherches sur les déformations no. 3

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Stanisław Bednarek

BASIC OPTICAL ILLUSIONS CAUSED BY MOTION

Summary

The basic optical illusions as: stroboscopic effect, bird in a cage phenomenon, effect of circulating pendulum, permanence of vision effect, waterfall effect, depth of seeing phenomenon and Benhams effect are described and its examples are given. The mentioned illusions are also explained and experimental method its demonstration are presented. Occurrence and significance this illusions in every day life and modern technology are also discussed.

Keywords and phrases: optical illusion, stroboscopic effect, waterfall effect, Benhams effect

1. Introduction

Sight is one of the five senses that a human possesses, apart from hearing, smell, taste and touch. Undoubtedly, it is one of the most significant senses when it comes to the amount of the transferred data. it is estimated that 90 % of information is perceived through sight. This sense presents an adaptability skill in a broad range to different conditions. Thanks to this, correct perception of things at e.g. various lightning and distance is possible.

However in some cases, the sense of sight may cause the perceived sensations to distort, and then we experience optical illusions. One of the factors causing optical illusions is motion. Aim of this article is to present optical illusions caused by motion. It turns out that these illusions play an important role in everyday life, while watching movies or playing computer games.

2. Stroboscopic effect

Image created by a light stimulus on an eyes retina does not fade out right after this stimulus stops working. Extinction of the image that emerged takes place after a period of several hundredths of a second to one tenth of a second after the stimulus has faded out. This fact, often called “persistence of eye”, has mutual effects on the vision process [1]. The first one includes causing a sense of motion while watching images that come after each other quickly, presenting subsequent positions of a given item.

In order to realize this fact, we will conduct an experiment using Fig. 1. We can see here the subsequent positions of a simple pendulum that vibrates. How about copying this picture and cutting it into eight parts. Lets put these parts on each other – according to the given numbers – and staple, putting the stitch on the left side, on the marked spots. In this way, we have a small book comprising eight pages.

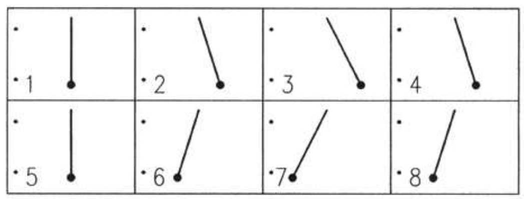

Fig. 1: Subsequent phases of a simple pendulum motion.

We hold the backbone in fingers of the left hand, and we leaf it through with fingers of the right hand. Looking at the pages of the book being leafed through, we experience an illusion that the pendulum vibrates. Our eye perceive the images of pendulums positions quickly coming after each other as a one picture of a moving pendulum. The observed effected has been used to project moving images in the cinema, TV and games. In order to provide the continuity of motion, the cinema usually uses the projection of 24 frames per second, and TV 50 frames or more.

The second type of the persistence of the eye effects can be noticed while illuminating the moving objects with short, periodical, bursts of lights. It includes causing a sensation of an item immovability when frequency of the bursts equals with or is an integral multiple of the items motion frequency. We can spot this effect by waving a hand quickly, in front of a TV or computer screen. When we wave at a frequency which is a multiplication of the projected image frequency, we will be able to see a multiplied, motionless image of the hand. This effect allows to establish the frequency of an item motion, e.g. of a rotating or vibrating part of machine, based on the frequency of the bursts of light.

A TV set or computer screen can be also used as an stroboscopic illuminator in a school physics lab or at home. During some experiments, it allows to watch motionless or slowed down images of the observed phenomenon, e.g. falling bodies or wave motion. The way of observing the falling drops is presented in Fig. 2. The speed of the creation of drops is regulated by a degree of opening a tap 2, being located at the output of the vessel with liquid. Time of burst illuminating the drops can be changed through increasing or decreasing the width of a gap between two sheets of black paper 5, hiding a screen.

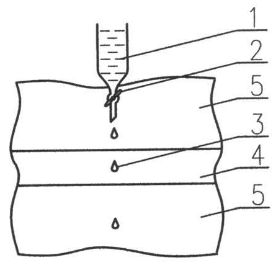

Fig. 2: A system for drops falling observations, in stroboscopic lightning; 1 – a vessel with liquid, 2 – tap, 3 – drop, 4 – TV or computer screen, 5 – black paper.

