B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES
ET DES LETTRES DE ŁÓDŹ

SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXIV, no. 1

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Volume LXIV, no. 1

ŁÓDŹ 2014

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**Wydano z pomocą finansową Ministerstwa Nauki
i Szkolnictwa Wyższego**

PL ISSN 0459–6854

Wydanie 1.
Nakład 200 egz.
Skład komputerowy: Zofia Fijarczyk
Druk i oprawa: Drukarnia Wojskowa
Łódź, ul. Gdańska 130
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The journal appears in the bases *Copernicus* and *EBSCOhost*

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SUBMITTING THE PAPERS FOR BULLETIN

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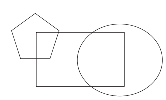

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References

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Affiliation/Address

TABLE DES MATIÈRES

Professor Claude Surry among the participants of the Łódź-Lyon/Saint Etienne/Villeurbanne Seminar at Malinka (Mazurian Lakeland, Poland), 1992

Seminar participants
Front row (from left to right): Robert Redon (Villeurbanne), Leszek Wojtczak (Łódź), Ilona Zasada (Łódź), Elżbieta Żegalska (Łomża), Zofia Fijarczyk (Łódź), Piotr Modrak (Warszawa).
Back row (from left to right): Julian Ławrynowicz (Łódź), Claude Surry (Saint Etienne), William Pezzaglia, Jr. (San Fracisco, CA), Guy Bertholon (Saint Etienne), Jerzy Rutkowski (Łódź), Madeleine Bertholon (Saint Etienne), Jacques López (Lyon), Yves Robach (Lyon), Louis Porte (Lyon)

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 11–24

In memory of
Professor Claude Surry

Nak Eun Cho, Oliwia Chojnacka and Adam Lecko

ON DIFFERENTIAL SUBORDINATION OF GEOMETRIC MEAN

Summary

For $t \in [0,1]$ and a, $b \in \mathbb{C} \setminus \{0\}$, let $\mathrm{G}_t(a,b) := a^{1-t}b^t$ be a geometric mean of a and b. Given $\gamma \in [0,1]$, we study the differential subordination of the following form:

$$\mathrm{G}_\gamma\left(p(z), p(z) + zp'(z)\Phi(p(z))\right) \prec h(z) \Rightarrow p(z) \prec h(z)$$

for $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, where p, h and Φ satisfy required assumptions.

Keywords and phrases: differential subordination, geometric mean, convex function

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ of functions f normalized by $f(0) := 0$ and $f'(0) := 1$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions.

It is well known that a function $f \in \mathcal{H}$ is said to be subordinate to a function $F \in \mathcal{H}$ if there exists $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = F \circ \omega$ in $\mathbb{D}$. We write then $f \prec F$. When F is univalent, then

$$f \prec F \Leftrightarrow (f(0) = F(0) \wedge f(\mathbb{D}) \subset F(\mathbb{D})). \tag{1.1}$$

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let $h \in \mathcal{H}$ be univalent. We say that a function $p \in \mathcal{H}$ satisfies the first-order subordination if a function

$$\mathbb{D} \ni z \mapsto \psi\left(p(z), zp'(z)\right)$$

is well defined and analytic, and

$$\psi\left(p(z), zp'(z)\right) \prec h(z), \quad z \in \mathbb{D}. \tag{1.2}$$

The question when (1.2) yields $p \prec h$ is the basis for the theory of differential subordinations (see Lewandowski, Miller and Złotkiewicz [9], Miller, Mocanu [10] and [11]). For further details and references see the book of Miller and Mocanu [13], Biernacki [16, 17], and Rogosinski [18].

For further discussion we need the following definition.

Definition 1.1. Let $\Phi : D \to \mathbb{C}$ be an analytic function in a domain D in $\mathbb{C}$. By $\mathcal{H}(\Phi)$ we denote the subclass of $\mathcal{H}$ of all functions p such that $p(\mathbb{D}) \subset D$, the function

$$P_{\Phi,p}(z) := \begin{cases} 1 + \dfrac{zp'(z)}{p(z)}\Phi(p(z)), & z \in \mathbb{D} \setminus p^{-1}(0), \\ \lim_{\mathbb{D}\ni\zeta\to z} P_{\Phi,p}(\zeta), & z \in p^{-1}(0), \end{cases} \tag{1.3}$$

is analytic and $P_{\Phi,p}(z) \neq 0$ for $z \in \mathbb{D}$.

By $\mathcal{H}^0(\Phi)$ we denote the subclass of $\mathcal{H}(\Phi)$ of all functions p such that $p(z) \neq 0$ for $z \in \mathbb{D}$.

Remark 1.2. 1. For $p \in \mathcal{H}^0(\Phi)$ we have $p^{-1}(0) = \emptyset$, so then the function

$$P_{\Phi,p}(z) = 1 + \frac{zp'(z)}{p(z)}\Phi(p(z)), \quad z \in \mathbb{D}, \tag{1.4}$$

is analytic and has no zero in $\mathbb{D}$.

2. Note that $p \in \mathcal{H}^0(\Phi)$ if and only if $p \in \mathcal{H}$, $p \not\equiv \text{const}$, $p(\mathbb{D}) \subset D$, and

$$p(z) \neq 0, \quad p(z) + zp'(z)\Phi(p(z)) \neq 0, \quad z \in \mathbb{D}. \tag{1.5}$$

3. Let $\Phi : D \to \mathbb{C}$ be an analytic function in a domain D in $\mathbb{C}$ with $0 \in D$. We now show that $\mathcal{H}^0(\Phi) \subsetneq \mathcal{H}(\Phi)$. Clearly, $\mathbb{D}_R \subset D$ for some $R > 0$. Given $n \in \mathbb{N}$ and $c \in \mathbb{D}_R$ with $0 < |c| =: r$, take the function

$$p_c(z) := cz^n, \quad z \in \mathbb{D}.$$

Note that

$$p_c(\mathbb{D}) = \mathbb{D}_r \subset \mathbb{D}_R \subset D. \tag{1.6}$$

By (1.3) we have

$$\begin{aligned} P_{\Phi,p_c}(z) &= 1 + \frac{zp_c'(z)}{p_c(z)}\Phi(p_c(z)) \\ &= 1 + cz\Phi(p_c(z)), \quad z \in \mathbb{D}. \end{aligned} \tag{1.7}$$

By (1.6), $\overline{\mathbb{D}}_r \subset D$, so for every $z \in \mathbb{D}$,

$$|\Phi(p_c(z))| = |\Phi(cz^n)| \leq \max_{w\in\overline{\mathbb{D}}_r} |\Phi(w)| =: M_r.$$

Since $M_r \to 0^+$ as $r \to 0^+$, we can take $r_0 \in (0, R)$ such that

$$r_0 M_{r_0} < 1.$$

Let $c_0 \in \mathbb{C}$ with $|c_0| =: r_0$ be any. Then

$$\begin{aligned} |1 + c_0 z \Phi(p_{c_0}(z))| &\geq 1 - r_0 \left|\Phi(p_{c_0}(z))\right| \\ &\geq 1 - r_0 M_{r_0} > 0, \quad z \in \mathbb{D}. \end{aligned} \tag{1.8}$$

Summarizing, since for $c := c_0$ and $r := r_0$ the inclusion (1.6) holds, by (1.7) for $c := c_0$ the function $P_{\Phi, p_{c_0}}$ is analytic and by (1.8) the function $P_{\Phi, p_{c_0}}$ has no zero in $\mathbb{D}$, so we conclude that $p_{c_0} \in \mathcal{H}(\Phi)$. On the other hand, $p_{c_0} \notin \mathcal{H}^0(\Phi)$.

4. If $p \in \mathcal{H}(\Phi)$, then the analytic branch $\log P_{\Phi,p}$, i.e., the analytic function

$$\mathbb{D} \ni z \mapsto \log P_{\Phi,p}(z),$$

exists. Thus for every $\gamma \in (0, 1)$ the analytic branch $P^{\gamma}_{\Phi,p}$ exists.

In [4] the authors started to consider (1.2) related to geometric mean, namely, for $\gamma \in [0, 1]$, an analytic function $\Phi : D \to \mathbb{C}$ in a domain D in $\mathbb{C}$, a univalent function $h \in \mathcal{H}$ and $p \in \mathcal{H}(\Phi)$, they studied the differential subordination of the type

$$pP^{\gamma}_{\Phi,p} \prec h. \tag{1.9}$$

When $p \in \mathcal{H}^0(\Phi)$, then, taking into account (1.4), the condition (1.9) is of the form

$$p(z) \left[1 + \frac{zp'(z)}{p(z)} \Phi(p(z))\right]^{\gamma} \prec h(z), \quad z \in \mathbb{D}, \tag{1.10}$$

which, by (1.5), is equivalent to

$$[p(z)]^{1-\gamma} \left[p(z) + zp'(z)\Phi(p(z))\right]^{\gamma} \prec h(z), \quad z \in \mathbb{D}. \tag{1.11}$$

Further studies in this subject were continued in [5], [6], [7] and [1].

Let us remark that the source of considering (1.9) is in the concept of γ-starlike functions introduced by Lewandowski, Miller, Złotkiewicz [8]. Recall that a function $f \in \mathcal{A}$ with

$$\frac{f(z)f'(z)}{z} \neq 0 \quad \frac{zf''(z)}{f'(z)} \neq -1, \quad z \in \mathbb{D} \setminus \{0\}, \tag{1.12}$$

is called γ-starlike if

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} > 0, \quad z \in \mathbb{D} \setminus \{0\}. \tag{1.13}$$

In particular, 1-starlike functions are convex functions and 0-starlike functions are starlike ones. Recall that $f \in \mathcal{H}$ is called convex if it is univalent and $f(\mathbb{D})$ is a convex set.

Let $\Phi(w) := 1/w,\ w \in \mathbb{C} \setminus \{0\}$. For $f \in \mathcal{A}$ satisfying (1.12) let

$$p(z) := \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D} \setminus \{0\}, \quad p(0) := 1.$$

Thus $p \in \mathcal{H}$ and $p \not\equiv \text{const}$. Moreover, by (1.12), $p(z) \neq 0$ for $z \in \mathbb{D}$ and

$$p(z) + zp'(z)\Phi(p(z)) = p(z) + \frac{zp'(z)}{p(z)}$$

$$= 1 + \frac{zf''(z)}{f'(z)} \neq 0, \quad z \in \mathbb{D}.$$

Hence and from (1.5) it follows that $p \in \mathcal{H}^0(\Phi)$. Consequently, the inequality in (1.13) is equivalent to the subordination (1.11) with

$$h(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

i.e., the subordination

$$[p(z)]^{1-\gamma}\left[p(z) + \frac{zp'(z)}{p(z)}\right]^{\gamma} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

The basis of this paper is Theorem 2.3, where the differential subordination of the form (1.9) with a convex function h is considered. This result is related to the similar one from [4] and [6]. In cited papers the differential subordination of the form (1.10) was examined. However, the method of proof here, based on Lemma 2.1, is essentially new. Moreover, Lemma 2.2, which in general is well known in the theory, was equipped with detailed proof in case when a dominant function h is convex with a piecewise smooth boundary curve of $h(\mathbb{D})$, namely, with h in the subclass $\mathcal{Q}$ of convex functions. Restricting our interest to the subclass $\mathcal{Q}$, we were able to present self-contained detailed proofs of results considered. On the other hand, the class $\mathcal{Q}$ is enough general for typical applications.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| < 1\}$. Given $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$.

The lemma below is a modification of Lemma 2.2c [13, p. 22]. However, in this form it follows directly from Jack's Lemma [3].

Lemma 1.1. *Let $h \in \mathcal{H}$ be univalent and assume that $h'(\zeta_0) \neq 0$ at $\zeta_0 \in \mathbb{T}$ exists. Let $p \in \mathcal{H}$ be a nonconstant function with $p(0) = h(0)$. If $z_0 \in \mathbb{D}$ is such that $p\left(\mathbb{D}_{|z_0|}\right) \subset h(\mathbb{D})$ and $p(z_0) = h(\zeta_0)$, then*

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) \tag{1.14}$$

for some $m \geq 1$.

2. Main result

Given $A \subset \mathbb{C}$, by $\overline{A}$ we denote the closure of A in $\mathbb{C}$. For $\mathbb{H}$ being an open half-plane in $\mathbb{C}$, let $\overline{\mathbb{H}}^0 := \overline{\mathbb{H}} \setminus \{0\}$. Given $z_0 \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and $C(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$. Define

$$\hbar(z) := \frac{1}{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

For $t \in [0,1]$ and $a, b \in \mathbb{C} \setminus \{0\}$ let

$$\mathrm{G}_t(a,b) := a^{1-t} b^t.$$

Lemma 2.1. *Let $\mathbb{H}$ be an open half-plane in $\mathbb{C}$ such that $0 \notin \mathbb{H}$. Then*

$$\bigwedge_{a,b \in \overline{\mathbb{H}}^0} \bigwedge_{t \in [0,1]} \mathrm{G}_t(a,b) \in \overline{\mathbb{H}}^0.$$

Proof. Since $0 \notin \mathbb{H}$, either $0 \in \partial\mathbb{H}$, or $0 \notin \overline{\mathbb{H}}$. If $0 \in \partial\mathbb{H}$, then $\hbar(\mathbb{H})$ is an open half-plane, say $\mathbb{E}$, with $0 \in \partial\mathbb{E}$. Moreover

$$\hbar\left(\overline{\mathbb{H}}^0\right) = \overline{\mathbb{E}}^0. \tag{2.1}$$

If $0 \notin \overline{\mathbb{H}}$, then

$$\hbar\left(\overline{\mathbb{H}}\right) = \overline{\mathbb{D}}(\xi, |\xi|) \setminus \{0\} =: \overline{\mathbb{D}}^0(\xi) \tag{2.2}$$

for some $\xi \in \mathbb{C} \setminus \{0\}$. We can choose a half-line l with end point at $0 \in l$ such that

$$\overline{\mathbb{E}}^0 \subset \mathbb{C} \setminus l \tag{2.3}$$

in case of (2.1), or

$$\overline{\mathbb{D}}^0(\xi) \subset \mathbb{C} \setminus l \tag{2.4}$$

in case of (2.2). Clearly, the branch of logarithm

$$\mathbb{C} \setminus l \ni w \mapsto \log w \tag{2.5}$$

exists.

Thus in case of (2.1), $\log\left(\overline{\mathbb{E}}^0\right)$ is a horizontal strip of width π, so a convex set.

Now we prove that in case of (2.2), $\log\left(\overline{\mathbb{D}}^0(\xi)\right)$ is a convex set. To this end, we show that $\partial \log\left(\overline{\mathbb{D}}^0(\xi)\right)$ is a convex curve. Let $C^0(\xi) := C(\xi, |\xi|) \setminus \{0\}$. Since

$$\overline{\mathbb{D}}^0(\xi) = \mathbb{D}(\xi, |\xi|) \cup C^0(\xi),$$

so

$$\begin{aligned} \log\left(\overline{\mathbb{D}}^0(\xi)\right) &= \log\left(\mathbb{D}(\xi, |\xi|) \cup C^0(\xi)\right) \\ &= \log\left(\mathbb{D}(\xi, |\xi|)\right) \cup \log\left(C^0(\xi)\right). \end{aligned} \tag{2.6}$$

But

$$\mathbb{D}(\xi, |\xi|) \cap C^0(\xi) = \emptyset,$$

so the univalence of the function (2.5) yield

$$\log\left(\mathbb{D}(\xi, |\xi|)\right) \cap \log\left(C^0(\xi)\right) = \log\left(\mathbb{D}(\xi, |\xi|) \cap C^0(\xi)\right) = \emptyset.$$

Hence and by (2.6) we get

$$\partial \log\left(\overline{\mathbb{D}}^0(\xi)\right) = \log\left(C^0(\xi)\right).$$

As $\xi \neq 0$, set $\xi := |\xi|\mathrm{e}^{\mathrm{i}\tau}$, where $\tau \in [0, 2\pi)$. Let

$$z(t) := |\xi| \left(\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{\mathrm{i}t}\right), \quad t \in (\tau - \pi, \tau + \pi),$$

be a parametrization of $C^0(\xi)$ and

$$w(t) := \log z(t), \quad t \in (\tau - \pi, \tau + \pi),$$

be a parametrization of $\log\left(C^0(\xi)\right)$. It suffices to show that the function

$$(\tau - \pi, \tau + \pi) \ni t \mapsto \arg w'(t) \tag{2.7}$$

is monotonic (see e.g., [2, Vol. I, p. 110]). Indeed, since

$$z'(t) = \mathrm{i}|\xi|\mathrm{e}^{\mathrm{i}t}, \quad z''(t) = -|\xi|\mathrm{e}^{\mathrm{i}t}, \quad t \in (\tau - \pi, \tau + \pi),$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \arg w'(t) = \operatorname{Im} \frac{w''(t)}{w'(t)} &= \operatorname{Im}\left\{\frac{z''(t)}{z'(t)} - \frac{z'(t)}{z(t)}\right\} \\ = \operatorname{Im}\left\{\mathrm{i} - \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{\mathrm{i}t}}\right\} &= \operatorname{Im} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\tau}}{\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{\mathrm{i}t}} \\ = \operatorname{Re} \frac{\mathrm{e}^{\mathrm{i}\tau}}{\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{\mathrm{i}t}} = \frac{1 + \cos(\tau - t)}{|\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{\mathrm{i}t}|^2} &> 0, \quad t \in (\tau - \pi, \tau + \pi).\end{aligned}$$

This shows that the function (2.7) is increasing.

Summarizing, we proved that the sets $\log\left(\overline{\mathbb{E}}^0\right)$ and $\log\left(\overline{\mathbb{D}}^0(\xi)\right)$ are convex.

Fix $a, b \in \overline{\mathbb{H}}^0$. We consider the case (2.1). The case (2.2) follows analogously. Since

$$\{\hbar(a), \hbar(b)\} \subset \hbar\left(\overline{\mathbb{H}}^0\right) = \overline{\mathbb{E}}^0,$$

so

$$\{\log(\hbar(a)), \log(\hbar(b))\} \subset \log\left(\overline{\mathbb{E}}^0\right).$$

Hence and from the convexity of $\log\left(\overline{\mathbb{E}}^0\right)$ it follows that

$$(1 - t)\log(\hbar(a)) + t\log(\hbar(b)) \in \log\left(\overline{\mathbb{E}}^0\right), \quad t \in [0, 1].$$

Consequently,

$$\log\left(\hbar(a)^{1-t}\hbar(b)^t\right) \in \log\left(\overline{\mathbb{E}}^0\right), \quad t \in [0, 1].$$

This and the univalence of the function (2.5) yield

$$\hbar(a)^{1-t}\hbar(b)^t \in \overline{\mathbb{E}}^0, \quad t \in [0, 1]. \tag{2.8}$$

Since

$$\hbar(a)^{1-t}\hbar(b)^t = \frac{1}{a^{1-t}b^t} = \hbar\left(a^{1-t}b^t\right) = \hbar(\mathrm{G}_t(a, b)), \quad t \in [0, 1],$$

so by (2.8) and (2.1) we get

$$\hbar(\mathrm{G}_t(a, b)) \in \overline{\mathbb{E}}^0 = \hbar\left(\overline{\mathbb{H}}^0\right), \quad t \in [0, 1].$$

Consequently,

$$\mathrm{G}_t(a,b) = \hbar \circ \hbar(\mathrm{G}_t(a,b)) \in \hbar \circ \hbar\left(\overline{\mathbb{H}}^0\right) = \overline{\mathbb{H}}^0, \quad t \in [0,1],$$

which ends the proof of the lemma. □

Now we introduce the class $\mathcal{Q}$ of convex functions h with some natural regularity of the boundary $\partial h(\mathbb{D})$ (for details on corners of curves see e.g., [14, pp. 51-65]).

Definition 2.2. By $\mathcal{Q}$ we denote the class of convex functions $h \in \mathcal{H}$ with the following properties:

(a) $h(\mathbb{D})$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity),

(b) $E(h)$ is the set of all points $\zeta \in \mathbb{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbb{D})$,

(c) $h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbb{T} \setminus E(h)$.

The lemma below is similar to Lemma 2.3d of [13, p. 24]. However we prove it in details when a dominant h is a function from the class $\mathcal{Q}$, so $\partial h(\mathbb{D})$ is a piecewise smooth boundary curve having a finite number of corners. Therefore this lemma is useful for applications. Using it we omit a limiting procedure standardly used in argumentation which can fail to be apply in some situations.

Lemma 2.2. *Let $h \in \mathcal{Q}$ and $p \in \mathcal{H}$ be a nonconstant function with $p(0) = h(0)$. If p is not subordinate to h, then there exist $z_0 \in \mathbb{D}$, $z_0 \neq 0$, and $\zeta_0 \in \mathbb{T} \setminus E(h)$ such that*

$$p\left(\mathbb{D}_{|z_0|}\right) \subset h(\mathbb{D}) \tag{2.9}$$

and

$$p(z_0) = h(\zeta_0). \tag{2.10}$$

Proof. Assume that p is not subordinate to h. Then, in view of (1.1),

$$p(\mathbb{D}) = p\left(\bigcup_{r\in(0,1)} \overline{\mathbb{D}}_r\right) = \bigcup_{r\in(0,1)} p\left(\overline{\mathbb{D}}_r\right) \not\subset h(\mathbb{D}). \tag{2.11}$$

Define

$$r_0 := \inf\left\{r \in (0,1) : p\left(\overline{\mathbb{D}}_r\right) \not\subset h(\mathbb{D})\right\}.$$

Since, in view of (2.11), $p\left(\overline{\mathbb{D}}_r\right) \not\subset h(\mathbb{D})$ for some $r \in (0,1)$, so r_0 well defined. We show that $r_0 \in (0,1)$. Note first that

$$p\left(\overline{\mathbb{D}}_{r_1}\right) \subset p\left(\overline{\mathbb{D}}_{r_2}\right), \quad 0 < r_1 \le r_2 < 1. \tag{2.12}$$

By the continuity of p at zero, by the fact that $p(0) = h(0)$ and $h(\mathbb{D})$ is an open set, it follows that there exists $0 < \varrho_1 < 1$ such that

$$p\left(\overline{\mathbb{D}}_{\varrho_1}\right) = \overline{p\left(\mathbb{D}_{\varrho_1}\right)} \subset h(\mathbb{D}).$$

Thus, taking into account (2.12), we have

$$p\left(\overline{\mathbb{D}}_r\right) \subset p\left(\overline{\mathbb{D}}_{\varrho_1}\right) \subset h(\mathbb{D}), \quad r \in (0, \varrho_1].$$

Consequently, $0 < \varrho_1 \le r_0 \le 1$.

On the other hand, from (2.11) it follows that for some $\varrho_2 \in (0,1)$,

$$p\left(\overline{\mathbb{D}}_{\varrho_2}\right) \not\subset h(\mathbb{D}).$$

Hence and by (2.12) we have

$$p\left(\overline{\mathbb{D}}_r\right) \not\subset h(\mathbb{D}), \quad r \in [\varrho_2, 1).$$

Consequently, $0 < \varrho_1 \le r_0 \le \varrho_2 < 1$, so $r_0 \in (0,1)$.

Observe now that

$$p\left(\overline{\mathbb{D}}_{r_0}\right) \not\subset h(\mathbb{D}). \tag{2.13}$$

Indeed, if otherwise

$$p\left(\overline{\mathbb{D}}_{r_0}\right) = \overline{p\left(\mathbb{D}_{r_0}\right)} \subset h(\mathbb{D}),$$

then

$$\overline{p\left(\mathbb{D}_{r_0}\right)} \subset U \subset h(\mathbb{D}) \tag{2.14}$$

for some open set U in $h(\mathbb{D})$. By the continuity of p, the set $p^{-1}(U)$ is open and by (2.14) we have

$$p^{-1}\left(\overline{p\left(\mathbb{D}_{r_0}\right)}\right) = p^{-1}\left(p\left(\overline{\mathbb{D}}_{r_0}\right)\right) = \overline{\mathbb{D}}_{r_0} \subset p^{-1}(U).$$

Thus there exists $r_1 \in (r_0, 1)$ such that

$$\overline{\mathbb{D}}_{r_0} \subset \overline{\mathbb{D}}_{r_1} \subset p^{-1}(U).$$

Hence and by (2.14) we have

$$p\left(\overline{\mathbb{D}}_{r_1}\right) \subset U \subset h(\mathbb{D}).$$

But $r_1 > r_0$, which contradicts the definition of r_0, so (2.13) holds. Thus

$$r_0 = \min\left\{r \in (0,1) : p\left(\overline{\mathbb{D}}_r\right) \not\subset h(\mathbb{D})\right\}.$$

Moreover we can state that

$$p\left(\mathbb{D}_{r_0}\right) \subset h(\mathbb{D}). \tag{2.15}$$

Otherwise, there exists $z_0 \in \mathbb{D}_{r_0}$ such that

$$p(z_0) \notin h(\mathbb{D}). \tag{2.16}$$

Since $|z_0| < r_0$, by (2.16) we have

$$p\left(\overline{\mathbb{D}}_{|z_0|}\right) \not\subset h(\mathbb{D}),$$

which contradicts the definition of r_0.

From (2.15) we have

$$\overline{p\left(\mathbb{D}_{r_0}\right)} \subset \overline{h(\mathbb{D})}.$$

Hence

$$p\left(\overline{\mathbb{D}}_{r_0}\right) = \overline{p\left(\mathbb{D}_{r_0}\right)} \subset \overline{h(\mathbb{D})} = h(\mathbb{D}) \cup \partial h(\mathbb{D}). \tag{2.17}$$

By (2.13) there exists $z_0 \in \overline{\mathbb{D}}_{r_0}$ such that (2.16) holds. In view of (2.15), $z_0 \in \mathbb{T}_{r_0}$. Since $p(z_0) \in p\left(\overline{\mathbb{D}}_{r_0}\right)$, by (2.17), $p(z_0) \in h(\mathbb{D}) \cup \partial h(\mathbb{D})$. Hence, taking into account (2.16), we deduce that $p(z_0) \in \partial h(\mathbb{D})$. Thus there there exists $\zeta_0 \in \mathbb{T}$ such that

$$p(z_0) = h(\zeta_0). \tag{2.18}$$

Summarizing, we proved that (2.15) and (2.18), so (2.9) and (2.10) hold.

It remains to prove that $\zeta_0 \in \mathbb{T} \setminus E(h)$. Suppose that $\zeta_0 \in E(h)$. By the convexity of $h(\mathbb{D})$, $h(\zeta_0)$ is the corner of opening $\alpha \in (0, \pi)$, i.e., the following limits

$$\lim_{t\to 0^+} \arg\left(h\left(\mathrm{e}^{\mathrm{i}t}\zeta_0\right) - h(\zeta_0)\right) =: \beta, \tag{2.19}$$

and

$$\lim_{t\to 0^+} \arg\left(h\left(\mathrm{e}^{-\mathrm{i}t}\zeta_0\right) - h(\zeta_0)\right) =: \beta + \alpha \tag{2.20}$$

exist (see [14, p. 51]). Since $h(\mathbb{D})$ is convex, so

$$\overline{h(\mathbb{D})} \subset V,$$

where V is the closed convex sector with vertex at $h(\zeta_0)$ bounded by two half-lines of the directions (2.19) and (2.20), respectively. Hence and from (2.17) it follows that $p(z) \in V$ for every $z \in \mathbb{T}_{r_0}$. Moreover, when $p(z) \neq p(z_0)$, in view of (2.18), we get

$$\beta \le \arg\left(p(z) - p(z_0)\right) = \arg\left(p(z) - h(\zeta_0)\right) \le \beta + \alpha.$$

Hence, for every z_1, $z_2 \in \mathbb{T}_{r_0}$, such that $p(z_1) \neq p(z_0)$ and $p(z_2) \neq p(z_0)$ we obtain

$$\left|\arg\left(p(z_2) - p(z_0)\right) - \arg\left(p(z_1) - p(z_0)\right)\right| \le \alpha < \pi. \tag{2.21}$$

Assume first that $p'(z_0) \neq 0$. Then there exists $\varepsilon \in (0, 1 - |z_0|)$ such that p is an invertible function in $\mathbb{D}(z_0, \varepsilon)$. Thus the arc $\mathbb{T}_{r_0} \cap \mathbb{D}(z_0, \varepsilon)$ is mapped univalently by p onto an analytic Jordan arc having a tangent line at $p(z_0)$. Thus

$$\lim_{t\to 0^+} \left(\arg\left(p\left(\mathrm{e}^{-\mathrm{i}t}z_0\right) - p(z_0)\right) - \arg\left(p\left(\mathrm{e}^{\mathrm{i}t}z_0\right) - p(z_0)\right)\right) = \pi.$$

Hence and from the fact that $\alpha < \pi$ it follows that there exists $t_1 > 0$ such that

$$\alpha < \left|\arg\left(p\left(\mathrm{e}^{-\mathrm{i}t_1}z_0\right) - p(z_0)\right) - \arg\left(p\left(\mathrm{e}^{\mathrm{i}t_1}z_0\right) - p(z_0)\right)\right|,$$

which contradicts (2.21) with

$$z_1 := \mathrm{e}^{\mathrm{i}t_1}z_0 \quad \text{and} \quad z_2 := \mathrm{e}^{-\mathrm{i}t_1}z_0.$$

When $p'(z_0) = 0$, then there exist $k \in \mathbb{N}$ and $\varepsilon \in (0, 1 - |z_0|)$ such that

$$p(z) = p(z_0) + q^k(z), \quad z \in \mathbb{D}(z_0, \varepsilon),$$

where q is an analytic invertible function in $\mathbb{D}(z_0, \varepsilon)$ such that $q(z_0) = 0$ and q' has no zero in $\mathbb{D}(z_0, \varepsilon)$ (see [15, p. 216, Theorem 10.32]). Then, p fails to satisfy (2.21), evidently.

In this way we proved that $\zeta_0 \in \mathbb{T} \setminus E(h)$, which ends the proof of the lemma. □

Theorem 2.3. *Let $\gamma \in [0,1]$, $h \in \mathcal{Q}$ be such that $0 \in \overline{h(\mathbb{D})}$ and $\Phi : D \to \mathbb{C}$ be an analytic function in a domain D in $\mathbb{C}$ such that $D \supset \overline{h(\mathbb{D})}$ and*

$$\operatorname{Re}\Phi(w) \geq 0, \quad w \in \overline{h(\mathbb{D})}. \tag{2.22}$$

If $p \in \mathcal{H}(\Phi)$, $p(0) = h(0)$ and

$$pP^{\gamma}_{\Phi,p} \prec h, \tag{2.23}$$

then

$$p \prec h. \tag{2.24}$$

Proof. Suppose, on the contrary, that p is not subordinate to h. From Lemma 2.2 it follows that there exist $z_0 \in \mathbb{D}$ and $\zeta_0 \in \mathbb{T} \setminus E(h)$ such that (2.9) and (2.10) hold. Hence, applying Lemma 1.1 we have

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) \tag{2.25}$$

for some $m \geq 1$.

Let $\mathbb{H}$ be the open half-plane supporting the convex domain $h(\mathbb{D})$ at $h(\zeta_0)$. Thus

$$h(\zeta_0) \in \partial\overline{\mathbb{H}} \tag{2.26}$$

and

$$h(\mathbb{D}) \cap \overline{\mathbb{H}} = \emptyset. \tag{2.27}$$

Assume first that $0 \in h(\mathbb{D})$. Hence and from (2.27) it follows that

$$0 \notin \overline{\mathbb{H}}. \tag{2.28}$$

Moreover, this and (2.26) yield $h(\zeta_0) \neq 0$. Thus

$$h(\zeta_0) \in \overline{\mathbb{H}}^0. \tag{2.29}$$

Since $\zeta_0 h'(\zeta_0) \neq 0$ is an outward normal to $\partial h(\mathbb{D})$ at $h(\zeta_0)$, so from (2.26) and (2.27) we have

$$h(\zeta_0) + m\zeta_0 h'(\zeta_0)\Phi(h(\zeta_0)) \in \overline{\mathbb{H}}. \tag{2.30}$$

Let

$$a := p(z_0), \tag{2.31}$$

and

$$b := p(z_0) + z_0 p'(z_0)\Phi(p(z_0)). \tag{2.32}$$

From (2.10) and (2.25) we have

$$a = h(\zeta_0), \quad b = h(\zeta_0) + m\zeta_0 h'(\zeta_0)\Phi(h(\zeta_0)).$$

Thus in view of (2.29), (2.30) and (2.28) we get

$$\{a, b\} \subset \overline{\mathbb{H}}^0. \tag{2.33}$$

But $z_0 \in \mathbb{D} \setminus p^{-1}(0)$ and by (2.33), $ab \neq 0$, so in view of (1.3), (2.31) and (2.32) we have

$$
\begin{aligned}
(2.34) \qquad p(z_0)P^{\gamma}_{\Phi,p}(z_0) &= p(z_0)\left[1+\frac{z_0p'(z_0)}{p(z_0)}\Phi(p(z_0))\right]^{\gamma} \\
&= [p(z_0)]^{1-\gamma}\,[p(z_0)+z_0p'(z_0)\Phi(p(z_0))]^{\gamma} \\
&= \mathrm{G}_{\gamma}(a,b).
\end{aligned}
$$

Hence and from (2.33), by applying Lemma 2.1 we get

$$p(z_0)P^{\gamma}_{\Phi,p}(z_0)=\mathrm{G}_{\gamma}(a,b)\in\overline{\mathbb{H}}^{0}.$$

Consequently, from (2.27) we have

$$p(z_0)P^{\gamma}_{\Phi,p}(z_0)\notin h(\mathbb{D}), \tag{2.35}$$

which contradicts (2.23).

Assume now that $0\in\partial h(\mathbb{D})$. Let a and b be given by (2.31) and (2.32), respectively, and suppose that (2.33) holds. Then we get again (2.35), so a contradiction.

Suppose that

$$a=h(\zeta_0)=p(z_0)=0. \tag{2.36}$$

Since $p\in\mathcal{H}(\Phi)$, and $z_0\in p^{-1}(0)$, by (1.3),

$$P_{\Phi,p}(z_0)=1+z_0\lim_{\mathbb{D}\ni z\to z_0}\frac{p'(z)\Phi(p(z))}{p(z)}$$

exists and is finite. This and (2.36) yield

$$p(z_0)P_{\Phi,p}(z_0)=0.$$

But $0\in\partial h(\mathbb{D})$, which contradicts (2.23).

At the end, observe that the case

$$a=p(z_0)\neq 0$$

and

$$b=p(z_0)+z_0p'(z_0)\Phi(p(z_0))=0$$

does not hold, since then $P_{\Phi,p}(z_0)=0$, which contradicts the definition of $P_{\Phi,p}$. This ends the proof of the theorem. □

Theorem 2.4. *Let $\gamma\in[0,1]$, $h\in\mathcal{Q}$ be such that $0\in\overline{h(\mathbb{D})}$ and $\Phi: D\to\mathbb{C}$ be an analytic function in a domain D in $\mathbb{C}$ such that $D\supset\overline{h(\mathbb{D})}$ and* (2.22) *hold. If $p\in\mathcal{H}^0(\Phi)$, $p(0)=h(0)$ and*

$$p(z)\left[1+\frac{zp'(z)}{p(z)}\Phi(p(z))\right]^{\gamma}\prec h(z),\quad z\in\mathbb{D},$$

then

$$p\prec h.$$

Remark 2.6. In typical applications, the convex function h is from the class $\mathcal{Q}$. Therefore Theorem 2.3 is enough general and useful to apply.

When h is arbitrary convex functions, then its boundary curve can contain at most countably many corners (see [14, p. 65]) which have accumulating points either finite or at the infinity.

For $\gamma := 1$ we get a weaker form of Theorem 1 of [12] (see [6, p. 379]). In this case the assumption $0 \in \overline{h(\mathbb{D})}$ is not needed.

Corollary 2.5. *Let $h \in \mathcal{Q}$, $\Phi : D \to \mathbb{C}$ be an analytic function in a domain D in $\mathbb{C}$ such that $D \supset h(\mathbb{D})$ and (2.22) hold. If $p \in \mathcal{H}$ is a nonconstant function with $p(0) = h(0)$ and*

$$p(z) + zp'(z)\Phi(p(z)) \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

Example 2.6. Functions h below are elements of $\mathcal{Q}$.

1. Given $\beta \in (0, 1]$, let

$$h(z) = h_\beta(z) := \left(\frac{1+z}{1-z}\right)^\beta, \quad z \in \mathbb{D}.$$

Then $E(h_\beta) = \{-1, 1\}$ for $\beta \in (0, 1)$ and $E(h_1) = \{1\}$.

2. Given $-1 \leq B < 1$, let

$$h(z) = h_B(z) := \frac{1+z}{1+Bz}, \quad z \in \mathbb{D}.$$

Then $E(h_B) = \emptyset$ for $-1 < B < 1$ and $E(h_{-1}) = \{1\}$.

3. Given $M > 0$ and $a \in \mathbb{D}_M$, let

$$h(z) = h_M(z) := M\frac{Mz + a}{M + \bar{a}z}, \quad z \in \mathbb{D}.$$

Then $E(h_M) = \emptyset$.

4. Given $n \geq 3$, let

$$h(z) = h_n(z) := \int_0^z (\zeta^n - 1)^{-2/n} \, \mathrm{d}\zeta, \quad z \in \mathbb{D}.$$

Then $h_n(\mathbb{D})$ is a regular convex polygon with n-sides and with

$$E(h_n) = \{\mathrm{e}^{2k\pi\mathrm{i}/n} : k = 0, \ldots, n-1\}.$$

Moreover $h_n(0) = 0$ is a center of the polygon $h_n(\mathbb{D})$.

Acknowledgements

This research was supported for the first author and partially for the third author by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0007037).

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

O PODPORZADKOWANIU RÓŻNICZKOWYM ŚREDNIEJ GEOMETRYCZNEJ

S t r e s z c z e n i e

Dla $t \in [0,1]$ i a, $b \in \mathbb{C} \setminus \{0\}$, niech $\mathrm{G}_t(a,b) := a^{1-t}b^t$ będzie średnią geometryczną liczb a i b. W pracy tej, dla ustalonego $\gamma \in [0,1]$, badane jest różniczkowe podporządkowanie następującej postaci:

$$\mathrm{G}_\gamma \left(p(z), p(z) + zp'(z)\Phi(p(z))\right) \prec h(z) \Rightarrow p(z) \prec h(z)$$

dla $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, gdzie p, h i Φ spełniają potrzebne założenia.

Słowa kluczowe: podporzadkowanie różniczkowe, średnia geometryczna, funkcje wypukłe

PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 25–34

In memory of
Professor Claude Surry

Emilia Fraszka-Sobczyk

ON SOME GENERALIZATION OF THE COX-ROSS-RUBINSTEIN MODEL AND ITS ASYMPTOTICS OF BLACK-SCHOLES TYPE

Summary

In this paper we present the generalization of the Cox-Ross-Rubinstein (CRR) model. We assume that upper and lower levels of the stock price do not satisfy the condition $u_n \cdot d_n = 1$, which was assumed in the CRR model. Next we demonstrate the convergence of option prices in the generalization of the CRR model to the price that is given by some formula that is corresponding to the Black-Scholes formula.

Keywords and phrases: Cox-Ross-Rubinstein model, CRR model, Binomial Model, Black-Scholes formula, option prices, convergence to the Black-Scholes formula, the pricing of a European call option, the limit case of the Binomial Model of Cox-Ross-Rubinstein model (CRR), convergence of option prices in the CRR model to the Black-Scholes formula

1. Introduction

In this chapter we demonstrate some notations connected with the pricing of a European call option and we recall the Black-Scholes formula and the CRR model [1–4].

1.1. European call option

A European call option is a contract which gives the holder the right but not the obligation to buy the underlying asset (for example stocks) for a strike price K (determined at $t = 0$) only at a future date T, which is called the expiry date. Since the holder has the right and not the obligation to buy the asset he will only exercise it if it is profitable to him. So he will exercise the option if the market price is greater than K. Of course, the option holder pays the premium and this premium is called the option pricing. Formally, we have the following definition.

Definition 1.1. A European call option is a pair (T, C_T) where $T > 0$ and $C_T(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is the function

$$C_T(s) = (s-K)^+ = \begin{cases} s-K & \text{if} \quad s > K, \\ 0 & \text{if} \quad s \leq K, \end{cases} \qquad \text{for some} \quad K \in \mathbb{R}_+.$$

1.2. Black-Scholes formula

For fixed numbers $K > 0,\ r > 0,\ \sigma^2 > 0,\ T > 0$ the following formula is very famous.

Definition 1.2. The function $C_t(s)$, $0 \leq t \leq T$, $s > 0$, is said the Black-Scholes price if

$$C_t(s) = s \cdot \phi(d_+) - Ke^{-r(T-t)} \cdot \phi(d_-),$$

where $\phi(\cdot)$ – the cumulative normal distribution function,

$$d_+ = \frac{\ln \frac{s}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_- = \frac{\ln \frac{s}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

The famous interpretation is following: K describes the price that a holder of a European call option will pay a seller if he wants to make this transaction in the expiry date T, r is the interest rate of the bank account (or credit) for one year or one other fixed period of time, σ^2 describes speed of random changes of stock prices, namely σ^2 is the variance of $\ln \frac{S(1)}{S_0}$, $S(1)$ is random stock price after one period of time and S_0 is the beginning stock price. Then, if $S(t)$ denotes the process of changes of stock prices, then

$$C_t(S(t)) = S(t) \cdot \phi(d_+) - Ke^{-r(T-t)} \cdot \phi(d_-)$$

is the pricing of a European call option that we describe in the Definition 1.1.

One of a way of an explanation for sense of this pricing is taking into consideration the limit case of the Binomial Model of Cox-Ross-Rubinstein (CRR).

1.3. CRR model

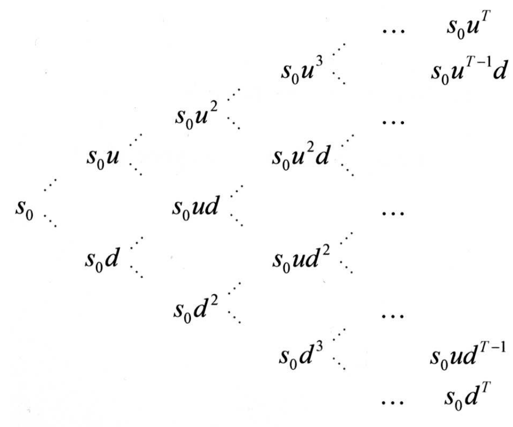
In this section $t \in N = \{0, 1, 2, \ldots\}$.

Definition 1.3.1. The system of all sequences $(S(0), S(1), \ldots, S(T))$ satisfying

$$S(0) = s_0$$

$$S(t) = S(t-1)u \quad \text{or} \quad S(t) = S(t-1)d, \quad 1 \leq t \leq T,$$

with some number $\hat{r}$ and fixed $T \in N$, $s_0 > 0$, $0 < d < \hat{r} < u$, is called the Binomial Model of Cox-Ross-Rubinstein (CRR). These sequences can be presented as paths in the following diagram:



Interpretation

The sequence $(S(0), \ldots, S(T))$ presents all possible changes of the stock price, $\hat{r} = 1 + r$, where r is the interest rate of the bank account (or credit) for a fixed **short** period of time, t is a number of short periods, u and d are the only possible changes of stock prices during one short period.

Definition 1.3.2. A European type option in CRR model is a pair (T, C_T) where $T > 0$ and C_T is a real function on the set $\{s_0d^T, s_0ud^{T-1}, \ldots, s_0u^{T-1}d, s_0u^T\}$.

The following pricing of such option is famous and important.

Definition 1.3.3 **(CRR-pricing)**. For the fixed European call option

$$C_T(\cdot) : \{s_0d^T, s_0ud^{T-1}, \ldots, s_0u^{T-1}d, s_0u^T\} \to \mathbb{R}$$

with the expiry date T the following formula depended on parameter s_0

$$C_0(s_0) = \frac{1}{\hat{r}^T} \sum_{k=0}^{T} \binom{T}{k} p^{*k} q^{*T-k} C_T(s_0u^kd^{T-k}),$$

where $p^* = 1 - q^*$ is so called martingale probability,

$$p^* = \frac{\hat{r} - d}{u - d},$$

is called the CRR price of the European call option.

As a particular case of a European type option we have:

Definition 1.3.4. A pair (T, C_T) is a European call option with a strike price K and an expiry date T if

$$C_T(s_0 u^k d^{T-k}) = (s_0 u^k d^{T-k} - K)^+.$$

Corollary 1.3.1. *For a European call option $C_T(\cdot)$ with the expiry date T and the strike price $K = s_0 u^{k_0} d^{T-k_0}$, $0 \le k_0 \le T$, the CRR-option pricing is described by the following formula:*

$$C_0(s_0) = s_0 \bar{D} - \frac{K}{\hat{r}^T} D^*, \tag{1.3}$$

where

$$\bar{D} = \sum_{k=k_0}^{T} \binom{T}{k} \cdot \bar{p}^k \cdot \bar{q}^{T-k}, \quad D^* = \sum_{k=k_0}^{T} \binom{T}{k} \cdot p^{*k} \cdot q^{*T-k},$$

$$k_0 = \frac{\ln \frac{K}{S_0} - T \cdot \ln d}{\ln \left(\frac{u}{d}\right)}, \quad p^* = \frac{\hat{r} - d}{u - d}, \quad q^* = \frac{u - \hat{r}}{u - d}, \quad \bar{p} = p^* \cdot \frac{u}{\hat{r}}, \quad \bar{q} = q^* \cdot \frac{d}{\hat{r}}.$$

1.4. Calibration of CRR model

In this section we adjust parameters of CRR Model to the financial market. Of course, on the financial market we can trade in stocks at any time during the day and stock prices can get more than two values during the day. That is why, the adjustment of parameters of CRR Model is necessary.

Let τ denotes the quantity of time units to expiry time (for example the quantity of months), and n – the quantity of moments of the portfolio's change at one time unit. Thus we shall discuss a sequence of CRR models, indexed by $n \ge 1$. In such model we have expiry date $T_n = n \cdot \tau$ and it proves to be natural to take an interest rate for one (short) period $\hat{r}_n = e^{r \cdot \frac{1}{n}}$, and possible price changes

$$u_n = e^{\sigma \frac{1}{\sqrt{n}}}, \quad d_n = \frac{1}{u_n} = e^{-\sigma \frac{1}{\sqrt{n}}}.$$

Corollary 1.4.1. *If we provide for all assumption given above and the formula* (1.3) *we get the corresponding formula for the pricing a European call option:*

$$C_{0,n}(s_0) = s_0 \bar{D}_n - \frac{K}{e^{r\tau}} D_n^*, \tag{1.4}$$

where

$$\bar{D}_n = \sum_{k=k_{0,n}}^{n\tau} \binom{n\tau}{k} \cdot \bar{p}_n^k \cdot \bar{q}_n^{n\tau-k}, \quad D_n^* = \sum_{k=k_{0,n}}^{n\tau} \binom{n\tau}{k} \cdot p_n^{*k} \cdot q_n^{*n\tau-k}$$

$$k_{0,n} = \frac{\ln \frac{K}{s_0} - \tau \cdot n \cdot \ln d_n}{\ln \left(\frac{u_n}{d_n}\right)} = \frac{\sqrt{n} \ln \frac{K}{s_0} + \sigma \cdot \tau \cdot n}{2\sigma},$$

$$p_n^* = \frac{\hat{r}_n - d_n}{u_n - d_n}, \quad q_n^* = \frac{u_n - \hat{r}_n}{u_n - d_n}, \quad \bar{p}_n = p_n^* \cdot \frac{u_n}{\hat{r}_n}, \quad \bar{q}_n = q_n^* \cdot \frac{d_n}{\hat{r}_n}.$$

1.5. Convergence of option prices in the CRR model to the Black-Scholes formula

We shall describe the convergence next to $n \to \infty$ of CRR model to Black-Scholes formula. For a sequence of function (1.4) we have

$$\lim_{n\to\infty} C_{0,n}(s_0) = s_0\phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right) - \frac{K}{e^{r\tau}}\phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right).$$

It is the famous Black-Scholes formula.

2. Generalization of the CRR model

In this chapter we shall research some generalization of the CRR Model and limit theorems that include generalization of the Black-Scholes formula.

2.1. Assumptions

Now we assume that in n-th CRR model

$$u_n = e^{\sigma\frac{1}{\sqrt{n}}} \cdot e^{\rho\frac{1}{\sqrt{n}}}, \quad d_n = \frac{e^{2\rho\frac{1}{n}}}{u_n} = e^{-\sigma\frac{1}{\sqrt{n}}} \cdot e^{\rho\frac{1}{n}}, \quad \hat{r}_n = e^{r\frac{1}{n}}.$$

The investigation of a European call option leads to limit theorems for the following system of functions.