3. A bird in a cage

This illusion consist in the fact that two images, quickly presented after each other, are perceived as a single picture. In order to find it out, lets copy the Fig. 3 and cut out the rectangle with the obtained images inside. This rectangle should be folded in half along the dashed line, and the open side should be taped with a transparent tape. The end of a pencil, with the images taped to it with the same tape, is put in the folded images. In this way we obtained an item resembling a lollipop [2].

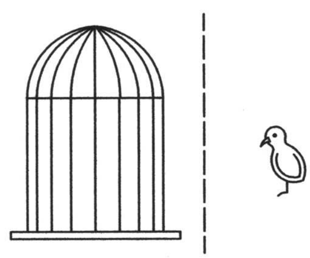

Fig. 3: Images for a bird in a cage observations.

Now, the free end of the pencil should be placed between hands, and the item should be propelled by moving hands one after another, in a way that makes it present a quick circular motion. While looking at the images, it will be noticed that the bird is inside the cage. Actually, it is an illusion caused by a short time interval, when the images of the cage and the bird are demonstrated. Because of that, the eyes are not able to differentiate these pictures, and as a result we see the bird in the cage, not realizing that it is a mix of two images.

4. Effect of a circulating pendulum

This effect can be easily checked in an experiment as well. We should prepare a pendulum, similar to a simple one, through attaching a modeling paste ball to a piece of thread. After hanging this pendulum and deviating it from the perpendicular direction, we can find out that it vibrates in one plane, which vanishes slowly. Therefore, this phenomena takes place according to our assumptions. Lets prepare glasses with one glass that is strongly dimmed. Such sunglasses can be easily obtained by taking one glass from sunglasses. Again, lets deviate the pendulum from the perpendicular direction and look at it through the prepared glasses.

This time, we will notice that while vibrating, the pendulum gets closer and moves away, making circles. Therefore, the pendulums motion seems to be a three-dimensional motion. However, it is just an illusion, because when we take off the glasses we see that the pendulum still vibrates in one plane. The observed illusion is cause by different times of perceiving dark and bright pendulum image by the eyes. Each eye receives different images of the pendulum – one receives a dark and the second one a bright picture. It turns out that the dark image is perceived slightly later than the bright one. These two images mix up in the sight center inside brain, which gives a sensation of three-dimensional motion. The described illusion is also called a Schwarzschild effect and is sometimes used in some cinemas for a three-dimensional visualization of an image through special glasses.

5. The permanence of vision

A plastic pipe of 2–3 cm dimension and 25–30 cm length will be necessary for the next experiment – presenting an optical illusion caused by motion. It may be a pipe from plastic or metal. Such a pipe can also be easily made by covering an item, e.g. a handle of a brush, of the given dimension with a sheet of paper, and gluing its edges. When the glue became and the paper is taken off the handle, the pipe is ready. One of the pipes ends needs to be closed with a non-transparent disk with a gap, showed in Fig. 4. Such a disk can be also successfully cut out of a sheet of paper. The teeth, which are visible on a disks edge, should be folded and glued to the pipe.

Now, we should apply the open end of the pipe to one eye and close the second eye. Through the gap, we can see just an oblong part of the surroundings in front of us. This part is limited by the gaps edges. However, if we start to move the pipe quickly with fingers, we will notice a part of surroundings limited by a round edge of the pipe, as there was no disk at all. This illusion is caused by the fact that subsequent images limited by the gap are present on retina, which gives a sensation of one image limited by the pipes edge.

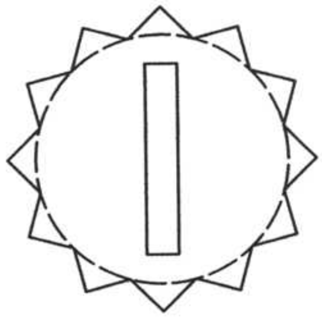

Fig. 4: A disk with a gap for permanence of seeing examination.