Definition 2.1. The formula for the pricing a European call option is the following

$$C_{0,n}(s_0) = s_0\bar{D}_n - \frac{K}{e^{r\tau}}D_n^*, \tag{2.1}$$

where

$$\bar{D}_n = \sum_{k=k_{0,n}}^{n\tau}\binom{n\tau}{k}\cdot\bar{p}_n^k\cdot\bar{q}_n^{n\tau-k}, \quad D_n^* = \sum_{k=k_{0,n}}^{n\tau}\binom{n\tau}{k}\cdot p_n^{*k}q_n^{*n\tau-k},$$

$$k_{0,n} = \frac{\ln\frac{K}{s_0} - \tau\cdot n\cdot\ln d_n}{\ln\left(\frac{u_n}{d_n}\right)} = \frac{\sqrt{n}\ln\frac{K}{s_0} + \sigma\cdot\tau\cdot n}{2\sigma},$$

$$p_n^* = \frac{\hat{r}_n - d_n}{u_n - d_n}, \quad q_n^* = \frac{u_n - \hat{r}_n}{u_n - d_n}, \quad \bar{p}_n = p_n^*\cdot\frac{u_n}{\hat{r}_n}, \quad \bar{q}_n = q_n^*\cdot\frac{d_n}{\hat{r}_n},$$

$$u_n = e^{\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}} \quad \text{and} \quad d_n = \frac{e^{2\rho\frac{1}{n}}}{u_n} = e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}.$$

2.2. Convergence to the Black-Scholes type formula

In this section we get the convergence next to $n \to \infty$ of generalization of CRR model to the formula that is corresponding to the Black-Scholes formula.

Theorem 2.2.1. *For a European call option with the strike price K, the expiry date τ, the beginning stock price s_0, the interest rate of the bank account (or credit) for one unit r and volatility σ, defined in* (2.1) *we have*

$$\lim_{n\to\infty} C_{0,n}(s_0) = s_0\phi\left(\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right) - \frac{K}{e^{r\tau}}\phi\left(\frac{\ln\frac{s_0}{K}+\tau\left(r-\rho-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right).$$

This theorem is interesting, because ρ can be interpreted as the average change of logarithm of stocks' prices after one time unit (for example after one month). Such parameter obviously should be considered in real markets. Before we prove the theorem 2.2.1 we present three lemmas.

Lemma 2.2.1. *We have the following convergence:*

$$\sqrt{p_n^*\cdot q_n^*}\overset{n\to\infty}{\to}\frac{1}{2},\qquad \sqrt{\bar{p}_n\cdot\bar{q}_n}\overset{n\to\infty}{\to}\frac{1}{2}.$$

Proof.

$$\lim_{n\to\infty}\sqrt{p_n^*\cdot q_n^*} = \lim_{n\to\infty}\sqrt{\frac{e^{r\frac{1}{n}}-e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}}{e^{\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}-e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}}}\cdot\sqrt{\frac{e^{\rho\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}-e^{r\frac{1}{n}}}{e^{\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}-e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}}}$$
$$=\left(\varepsilon:=\frac{1}{\sqrt{n}}\right)$$
$$=\lim_{\varepsilon\to 0}\sqrt{\frac{e^{r\cdot\varepsilon^2}-e^{-\sigma\cdot\varepsilon}\cdot e^{\rho\varepsilon^2}}{e^{\sigma\cdot\varepsilon}\cdot e^{\rho\cdot\varepsilon^2}-e^{-\sigma\cdot\varepsilon}\cdot e^{\rho\cdot\varepsilon^2}}}\cdot\sqrt{\frac{e^{\sigma\cdot\varepsilon}\cdot e^{\rho\cdot\varepsilon^2}-e^{r\cdot\varepsilon^2}}{e^{\sigma\cdot\varepsilon}\cdot e^{\rho\cdot\varepsilon^2}-e^{-\sigma\cdot\varepsilon}\cdot e^{\rho\cdot\varepsilon^2}}}\overset{H}{=}\dots$$
$$=\sqrt{\frac{\sigma}{\sigma+\sigma}}\cdot\sqrt{\frac{\sigma}{\sigma+\sigma}}=\frac{1}{2}.$$

$$\lim_{n\to\infty}\sqrt{\bar{p}_n\cdot\bar{q}_n} = \lim_{n\to\infty}\sqrt{p_n^*\cdot\frac{e^{\sigma\cdot\frac{1}{\sqrt{n}}}\cdot e^{\rho\cdot\frac{1}{n}}}{e^{r\cdot\frac{1}{n}}}\cdot q_n^*\frac{e^{-\sigma\cdot\frac{1}{\sqrt{n}}}\cdot e^{\rho\cdot\frac{1}{n}}}{e^{r\cdot\frac{1}{n}}}} = \lim_{n\to\infty}\sqrt{p_n^*\cdot q_n^*}=\frac{1}{2}.$$

□

In the following lemmas we use the symbol $o(n^\alpha)$, which means some sequence $(a_n)_{n\in N}$ that satisfies

$$\frac{a_n}{n^\alpha}\overset{n\to\infty}{\to}0,\qquad \alpha\in\mathbb{R}.$$

We have the following obvious properties:

a) $o(n^{\alpha+\beta})$ stands in for $o(n^\alpha)\cdot o(n^\beta)$,

b) for $\alpha \le \beta$, $o(n^\beta)$ stands in for $o(n^\alpha) + o(n^\beta)$,

c) $o(n^{\alpha+\beta})$ stands in for $o(n^\alpha) \cdot n^\beta$,

d) for $\alpha < 0$, $1 + o(n^\alpha)$ stands in for $\frac{1}{1+o(n^\alpha)}$.

Lemma 2.2.2. *We have the following asymptotes:*

$$p_n^* = \frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \overset{n\to\infty}{\to} \frac{1}{2},$$

$$\bar{p}_n = \frac{1}{2} + \left(\frac{r-\rho}{2\sigma} + \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \overset{n\to\infty}{\to} \frac{1}{2}.$$

Proof.

$$\begin{aligned}
p_n^* &= \frac{e^{r\frac{1}{n}} - e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}}{e^{\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}} - e^{-\sigma\frac{1}{\sqrt{n}}}\cdot e^{\rho\frac{1}{n}}} = \frac{e^{(r-\rho)\frac{1}{n}} - e^{-\sigma\frac{1}{\sqrt{n}}}}{e^{\sigma\frac{1}{\sqrt{n}}} - e^{-\sigma\frac{1}{\sqrt{n}}}} \\
&= \frac{1 + (r-\rho)\frac{1}{n} + o\left(\frac{1}{n}\right) - 1 + \sigma\frac{1}{\sqrt{n}} - \frac{1}{2}\sigma^2\frac{1}{n} + o\left(\frac{1}{n}\right)}{1 + \sigma\frac{1}{\sqrt{n}} + \frac{1}{2}\sigma^2\frac{1}{n} + o\left(\frac{1}{n}\right) - 1 + \sigma\frac{1}{\sqrt{n}} - \frac{1}{2}\sigma^2\frac{1}{n} + o\left(\frac{1}{n}\right)} \\
&= \frac{\sigma\frac{1}{\sqrt{n}} + \left(r-\rho-\frac{1}{2}\sigma^2\right)\frac{1}{n} + o\left(\frac{1}{n}\right)}{2\sigma\frac{1}{\sqrt{n}} + o\left(\frac{1}{n}\right)} = \frac{\frac{1}{2} + \left(\frac{r}{2\sigma} - \frac{\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + \frac{o\left(\frac{1}{n}\right)}{2\sigma\frac{1}{\sqrt{n}}}}{1 + \frac{o\left(\frac{1}{n}\right)}{2\sigma\frac{1}{\sqrt{n}}}} \\
&= \left[\frac{1}{2} + \left(\frac{r}{2\sigma} - \frac{\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]\cdot\left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

$$\begin{aligned}
\bar{p}_n &= p_n^*\cdot\frac{u_n}{\hat{r}_n} = \left[\frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]\cdot\frac{e^{\sigma\frac{1}{\sqrt{n}}}}{e^{(r-\rho)\frac{1}{n}}} \\
&= \left[\frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]\cdot\frac{1 + \sigma\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)}{1 + (r-\rho)\frac{1}{n} + o\left(\frac{1}{n}\right)} \\
&= \left[\frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]\cdot\left(1 + \sigma\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right)\cdot\left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \frac{1}{2} + \left(\frac{r-\rho}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{2}\sigma\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{2} + \left(\frac{r-\rho}{2\sigma} + \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Lemma 2.2.3. *We have the following convergence:* □

$$\sqrt{n}\,(1-2p_n^*) \overset{n\to\infty}{\to} \frac{1}{2}\sigma - \frac{r-\rho}{\sigma},$$

$$\sqrt{n}\,(1-2\bar{p}_n) \overset{n\to\infty}{\to} -\frac{1}{2}\sigma - \frac{r-\rho}{\sigma}.$$

Proof. From lemma 2.2.2 we get

$$\begin{aligned}\sqrt{n}\,(1-2p_n^*) &= \sqrt{n}\left(1-2\left(\frac{1}{2}+\left(\frac{r-\rho}{2\sigma}-\frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)\right)\\ &= \frac{1}{2}\sigma - \frac{r-\rho}{\sigma} + o(1) \overset{n\to\infty}{\to} \frac{1}{2}\sigma - \frac{r-\rho}{\sigma},\\ \sqrt{n}\,(1-2\bar{p}_n) &= \sqrt{n}\left(1-2\left(\frac{1}{2}+\left(\frac{r-\rho}{2\sigma}+\frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)\right)\\ &= -\frac{1}{2}\sigma - \frac{r-\rho}{\sigma} + o(1) \overset{n\to\infty}{\to} -\frac{1}{2}\sigma - \frac{r-\rho}{\sigma}.\end{aligned}$$

□

We shall use the following version of Central Limit Theorem. It is an immediate consequence of Lindeberg-Feller theorem.

Theorem 2.2.2. *For any sequences $p_n \to p$, $q_n = 1 - p_n$, $x_n \to x$ with $0 \le p_n \le 1$ and $0 < p < 1$, and for sequences of independent random variables $X_1^n, X_2^n, \ldots, X_n^n$ with $P(X_i^n = 1) = p_n$, $P(X_i^n = 0) = q$, we have*

$$P\left(\frac{S_n - ES_n}{\sqrt{D^2 S_n}} < x_n\right) \overset{n\to\infty}{\to} \phi(x),$$

where $S_n = X_1^n + X_2^n + \ldots + X_n^n$. It means that

$$\sum_{\substack{0\le k\le n\\ k<x_n\sqrt{np_nq_n}+np_n}} \binom{n}{k} p_n^k q_n^{n-k} \overset{n\to\infty}{\to} \phi(x).$$

Proof the theorem 2.2.1. We notice

$$\frac{k_{o,n} - n\tau\cdot\bar{p}_n}{\sqrt{n\tau\cdot\bar{p}_n\bar{q}_n}} = \frac{\ln\frac{K}{s_0} + \tau\sigma\sqrt{n}(1-2\bar{p}_n)}{2\sqrt{\tau}\sigma\sqrt{\bar{p}_n\bar{q}_n}},$$

$$\frac{k_{o,n} - n\tau\cdot p_n^*}{\sqrt{n\tau\cdot p_n^* q_n^*}} = \frac{\ln\frac{K}{s_0} + \tau\sigma\sqrt{n}(1-2p_n^*)}{2\sqrt{\tau}\sigma\sqrt{p_n^* q_n^*}}.$$

From Lemmas 2.2.1 and 2.2.3 we know that the limits next to $n \to \infty$ of the expressions above exist. Let denote:

$$\bar{y} = \lim_{n\to\infty} \frac{\ln \frac{K}{s_0} + \tau\sigma\sqrt{n}(1-2\bar{p}_n)}{2\sqrt{\tau}\sigma\sqrt{\bar{p}_n\bar{q}_n}},$$

$$y^* = \lim_{n\to\infty} \frac{\ln \frac{K}{s_0} + \tau\sigma\sqrt{n}(1-2p_n^*)}{2\sqrt{\tau}\sigma\sqrt{p_n^* q_n^*}},$$

$S_{n\tau}$ – the number of success in Bernoulli scheme with $n\tau$ samples and the probability of success in one sample $\bar{p}_n$.

Then from the Theorem 2.2.2 , Lemmas 2.2.1–2.2.3 and Definition 2.1 we have

$$\lim_{n\to\infty} \bar{D}_n = 1 - \phi(\bar{y}),$$

analogically

$$\lim_{n\to\infty} D_n^* = 1 - \phi(y^*),$$

and

$$\begin{aligned}\lim_{n\to\infty} C_{o,n}(s_0) &= s_0(1-\phi(\bar{y})) - \frac{K}{e^{r\tau}}\left(1-\phi(y^*)\right) \\ &= s_0\phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r-\rho+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right) - \frac{K}{e^{r\tau}}\phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r-\rho-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right).\end{aligned}$$

□

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

UOGÓLNIENIE MODELU COXA-ROSSA-RUBINSTEINA I JEGO PRZEJŚCIE GRANICZNE DO ANALOGU MODELU BLACKA-SCHOLESA

S t r e s z c z e n i e

W pracy zaprezentowano uogólnienie modelu Coxa-Rossa-Rubinsteina (CRR). Założono, że górne i dolne wzrosty ceny akcji nie spełniają warunku $u_n \cdot d_n = 1$, przyjętego w klasycznym modelu CRR. Następnie przedstawiono przejście graniczne przy $n \to \infty$ uogólnionego modelu Coxa-Rossa-Rubinsteina do analogu modelu Blacka-Scholesa. Można wówczas uzyskać wynik równie przejrzysty jak klasyczna formuła Blacka-Scholesa.

Słowa kluczowe: model CRR, model Coxa-Rossa-Rubinsteina, formuła Blacka-Scholesa, wzór Blacka-Scholesa, zbieżność do formuły Blacka-Scholesa, wycena opcji, zbieżność wyceny opcji w modelu CRR do formuły Blacka-Scholesa, graniczne przypadki modelu CRR

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 35–44

In memory of
Professor Luboš Valenta

Leszek Wojtczak

ON THE VALENTA MODEL AND ITS ACTUALITY III

Summary

The present article describes some events which characterized the development of the theory of magnetic thin films. In this context the present contribution can be treated as the third part of the previous papers *On the Valenta model and its actuality I* and *II* [1].

The model originally introduced by Luboš Valenta, in order to describe magnetic thin films properties, at the level of molecular field approximation (MFA) and, recently modified by the group working under the supervision of Leszek Wojtczak (together with B. Mrygoń, I. Zasada, B. Busiakiewicz), in particular, in terms of reaction field approach (RFA), is still of great interest for modern physics and technology. The present article is meant to be a report concerning the research cooperation schemes between the Charles University in Prague and the University of Łódź.

Keywords and phrases: ferromagnetic thin films, spin autocorrelation functions, Valenta model modified by Reaction Field Approach

1. Introduction

Fifty five years ago Luboš Valenta introduced the model describing properties of magnetic thin films, in particular, the spontaneous magnetization and its angular and spatial distribution. The model allows us to construct spin waves, their resonances including the instability conditions as well as the phase transitions theory based on the order-disorder crystalline lattice thermodynamics and electronic phenomena [1, 2].

The model for magnetic thin films known in literature as the Valenta model has been proceeded by a pioneering work on the angular distribution of magnetization in one-dimensional toroid [3]. The model, originally formulated by Valenta is equivalent to the approach at the level of molecular field approximation (MFA) [4]. This work has been found as a good starting point in order to explain the surface deformation and the role of the structure, in particular, characteristic for helimagnetism in rare-earth thin films, the Mössbauer effect as well as the neutron inelastic magnetic scattering [5].

It seems to us that the Valenta model is very convenient for interpretation of spin waves properties, first of all, the physical nature of standing spin waves in magnetic films propagating in the samples of metals: iron, nickel and cobalt when the structure with two sublattices is reflected by acoustic and optical branches. Moreover, a spin wave means a propagation spin deformation inside the network sites. The spins do not change their positions which form the ground state with respect to one or two-dimensional trajectory of a deformation responsible for the formation of spin waves, respectively.

Taking into account the Valenta model in the representation of localized spins we can see that the structure is well established for different types of exchange or anisotropy interactions which play an essential role for a given construction of occupied lattice sites.

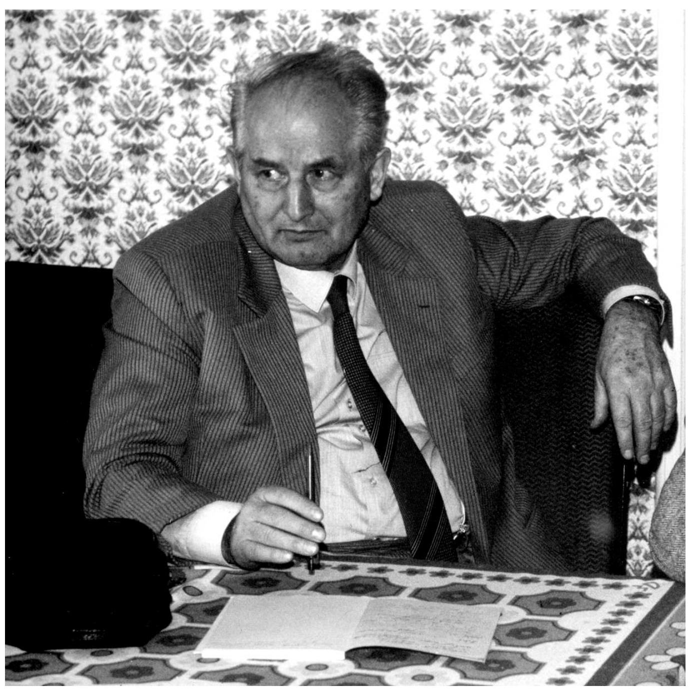

Fig. 1: Professor Luboš Valenta (1924–1994). Professor of the Charles University in Prague (Katedra fyziky pevných látek) and the Friedrich Schiller University in Jena, *Doctor honoris causa* of the University of Łódź.

Spin waves (or equivalent quasiparticles, i.e. magnons) belong to the class of particles which are of a quantum and collective nature at the same time.

In the case of magnetic thin films the relations between thermodynamic properties and geometry of samples play an essential role for the calculations of the average values, in particular, frequencies whose sequences are responsible for the spectrum properties. Spin waves in the direction perpendicular to the surface of a sample are standing waves and they are of discretized character in the momentum space.

The present paper is meant to commemorate the 50th anniversary of the beginning of the international cooperation between the Universities of Prague, Łódź, Košice and Jena. The research, devoted to the problems of magnetic thin films, develops fruitfully until the present moment.

2. The beginnings the research cooperation scheme (Prague-Łódź)

The above research achievements were fundamental for the cooperation scheme established between Charles University in Prague (earlier ČVUT in Prague) and the University of Łódź organized and headed by Professor L. Valenta who was preparing the co-workers in the first period of the research investigations in the field of magnetism. Speaking more precisely, he was preparing specialists in Prague and in Łódź for thin films magnetism.

The opening scenario began the first period of the research investigations (1964–1994). It was fifty years ago in 1964 in Prague, in the Czech Technical University (ČVUT), in Myslíkova street close to the Vltava river bank. In that time Professor L. Valenta was waiting in his laboratory of the Department of Solid State Physics (Katedra fyziky pevných laték) for two young postgraduate students, tentative co-workers in order to take part in the informal project on thin films. Professor L. Valenta was very well known in the international magnetic community as a physicist and a specialist in the theory of thin films physics and as the author of an important model of magnetic films which received the State Award in Physics. Two of Professor Valenta's former young postgraduate students are the authors of the present article – Štefan Zajac who fifty years ago was a student of ČVUT finishing his diploma thesis devoted to the contribution of inelastic scattering of polarized neutrons in Fe and Ni [5], and Leszek Wojtczak, a postgraduate student at the University of Łódź (Poland) who worked under the supervision of Professor L. Valenta in Prague. L. Wojtczak's doctoral dissertation was devoted to *The eigenfunctions and energy eigenvalues of atoms*. The supervisor of the dissertation, Professor Tadeusz Tietz was a very well known physicist among the specialists of the statistical theory of atoms. The program of the scholarship in Prague refers to the red shift of spin wave frequencies in Fe and Ni as well as to the characterization of the acoustic and optic branches in Co spectrum [6].

3. The development of the research cooperation scheme (Prague, Łódź, Jena, Košice)

The second period of the research cooperation between the universities in Prague and in Łódź (1994–2014) is mainly connected with the discussion of the reaction field approach (RFA) applied to the considerations in the case of magnetic thin films.

The method seems to be one of many evident calculations procedures in the case when the convergence of a number of magnons is discussed in the context of the construction which changes the static properties and influences the thermodynamics when the size plays an important and effective role. The spontaneous magnetization is given by a distribution dependent on the surface conditions.

In this context we can observe an extension of the anisotropy considerations which play an important role in the description of the nature of thin films and their surfaces.

The first period of the Prague-Łódź cooperation scheme was connected with the original version of the Valenta model in which we considered the sample geometry whose discretization reflects the crystallographic lattice properties. Moreover, in the case of small particles their thermodynamics is modified for inhomogeneous media.

The second period of the Prague-Łódź cooperation scheme was devoted to the presentation of a review by the modification of the original structure and it allowed us to consider some effective parameters by mutual relations (RFA); in particular, the kinetic equation for spins introduced by Oguchi in order to consider the damping term or the construction introduced by Néel in the form of sublattices. The properties of the (RFA) model are described more precisely in the present paper (part II) [1].

The report concerning the international cooperation on magnetic thin films contains also information on the accompanying events which created the special atmosphere and intellectual climate of the discussions [7].

Gradually, the number of research groups was strengthened by inviting colleagues from the Friedrich Schiller University in Jena. Continuing research on directly applied transformations (Holstein-Primakoff transformations) the participants of the project could organize, more or less formally, seminars, meetings and develop other conditions to make the cooperation more fruitful.

The Quantum Chemistry Group was supervised by Professor L. Valenta with the help of Professor Hans Müller from Jena while Professor L. Wojtczak (Łódź) was responsible for the cooperation links headed in the second period by Professor L. Skála (Prague) and Professor S. Romanowski (Łódź).

Professor L. Valenta was elected to be chairman of the International Organizing Committee (IOC) for the International Colloquium on Magnetic Films and Surfaces (ICMFS). Trilateral Seminars on Contemporary Quantum Chemistry were devoted to the problems of fundamental elements serving the proper structure.

In 1966 the Prague-Łódź team accepted dr. Andrzej Sukiennicki from the University of Technology in Warsaw who was working under the supervision of Professor

Fig. 2: During the 6th General Conference of the European Physical Society (1984) in Prague. From the left: Dr O. Bílek (Charles University, Prague), Professor C. Rau (Rice University, Houston), Professor Luboš Valenta (Charles University, Prague), Professor Š. Zajac (Charles University, Prague) and Professor L. Wojtczak (University of Łódź).

Valenta. Taking part in the project he obtained his excellent result, the so-called Valenta-Sukiennicki model which was very often applied in literature as a model describing an order-disorder transition in the case of an inhomogeneous system determined by the structural network. Recently, an extended version of this model confirmed the actuality of the Valenta-Sukiennicki approach to the considered problems. At the same time the surface model is an example of construction by means of application to the extended version of the model.

Next, we consider the achievements obtained within the Prague-Košice project on the basis of which a cooperation between the University of Łódź and the P. J. Šafárik University in Košice began. The first period of the informal project development was a time when the University in Košice was founded. Physics, which was included to the Faculty of Sciences, directed its research towards magnetism taking into account both the aspects, namely, the theoretical approach as well as the experiment.

Creating the initial basis conditions for the construction of laboratories in Košice was possible thanks to the activity of two people – Professor L. Valenta, who was a permanent guest in Košice, and Professor Vladimir Hajko, physicist, who was the first rector of the P. J. Šafárik University in Košice.

Thirty years ago the University od Łódź was included to the cooperation in the Łódź-Košice group starting with a personal exchange between the research groups headed by Professor T. Balcerzak (Łódź) and Professor A. Bobák (Košice). One of the recent topics of professor Balcerzak's research is the antiferromagnetic interlayer coupling in diluted magnetic thin films with RKKY interaction confirmed by the excellent experiment performed in the group of Professor J. Furdyna [8].

Another group of problems reflected in a series of papers concerns the anisotropic Heisenberg model studied for the bilayer multilayer as well as the bulk diluted system. Other papers discuss approaches convenient in the context of the magnetocaloric effect studies.

The thermodynamic properties resulting from the non-magnetic Gibbs energy analysis have been calculated and successfully compared with experimental data for argon. In the magnetic materials studies, in particular, the compensation effect for the Curie temperature has been found for asymmetric interactions within the neighbouring planes of the bilayer.

The number of papers and their relevance to some of the most topical problems of magnetism are the most tangible results of the Łódź-Košice cooperation.