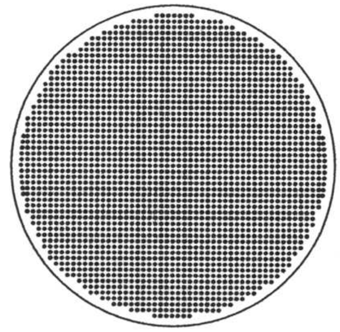

Fig. 5: A net of dots for waterfall effect observation.

6. Waterfall effect

In order to see this effect, we make a copy of a net of dots presented in Fig. 5. Then we cut the copied disk out of a piece of paper. We will have to make it rotate at a speed of about 1–2 rotations per second. Then, lets have a look at the unfolded fingers of a hand. We will notice that they seem to twist in a direction opposite to this of the disk. This is a waterfall effect, because it is similar to the phenomenon that takes place when a person stares at falling water, with rocks in the background. If at some point we take a look at the rocks surrounding the waterfall, we will have a sensation that they move upwards. It is worth noticing that if the speed of the disk rotation is too high, the dots will created circles, and we will not spot the described effect. In order to explain this effect, we should take into account the mutual interaction of the eyes and sight centers inside the brain. It is assumed that certain parts of the cortex are stimulated when eyes follow the e.g. right motion. Whereas other parts are activated when the eyes look at the e.g. left motion. If an item is motionless, or two objects move in opposite direction at the same speed, the stimulation level of both centers is the same, and we do not experience the sensation of motion. If then, we have a look on a motionless object, the centers that are stimulated faster will present a shorter time of stimulation vanishing than the centers stimulated slower. As a result, we have a sensation of rotation in opposite direction.

7. The depth of seeing

In order to see this effect, we also copy the object in Fig. 7, and we prepare a disk, which is rotated in a way described in the case of waterfall effect. Looking at the surface of the rotating disk, we can spot colorful strips in a shape of an arc. This effect is explained in the following way. There are three kinds of photoreceptor cells sensitive to red, green and blue color. Eyes and the brain record the white color only when they are stimulated by light in a way that all kinds of photoreceptor cells show the similar response times. White arc move at different speed on the rotating disk – the greater the speed, the greater their distance from the center of the disk. Length of the arcs are also different. As a result, times of response to stimulation and its vanishing for particular kinds of photoreceptor cells are different, and the arcs seems to be colored. It is worth adding that the descried effect was discovered in 19th century by Benham – producer of toys, who spotted colorful arcs on a surface of a spinning top, painted in black and white stripes.

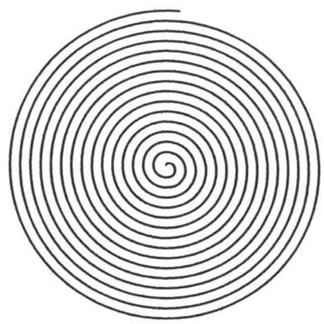

Fig. 6: A spiral for depth of seeing presentation.

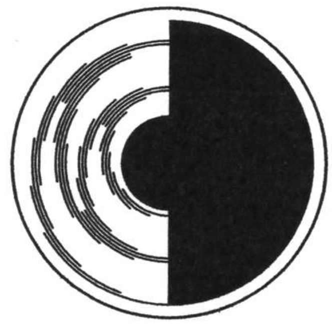

Fig. 7: A figure for Benhams effect observations.

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PODSTAWOWE ZŁUDZENIA OPTYCZNE SPOWODOWANE RUCHEM

S t r e s z c z e n i e

W artykule opisane zostały podstawowe złudzenia optyczne spowodowane ruchem, takie jak: efekt stroboskopowy, zjawisko ptaka w klatce, efekt krążacego wahadła, trwałość widzenia, efekt wodospadu, głębokość widzenia oraz dysk Benhams. Wspomniane złudzenia zostały również wyjaśnione i podano sposoby przeprowadzenia demonstrujących je doświadczeń. Omówiono także występowanie i znaczenie tych złudzeń w życiu codziennym oraz we współczesnej technice.