4. Final conclusions

The main result of the paper II is finding a comparison between the Valenta model originally applied in MFA approach and the model modified in terms of RFA, introduced to the theoretical construction considered in both cases – thin films as well as nanoparticles structures.

An evident advantage in the case of RFA method is observed when a generalized susceptibility considerations are included to the sample energy minimization and lead to the conclusion that the convergence of a mean number of magnons is obtained even in thin films, which differs from the result of MFA calculations.

We consider the above problem as explanation of the spontaneous magnetization in some isotropic layered system which gives the average magnetization vanishing at the temperature assumed to be different from zero.

For this purpose, we remember that a thin film in the Valenta model is treated as a set of n monoatomic layers parallel with the film surfaces. The set of layers is equivalent in its interpretation to Néel sublattices embedded in the limited space of the discrete geometry. Of course, the construction of the lattice for the structural form in the case of RFA is the same.

In terms of thermodynamics we consider the properties of a sample treated as a composition of layers which form homogeneous independent subsystems. Thus, the relation between the main values of spontaneous magnetization and the effective number of magnons is different when MFA or RFA are applied [9].

Concluding we can see that the mean number of particles vanishes when $T \neq 0$ and it takes the value different from zero when $T = 0$.

Fig. 3: Professor L. Wojtczak during a lecture in 1994.

The second conclusion which is important for the present paper and brings interpretations of great interest for physics methodology refers to the interplay between theory and experiment.

The theory and, first of all, its development from MFA to RFA shows the interpretation of the considered effects at the surface. At the same effects time, the theoretical description is an inspiration of new experimental techniques based on the investigated effects. This interdependence is seen particularly in the surface physics domain. The relation between theory and experiment is a leading factor in the progress of coherent and successive interpretations. The method applied to the long and short-ranged ferromagnetic order at the topmost surface layer as well as layer in the middle of the interface is an example of mutual considerations.

Next, we have applied the Valenta model to the description of the electron-phonon system (Zajac and Wojtczak) which is very convenient for the model of electrical conductivity. We plan at this point to reconstruct the transport theory allowing us to describe the giant magnetoresistance.

The model and its actuality is still accepted in the case of order disorder phase transition (Valenta and Sukiennicki). A comparison of experimental data and theoretical results shows, however, that the use of the standard MFA is insufficient for the description of the nanoparticles. This picture needs the stochastic and stoichiometric distribution when we use RFA [9].

Finally, we can conclude that the Valenta model can be useful not only in the case of localized spins in the ferromagnetic structure but also in the systems of itinerant electrons in the band theory applied for magnetic samples, rare earth metals, with special emphasis on the surface and interface properties like surface magnetisation, contact potentials or sorption effects (Valenta and Wojtczak). The proposed formalism works, roughly speaking, on the Hartee-Fock level of accuracy, as shown in some earlier papers whose applicability has been tested on Ni films and on Ni-Fe alloys (Valenta and Wojtczak with the assistance of Nhan and Khan, students from Viet-Nam).

Taking into account the present experience in the field we expect that the band model can be extended and modified to RFA considerations [9].

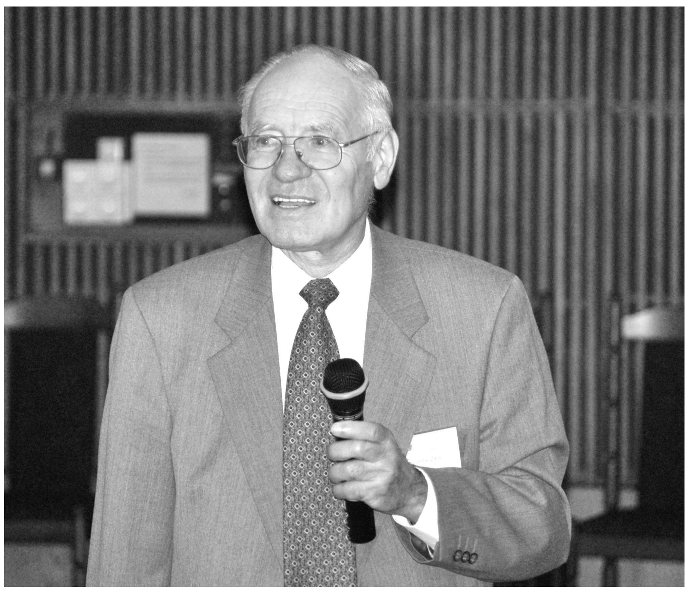

Fig. 4: Professor Štefan Zajac from the Czech Technical University in Prague [10].

Let us remark that the important biographical facts mentioned in the first part of our article are also reflected in the main achievements of Professor L. Valenta presented by him during the honoris causa doctor [11] ceremony which was held in the University of Łódź. The Senate of the University of Łódź conferred the title and the degree of the honoris causa doctor upon Professor L. Valenta in the light of the topicality and practical applicability of his achievements as well as of the cooperation the survey of which was presented during the ceremony, in 1994. The lecture delivered by the laureate showed the discovery of thermodynamics and geometry links and their importance for the modified versions of the fundamental model.

The lecture, delivered in Polish, focussed on some of the last original ideas introduced by Professor Valenta and ended with prospective expectations and developments of surface physics.

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Presented by Leszek Wojtczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2013

O MODELU VALENTY I JEGO PRZYSTOSOWANIU DO RZECZYWISTOŚCI III

S t r e s z c z e n i e

Niniejszy artykuł jest kontynuacją dwóch poprzednich prac i opisuje pewne wydarzenia, które charakteryzują rozwój teorii magnetycznych cienkich warstw. W tym kontekście obecna praca może być traktowana jako trzecia część poprzednich prac traktujących *O modelu Valenty i jego przystosowaniu do rzeczywistości III* i jako uzupełnienie części II i części I.

Rozpatrywany model służy do opisu własności warstw magnetycznych. Był on oryginalnie wprowadzony przez Luboša Valentę na poziomie aproksymacji pola molekularnego (MFA) oraz ostatnio modyfikowany przez kilku autorów pracujących pod kierunkiem L. Wojtczka (M. Mrygoń, I. Zasada, B. Busiakiewicz) przy założeniu metodologicznego ujęcia w terminach pola reakcji (RFA). Zmodyfikowana metoda budzi wciąż duże zainteresowanie fizyków i technologów.

Niniejszy artykuł jest napisany w taki sposób, aby razem z artykułami I i II stanowił raport dotyczący badań we współpracy czterech ośrodków. Uniwersytet Karola w Pradze i Uniwersytet Łódzki prowadzą nieformalną współpracę bezpośrednią, uwzględniajacą także udział pośredni Uniwersytetu Friedricha Schillera w Jenie oraz Uniwersytetu P. J. Šafarika w Košicach.

Przedstawiony raport został oparty na kolejnych przybliżeniach modelu oraz kolejnych opisach zjawisk. Szczegółowe odnośniki wskazują na przytaczane artykuły, z których najważniejsze były prezentowane na międzynarodowych konferencjach o magnetyzmie.

Słowa kluczowe: cienkie warstwy ferromagneyczne, funkcje spinowe autokorelacji, model Valenty modyfikowany przez pole reakcji

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 45–60

In memory of
Professor Claude Surry

Stanisław Bednarek and Paweł Tyran

SYSTEMATIC RESEARCH OF FACTORS DETERMINING OF THE GIANT MAGNETORESISTANCE IN MAGNETORHEOLOGIACAL SUSPENSIONS

Summary

The article describes a way of producing conductive magnetorheological suspensions, containing submillimeter particles of chemically pure iron, with sizes between 0.05–0.30 mm, and colloidal particles of graphite, with seizes of 0.5 μm, dispersed in cedar wood oil. It also presents a construction of an experiment system used to measure the magnetoresistance in these suspensions, in three perpendicular directions within a high magnetic field with induction level of 0–8 T. The work demonstrates results of the magnetoresistance measurements of the described suspensions, depending on: induction and speed of field changes, sizes and content of iron particles, graphite particles content and viscosity of disperse liquid. The obtained results shows giant magnetoresistance in three directions of measurement and its hysteresis.

Keywords and phrases: suspension, magnetoresistance

1. Introduction

Magnetorheologiacal suspensions comprise ferromagnetic particles with sizes of tenths or hundredths parts of a millimetre, dispersed in liquids with adequately high viscosity [1–3]. Possibilities to control viscosity or to achieve a non-zero module of stiffness [4–7] are well known. It happens thanks to strong interactions of dispersed particles with an external magnetic field [4–7]. A phenomena of resistance within magnetorheological suspensions is researched to a significantly lower extent. Presence of high level of magnetoresistance in these suspensions was proved at the end of the 20th century [8]. Aim of this article is to present systematic and detailed results of investigation on factors, which influence to magnetoresistance in magnetorheological suspensions. A new feature of this research is a measurement of magnetoresistance for a particular induction of a magnetic field in three perpendicular directions; cf. also [15, 16].

2. Preparation of samples

The investigation uses the magnetorheological suspensions, comprising chemically pure particles of iron, dispersed in cedar wood oil, doped with graphite particles. Graphite doping made it able to increase an effective electric conductivity of the suspensions. Structure of the investigated suspension is presented on Fig. 1. Particles of chemically pure iron were used as a reagent to produce the suspensions. Initial sizes of the particles were 0.4–0.6 mm. These particles underwent grounding in an electric grinder multiple times, in order to decrease their sizes, and sifting through sets of double sieves. In this way, three fractions of particles with sizes: 0.25–0.30 mm, 0.20–0.25 mm and 0.15–0.20 mm, were selected. Cedar wood oil – used as immersion liquid in optical microscopy – was used as disperse liquid. This oil was characterised by high viscosity of 360 Pa·s in a temperature of 20°C. Viscosity of the oil was determined with a flow-through Ostwald type viscometer, of own construction. Such a high viscosity prevented the particles dispersed in the oil from sedimentation.

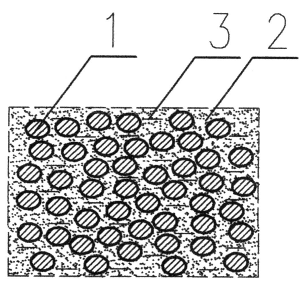

Fig. 1: Structure of suspension before applying a magnetic field; 1 – ferromagnetic particle, 2 – graphite particle, 3 – cedar wood oil.

Both pure oil and its solutions, comprising 10 % or 20 % of ethanol, were used to produce the suspensions. Ethanol caused a decrease of oil viscosity, respectively to 0.9 or to 0.8 of the initial value. Used graphite was in a form of dust, comprised of colloidal particles, whose sizes (determined with a measurement microscope) were 2–4 μm. Content of iron and graphite particles within suspensions were determined by filling factors, marked respectively as p_f and p_g. These factors are defined with the following formulas:

$$p_f = \frac{V_f}{V}, \qquad p_g = \frac{V_g}{V} \tag{1}$$

where V_f, V_g mean volume sums of iron and graphite particles, and V means volume of the suspension.

In order to produce the suspensions, masses of iron, graphite and cedar oils were weighed out. These masses were calculated on the basis of required filling and volume factors of the final suspension. Then, the weighed out mass of the graphite particles was added to the cedar oil and stirred for 40 minutes. After that, a weighed out mass of the iron particles was added and stirred for 10 minutes more. An electric stirrer was used for stirring. In the obtained suspensions, factors of p_f, p_g was 0.15, 0.25 or 0.35 p_g. The produced suspensions completely filled cubical closed containers made from plastics, Fig. 2. The external size of the container was 29 mm, and the walls were 2 mm thick. A narrow canal in the upper part of the container made it easier for air to flow while closing the container, and facilitated its filling up. This canal, after filling the container up and attaching the upper wall to it, was closed as well. A symmetrically square electrode, with a side length of 23 mm, made from copper foil, was also attached to every wall inside the container. Every electrode had a cable soldered on, going outside the container through a hole within its wall.

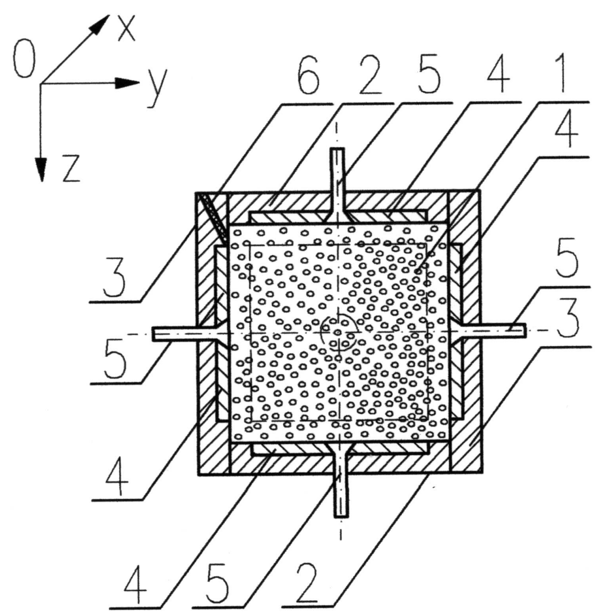

Fig. 2: Cross section of a container with a sample; 1 – magnetorheologiacal suspension, 2, 3 – walls of the container, respectively: vertical and horizontal, 4 – electrode, 5 – lead of the electrode, 6 – canal.

3. Experimental system

The produced samples were placed within a magnetic field, produced by a Bitter's magnet. Induction of the magnetic field was altered within a range of 0 T to ±8 T. Speeds of changes to the field induction were 0.1 T/s, 0.33 T/s or 0.011 T/s. Inhomogenity of the magnetic field in a sample did not exceed ±2 %. Vector of the field induction was parallel to the side walls of the sample and directed vertically, Fig. 3. Measurements of the electrical resistance of the samples were conducted after the field induction was changed for ±0.5 T in three perpendicular directions $0x$, $0y$, and $0z$ (Fig. 3). For this purpose, three digital mutlimeters, working as ohmmeters, were used, and subsequently connected to the sample, after achieving a given value of the field induction. During the measurement, only one ohmmeter was connected to a chosen pair of electrodes, corresponding to a given direction.

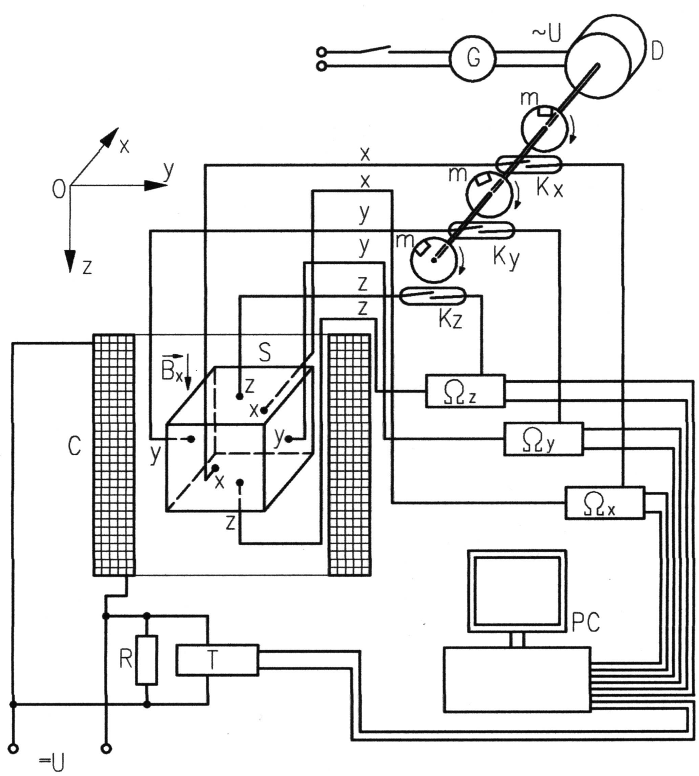

Fig. 3: Scheme of a measurement system; S – a container with sample, C – Bitter's magnet winding, T – teslameter, K_x, K_y, K_z – reed relay, m – neodymium magnet, d – synchronic engine, G – generator, Ox, Oy, Oz – ohmmeters, PC – personal computer.

Connection of the ohmmeters was carried out through using a special multiplexer of own construction, which exploited reed relays. A reed relay was installed within a circuit of each ohmmeter, closed with a neodymium magnet, attached to one of three plates. All of the three plates were placed on a common axis, and moved rotationally by a synchronic motor, powered by a generator with regulated frequency. When magnets were far enough from the reed relay, all circuits of the ohmmeters were open. Apart from the plates, also one magnet approaching a reed relay, e.g. K_x, cause closure of the ohmmeter circuit, connected with this reed relay. It enabled a measurement of resistance in one direction (in this case – $0x$). Thanks to a proper angular shift of attachment places of the magnets on the plates towards each other, the remaining reed relays were opened.

Further movement of the plate caused the magnet mentioned above to move away from the reed relay, and therefore open the circuit. Another magnet approached the next contractor (K_y), which caused its closing. Thanks to this, it was able to measure resistance in the direction $0y$. After that, the described situations repeated. The obtained results of measuring the electric resistance were recorded through a personal computer PC. Simultaneously, a measurement of a magnetic field induction was carried out. It was realized through measuring a voltage drop on an R resistor, attached as a shunt to the Bitter's magnet winding. This voltage drop was directly proportional to the field induction, and measured with a T multimeter, playing a role of a teslameter.

4. Discussion of the results

The obtained results of measurements were presented on Fig. 4–9. Analysis of the provided charts leads to a discovery of several general regularities, which are as follows. The measurements showed a decreasing in specific resistance of all researched samples of the suspension, together with an increase of induction of the applied magnetic field, and a growth of this resistance when the field induction was lowered. It means that all suspensions, which were investigated, show negative magnetoresistance. A characteristic property of all samples is also a non-linear relation of the specific resistance with the induction of the applied magnetic field. While decreasing the magnetic field induction, growth speed of the specific resistance for each suspension was lower than the drop speed when the field induction is increasing. As a result, the specific resistance of the suspensions after finishing the cycle of changes to the field induction did not go back to the initial value. This resistance was lower than the initial one. It means a presence of magnetoresistance hysteresis in all researched samples. The described regularities occur for all three perpendicular directions, within which measurements of the specific resistance of the samples according to the magnetic field induction were carried out. However, it is easy to notice that these changes towards an induction vector of the applied magnetic field $\mathbf{B}$ ($x - x$ direction) are significantly bigger than in case of two remaining direction perpendicular towards $\mathbf{B}$ ($y - y$ and $z - z$ directions).

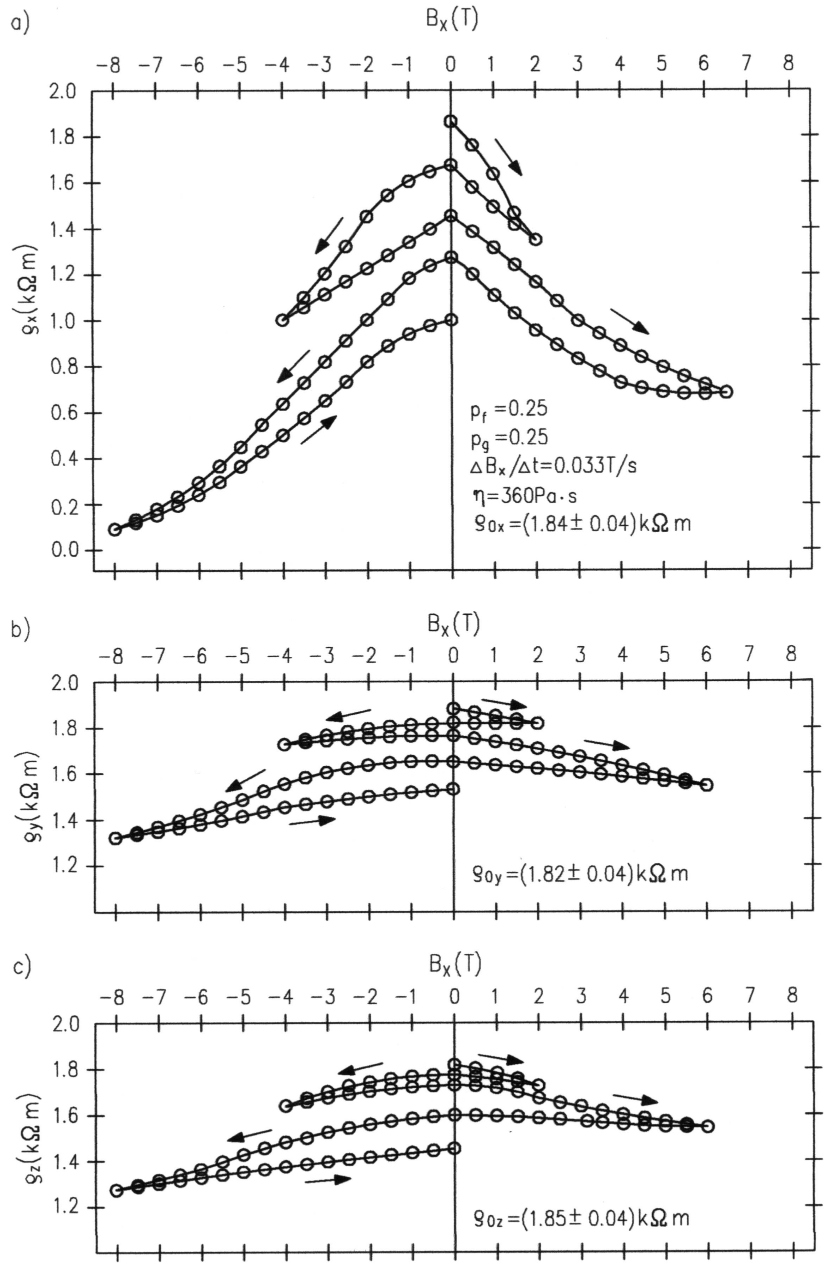

Fig. 4: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the magnetic field induction B_x, applied along the x axis.

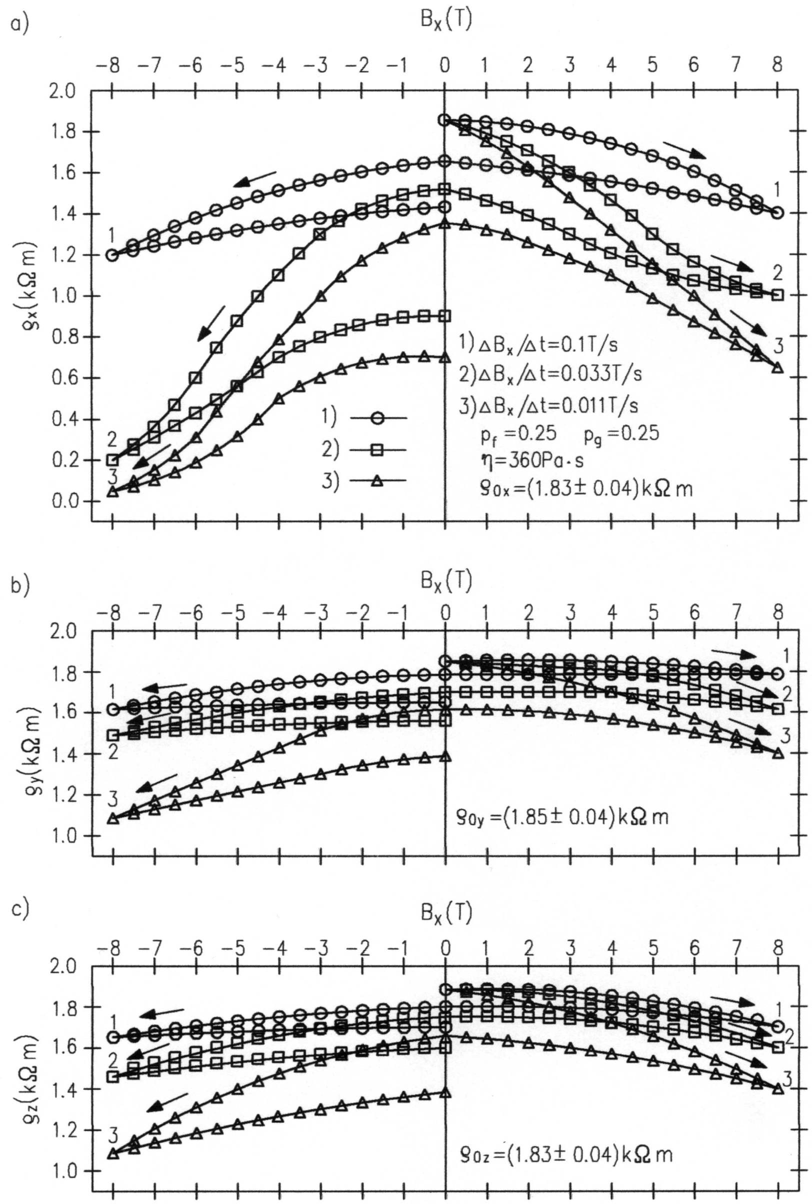

Fig. 5: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the speed of the magnetic field induction changes $\Delta B_x/\Delta t$ applied along the x axis.