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Romuald Brzozowski, Ewelina Z. Frątczak, Małgorzata Antoszewska-Moneta, Tomasz Gwizdalla, Bogdan Pawłowski, Józef Balcerski, Kazimierz Dolecki, and Marek Moneta

PHASE TRANSFORMATIONS OF VP800 SURFACE BY IMPACT OF SLOW HEAVY ION ANALYZED WITH CEMS

Summary

Structural transformation of amorphous material surface is related to energy thresholds, either in the potential energy, 10 keV deposited by Highly Charged Ion (HCI) or with the threshold in electronic energy loss, 5 keV/nm transferred by Swift Heavy Ion (SHI). In this work, in order to look for structural and magnetic transformations, thin foils of amorphous alloy $Fe_{73}Si_{16}B_7Cu_1Nb_3$ (VP800) were irradiated with slow heavy ions (200 keV Ar and N) at doses 10^{10} and 10^{11} Ar/cm^2. With the use of *ex-situ* Mössbauer spectroscopy (CEMS) Fe and Fe(Si) clusters accompanied by Fe_3Si nano-crystals were found in the samples irradiated at lower ion doses, whereas rather amorphous structures can be spotted in samples more heavily implanted ions.

Keywords and phrases: crystallization, phase transformations, Mössbauer spectroscopy, heavy ions

1. Introduction

Amorphous alloys can crystallize partially, if an appropriate amount of energy is supplied [1–3]. Recently, crystallization induced by 5 GeV Pb ions at low fluency in amorphous alloys which exhibit two steps thermal crystallization (like Finemet) was reported [4], against absence of the effect in alloys suffering only a single step thermal crystallization. In this case only secondary (without primary) crystallization phase was observed, probably in some correlation with absence of Cu. Crystallites of 1–4 nm size were formed around amorphous ion track of 6–8 nm in diameter, thus a single ion converts material from initially amorphous to other amorphous and crystalline structures roughly within 100 nm^2 area [4].

During slowing down of a SHI or impact of a HCI its energy is faster deposited to the material through electronic excitations (10^{-15} s), than subsequently transferred to the lattice through electron-phonon coupling (10^{-13} s) [5–6]. This can locally cause rapid heat-cooling pulse, with the peak temperature of thousands of K.

We look for a similar phase transformation induced in surface of amorphous alloy by impact of slow heavy ion [7].

2. Experimental setup

The samples used in this study were 20 μm thick, amorphous ribbons prepared by a rapid cooling (10^7 K/s) with the melt-spinning technique. It was checked with X-ray diffraction (XRD) and with transmission Mössbauer spectroscopy (TMS) that initially the foils were amorphous (with small addition of nano-crystals of Fe_3Si) and with differential scanning calorimetry (DSC) that they exhibit two step crystallization [2–3]. The irradiation with Ar and N ions were performed at normal incidence under a controlled ion flux lower than 10^9 ions/cm^2/s with Cockroft-Walton type accelerator working effectively in the voltage range from 50 to 300 kV. It is capable to provide on the sample ion beam current density from 1 nA/cm^2 up to 1 μA/cm^2, measured by the Faraday cup. In VP800 alloy Ar ions have a rate of energy deposition into electronic and nuclear processes of Se$\sim$Sn$\sim$778 eV/nm [8], in a very short time scale of 10^{-13} s, required to stop the ion.

MS experiments were performed in the constant acceleration mode with ^{57}Co: Rh (6.3 keV X-rays and 14.4 keV γ-rays calibration lines) source of 50 mCi activity. CEMS was based on detection of 7.3 keV Fe conversion e^-, penetrating roughly 200 nm surface layer by LEK-2 He+5 % CH_4 gas flow detector at room temperature. The CEMS data acquisition was based on MOSIEK analyzer and for signal processing the CAMAC system combined with Tukan 8k analyzer was used. Spectra were analyzed with RECOIL [9].

The 20 μm thick $Fe_{73}Si_{16}B_7Cu_1Nb_3$ foils after surface cleaning with Ar ion sputtering were transferred through the air to the reaction chamber of 10^{-6} hPa, fixed on LN_2 cryostat and irradiated at low current of 1 nA/cm^2 with 200 keV Ar ions at the fluency 10^{10} and 10^{11} Ar/cm^2. Subsequently, the foils were again transferred through the air to the CEMS reaction chamber of 10^{-9} hPa for the *ex-situ* Mössbauer analysis.

In this work are presented results for Ar incident beam at 200 keV energy and simulations from SRIM for bought Ar and N for 200 keV.