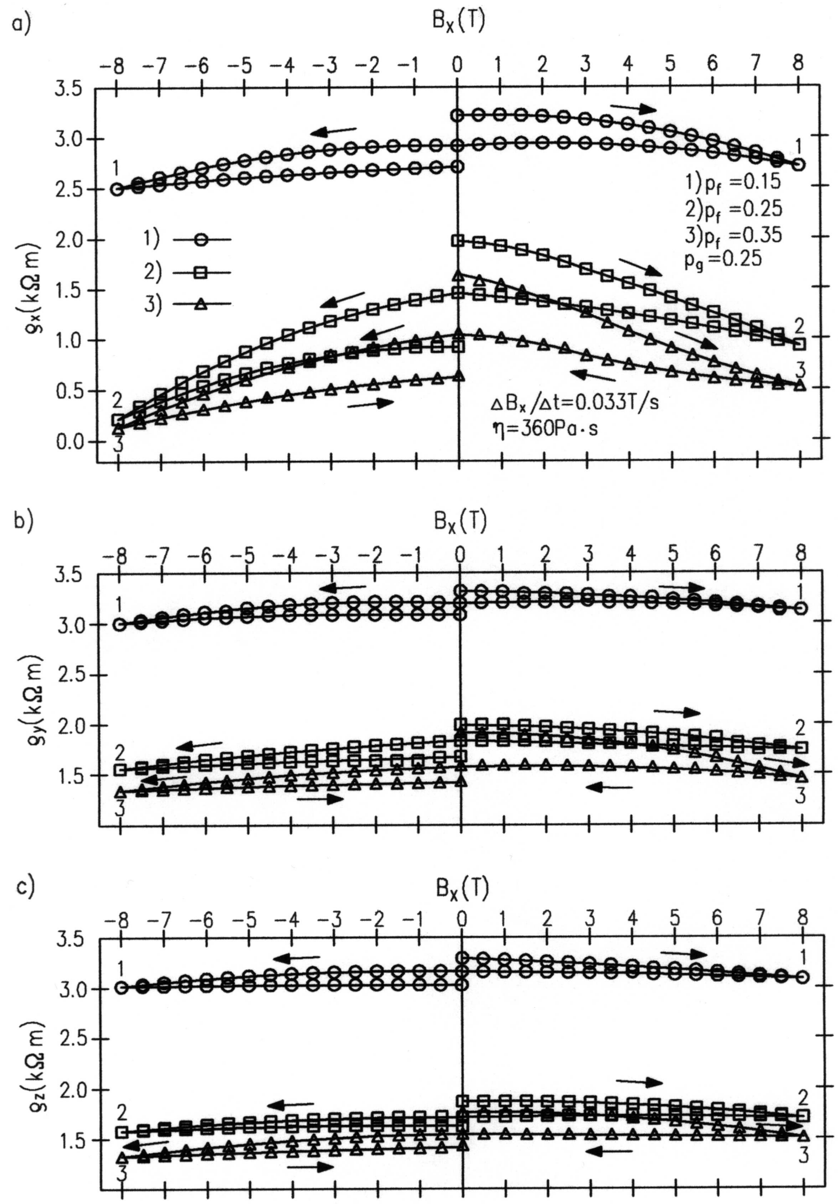

Fig. 6: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the magnetic field induction B_x, applied along the x axis, for different factors of filling by iron particles p_f.

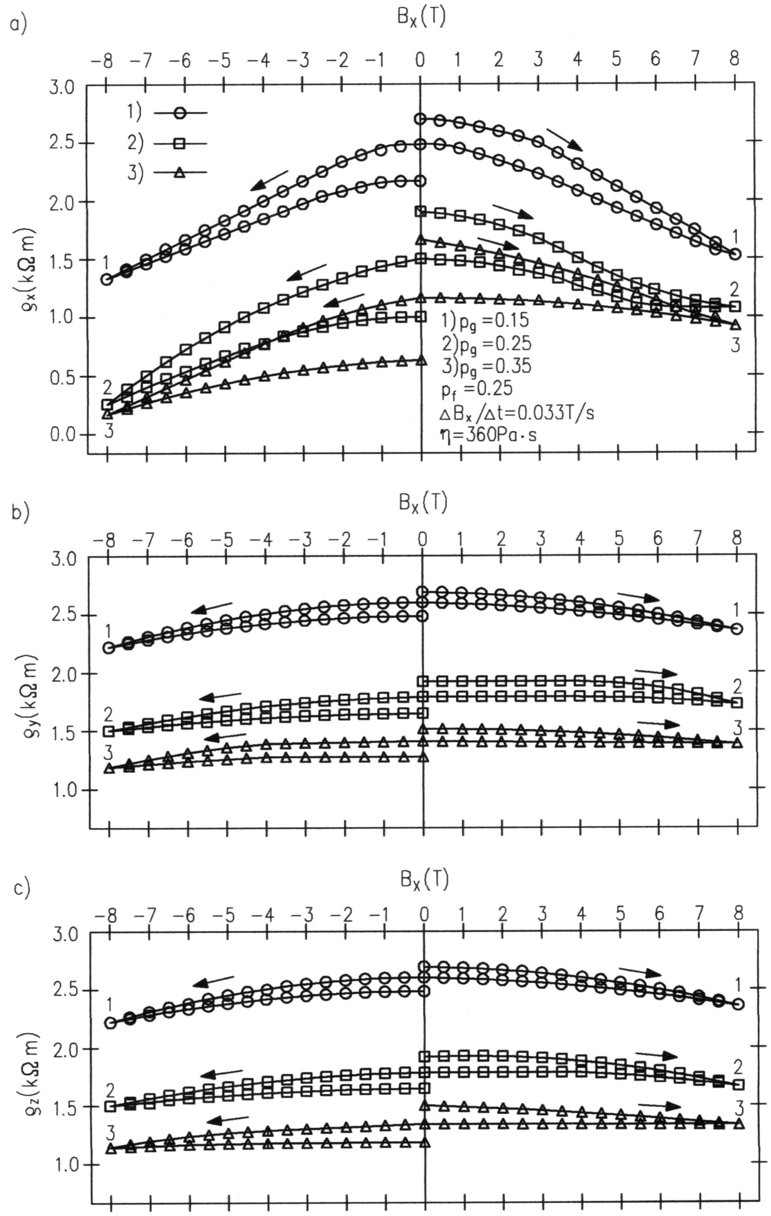

Fig. 7: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the magnetic field induction B_x, applied along the x axis, for different factors of filling by graphite particles p_g.

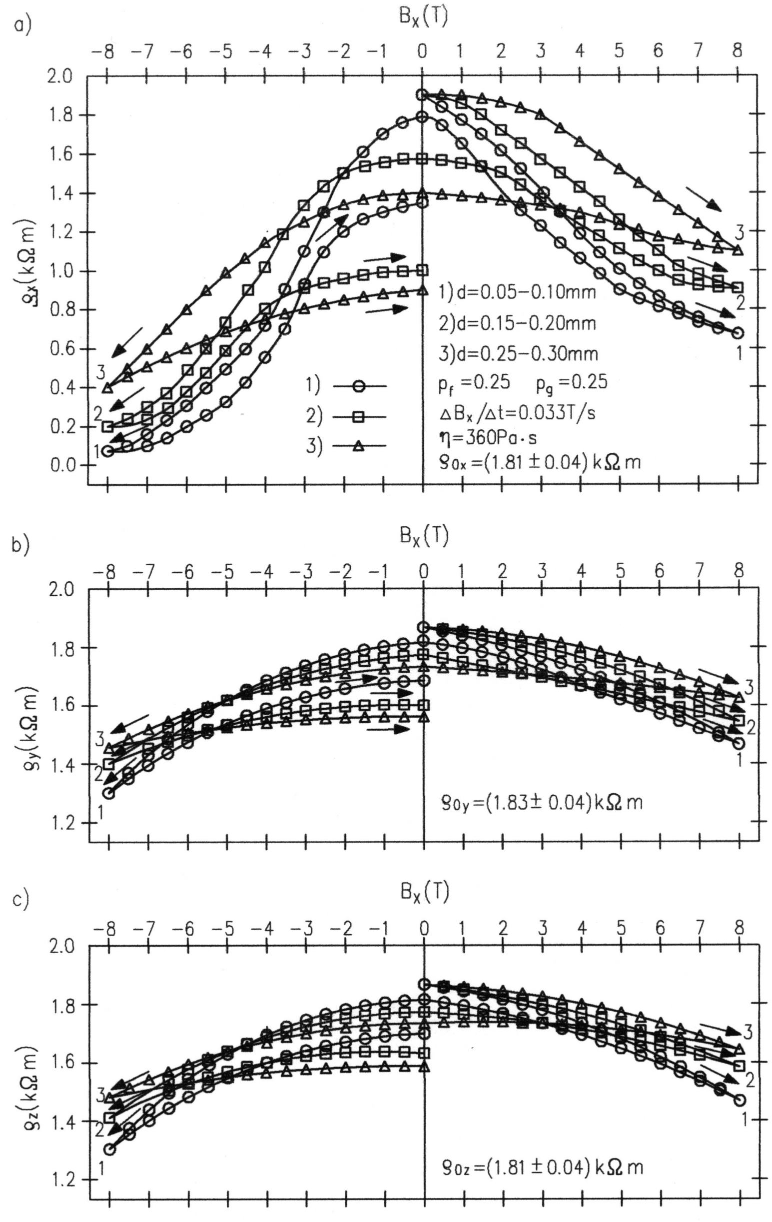

Fig. 8: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the magnetic field induction B_x, applied along the x axis, for different sizes of iron particles d.

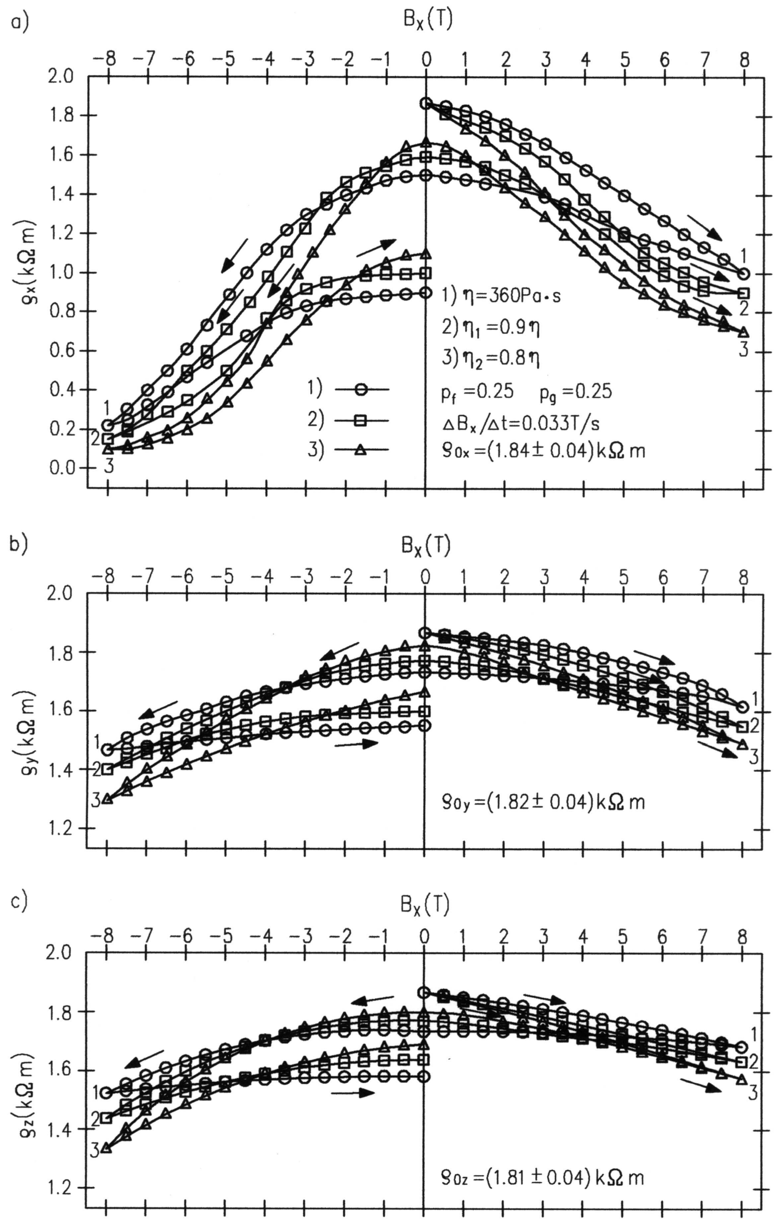

Fig. 9: Dependence of the specific resistance of the suspension sample ρ_x, ρ_y, ρ_z measured respectively: in directions of an x, y, z axis on the magnetic field induction B_x, applied along the x axis, for different coefficient viscosity η of disperse liquid.

What is more, changes within directions of $y - y$ and $z - z$ have similar values. It means that both longitudinal magnetoresistance (towards the applied field) and transverse magnetoresistance (in directions that are perpendicular towards this field) occurs.

In order to compare the detected changes in case of different samples thoroughly, we can introduce a factor of a relative alteration of the specific resistance in a chosen direction, e.g. W_x for $x - x$ direction. This factor is expressed by a relation of change of the initial specific resistance of the sample, measured in this direction before applying a magnetic field ($B = 0$), e.g. ρ_{0x} for $x - x$ direction, to the specific resistance ρ_0 measured in a field, whose induction value is B.

$$w_x = \frac{\rho_x - \rho_{0x}}{\rho_{0x}}. \tag{2}$$

The maximal value of the w_x factor defined in this way is 60.2 and was found within a magnetic field with an induction value of 8 T for a sample with filling factors of $p_f = p_g = 0.25$ (Fig. 5.a, line No. 3). This sample was earlier subjected to the changes of magnetic field within a range of $0 - 8 - 0$ T. Speed of changes of the magnetic field was 0.011 T/s. For the same samples, values of the analogically defined factors w_y and w_z were -0.73 (Fig. 5.b, c). Detected changes of the specific resistance are multiple times bigger, than changes of the specific resistance occurring in case of metals, with a comparable field induction value, not exceeding several % [9, 10]. The revealed changes are even bigger than giant magnetoresistance present within systems of thin layers, including ferro- and paramagnetic metals [11–14]. The detected changes are comparable with the colossal magnetoresistance. However, reason of presence of magnetoresistance in the researched suspensions is completely different than that in case of thin layers systems and metals.

In the case of metals, the reason of magnetoresistance is influence of electrodynamics force at electrons moving in the applied magnetic field. An effect of this force comprises deviations of the electrons' trajectories from the direction of the electric field, causing current to flow. Because of that, a number of electrons moving in a direction established initially by an electric field are changed, as well as current intensity. Change of electric resistance is a macroscopic indication of such a situation [9]. In the case of thin layers system a significant meaning is also attached to an orientation change of electrons' spin towards the layers within the applied magnetic field and quantum effects. It influences the movement of electrons and current intensity in such systems [11]. Within the researched suspensions, the main reason of magnetoresistance is change of spatial distribution of iron and graphite particles dispersed in cedar wood oil.

Before applying the magnetic field, the spatial distribution of both iron and graphite particles within suspension is disordered. In such a situation, a relatively small number of particles have contact with each other. The contacting particles create connections between electrodes with a small cross section and significant length. As a result, the initial specific resistance is relatively high. Application of the magnetic field causes induction of magnetic moments in the particles of iron. Opposite poles of magnetized particles attract each other. It leads to a situation, when iron particles are closer and form thin fibres created along the line of the magnetic field. The moving iron particles cause the graphite particles to move as well through forces of viscosity and cedar wood oil. Because of the fact that sizes of the graphite particles are much smaller than the iron ones, graphite particles fill the spaces between iron particles. As a result, connections between electrodes with smaller length and bigger cross section are established. They cause a drop in the specific resistance of a sample of the applied magnetic field direction, observed as negative longitudinal magnetoresistance. Iron particles attracting each other decrease contact resistance at the surfaces of both particles. Previous studied revealed that the fibres group themselves into chains and columns connected with each other. This effect is known as a production of the Winslow's column and fibre structure [17, 18]. Such a structure is presented in a scheme manner by a Fig. 10. Presence of this structure was also proved by X-ray inspections of the suspensions sample placed within the magnetic field [19]. As a result of these connections between fibres, the specific resistance is decreased also in directions perpendicular towards the applied magnetic field. It is observed as negative transverse magnetoresistance. Because of the fact of very high viscosity of the cedar wood oil, a process of creating a more ordered structure is delayed in relation to the changes of the magnetic field. This is a reason for the observed hysteresis of magnetoresistance. When the field is switched off, the described structure remains maintained thanks to very high viscosity of the cedar wood oil. As a result, remanence of magnetoresistance is observed, i.e. maintaining its lowered level, in relation to the initial value.

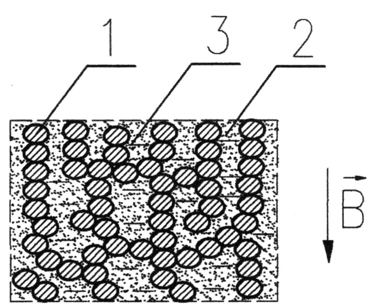

Fig. 10: Structure of the suspension after applying the magnetic field – numbers mean the same as in Fig. 1.

Apart from general regularities characterising magnetoresistance within the investigated suspensions, also more detailed regularities are observed, occurring while changing particular parameters of the suspensions or conditions of conducing the studies. Charts presented on Fig. 5 inform that lowering the sped of changes of magnetic field induction, while maintaining the remaining parameters of the suspension, cause bigger changes to the specific resistance, corresponding to the same change of the field induction. A higher hysteresis of magnetoresistance occurs when the field is switched off. This effect can be explained by a fact known from rheology, telling that value of some changes within a viscoelastic material, e.g. of a deformation, depends on the speed of the applied intensity of stress. Usually, the final deformation is bigger when the stress is applied slower. What is more, the viscoelastic material shows a tendency for enhancing the deformation with an established stress, i.e. for deformation flow [20]. The investigated suspensions interact as viscoelastic materials, where the stress is applied through an external magnetic field. This observation is also confirmed by charts presented in Fig. 4. They show that changes of induction of the magnetic field conducted several times, in an increasing range, but applied to the same sample with the same speed, cause a higher level of the final magnetoresistance. It is also noticed that while induction of the magnetic field approaches the final values, speed of changes of the specific resistance is lower. It points to the fact that the structure of suspension approaches the highest state of order, which can be obtained for a given value of the magnetic field induction.

Charts presented on Fig. 6 and 7 demonstrated presence of volume effect within the investigated suspensions. The volume effect consists in decreasing the initial specific resistance of the suspensions, together with increasing their filling factors, both by iron p_f and graphite p_g particles. Growth in the filling factors means an increase of volume of the conductive phase within the suspensions. In case of bigger volumes of this phase, a huge cluster with many branches ending at opposite electrodes is created in an easier way [21]. From this reason, even before applying the magnetic field, i.e. in an unordered state, there are more percolation paths within the suspension, through which an electric current may flow. A final effect is that the specific resistance of a suspension is decreased by this. Analysis of the charts presented in Fig. 6 and 7, allows also noticing that in the case of bigger values of factors p_f and p_g, changes of the specific resistance take place quicker together with alterations to induction of the magnetic field, and hysteresis of magnetoresistance is increased as well. It is caused by stronger interaction of ferromagnetic phase with bigger volume and the applied magnetic field, and also more effective interaction ordering iron and graphite particles through forces of viscosity.

Analysis of the charts presented on Fig. 8 shows that there is also a size effect within the investigated suspensions. It consists in increasing the speed of changes of the specific resistance together with decreasing the size of iron particles. What is more, if size of particles is smaller, hysteresis of magnetoresistance is smaller as well. A reason for such effect is direct proportionality of the viscosity force influencing the particles to their sizes. This is why, smaller particles are met with smaller resistance of motion, and it is easier for them to fill spaces between given particles. Analysis of the charts presented on the Fig. 9 implies that a decrease in the viscosity factor η of disperse liquid causes an increase in the speed of changes to the specific resistance and an increase of the magnetoresistance hysteresis. These effects may be also explained by easier ordering of the particles by the magnetic field within disperse liquid characterized by lower viscosity.

The conducted investigation allowed not only recognizing more and systematizing the effects within magnetorheologiacal suspensions with a conducting carrier. Results of the studies may turn out to be useful in improving those suspensions and their practical applications, e.g. in sensors of magnetic field, switching or memory elements [1–3, 6].

Acknowledgment

At the end, the authors would like to express their gratitude towards Prof. Tomasz Palewski and Prof. Krzysztof Rogacki -– directors of the International Laboratory of High Magnetic Fields and Low Temperatures in Wroclaw — for providing access to the Bitter's magnet working in the laboratory, for the using of the conducted investigations.

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Presented by Leszek Wojtczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 18, 2013

SYSTEMATYCZNE BADANIA CZYNNIKÓW DETERMINUJĄCYCH GIGANTYCZNY MAGNETOOPÓR W ZAWIESINACH MAGNETOREOLOGICZNYCH

S t r e s z c z e n i e

W artykule opisano sposób wytwarzania przewodzących zawiesin magnetoreologicznych, zawierających submilimetrowe cząstki chemicznie czystego żelaza o rozmiarach 0.05–0.30 mm i koloidalne cząstki grafitu o rozmiarach 0.5 μm, zdyspergowane w oleju cedrowym. Przedstawiono też budową układu doświadczalnego do pomiaru magnetooporu tych zawiesin w trzech wzajemnie prostopadłych kierunkach w silnym polu magnetycznych o indukcji 0–8 T. Podano wyniki pomiarów magnetooporu opisanych zawiesin w zależnośći od: indukcji i szybkości zmian pola, rozmiarów i zawartości cząstek żelaza, zawartości cząstek grafitu oraz lepkości cieczy dyspersyjnej. Uzyskane wyniki wykazują gigantyczny magnetoopór w trzech kierunkach pomiaru oraz jego histerezę.

Słowa kluczowe: zawiesina, magnetoopór

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 61–66

In memory of
Professor Claude Surry

Stanisław Bednarek

GREAT HUMAN BATTERY

Summary

The interesting educational experiments in physics are described in this paper. In these experiments a voltage and electric current are generated in the chain of human connected by special electrodes. Every electrode consist a two elements produced from different metal and mutually connected.

Keywords and phrases: galvanic cell, conection, electromotive force, current intensity

1. Introduction

Experiments called "a fruit battery" or "a vegetable battery" are well known. They include electrodes made from two different kinds of metal stuck in some fruits or vegetables, e.g. in a pickled cucumber, (Fig. 1). As a result of electrochemical reactions electromotive force (emf) is generated [1]. Several such batteries joined together are able to efficiently supply electricity to a receiver of low power, e.g. a clock with an liquid crystal display (LCD) screen or a light emitting diode (LED).

Aim of this work is to present a description of an interesting battery, where electromotor power is generated as a result of a human hand touching sets of electrodes, made from different metals. Such a battery may be called "a hand battery" or "a human battery". The following tools and materials are needed to construct such a battery: universal meter (a digital or analogue multimeter) with a range of a millivoltmeter and a voltmeter, and optionally a microammeter, a piece of plywood or other insulating plate with dimensions about $30 \times 20\,\mathrm{cm}^2$, pieces of sheet from different metals, e.g. copper and zinc or copper and aluminum – two bigger ones (about $15 \times 10\,\mathrm{cm}^2$) and several dozens of smaller ones (about $10 \times 2\,\mathrm{cm}^2$, numbers of pieces from both kinds of metal should be equal), epoxy glue, connection wires in insulation, banana plugs, tin and a soldering tool.

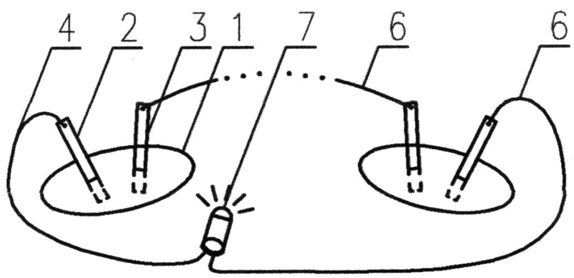

Fig. 1: The example of vegetable battery; 1 – cucumber, 2, 3 – electrodes made of different metals, 4–6 – wires, 7 – light emitting diode (LED).