3. Results of SRIM calculations

The plot in Fig. 1 shows the final distribution of the ions in the target calculated by STRIM [7]. The average ion ranges in VP800 are about 100 nm for Ar ions and 230 nm for N ions.

The final distribution of the recoils atoms of Fe, Si and B from the VP800 target structure are presented in Fig. 2. This plot is updated after each ion. For Ar 200 keV ions the distribution has maximum at about 70 nm for each recoiled atoms. For N 200 ions the maximum are shifted to 200 nm for the same incident energy of ions. Also the shape of distribution is much different.

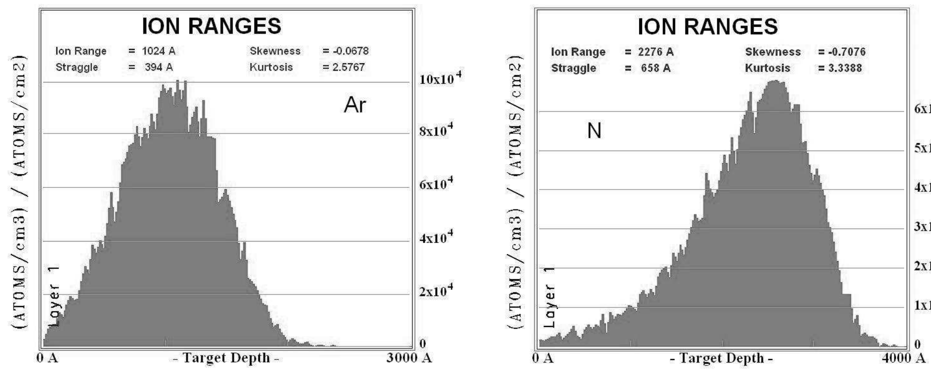

Fig. 1: Depth distribution of energy absorbed from 200 keV Ar and N ions by atoms in $Fe_{73}Si_{16}B_7Cu_1Nb_3$ amorphous alloy.

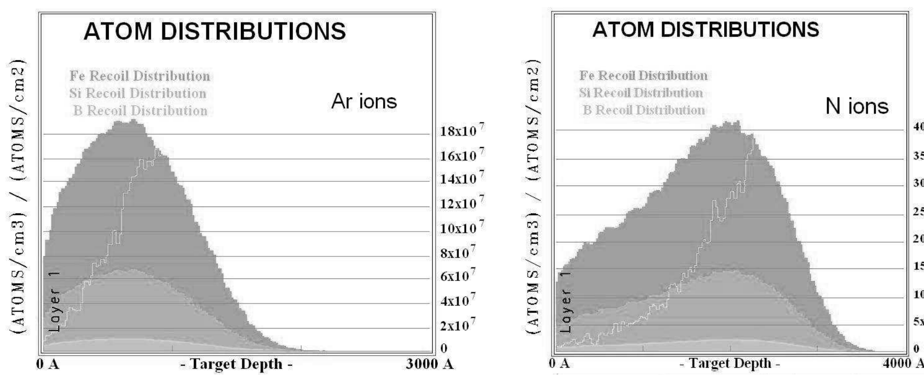

Fig. 2: The distribution of the recoils atoms of Fe, Si and B from the VP800 for 200 keV energy of Ar and N ions beam.

In Fig. 3 are the direct energy loss by the ion to the various target atoms. This energy loss, plus the direct energy loss of the ion to the target electrons, sum to the energy loss of the ion into the target. The distribution displays maxima at about 70 nm for Ar ions and about 200 nm for N baem. The efficiency of the energy absorption ranges from 3.57 keV/atom-B to 103 keV/atom-Fe per single Ar ion impact. Relatively smaller values are obtained for N ions: 1.47 keV/atom-B to 38.6 keV/atom-Fe. These two distributions show the channels of local heating.

The Fig. 4 contains the energy given up to the target electrons. The data relating to "Ions" is the direct energy transferred from the ion to the target electrons. The data relating to the "Recoils" is energy transferred from recoiling target atoms to the target electrons. For 200 keV Ar up to 130 nm more energy for ionization of target atoms are given by incident ions, up to 60 nm it is more then 50 %. For 200 keV N ions this almost all energy for ionization of target atoms are given by incident ions for all depth.

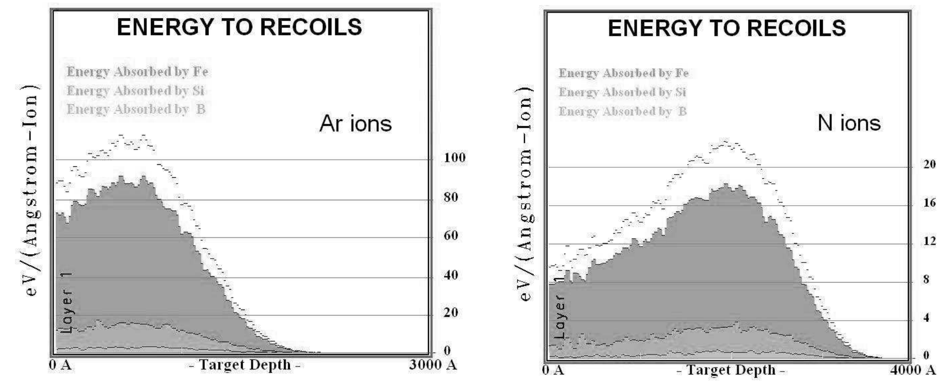

Fig. 3: The direct energy loss by the 200 keV Ar and N ions to the various target atoms.

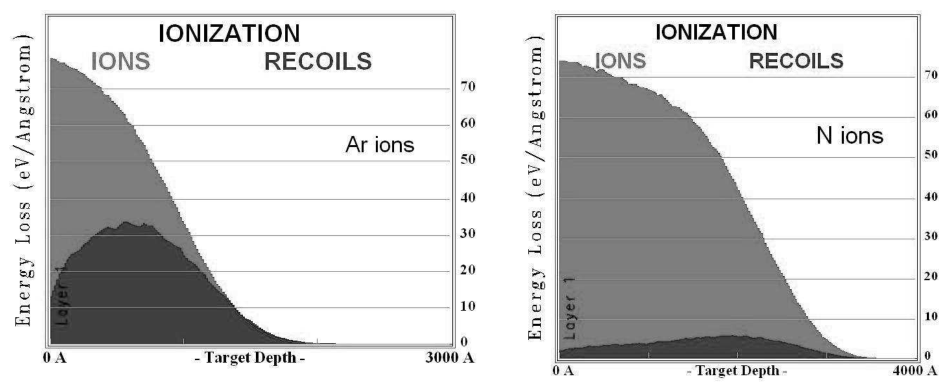

Fig. 4: The energy loss by 200 keV Ar and N ions for ionization of the target atoms.

4. Results of CEMS experiments

CEMS spectra obtained for pristine foil reveal amorphous structure which is characterized by set of hyperfine parameters and structure factors analyzed previously in detail [2, 3] in dependence on temperature ranging from 70 K up to 1200 K and backwards.

With the use of scanning calorimetry, scanning thermo-magnetometry (TM), X-ray diffraction and with transmission Mössbauer spectrometry it was shown that during conventional thermal treatment the bulk thick VP800 suffers a set of structural and magnetic transformations correlated with each other [3] and is characterized by set of specific parameters, like activation energy for crystallization, the Curie temperatures or hyperfine fields and isomer shifts, and other parameters [3].

The surfaces irradiated at the dose of 10^{10} Ar/cm^2 are statistically within 1 % covered with 100 nm^2-size spots, where ions hit the surface, whereas the spots overlapping begins at 10^{12} Ar/cm^2 causing massive structural and magnetic destruction. This latter implanted dose, distributed over 100 nm penetration depth, results in the average Ar concentration of 1 ppm, which can influence phase transformations. During stopping within 10^{-13} s the whole 200 keV energy of each Ar ion is transferred to electrons contained in $8 \cdot 10^3$ nm^3 and subsequently shared among $5 \cdot 10^5$ atoms. It results in increase of lattice temperature to $3 \cdot 10^3$ K, which in turn induces magnetic and structural phase transitions of the subsurface region. The penetration depth of 100 nm for 200 keV Ar ion is comparable to the mean range of conversion electrons, which allows for CEMS analysis of the whole irradiation region.