2. Description of experiments

The simplest version of a battery is presented on Fig. 2. Two rectangular pieces of sheet 2, 3, made from different kinds of metal, e.g. from copper and zinc with dimensions of $15 \times 10\,\mathrm{cm}^2$ are attached with epoxy glue to the plywood or insulating plate 1. Both sheets should be placed with a 1 cm distance from each other. Each sheet is joined through one wire 4 or 5 with a universal meter. The meter should be switched to the range of the millivoltmeter or the microammeter. You can also use analogue meters with indicating needles. Joints are made by soldering one end of each cord to the sheets, and equipping the remaining ends with banana plugs, destined to put them into adequate sockets of the meter. When nothing touches both sheets, the meter indicates 0. If on each plate a human 6 puts one hand, the meter will indicate voltage or current intensity (Photo 1). A typical value of the indicated voltage in case of zinc and copper electrodes in about 0.3–0.7 V, and current intensity equals several dozens of μA because of the fact that his current intensity is very low and resistance of human body very high when compared with resistance of connection cords and the meter, the indicated voltage practically equals the electromotor power.

A reason for the current flow is creation of a galvanic cell within a system of the sheets (playing a role of electrodes) and skin of a hand. There is always a certain amount of sweat on skin with salt dissolved in water (mainly sodium chloride), which are electrolyte. Indications of the meter depend on many factors, among others on the surface area of the hands, their pressure force to the electrodes, skin moisture, materials that electrodes are made of. Skin moisture depends for instance on age (it is usually lower in case of older people), health condition and an emotional state of a person performing the experiment. This last relation is used in so called lie detectors, which are also called polygraphs. These devices make use of a fact that stress caused by lying usually causes an increase in skin moisture and decrease of its electric resistance. In our experiment, taking even one hand away from the electrodes causes circuit opening, and indications of the meter will come back to zero.

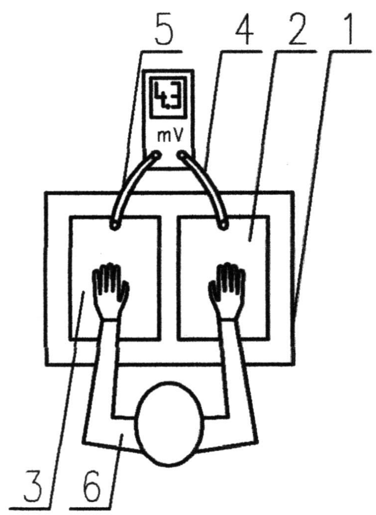

Fig. 2: "Hand battery" in the top view; 1 – insulating plate, 2, 3 – pieces of sheet made of different metals, 4, 5 – wires, 6 – a human, mV – millivoltmeter.

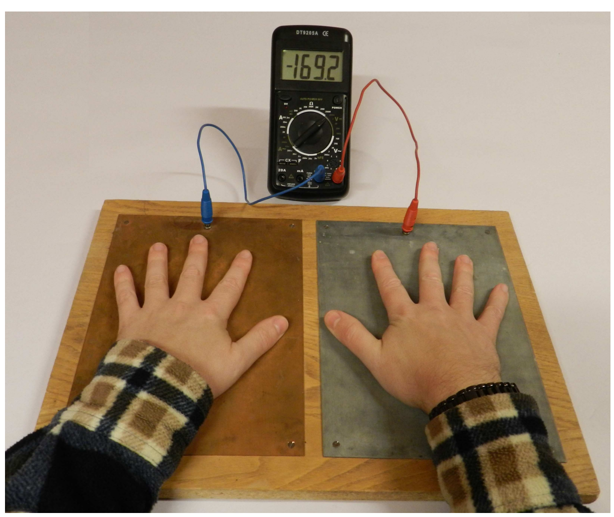

Photo 1. One of the examples of "a hand battery" in action.

The previous version of the experiment may be easily developed in a way that many persons may take part in it. For that purpose, several persons should stand in an arc. A person staying at the beginning of the arc puts one hand on one of the electrodes joined with the meter, and gives another hand to the next person. This second person gives the second hand to the next person and so on. A person standing at the end of the arch puts the free hand on the second electrode, which is also attached to the meter. Therefore, a chain closing the electric circuit is created. This circuit will host current flow caused by electromotive force, generated by the first and by the last person, keeping their hands on the electrodes. Value of the electromotive force is similar as in the previous version of the experiment, but current intensity is lowered, because resistance of separate persons is added and the total resistance of the circuit increases. For comparison, it needs to be mentioned that a typical value of human body resistance is about $1\,\mathrm{k\Omega}$. In order to lower resistance of the chain you can try to apply a mixed connection, i.e. to place several shorter chains next to each other, each created by several people. Obviously, everyone holds their neighbour's hand strongly.

Even more effective version of the experiment is presented on Fig. 3.

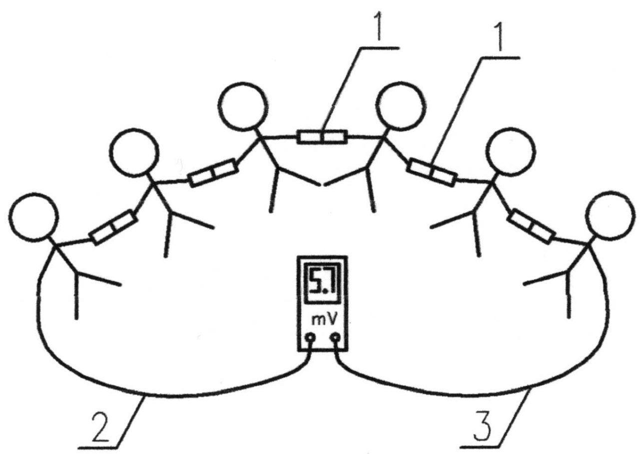

Fig. 3: Group "hand battery"; 1 – a set of electrodes, 2, 3 – wires, mV – millivoltmeter.

In this case, persons creating a chain hold hands of their neighbours through sets of electrodes 1, made from two different kinds of metals, joined to each other in any way, but it needs to provide good electric contact. It is important that part of each sets of electrodes made from the same type of metal, e.g. copper, be turned into the same direction, e.g. clockwise. The first and the last human in the chain keeps the insulated ends of the wires 2, 3, attached to the multimeter. In this case, electromotive forces generated by separate persons in the chain are also added, and high voltage may create. Current intensity depends, on among others, on area of the metal surface touching hands at every set of electrodes. Also in this experiment, the previously mentioned mixed connection of human taking part may be applied. Three simple examples of preparing sets of electrodes, constructed from stripes of sheet or bars, are shown in Fig. 4. Stripes of sheet or bars, creating the sets should have dimensions allowing to hold them comfortably in a hand – width or diameter about 2–3 cm and length about 10–15 cm. Instead of bars you can also use pipes. The least expensive ones are sets including stripes of thin sheet, which can be joined through soldering or riveting (Fig. 4a, b). While using pieces of sheet, their sharp edges needs to be smoothened to avoid potential cutting of hands skin. Pieces of bars may be joined through screws (Fig. 4c), and pipes with proper diameters may be put one into another in particular sections.

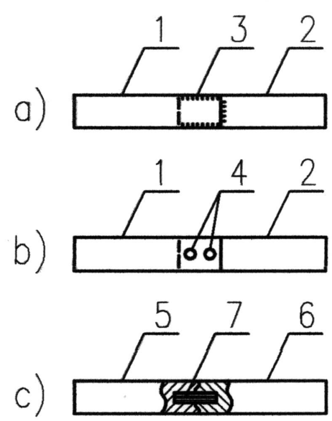

Fig. 4: Examples of method of joining the electrodes: a) soldiered, b) riveted, c) twisted; 1, 2 – stripes of sheet made of different metals, 3 – soldiering alloy, 4 – rivets, 5, 6 – pieces of a pipe made of different metals, 7 – screw.

3. Conclusions

The described experiments enable pictorial verification of the Ohm's law for the whole circuit or the Kirchhoff's law. These experiment are perfect to be used during different kinds of scientific festivals or picnics, as they are highly entertaining, educative and integrating for participants. They may also serve for setting individual records, e.g. which group will generate the highest voltage or current intensity. For that purpose, instead of zinc and copper elements, one may use elements from different metals, e.g. aluminium and copper – a clue for optimum choice is an electrochemical row of metals, present in physical or chemical tables. Regarding its uniqueness, such records have a chance to be registered as Guinness World Records.

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Presented by Leszek Wojtczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2013

WIELKA LUDZKA BATERIA

S t r e s z c z e n i e

W artykule opisano interesujące edukacyjne doświadczenia z fizyki. W tych doświadczeniach napięcie elektryczne i prąd są wytwarzane przez łańcuch ludzi połączonych ze sobą za pomocą specjalnych elektrod. Każda elektroda składa się z dwóch elementów wykonanych z różnych metali i wzajemnie połączonych.

Słowa kluczowe: ogniwo galwaniczne, złącze, siła elektromotoryczna, natężenie prądu

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 67–73

In memory of
Professor Claude Surry

Agnieszka Niemczynowicz

MODEL OF COUPLED HARMONIC OSCILLATOR IN A ZWANZIG-TYPE CHAIN. REMARKS ON ROWLANDS APPROACH

Summary

The aim of the present paper is to consider possibility to apply our previous ideas [7,8] and to extend the model of coupled harmonic oscillations in a Zwanzig-type chain [1,6–8] to the case of the one-dimensional *finite* stochastic chain. The classical equations of motion are simplified to the system of equations of motion for relative displacement. We discuss the the first order differential equation of generating function which can help us to find the general form of solutions in the case of stochastic chain.

Keywords and phrases: harmonic oscillator, Zwanzig-type chain atoms, damping of acoustic modes

1. Introduction

The simple theoretical model of a gas atom interactions with a solid surface appears in many investigations coming from the different branches of physics, chemistry and biology, as well. In particular, very useful and interesting model was study by Zwanzig [1] and Cabrera [2]. Both of them considered the interaction between collision of a single atom (e.g. a gas atom) with a solid represented by one-dimensional semi-infinite harmonic chain. This simple model assumed that collision atom interact with the sample via a truncated parabolic potential.

In the next investigations of many researches (e.g. [3, 4, 9]) we can discover more precisely (or not) consideration or extension above model. The illustrative examples can be given in the papers [3, 4], where the authors approximate this problem in the case of homogenous chain or chain with impurities and presented the computer simulations of this problem. Another very curious example we can find in the paper [5], where damping is discussed as the effect of surface disorder.

Motivated by mentioned considerations we write the system of equations for the relative displacement of atoms in stochastic chain and study the first order differential equation of generating function, which can leads us to the general form of solutions for relative displacement of atoms in stochastic chain. We compare solutions obtained by the generating technique function in [7, 8] with the known solutions for the homogenous chain (a Zwanzig-type chain) [3, 4] and chain with one impurity (different mass of one atom) obtained by "two" generating functions or by using the convolution of functions [4, 9].

2. Equations of motion for a stochastic chain

Let us consider the *finite* chain of atoms colliding with the gas atom of a mass M_0 (Fig. 1). The atoms of the chain are not identical, so $M_i \neq M_{i+1}$ $i = 1, ..., N-1$. The motion is characterized by the system of equations

$$\begin{aligned} &M_1\ddot{r}_1(t) = -K_1(r_1 - r_2) \\ &M_i\ddot{r}_i(t) = K_{i-1}(r_{i-1} - r_i) - K_i(r_i - r_{i+1}),\ i = 2, ..., N-1, \\ &M_i\ddot{r}_i(t) = K_i(r_{i-1} - r_i),\ i = N, \end{aligned} \tag{1}$$

where $r_i(t)$ denotes the displacement of ith atom from its equilibrium position, K_i is the force constants for the harmonic interaction between particles i and $i+1$, in general differ from atom to atom in the unit cell.

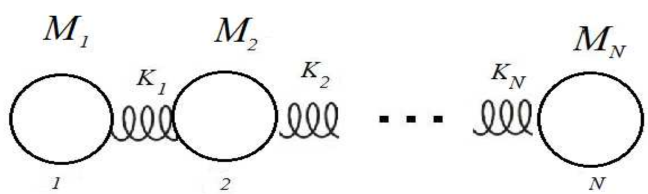

Fig. 1: One dimensional stochastic finite chain of atoms of mass $M_i,\ i = 1, 2, ..., N$.

The system (1) is equivalent to

$$\begin{aligned} &\ddot{r}_1(t) = -\omega^2\alpha_1^1(r_1 - r_2) \\ &\ddot{r}_i(t) = \omega^2\left[\alpha_i^{i-1}(r_{i-1} - r_i) - \alpha_i^i(r_i - r_{i+1})\right],\ i = 2, 3, ..., N-1 \\ &\ddot{r}_N(t) = \omega^2\alpha_N^N(r_{N-1} - r_N). \end{aligned} \tag{2}$$

Here, we suppose that $\omega = \sqrt{K/M}t$, where K is responsible for a interaction between pair of homonuclear atoms, each of mass M and $\alpha_j^i = K_iM/KM_j$ for $i, j = 1, ..., N$.

2.1. Disscusion of optimality of substitution

For simplify of (2) we introduce the following substitution [1]

$$u_{2i}(\tau) = 2(d/d\tau)r_i(\tau),\ i = 1, 2, ..., N, \tag{3}$$

$$u_{2i+1}(\tau) = r_i(\tau) - r_{i+1}(\tau),\ i = 1, 2, ..., N-1, \tag{4}$$

$$u_{2N+1}(\tau) = r_N \text{ and } \tau = 2\omega t. \tag{5}$$

Now, we can rewrite the system of equation (2) in the form

$$\frac{d}{d\tau}u_2(\tau) = -\frac{1}{2}\alpha_1^1 u_3(\tau) \tag{6}$$

$$\frac{d}{d\tau}u_k(\tau) = \frac{1}{2}\left[\alpha^{\frac{1}{2}k}_{\frac{1}{2}k-1}u_{k-1}(\tau) - \alpha^{\frac{1}{2}k}_{\frac{1}{2}k}u_{k+1}(\tau)\right],\ k = 4, 6, ..., 2N-2, \tag{7}$$

$$\frac{d}{d\tau}u_k(\tau) = \frac{1}{2}\left[u_{k-1}(\tau) - u_{k+1}(\tau)\right],\ k = 3, 5, ..., 2N-1 \tag{8}$$

$$\frac{d}{d\tau}u_{2N}(\tau) = \frac{1}{2}\alpha_N^N u_{2N-1}(\tau), \tag{9}$$

$$\frac{d}{d\tau}u_{2N+1}(\tau) = \frac{1}{2}u_{2N}(\tau). \tag{10}$$

Using equations (6), (7) for $k = 4$ and (8) for $k = 3$ we can easily express the second derivative $(d^2/d\tau^2)u_3$ as a linear function of u_5, namely

$$(4D^2 + \gamma^1_{1,2})u_3 = \alpha_2^2 u_5, \tag{11}$$

where $\gamma^1_{1,2} = \alpha_1^1 + \alpha_2^1$ and $D^2 = d^2/d\tau^2$.

It is natural to ask is the substitution (4)-(6) is optimal? In order to examine mentioned substitution we replace (5) by the following

$$A_n u_{2i+1}(\tau) = r_i(\tau) - r_{i+1}(\tau),\ i = 1, 2, ..., N-1, \tag{12}$$

where

$$A_n = \begin{cases} 1/\alpha_i^{i-1} & \text{for} \quad n = 2i-1, \\ 1/\alpha_i^i & \text{for} \quad n = 2i+1. \end{cases}$$

Further calculations lead to the following system of equations

$$\frac{d}{d\tau}u_2(\tau) = -\frac{1}{2}u_3(\tau), \tag{13}$$

$$\frac{d}{d\tau}u_k(\tau) = \frac{1}{2}\left[u_{k-1}(\tau) - u_{k+1}(\tau)\right], \quad k = 4, 6, ..., 2N-2, \tag{14}$$

$$\frac{d}{d\tau}u_k(\tau) = \frac{1}{2}\alpha^{\frac{1}{2}(k-1)}_{\frac{1}{2}(k-1)}\left[u_{k-1}(\tau) - u_{k+1}(\tau)\right], \quad k = 3, 5, ..., 2N-1, \tag{15}$$

$$\frac{d}{d\tau}u_{2N}(\tau) = \frac{1}{2}\frac{\alpha^N_N}{\alpha^{N-1}_{N-1}}u_{2N-1}(\tau), \tag{16}$$

$$\frac{d}{d\tau}u_{2N+1}(\tau) = \frac{1}{2}u_{2N}. \tag{17}$$

As we can see the equation (12) corresponds to equation (6) for $\alpha_1^1 = 1$, and equations (13) corresponds to equations (8) in the case of $k = 3, 5, ..., 2N-1$. The equations for the second derivative $(d^2/d\tau^2)u_3$ takes the form

$$(2D^2 + \alpha_1^1)u_3 = \alpha_1^1 u_5, \tag{18}$$

where D^2 is define as previously. Similarly, like in paper [7], we can see by (11) and (18) that the relative displacement u_5 plays *specific role.*

2.2. Generating function method for stochastic chain

Let us write the system of equations (6)-(10) to a more convenient form by setting $x_k = x_k(\tau) = u_{k+2}(\tau)$ for $k = 0, ..., 2N-2$. Then we obtain

$$\frac{d}{d\tau}x_0(\tau) = -\frac{1}{2}\alpha_1^1 x_1(\tau), \tag{19}$$

$$\frac{d}{d\tau}x_k(\tau) = \frac{1}{2}\left[\alpha^k_{k-1}x_{k-1}(\tau) - \alpha^k_k x_{k+1}(\tau)\right], \; k = 2, 4, 6, ..., 2N-4, \tag{20}$$

$$\frac{d}{d\tau}x_k(\tau) = \frac{1}{2}\left[x_{k-1}(\tau) - x_{k+1}(\tau)\right], \; k = 1, 3, 5, ..., 2N-3, \tag{21}$$

$$\frac{d}{d\tau}x_{2N-2}(\tau) = \frac{1}{2}\alpha^N_N x_{2N-3}(\tau), \tag{22}$$

$$\frac{d}{d\tau}x_{2N-1}(\tau) = \frac{1}{2}x_{2N-2}(\tau). \tag{23}$$

Let us consider the generating function in the form

$$\theta(z,\tau)=\theta^{+}(z,\tau)+\theta^{-}(z,\tau), \tag{24}$$

where

$$\theta^{+}(z,\tau)=\sum_{\substack{k=0,\\ k\text{ even}}}^{2N-2} x_k z^k \quad\text{and}\quad \theta^{-}(z,\tau)=\sum_{\substack{k=1,\\ k\text{ odd}}}^{2N-1} x_k z^k.$$

Using (19)–(23) we obtain the derivative of generating function (24)

$$\begin{aligned}2\frac{\partial}{\partial\tau}\theta &= (z-z^{-1})\theta^{+}+\sum_{\substack{k=1,\\ k\text{ odd}}}^{2N-1} x_k z^k\left(\alpha_{k+1}^{k}z-\alpha_{k}^{k+1}z^{-1}\right)\\ &+x_1 z\left(\alpha_1^2 z-\alpha_1^1 z^{-1}\right)+x_0 z^{-1}+\alpha_{2N-3}^{2N-2}x_{2N-3}z^{2N-2},\end{aligned} \tag{25}$$

with the extra condition $\alpha_{2N-1}^{2N}=\alpha_{2N}^{2N-1}$.

2.2.1. Remaks on homogeneous chain

Let us consider homogeneous chain of atoms, it means the chain of the same atoms. In this case we suppose that $M_i = M_j$ and $K_i = K_j$ for every i, j. Equations of motions and further calculations using the generating technique function provide to the same results as in the paper [7, 8].

2.2.2. Remaks on chain with one impurity

Let us consider the system of equations of motion (1) in the following form

$$\begin{aligned}&M_1\ddot{r}_1(t)=-K_1(r_1-r_2)\\ &M\ddot{r}_2(t)=K_1(r_1-r_2)-K(r_2-r_3),\\ &M\ddot{r}_i(t)=K(r_{i-1}-2r_i+r_{i+1}),\ i=3,...,N-1,\\ &M\ddot{r}_i(t)=K(r_{i-1}-r_i),\ i=N,\end{aligned} \tag{26}$$

where we assume that $M_i = M$ and $K_i = K$ for $i = 2, ..., N$. Using the substitutions (3)–(5) and introduce new notations $\alpha = K_1/K$ and $\beta = M_1/M$ we can simplify the system (26) into the system of equations for relative displacement considered by Dolgov and Khizhnyak [9]

$$\frac{d}{d\tau}x_0(\tau)=-\frac{1}{2}\frac{\alpha}{\beta}x_1(\tau) \tag{27}$$

$$\frac{d}{d\tau}x_2(\tau) = \frac{1}{2}\left[\alpha x_1(\tau) - x_3(\tau)\right], \tag{28}$$

$$\frac{d}{d\tau}x_k(\tau) = \frac{1}{2}\left[x_{k-1}(\tau) - x_{k+1}(\tau)\right], \ k = 3, 4, ..., 2N-1, \tag{29}$$

$$\frac{d}{d\tau}x_{2N-1}(\tau) = \frac{1}{2}x_{2N-2}(\tau). \tag{30}$$

In this case the derivative of generating function take the form

$$2\frac{\partial}{\partial}\tau\theta = \theta(z - z^{-1}) + x_1 z(z - \frac{\alpha}{\beta}z^{-1}) + x_0 z^{-1} + x_{2N-3}z^{2N-2},$$

which is different from the result obtained in [9]. This difference appears because in our consideration we use calculations based on one generating function, but technique applied by Dolgov and Khizhnyak is based on introducing two generating functions. In consequence, the solutions for the relative displacement in the simple homogeneous model of chain are similar but not the same. In both case approximation methods the results are presented in the terms of Bessel functions.

3. Conclusion

In relations to the Zwanzig's model [1], we compared differential equation of generating function with the knowing solutions for the homogeneous chain and chain with one impurity. Mathematically, we obtain the system of differential-difference equations of motions for relative displacement and differential equation for generating function. We shown the difference between the solutions and we can see that use of our approximation method can give us a more precisely (general) results. In discussion of optimality of substitution we notice that the u_5 plays *specific role* like u_3 in the Zwanzig's model.

Acknowledgment

The author should like to express their cordial thanks to Professors Julian Ławrynowicz and Leszek Wojtczak for suggesting of problems and stimulating discussions. This work was supported by Department of Mathematics and Computer Science, University of Warmia and Mazury through the grant 528-1310-0881.

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 24, 2011

MODEL OSCYLATORA HARMONICZNEGO W ŁAŃCUCHU ATOMÓW TYPU ZWANZIGA. UWAGI NA TEMAT PODEJŚCIA ROWLANDSA

S t r e s z c z e n i e

W pracy podjęto próbę zastosowania metody aproksymacyjnej R. W. Zwanziga (1960) oraz rozwiązania układu równań różniczkowych ruchu w przypadku skończonego stochastycznego łańcucha atomów [5]. Porównano równania funckji generującej dla znanych przypadków: łańcucha jednorodnego oraz łańcucha z lokalną niejednorodnością z dotychczas otrzymanymi wynikami [3, 4, 9].

Słowa kluczowe: oscylator harmoniczny, łańcuch atomów typu Zwanziga, tłumienie modów akustycznych

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 75–82

In memory of
Professor Claude Surry

Arezki Touzaline

ADHESIVE CONTACT OF AN ELASTIC BODY WITH PRESCRIBED NORMAL STRESS AND TOTAL SLIP-DEPENDENT FRICTION I

PROBLEM STATEMENT AND VARIATIONAL FORMULATION

Summary

We consider a mathematical model which describes the frictional contact between a nonlinear elastic body and a foundation. The normal stress on the contact surface is prescribed, the coefficient of friction depends on the total slip and adhesion of contact surfaces is taken into account. In the first part of the paper the evolution of the bonding field is discribed by a first order differential equation. We derive a variational formulation of the mechanical problem.

Keywords and phrases: elastic, adhesion, total slip-dependent friction, fixed point, weak solution

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [17]. The frictional contact problem with adhesion and normal compliance for nonlinear elastic materials was studied in [12]. The static frictional contact probem with a slip dependent coefficient of friction and a prescribed normal stress for elastic materials was studied in [5]. This model, when the adhesion of contact surfaces is taken into account, was studied in [23] where an existence and uniqueness result was established under the hypothesis of smallness of the coefficient of friction and the prescribed normal stress. In this paper we deal moreover with the study of this last model which was recently studied in [20] but when the coefficient of friction depends on the total slip. Indeed, recently in [20] the authors have proved new existence, uniqueness and regularity results in the study of a class of quasivariational inequalities defined on an unbounded interval of time. They have also applied these results in the analysis of several quasistatic contact problems. In the next, recall that models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 8, 17–19, 21, 22]. In [2, 6, 12–15,23–25] some frictional contact problems with adhesion were studied. Indeed, in [2, 6, 14], quasistatic frictional contact problems with adhesion in elasticity were studied and the existence results for a friction coefficient small enough were established while under the same condition, the existence and uniqueness results were proved in [12, 23]. Also in [24, 25] quasistatic frictional contact problems with adhesion in viscelasticity were studied and the existence and uniqueness of solutions under a smallness assumption on the coefficient of friction was proved. In this paper, as in [9, 10] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [1, 11, 14–19]. In this work we provide the variational formulation of the mechanical problem for which we prove the existence of a unique weak solution and obtain a partial regularity result. Unlike the result obtained in [23], we observe that the hypothesis of smallness on the coefficient of friction was removed because this latter depends on the total slip and is a term "history-dependent" while the term containing the slip-dependent friction is not.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result.