CEMS spectra from foils irradiated at doses from 10^{10} Ar/cm^2 to 10^{12} Ar/cm^2, reveal tiny effect which can be unambiguously correlated with impact of ion beam.

For untreated foil the CEMS spectra reveal a broadened Zeeman sextet ascribed to the primary amorphous phase and 5 % of Fe_3Si nano-crystals Fig. 5. For foil radiated at $3 \cdot 10^{10}$ Ar/cm^2 and $3 \cdot 10^{11}$ Ar/cm^2, with the remaining of amorphous phase the Fe_3Si phase have bigger contribution, relative to dose Fig. 6. This effect can be correlated to grown of Fe_3Si crystals in amorphous matrix or creation of new Fe_3Si crystals in the neighborhoods of ion trace, but this should be investigated by XRF in the future. The dependence on irradiation dose suggests that the sextets are related to crystalline parts of the alloy produced under impact of Ar ions. Based on hyperfine parameters the multiplets were identified to come from Fe_3Si nano-crystals.

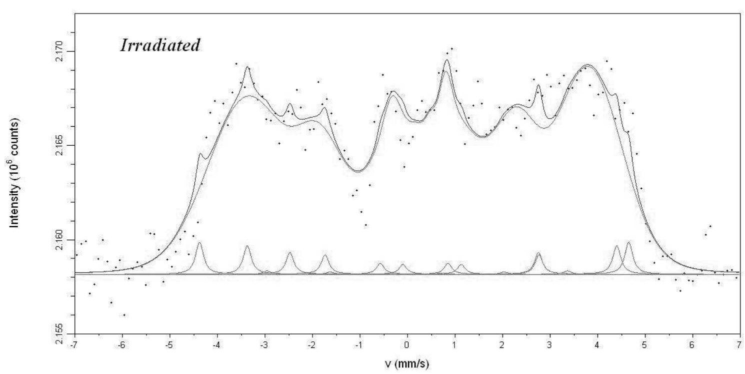

Fig. 5: CEMS from $Fe_{73}Si_{16}B_7Cu_1Nb_3$ (amorphous phase with sextets from Fe_3Si) irradiated with 200 keV Ar ions, analysed with RECOIL [9].

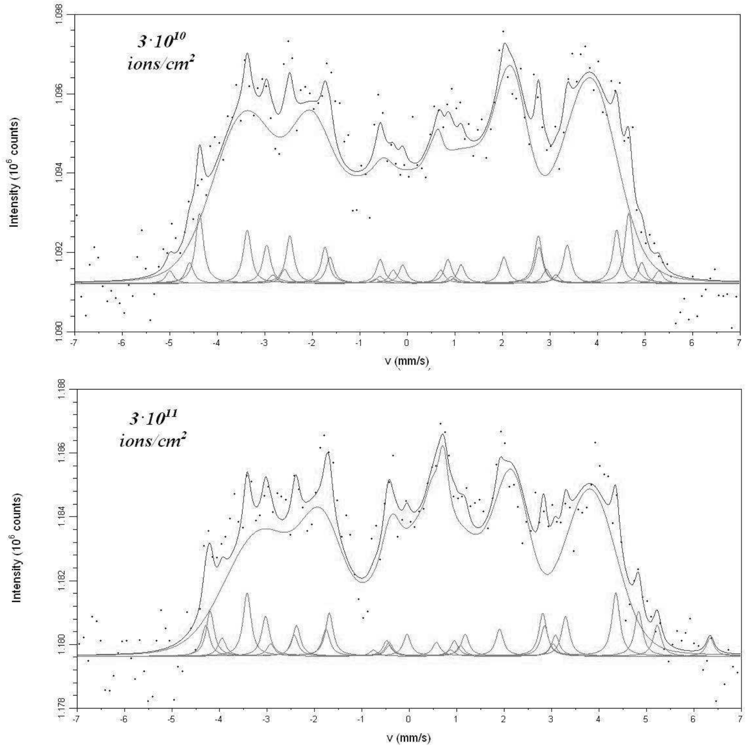

Fig. 6: CEMS from $Fe_{73}Si_{16}B_7Cu_1Nb_3$ radiated at $3 \cdot 10^{10}$ Ar/cm^2 and $3 \cdot 10^{11}$ Ar/cm^2 (amorphous phase with sextets from Fe_3Si), analysed with RECOIL.

The amorphous phase (and amorphous remainder) is characterized mainly by distribution of the hyperfine magnetic field which reveals weak (10 T) and strong (22 T) field components related to two basically distinct magnetic neighbourhood of Fe nuclei. Table 1 shows the calculated from CEMS spectra percentage values amorphous and crystalline phases in the sample before and after the irradiation of argon ions beam.

Tab. 1: Percentage values amorphous and crystalline phases in the sample VP800.

Sample	Amorphous phase	Crystalline phase (Fe_3Si)
Irradiated	*95 %*	*5 %*
$3 \cdot 10^{10}$ ions/cm^2	*85 %*	*15 %*
$3 \cdot 10^{11}$ ions/cm^2	*82.5 %*	*17.5 %*

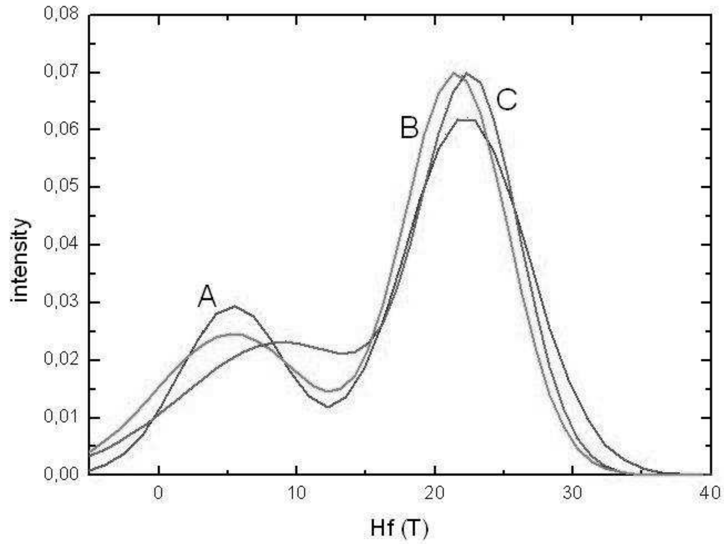

Fig. 7: Distribution of the hyperfine field of amorphous phase and amorphous remainder in VP800 irradiated (A) and radiated (B) – $3 \cdot 10^{10}$ Ar/cm^2 and (C) $3 \cdot 10^{11}$ Ar/cm^2, analysed with RECOIL.

It can be seen that the actual structural and magnetic composition of the surface is a result of at least two concurrent processes: creation of structures caused by heating-cooling pulses due to energy lost by the ion and disintegration of the structures due to kinematic amorphisation by HI impact.

5. Conclusions

It was demonstrated that impact of a relatively slow and relatively heavy ion can cause structural and also magnetic transformations of the amorphous alloys surface, the effect which up to now, in metals and isolators [5, 6], has been reserved only for slow HCI and fast HI and restricted by energy thresholds.

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ANALIZA METODĄ CEMS PRZEMIAN FAZOWYCH WYWOŁANYCH BOMBARDOWANIEM NISKOENERGETYCZNYMI CIĘŻKIMI JONAMI POWIERZCHNI VP800

S t r e s z c z e n i e

Przejścia fazowe na powierzchni materiałów amorficznych związane są z przekroczeniem energii progowej, ok. 10 keV dla energii potencjalnej przekazywanej przez wysokonaładowany jon lub ok. 5 keV/nm strat energii kinetycznej dla przyspieszonych ciężkich jonów. W pracy użyty został stop amorficzny $Fe_{73}Si_{16}B_7Cu_1Nb_3$ (VP800) w postaci cienkiej folii (20 μm), który następnie naświetlono jonami Ar o energii 200 keV dla dwóch dawek: 10^{10} jonów/cm^2 i 10^{11} jonów/cm^2. Następnie przeprowadzono pomiary za pomocą spektrometru mössbauerowskiego elektronów konwersji wewnetrznej CEMS, które ujawniły powstałe w wyniku bombardowania powierzchni VP800 jonami Ar krystalitów Fe_3Si w przeciwieństwie do amorfizacji powodowanej przez bombardowanie jonami o wyższych energiach.

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