2. Problem statement and variational formulation

A nonlinear elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. A volume force of density φ_1 acts in Ω and surface tractions of density φ_2 act on Γ_2. In the reference configuration the body is in an adhesive frictional contact with a foundation, over the potential contact surface Γ_3.

Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that, for all $t \in [0,T]$,

$$div\sigma (t) + \varphi_1 (t) = 0 \text{ in } \Omega, \tag{2.1}$$

$$\sigma (t) = F\varepsilon (u (t)) \text{ in } \Omega, \tag{2.2}$$

$$u (t) = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma (t) \nu = \varphi_2 (t) \quad \text{on } \Gamma_2, \tag{2.4}$$

$$-\sigma_\nu (t) = S - c_\nu \beta^2 (t) R_\nu (u_\nu (t)) \quad \text{on } \Gamma_3, \tag{2.5}$$

(2.6)
$$\left.\begin{cases} \left\|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))\right\| \le \mu \left(\int_0^t \|u_\tau (s)\| ds\right) S \\ \left\|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))\right\| < \mu \left(\int_0^t \|u_\tau (s)\| ds\right) S \Longrightarrow u_\tau (t) = 0 \\ \left\|\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))\right\| = \mu \left(\int_0^t \|u_\tau (s)\| ds\right) S \Longrightarrow \\ \exists \lambda \ge 0 \text{ s.t. } u_\tau (t) = -\lambda \left(\sigma_\tau (t) + c_\tau \beta^2 (t) R_\tau (u_\tau (t))\right) \end{cases}\right. \text{on } \Gamma_3,$$

$$\dot\beta (t) = -[\beta (t) (c_\nu (R_\nu (u_\nu (t)))^2 + c_\tau \|R_\tau (u_\tau (t))\|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3, \tag{2.7}$$

$$\beta (0) = \beta_0 \text{ on } \Gamma_3. \tag{2.8}$$

We denote by σ the stress field and $\varepsilon (u)$ the linearized strain tensor. Equality (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given nonlinear function while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents the prescribed normal stress S with adhesion and (2.6) is the associated Coulomb's law of dry friction on the contact surface Γ_3. Here the parameters c_ν, c_τ and ε_a are adhesion coefficients. As in [17], R_ν, R_τ are truncation operators defined by

$$R_\nu (s) = \begin{cases} L \text{ if } s < -L \\ -s \text{ if } -L \le s \le 0 \\ 0 \text{ if } s > 0 \end{cases}, \quad R_\tau (v) = \begin{cases} v & \text{if } \|v\| \le L \\ L\dfrac{v}{\|v\|} & \text{if } \|v\| > L \end{cases},$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [17] where $[s]_+ = \max (s, 0)\ \forall s \in \mathbb{R}$. Since $\dot\beta \le 0$ on $\Gamma_3 \times (0,T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [13] it follows that the model does not allow for complete debonding field in finite time. Finally, (2.8) represents the initial bonding field. Recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S_d are given by

$$u.v = u_i v_i, \quad \|v\| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \ \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\sigma \in Q : div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

For all $v \in H_1$, the linearized strain tensor is defined as

$$\varepsilon(v) = (\varepsilon_{ij}(v)) = \frac{1}{2}(v_{i,j} + v_{j,i})$$

and for a all function $\tau \in Q_1$, $div\tau = (\tau_{ij,j})$ is the divergence of τ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similary, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v da \ \ \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|.\|_{H_1}$and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space.

Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in C([0,T];H), \qquad \varphi_2 \in C\left([0,T];\left(L^2(\Gamma_2)\right)^d\right) \tag{2.11}$$

and, moreover, by Riesz's representation theorem, we define a function

$$f : [0,T] \to V$$

by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da - \int_{\Gamma_3} S v_\nu da \ \forall v \in V,\ t \in [0,T]. \tag{2.12}$$

We see that (2.11) and (2.12) imply

$$f \in C([0,T];V).$$

Also we define the functional $j_{fr} : L^2(\Gamma_3) \times V \to \mathbb{R}$ by

$$j_{fr}(\xi, w) = \int_{\Gamma_3} \mu(\xi) S \|w_\tau\| da \quad \forall \xi \in L^2(\Gamma_3) \quad \forall w \in V,$$

where the coefficient of friction μ is assumed to satisfy

$$\left.\begin{array}{l}
(a)\ \ \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\[2ex]
(b)\ \ \text{there exists } L_\mu > 0 \text{ such that} \\
\qquad |\mu(x, r_1) - \mu(x, r_2)| \leq L_\mu |r_1 - r_2| \ \ \forall r_1, r_2 \in \mathbb{R}_+,\ a.e.\ x \in \Gamma_3; \\[2ex]
(c)\ \ \text{for any } r \in \mathbb{R}_+, \text{ the mapping } x \to \mu(x, r) \text{ is measurable on } \Gamma_3; \\[2ex]
(d)\ \ \text{the mapping } x \to \mu(x, 0) \in L^2(\Gamma_3).
\end{array}\right\} \tag{2.13}$$

We suppose that S satisfies

$$S \in L^\infty(\Gamma_3) \ \text{ and } S \geq 0 \ \ a.e. \text{ on } \Gamma_3. \tag{2.14}$$

In the study of Problem P_1 we assume that the elasticity operator A satisfies

$$(2.15)\quad \left.\begin{array}{l}(a)\ \ F:\Omega\times S_d\to S_d;\\ \\ (b)\ \text{there exists } M_F>0 \text{ such that}\\ \qquad \|F(x,\varepsilon_1)-F(x,\varepsilon_2)\|\leq M_F\|\varepsilon_1-\varepsilon_2\|,\\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x\ \in\ \Omega;\\ \\ (c)\ \text{there exists } m_F>0 \text{ such that}\\ \qquad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\geq m_F\|\varepsilon_1-\varepsilon_2\|^2,\\ \qquad \text{for all}\ \ \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ \ a.e.\ x\ \in\Omega;\\ \\ (d)\ \text{the mapping } x\to F(x,\varepsilon)\ \text{ is Lebesgue measurable on } \Omega,\\ \qquad \text{for any } \varepsilon \text{ in } S_d;\\ \\ (e)\ x\to F(x,0)\in Q.\end{array}\right\}$$

We suppose that the adhesion coefficients c_ν, c_τ and ε_a satisfy the conditions

$$(2.16)\qquad c_\nu,\ c_\tau\in L^\infty(\Gamma_3),\ \varepsilon_a\in L^2(\Gamma_3),\ \ c_\nu,\ c_\tau,\ \varepsilon_a\geq 0\ a.e.\text{ on }\Gamma_3.$$

We also consider the functional $h:C([0,T];V)\to C\left([0,T];L^2(\Gamma_3)\right)$ defined by

$$hv(t)=\int_0^t\|v_\tau(s)|_{\Gamma_3}\|\,ds\qquad\forall v\in C([0,T];V),\ t\in[0,T].$$

We assume that the initial data satisfies

$$(2.17)\qquad \beta_0\in L^2(\Gamma_3):0\leq\beta_0\leq 1,\ \ a.e.\text{ on }\Gamma_3.$$

Next, we define the functional $j_{ad}:L^2(\Gamma_3)\times V\times V\to\mathbb{R}$ by

$$j_{ad}(\beta,u,v)=\int_{\Gamma_3}(-c_\nu\beta^2R_\nu(u_\nu)\,v_\nu+c_\tau\beta^2R_\tau(u_\tau).v_\tau)da.$$

Finally, we need the following set for the bonding field:

$$\mathcal{B}=\left\{\theta:[0,T]\to L^2(\Gamma_3):0\leq\theta(t)\leq 1\ \forall t\in[0,T],\ \ a.e.\text{ on }\Gamma_3\right\}.$$

Now, a straightforward application of the Green formula yields the following variational formulation of Problem P_1, in terms of displacement and bonding fields.

Problem P_2. Find a displacement field $u\in C([0,T];V)$ and a bonding field $\beta\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{B}$ such that,

$$(2.18)\qquad \begin{aligned}&\langle F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t))\rangle_Q+j_{fr}(hu(t),v)-j_{fr}(hu(t),u(t))\\ &+j_{ad}(\beta(t),u(t),v-u(t))\geq(f(t),v-u(t))_V\quad\forall v\in V,\ t\in[0,T],\end{aligned}$$

$$(2.19)\quad \dot\beta(t)=-[\beta(t)(c_\nu(R_\nu(u_\nu(t)))^2+c_\tau\|R_\tau(u_\tau(t))\|^2)-\varepsilon_a]_+,\ \ a.e.\ t\in(0,T),$$

$$(2.20)\qquad \beta(0)=\beta_0.$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.1. *Let* $T > 0$ *and assume that* (2.11) , (2.13)*,* (2.14)*,* (2.15)*,* (2.16) *and* (2.17) *hold. Then there exists a unique solution to Problem* P_2*.*

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Presented by Ilona Zasada at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

OPIS ADHEZYJNEGO KONTAKTU CIAŁA ELASTYCZNEGO Z PODŁOŻEM W PRZYPADKU PROSTOPADŁEGO NACISKU I TARCIA ZALEŻNEGO OD POWIERZCHNI ŚLIZGU I

POSTAWIENIE PROBLEMU I SFORMUŁOWANIE WARIACYJNE

S t r e s z c z e n i e

W pracy został przedstawiony teoretyczny model opisujący tarcie pomiędzy nieliniowym ciałem elastycznym a podłożem. Rozważania dotyczą przypadku prostopadłego nacisku na powierzchnie styku i biorą pod uwagę zależność współczynnika tarcia od powierzchni ślizgu oraz adhezji powierzchni styku. W pierwszej części pracy ewolucja pola styku jest opisana przez równanie różniczkowe pierwszego rzędu. Autorzy uzyskują wariacyjne sformułowanie problemu wariacyjnego.

Słowa kluczowe: elastyczny, adhezja, tarcie zależne od całkowitego styku, punkt stały, słabe rozwiązanie

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 83–90

In memory of
Professor Claude Surry

Arezki Touzaline

ADHESIVE CONTACT OF AN ELASTIC BODY WITH PRESCRIBED NORMAL STRESS AND TOTAL SLIP-DEPENDENT FRICTION II

EXISTENCE AND UNIQUENESS OF SOLUTION

Summary

We consider a mathematical model which describes the frictional contact between a nonlinear elastic body and a foundation. The normal stress on the contact surface is prescribed, the coefficient of friction depends on the total slip and adhesion of contact surfaces is taken into account. In connection with a variational formulation of the mechanical problem we prove the existence, uniqueness and regularity result. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

Keywords and phrases: elastic, adhesion, total slip-dependent friction, fixed point, weak solution

3. Existence and uniqueness of solution

Let $\mathcal{Z}$ be the closed set defined as

$$\mathcal{Z} = \left\{\theta \in C\left([0,T]\,; L^2(\Gamma_3)\right) \cap \mathcal{B};\ \theta(0) = \beta_0\right\},$$

where the Banach space $C\left([0,T]\,; L^2(\Gamma_3)\right)$ is endowed with the norm

$$\|\theta\|_k = \sup_{t\in[0,T]} \left[\exp(-kt)\, \|\theta(t)\|_{L^2(\Gamma_3)}\right]$$

$$\text{for all} \quad \theta \in C\left([0,T]\,; L^2(\Gamma_3)\right),\ k > 0.$$

The goal of this section is the proof of Theorem 2.1 which will be carried out in several steps. In the first step, for a given $\beta \in \mathcal{Z}$, we consider the following variational problem.

Problem P_β. Find $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &\langle A\varepsilon\left(u_\beta\left(t\right)\right), \varepsilon\left(v\right) - \varepsilon(u_\beta\left(t\right))\rangle_Q + j_{fr}\left(hu_\beta\left(t\right), v\right) - j_{fr}\left(hu_\beta\left(t\right), u_\beta\left(t\right)\right) \\ &+j_{ad}\left(\beta\left(t\right), u_\beta\left(t\right), v - u_\beta\left(t\right)\right) \geq \left(f\left(t\right), v - u_\beta\left(t\right)\left(t\right)\right)_V \\ &\forall v \in V,\ t \in [0,T]. \end{aligned} \tag{3.21}$$

We show the following result.

Proposition 3.1. *Problem* P_β *has a unique solution and it satisfies*

$$u_\beta \in C\left([0,T]; V\right).$$

The proof of this lemma is based on fixed point arguments and will be established in several steps. In the first step let $\eta \in C\left([0,T]; V\right)$ and denote by $y_\eta \in C\left([0,T]; L^2\left(\Gamma_3\right)\right)$ the function

$$y_\eta\left(t\right) = h\eta\left(t\right) \quad \forall t \in [0,T]. \tag{3.22}$$

Consider now the following auxiliary problem.

Problem $P_{\beta\eta}$. Find $u_{\beta\eta} : [0,T] \to V$ such that

$$\begin{aligned} &\langle A\varepsilon\left(u_{\beta\eta}\left(t\right)\right), \varepsilon\left(v\right) - \varepsilon(u_{\beta\eta}\left(t\right))\rangle_Q + j_{fr}\left(y_\eta\left(t\right), v\right) - j_{fr}\left(y_\eta\left(t\right), u_{\beta\eta}\left(t\right)\right) \\ &+j_{ad}\left(\beta\left(t\right), u_{\beta\eta}\left(t\right), v - u_{\beta\eta}\left(t\right)\right) \geq \left(f\left(t\right), v - u_{\beta\eta}\left(t\right)\left(t\right)\right)_V \\ &\forall v \in V,\ t \in [0,T]. \end{aligned} \tag{3.23}$$

We have the following result.

Lemma 3.2. *There exists a unique solution* $u_{\beta\eta} \in C\left([0,T]; V\right)$ *to the problem* $P_{\beta\eta}$.

Proof. For each $t \in [0,T]$, we consider the nonlinear operator $A_t : V \to V$ defined by

$$\left(A_t v, w\right)_V = \langle F\varepsilon\left(v\right), \varepsilon(w)\rangle_Q + j_{ad}\left(\beta\left(t\right), v, w\right) \quad \forall v, w \in V.$$

Using (2.15) and the properties of the operators R_ν and R_τ (see [17]), we see that the operator A_t is Lipschitz continuous and strongly monotone. Then it follows from classical results for elliptic varaitional inequalities (see [2]) that there exists a unique element $u_\beta\left(t\right)$ that solves (3.23). Let us show that $u_{\beta\eta} \in C\left([0,T]; V\right)$. Indeed, for t_1, $t_2 \in [0,T]$, take $t = t_1$ and $v = u_{\beta\eta}\left(t_2\right)$ in inequality (3.3); then $t = t_2$ and $v = u_{\beta\eta}\left(t_1\right)$, by adding the resulting inequalities we obtain

$$
\begin{aligned}
&\langle F\varepsilon\left(u_{\beta\eta}\left(t_1\right)\right)-F\varepsilon\left(u_{\beta\eta}\left(t_2\right)\right), \varepsilon\left(u_{\beta\eta}\left(t_1\right)\right)-\varepsilon(u_{\beta\eta}\left(t_2\right))\rangle_Q \\
&\leq \left(f\left(t_1\right)-f\left(t_2\right), u_{\beta\eta}\left(t_1\right)-u_{\beta\eta}\left(t_2\right)\right)_V \\
&+j_{fr}\left(y_\eta\left(t_1\right), u_{\beta\eta}\left(t_2\right)\right)-j_{fr}\left(y_\eta\left(t_1\right), u_{\beta\eta}\left(t_1\right)\right)+j_{fr}\left(y_\eta\left(t_2\right), u_{\beta\eta}\left(t_1\right)\right) \\
&-j_{fr}\left(y_\eta\left(t_2\right), u_{\beta\eta}\left(t_2\right)\right)+j_{ad}\left(\beta\left(t_1\right), u_{\beta\eta}\left(t_1\right), u_{\beta\eta}\left(t_2\right)-u_{\beta\eta}\left(t_1\right)\right) \\
&+j_{ad}\left(\beta\left(t_2\right), u_{\beta\eta}\left(t_2\right), u_{\beta\eta}\left(t_1\right)-u_{\beta\eta}\left(t_2\right)\right)
\end{aligned} \tag{3.24}
$$

Using now the properties of the operators R_ν and R_τ (see [16]), (2.13) (b), (2.14), (2.15) (c) and (2.10), we obtain the following estimate

$$
\begin{aligned}
&m_F \left\|u_{\beta\eta}\left(t_1\right)-u_{\beta\eta}\left(t_2\right)\right\|_V \\
&\leq L d_\Omega \left(\left\|c_\nu\right\|_{L^\infty(\Gamma_3)}+\left\|c_\tau\right\|_{L^\infty(\Gamma_3)}\right)\left\|\beta\left(t_1\right)-\beta\left(t_2\right)\right\|_{L^2(\Gamma_3)} \\
&+d_\Omega L_\mu \|S\|_{L^\infty(\Gamma_3)}\left\|y_\eta\left(t_1\right)-y_\eta\left(t_2\right)\right\|_{L^2(\Gamma_3)}+\left\|f\left(t_1\right)-f\left(t_2\right)\right\|_V .
\end{aligned} \tag{3.25}
$$

Then, we deduce from (3.25) that $t \rightarrow u_{\beta\eta}(t) : [0, T] \rightarrow V$ is a continuous function which concludes the proof.

In the sequel we use Lemma 3.2 to define the operator $\Lambda_\eta : C([0, T]; V) \rightarrow C([0, T]; V)$ by equality

$$
\Lambda\eta = u_{\beta\eta} \quad \forall \eta \in C([0, T]; V). \tag{3.26}
$$

We have the following result.

Lemma 3.3. *The operator Λ has a unique fixed point $\eta^* \in C([0, T]; V)$.*

Proof. Let η_1, $\eta_2 \in C([0, T]; V)$, we have

$$
\left\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\right\|_V=\left\|u_{\beta\eta_1}(t)-u_{\beta\eta_2}(t)\right\|_V .
$$

Furthermore, by using an argument similar to that in the proof of (3.25), we obtain

$$
m_F\left\|u_{\beta\eta_1}(t)-u_{\beta\eta_2}(t)\right\|_V \leq d_\Omega L_\mu \|S\|_{L^\infty(\Gamma_3)}\left\|y_{\eta_1}(t)-y_{\eta_2}(t)\right\|_{L^2(\Gamma_3)} . \tag{3.27}
$$

On the other hand we have

$$
\left\|y_{\eta_1}(t)-y_{\eta_2}(t)\right\|_{L^2(\Gamma_3)} \leq c \int_0^t\left\|\eta_1(s)-\eta_2(s)\right\|_V ds, \tag{3.28}
$$

We combine now (3.27) and (3.28) to deduce the following inequality

$$
\left\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\right\|_V \leq \frac{c d_\Omega L_\mu \|S\|_{L^\infty(\Gamma_3)}}{m_F} \int_0^t\left\|\eta_1(s)-\eta_2(s)\right\|_V ds. \tag{3.29}
$$

Then it follows from (3.29) (see [25]) that the operator Λ has a unique fixed point η^*.

We have now all the ingredients to prove Proposition 3.1.

Proof. Existence. Let $\eta^* \in C([0,T];V)$ be the fixed point of the operator Λ. It follows from (3.22) and (3.26) that we have

$$y_{\eta^*}(t) = h\eta^*(t), \; u_{\beta\eta^*}(t) = \eta^*(t) \quad \forall t \in [0;T]. \tag{3.30}$$

We now take $\eta = \eta^*$ in inequality (3.3) and keeping in mind the equalities (3.11), we conclude that the function $\eta^* \in C([0,T];V)$ is a solution to the inequality (3.1).

Uniqueness. Let $\eta \in C([0,T];V)$ be a different solution of the inequality (3.21). Then by (3.22), it follows that η is a solution to the variational inequality (3.23) which has a unique solution, denoted $u_{\beta\eta}$. Thus, we conclude hat $\eta = u_{\beta\eta}$. This equality implies that $\Lambda\eta = \eta$ and by Lemma 3.3 we deduce that $\eta = \eta^*$.

Next, we consider the following problem.

Problem 3. Find $\beta^* : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}^*(t) = -[\beta^*(t)(c_\nu(R_\nu(u_{\beta^*\nu}(t)))^2 + c_\tau \|R_\tau(u_{\beta^*\tau}(t))\|^2) - \varepsilon_a]_+ \quad a.e.\ t \in (0,T), \tag{3.31}$$

$$\beta^*(0) = \beta_0. \tag{3.32}$$

Proposition 3.4. *There exists a unique solution to Problem 3 and it satisfies*

$$\beta^* \in W^{1,\infty}\left(0,T; L^2(\Gamma_3)\right) \cap \mathcal{B}.$$

Proof. Consider the mapping $T : \mathcal{Z} \to \mathcal{Z}$ defined by

$$T\beta(t) = \beta_0 - \int_0^t [\beta(s)(c_\nu(R_\nu(u_{\beta\nu}(s)))^2 + c_\tau \|R_\tau(u_{\beta\tau}(s))\|^2) - \varepsilon_a]_+ ds \quad \forall t \in [0,T],$$

where u_β is a solution of Problem P_β. Let $\beta_1, \beta_2 \in \mathcal{Z}$, then as in [25], there exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \\ &\le c_1 \textstyle\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} + c_1 d_\Omega \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V \, ds. \end{aligned} \tag{3.33}$$

Now we need to show the following lemma.

Lemma 3.5. *There exists a constant $c > 0$ such that*

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \le c \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \quad \forall t \in [0,T].$$

Proof. Let $t \in [0,T]$. Take $u_{\beta_2}(t)$ in the inequality (3.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$, we find after adding the two inequalities that

$$\left\langle F\varepsilon\left(u_{\beta_1}(t)\right)-F\varepsilon\left(u_{\beta_2}(t)\right),\varepsilon\left(u_{\beta_1}(t)\right)-\varepsilon\left(u_{\beta 2}(t)\right)\right\rangle_Q$$

$$\leq j_{ad}\left(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\right)+j_{ad}\left(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)$$

$$+j_{fr}\left(u_{\beta_1}(t),u_{\beta_2}(t)\right)-j_{fr}\left(u_{\beta_1}(t),u_{\beta_1}(t)\right)$$

$$+j_{fr}\left(u_{\beta_2}(t),u_{\beta_1}(t)\right)-j_{fr}\left(u_{\beta_2}(t),u_{\beta_2}(t)\right)$$

Using $(2.13)\,(b)$ this inequality implies

$$\begin{aligned}
&m_F\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2\leq\\
&j_{ad}\left(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\right)\\
&+j_{ad}\left(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)\\
&+j_{fr}\left(u_{\beta_1}(t),u_{\beta_2}(t)\right)-j_{fr}\left(u_{\beta_1}(t),u_{\beta_1}(t)\right)\\
&+j_{fr}\left(u_{\beta_2}(t),u_{\beta_1}(t)\right)-j_{fr}\left(u_{\beta_2}(t),u_{\beta_2}(t)\right).
\end{aligned}\tag{3.34}$$

Using the properties of R_ν and R_τ (see [17]), as in [25] we have

$$\begin{aligned}
&j_{ad}\left(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\right)\\
&+j_{ad}\left(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\right)\\
&\leq Ld_\Omega\left(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)}\right)\\
&\times\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V.
\end{aligned}\tag{3.35}$$

On the other hand using (2.17), $(2.18)\,(b)$ and $(2.18)\,(c)$ yields

$$\begin{aligned}
&j_{fr}\left(u_{\beta_1}(t),u_{\beta_2}(t)\right)-j_{fr}\left(u_{\beta_1}(t),u_{\beta_1}(t)\right)+j_{fr}\left(u_{\beta_2}(t),u_{\beta_1}(t)\right)\\
&-j_{fr}\left(u_{\beta_2}(t),u_{\beta_2}(t)\right)\leq d_\Omega^2L_\mu\|S\|_{L^\infty(\Gamma_3)}\\
&\times\left\|u_{\beta_2}(t)-u_{\beta_1}(t)\right\|_V\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}.
\end{aligned}\tag{3.36}$$

We now combine inequalities (3.34), (3.35) and (3.36) to deduce that for all $t\in[0,T]$,

$$\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V\leq c\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)},\tag{3.37}$$

where

$$c=\left[d_\Omega^2L_\mu\|S\|_{L^\infty(\Gamma_3)}+Ld_\Omega\left(\|c_\nu\|_{L^\infty(\Gamma_3)}+\|c_\tau\|_{L^\infty(\Gamma_3)}\right)\right]/m_F.$$

Now to end the proof of Lemma 3.4 we use (3.33) and (3.37) to obtain

$$\left\|T\beta_1(t)-T\beta_2(t)\right\|_{L^2(\Gamma_3)}\leq c_2\int_0^t\left\|\beta_1(s)-\beta_2(s)\right\|_{L^2(\Gamma_3)}ds\quad\forall t\in[0,T],\tag{3.38}$$

where $c_2 > 0$. As in [25], the inequality (3.38) implies

$$\|T\beta_1 - T\beta_2\|_k \leq \frac{c_2}{k} \|\beta_1 - \beta_2\|_k . \tag{3.39}$$

Thus for $k > c_2$, it follows from (3.29) that T is a contraction, then it admits a unique fixed point $\beta^* \in \mathcal{Z}$ which satisfies (3.31) and (3.32).

Now, the proof of Theorem 2.1 is a consequence of the previous lemmas and propositions.

Proof. Existence. Let $\beta = \beta^*$ and let u_{β^*} the solution of Problem P_β. We conclude by (3.21), (3.31) and (3.32) that (u_{β^*}, β^*) is a solution to Problem P_2.

Uniqueness. Suppose that (u, β) is a solution of Problem P_2. It follows from (2.18) that u is a solution to Problem P_β, and from Proposition 3.1 that $u = u_\beta$. Take $u = u_\beta$ in (2.19) and use the initial condition (2.20), we deduce that β is a solution to Problem P_3. Therefore, we obtain from Proposition 3.4 that $\beta = \beta^*$ and then we conclude that (u_{β^*}, β^*) is a unique solution to Problem P_2.

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Presented by Ilona Zasada at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

OPIS ADHEZYJNEGO KONTAKTU CIAŁA ELASTYCZNEGO Z PODŁOŻEM W PRZYPADKU PROSTOPADŁEGO NACISKU I TARCIA ZALEŻNEGO OD POWIERZCHNI ŚLIZGU II

ISTNIENIE I JEDNOZNACZNOŚĆ ROZWIĄZAŃ

S t r e s z c z e n i e

W pracy został przedstawiony teoretyczny model opisujący tarcie pomiędzy nieliniowym ciałem elastycznym a podłożem. Rozważania dotyczą przypadku prostopadłego nacisku na powierzchnie styku i biorą pod uwagę zależność współczynnika tarcia od powierzchni ślizgu oraz adhezji powierzchni styku. W związku z wariacyjnym sformułowaniem problemu autorzy dowodzą, że wariacyjne sformułowanie problemu mechanicznego prowadzi do jednoznacznych i prawidłowych wyników. Dowód oparty jest na argumentach wynikających z rozważania niezależnych od czasu nierówności wariacyjnych równań różniczkowych oraz twierdzenia Banacha o punkcie stałym.

Słowa kluczowe: elastyczny, adhezja, tarcie zależne od całkowitego styku, punkt stały, słabe rozwiązanie

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 91–98

In memory of
Professor Claude Surry

Arezki Touzaline and Rachid Guettaf

ANALYSIS AND APPROXIMATION OF A UNILATERAL CONTACT PROBLEM WITH ADHESION I

PROBLEM STATEMENT AND VARIATIONAL FORMULATION

Summary

The paper deals with the study of a quasistatic frictionless contact problem between a nonlinear elastic body and a foundation. The contact is modelled with a normal compliance condition associated to Signorini's unilateral constraint. The adhesion between contact surfaces is taken into account and is modelled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. In the first part of the paper we establish a variational formulation of the mechanical problem.

Keywords and phrases: elastic, normal compliance, adhesion, frictionless, unilateral, weak solution

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled with highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [8]. The mathematical, mechanical and numerical state of the art can be found in [20] where we find mathematical and numerical studies of the adhesive contact problems. The analysis and the approximation by finite element methods of the unilateral contact models take an important place (see [13, 16]). The numerical studies of the Sigonrini contact problem were made in [1–3, 13, 16]. In [16] we find a detailed analysis and numerical studies of elastic unilateral contact problems.

In this paper, we study a mathematical model which describes a frictionless quasistatic contact problem with adhesion between a nonlinear elastic body and a deformable foundation. Following [14, 15] the contact is modelled with a normal compliance condition associated to unilateral constraint with finite penetration. For instance recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [4–7, 9, 12, 15, 17–23].

Here as in [10, 11] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds, when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place.

This work is a continuation and an extension of the results established in [21, 23], where the frictionless and unilateral contact problems for elastic materials were studied. We establish a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution and obtain a partial regularity result for the solution. We also study the numerical approximation of the problem.

The paper is structured as follows. In section 2 we present some notations and give the variational formulation. In section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. Finally, in section 4 a fully discrete scheme is introduced , based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. A main error estimates result is proved, Theorem 4.1.

2. Problem statement and variational formulation

We consider a nonlinear elastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and assume that its boundary Γ is regular and partitioned into three measurable and disjoint parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is clamped on Γ_1 and thus the displacement field vanishes here. A volume force of density φ_1 acts in Ω and a surface traction of density φ_2 acts on Γ_2. In the reference configuration the body is in adhesion frictionless contact on Γ_3 with a deformable foundation. The contact is modelled with a normal compliance in such a way that the penetration is limited and the evolution of the bonding field is given by a differential equation of the first order. Inder these conditions, the classical formulation of the mechanical problem is the following.

Problem P_1. Find a displacement $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$ such that

$$div\sigma + \varphi_1 = 0 \text{ in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \leq g,\ \sigma_\nu + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \leq 0 \\ \\ \big(\sigma_\nu + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu)\big)(u_\nu - g) = 0 \end{array}\right\} \text{ on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\sigma_\tau = 0 \text{ on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\dot{\beta} = -\left(c_\nu \beta (R_\nu(u_\nu))^2 - \varepsilon_a\right)_+ \text{ on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.8}$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The condition (2.5) represents the unilateral contact with adhesion in which p and $-c_\nu \beta^2 R_\nu(u_\nu)$ are the normal contact functions where c_ν is a given adhesion coefficient and R_ν is a truncation operator given by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L \\ -s & \text{if } -L \leq s \leq 0 \\ 0 & \text{if } s > 0 \end{cases}.$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [20]) and p is a normal compliance function which satisfies the assumption beow (2.14). The condition (2.6) represents the frictionless contact in which the tangential traction vanishes.We denote by the positive constant g the maximum value of the penetration. When $u_\nu < 0$ i.e. when there is separation between the body and the foundation then the condition (2.5) combined with hypotheses (2.14) on the function p shows that $\sigma_\nu = -p_\nu(u_\nu, \beta)$ and by assumption (2.14) below, it does not exeed the value $L_\nu(1 + g)$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_\nu \leq g$. In this case of penetration (i.e. $u_\nu \geq 0$), when $0 \leq u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \leq 0$. Since p is an increasing function then the reaction of the foundation is increasing with the penetration and when $u_\nu = g$, then $-\sigma_\nu \geq p(g)$

and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, condition (2.5) becomes the Signorini contact condition with adhesion with a gap function,

$$u_\nu \leq g,\ \sigma_\nu \leq 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, the condition (2.5) combined with hypothese (2.14) becomes the Signorini contact condition with adhesion with a zero gap function, given by

$$u_\nu \leq 0,\ \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \leq 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))u_\nu = 0.$$

This contact condition was used in [20, 21, 23]. The condition (2.6) represents the frictionless contact in which the normal constraint vanishes. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field.where ε_a is an adhesion coefficient and $\beta_+ = \max(0, \beta)$. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [17] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ and $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d,\ H_1 = \left(H^1(\Omega)\right)^d,\ Q = \left\{\tau = (\tau_{ij});\tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\tau \in Q;\ div\tau \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq g \ a.e \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \qquad \varphi_2 \in W^{1,\infty}\left(0, T; (L^2(\Gamma_2))^d\right) \tag{2.11}$$

and denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t) .vdx + \int_{\Gamma_2} \varphi_2(t) .vda \quad \forall v \in V,\ t \in [0, T]. \tag{2.12}$$

Using (2.11) and (2.12) yield

$$f \in W^{1,\infty}(0, T; V).$$

In the study of the mechanical problem P_1 we assume that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ satisfies:

$$
(2.13)\qquad \left.\begin{array}{l}
(a) \text{ There exists } M>0 \text{ such that}\\
\quad |F(x,\varepsilon_1)-F(x,\varepsilon_2)| \le M|\varepsilon_1-\varepsilon_2| \quad \forall\, \varepsilon_1,\varepsilon_2 \in S_d,\\
\quad a.e.\ x\in\Omega;\\
(b) \text{ there exists } m>0 \text{ such that}\\
\quad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m|\varepsilon_1-\varepsilon_2|^2,\\
\quad\quad \forall\varepsilon_1,\varepsilon_2\in S_d,\ a.e.\ x\in\Omega;\\
(c) \text{ the mapping } x\to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\\
\text{ for any } \varepsilon\in S_d;\\
(d)\ F(x,0)=0 \text{ for } a.e.\ x\in\Omega.
\end{array}\right\}
$$

Next we define the functional $j : L^2(\Gamma_3)\times V\times V\to\mathbb{R}$ by

$$
j(\beta,u,v)=\int_{\Gamma_3}(p(u_\nu)-c_\nu\beta^2R_\nu(u_\nu))v_\nu da,
$$

$$
\forall(\beta,u,v)\in L^2(\Gamma_3)\times V\times V,
$$

where we assume that the normal compliance function $p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies:

$$
(2.14)\qquad \left\{\begin{array}{l}
(a)\ \text{ There exists } L_p>0 \text{ such that}\\
\quad |p(x,r_1)-p(x,r_2)|\le L_p|r_1-r_2|\\
\quad \forall r_1,\ r_2\in\mathbb{R},\ a.e.\ x\in\Gamma_3;\\
(b)\ \ (p(x,r_1)-p(x,r_2))(r_1-r_2)\ge 0\\
\quad\quad \forall r_1,\ r_2\in\mathbb{R},\ a.e.\ x\in\Gamma_3;\\
(c)\ \text{ the mapping } x\to p_\nu(x,r) \text{ is measurable on } \Gamma_3, \text{ for any } r\in\mathbb{R};\\
(d)\ \ p(x,r)=0\ \forall r\le 0,\ a.e.\ x\in\Gamma_3.
\end{array}\right.
$$

Finally we assume that the initial bonding field satisfies:

$$
(2.15)\qquad \beta_0\in L^2(\Gamma_3);\ 0\le\beta_0\le 1\ a.e. \text{ on } \Gamma_3
$$

and we need to introduce the set of the bonding field:

$$
B=\left\{\theta:[0,T]\to L^2(\Gamma_3);0\le\theta(t)\le 1,\ \forall t\in[0,T],\ a.e. \text{ on } \Gamma_3\right\}.
$$

Now, assuming the solution to be sufficiently regular and applying Green's formula, we deduce the following variational formulation of the mechanical problem P_1.

Problem P_2. Find a displacement field $u:[0,T]\to V$ and a bonding field $\beta:[0,T]\to L^2(\Gamma_3)$ such that

$$
(2.16)\qquad \begin{array}{l}
u(t)\in K,\ (F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t)))_Q+j(\beta(t),u(t),v-u(t))\\[2ex]
\ge(f(t),v-u(t))_V\quad\forall\, v\in K,\ t\in[0,T],
\end{array}
$$

$$
(2.17)\qquad \dot\beta(t)=-\left(c_\nu\beta(t)(R_\nu(u_\nu(t)))^2-\varepsilon_a\right)_+\quad a.e.\ t\in(0,T),
$$

$$
(2.18)\qquad \beta(0)=\beta_0.
$$

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Presented by Marek Moneta at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

ANALIZA I APROKSYMACJA W JEDNOSTRONNYM PROBLEMIE PRZYLEGANIA Z ADHEZJĄ I

POSTAWIENIE PROBLEMU I SFORMUŁOWANIE WARIACYJNE

S t r e s z c z e n i e

Artykuł poświęcony jest problemowi kwazi-statycznego, beztarciowego kontaktu pomiędzy nieliniowo elastycznym ciałem a podłożem. Kontakt jest modelowany za pomocą normalnych warunków odkształcenia związanych z jednostronnymi więzami Signoriniego. Przyleganie pomiędzy stykającymi się powierzchniami zostało uwzględnione i wymodelowane za pomocą pola powierzchniowych oddziaływań, którego zmiana opisywana jest równaniami różniczkowymi 1. stopnia. W pierwszej części pracy przedstawiamy sformułowanie wariacyjne odpowiedniego problemu mechanicznego.

Słowa kluczowe: elastyczny, normalne warunki odkształcenia, adhezja, pozbawiony tarcia, jednostronny, słabe rozwiązanie

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 99–108

In memory of
Professor Claude Surry

Arezki Touzaline and Rachid Guettaf

ANALYSIS AND APPROXIMATION OF A UNILATERAL CONTACT PROBLEM WITH ADHESION II
EXISTENCE, UNIQUENESS RESULT, AND NUMERICAL APPROACH

Summary

The paper deals with the study of a quasistatic frictionless contact problem between a nonlinear elastic body and a foundation. The contact is modelled with a normal compliance condition associated to Signorini's unilateral constraint. The adhesion between contact surfaces is taken into account and is modelled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. In the second part of the paper we problem and prove an existence and uniqueness result.The technique of the proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed-point theorem. We also introduce the fully discrete scheme based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. We derive error estimates on the approximate solutions under suitable regularity conditions.

Keywords and phrases: elastic, normal compliance, adhesion, frictionless, unilateral, weak solution

3. Existence and uniqueness result

Our main result which will be established in this section is the following theorem.

Theorem 3.1. *Let* (2.11), (2.13)–(2.17) *and* (218) *hold. Then Problem P_2 has a unique solution which satisfies*

$$u \in W^{1,\infty}(0,T;V) \cap C([0,T];K), \tag{3.1}$$

$$\beta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap B. \tag{3.2}$$

The proof of Theorem 3.1 is carried out in several steps. In the first step, let $k > 0$ and consider the closed subset X of $C\left([0,T];L^2(\Gamma_3)\right)$ defined as

$$X = \left\{\theta \in C\left([0,T];L^2(\Gamma_3)\right) \cap B,\ \theta(0) = \beta_0\right\},$$

where the Banach space $C\left([0,T];L^2(\Gamma_3)\right)$ is endowed with the norm

$$\|\theta\|_X = \max_{t\in[0,T]}\left[\exp(-kt)\,\|\theta(t)\|_{L^2(\Gamma_3)}\right] \text{ for all } \theta \in C\left([0,T];L^2(\Gamma_3)\right).$$

Next for a given $\beta \in X$, we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &u_\beta(t) \in K,\ \ (F\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + j(\beta(t), u_\beta(t), v - u_\beta(t)) \\ &\geq (f(t), v - u_\beta(t))_V \quad \forall v \in K,\ t \in [0,T]. \end{aligned} \tag{3.3}$$

We have the following result.

Lemma 3.2. *Problem* $P_{1\beta}$ *has a unique solution*

$$u_\beta \in C([0,T];K). \tag{3.4}$$

Proof. Let the operator $A_{\beta(t)} : V \to V$ defined by

$$\left(A_{\beta(t)}u, v\right)_V = (F\varepsilon(u), \varepsilon(v))_Q + j(\beta(t), u, v),\ \forall u, v \in V.$$

We use (2.11), (2.13), (2.14) and (2.15) to show that the operator $A_{\beta(t)}$ is strongly monotone and Lipschitz continuous; then by a standard existence and uniqueness result for elliptic quasivariational inequalities (see [4]), it follows that there exists a unique element $u_\beta(t) \in K$ which satisfies the inequality (3.3) since K is a non-empty, closed convex subset of V.

To see that $u_\beta \in C([0,T];K)$, it suffices to see after easy calculations (see [20]) that there exists a positive constant c such that

$$\begin{aligned} &\|u_\beta(t_1) - u_\beta(t_2)\|_V \leq \\ &\frac{c}{m}\left(\|f(t_1) - f(t_2)\|_V + \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}\right) \quad \forall t_1, t_2 \in [0,T]. \end{aligned} \tag{3.5}$$

Therefore, as $f \in C([0,T];V)$ and $\beta \in C\left([0,T];L^2(\Gamma_3)\right)$, we immediately conclude (3.4). $\square$

In the second step, we consider the following problem.

Problem $P_{2\beta}$. Find $\chi_\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\chi}_\beta(t) = -\left(c_\nu \chi_\beta(t) \left(R_\nu(u_{\beta\nu}(t))\right)^2 - \varepsilon_a\right)_+ \quad a.e.\ t \in (0,T), \tag{3.6}$$

$$\chi_\beta(0) = \beta_0. \tag{3.7}$$

We obtain the following result.

Lemma 3.3. *Problem $P_{2\beta}$ has a unique solution χ_β which satisfies*

$$\chi_\beta \in W^{1,\infty}\left(0,T; L^2(\Gamma_3)\right) \cap B.$$

Proof. Consider the mapping $F_\beta(t,\theta) : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\beta(t,\theta) = . - \left(c_\nu \theta(t) \left(R_\nu(u_{\beta\nu}(t))\right)^2 - \varepsilon_a\right)_+$$

It follows from the properties of the truncation operator R, that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \to F_\beta(t,\theta)$ belongs to $L^\infty\left(0,T; L^2(\Gamma_3)\right)$. Then, from a version of Cauchy-Lipschitz theorem, we deduce the existence of a unique fonction $\chi_\beta \in W^{1,\infty}\left(0,T; L^2(\Gamma_3)\right)$, which satisfies (3.6), (3.7). The regularity $\chi_\beta \in B$, follows from (3.6), (3.7) and (2.18), (see [20, 21]). Therefore, from Lemma 3.5, we deduce that for all $\beta \in X$, the solution χ_β of Problem $P_{2\beta}$ belongs to X. Next, we define the mapping $\Lambda : X \to X$ by

$$\Lambda\beta = \chi_\beta$$

The third step consists of the following lemma. □

Lemma 3.4. *The mapping Λ has a unique fixed point β^*.*

Proof. We have

$$\Lambda\beta(t) = \beta_0 - \int_0^t \left(c_\nu \left(\chi_\beta(s) \left(R_\nu(u_{\beta\nu}(s))\right)^2 - \varepsilon_a\right)_+ ds,$$

where u_β is the solution of Problem $P_{1\beta}$. Then for β_1, $\beta_2 \in X$, using (2.10) and the definition of R_ν, it follows that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} & \left\|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)\right\|_{L^2(\Gamma_3)} \\ \leq\ & c_1 \int_0^t \left(\left\|\chi_{\beta_1}(s) - \chi_{\beta_2}(s)\right\|_{L^2(\Gamma_3)} + \left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V\right) ds. \end{aligned}$$

Then using Gronwall's inequality we obtain

$$\left\|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)\right\|_{L^2(\Gamma_3)} \leq c_2 \int_0^t \left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V ds,$$

where $c_2 > 0$. Now let $t \in [0,T]$. Then, using the inequality (3.3), (2.13), (2.14) and (2.15), (see [21]) we deduce that there exists a constant $c_3 > 0$ such that

$$\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V \leq c_3 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)}.$$

Hence, it follows that

$$\|\Lambda\beta_1(t)-\Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \le c_4 \int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\, ds \quad \forall t\in[0,T],$$

where $c_4>0$. Therefore, we obtain

$$\|\Lambda\beta_1-\Lambda\beta_2\|_X \le \frac{c_4}{k}\|\beta_1-\beta_2\|_X, \ \forall\beta_1,\beta_2\in X.$$

Thus, this previous estimate shows that for $k>c_4$, Λ is a contraction. Then it has a unique fixed point β^* which satisfies (3.6) and (3.7). On the other hand from (3.5) we deduce that $u_{\beta^*}\in W^{1,\infty}(0,T;V)$. □

Proof of Theorem 3.1. Let $\beta=\beta^*$ and let u_{β^*} the solution to Problem $P_{1\beta}$. We conclude by (3.3), (3.6) and (3.7) that (u_{β^*},β^*) is a solution of Problem P_2. Now to prove the uniqueness of the solution, suppose that (u,β) is a solution of Problem P_2 which satisfies (2.16), (2.17) and (2.18). It follows from (2.16) that u is a solution of Problem $P_{1\beta}$ and by Lemma 3.2 we get $u=u_\beta$. Taking $u=u_\beta$ in (2.17) and using the initial condition (2.18), we deduce that β is a solution of Problem $P_{2\beta}$. Finally, using Lemma 3.5, we obtain $\beta=\beta^*$ and then (u_{β^*},β^*) is a unique solution to Problem P_2 which satisfies (3.1), (3.2).

4. Numerical approach

We now introduce a finite element method to approximate solutions of Prolem P_2 and derive an error estimate on them. We denote by $h>0$ the parameter of discretization which is done as follows. First, we consider a finite dimensional spaces $V^h\subset V$ and $B^h\subset L^2(\Gamma_3)$ associated with a partition $\mathcal{T}_h$, approximating respectively the spaces V and $L^2(\Gamma_3)$, where

$$V^h=\left\{v^h\in\left[C\left(\overline{\Omega}\right)\right]^d;\ v^h|T_r\in[P_1(T_r)]^d,\ T_r\in\mathcal{T}^h,\ v^h=0 \text{ on } \Gamma_1\right\},$$

$$B^h=\left\{\beta^h\in L^2(\Gamma_3);\ \beta^h|_\gamma\in\mathbb{R}\ \forall\gamma\in\mathcal{T}^h_{\Gamma_3}\right\},$$

and we recall (see [20], page 55) that $\mathcal{T}^h_{\Gamma_3}$ is a partition induced by the triangulation $\mathcal{T}^h$. Let also $\mathcal{P}_{B^h}: L^2(\Gamma_3)\to B^h$ be the orthogonal projection operator on B^h. Moreover, let K^h be the discrete convex set of admissibles displacements defined as $K^h=K\cap V^h$. Secondly, the time derivatives are discretized by using a uniform partition of $[0,T]$ denoted as

$$0=t_0<t_1<...<t_N=T.$$

Let k be the time step size, $k=T/N$, and for a continuous function $\phi(t)$ let $\phi_n=\phi(t_n)$. Finally, for a sequence $(w_n)_{n=0}^N$, we denote by $\delta w_n=(w_n-w_{n-1})/k$ the finite differences. In this section, no summation is assumed over a repeated index. Using the backward Euler scheme, the fully discrete approximation of Problem P_2 is the following.

Problem P_2^{hk}. Find a discrete displacement field

$$u^{hk} = \left(u_n^{hk}\right)_{n=1}^{N} \subset K^h$$

and a discrete bonding field

$$\beta^{hk} = \left(\beta_n^{hk}\right)_{n=1}^{N} \subset B^h$$

such that $\beta_0^{hk} = \beta_0^h$ and for all $n = 1, ..., N$

$$\text{(4.1)} \qquad \begin{cases} (F\varepsilon\left(u_n^{hk}\right), \varepsilon\left(v^h - u_n^{hk}\right))_Q + j\left(\beta_n^{hk}, u_n^{hk}, v^h - u_n^{hk}\right) \\ \geq \left(f_n, v^h - u_n^{hk}\right)_V \ \ \forall v^h \in K^h, \end{cases}$$

$$\text{(4.2)} \qquad \delta\beta_n^{hk} = -\mathcal{P}_{B^h}\left(c_\nu \beta_{n-1}^{hk}\left(R_\nu\left(u_{(n-1)\nu}^{hk}\right)\right)^2 - \varepsilon_a\right)_+.$$

and β_0^h is an appropriate approximation of the initial condition β_0. The same arguments used in the proof of Theorem 3.1 yield that Problem P_2^{hk} admits a unique solution. Our interest in this section lies in estimating the numerical errors $\left\|u_n - u_n^{hk}\right\|_V$ and $\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}$. We have the following main error estimates result.

Lemma 4.5. *Let the assumptions of Theorem* 3.1 *hold. Let* (u, β) *and* $\left(u_n^{hk}, \beta_n^{hk}\right)$ *denote the solution to Problem* P_2 *and Problem* P_2^{hk}*, respectively. Assume that* $\sigma = F\varepsilon(u)$*,* β *and* β_0 *satisfy the regularity*

$$\text{(4.3)} \qquad \sigma \in C\left([0,T]; \left(H^1(\Omega)\right)^{d\times d}\right) \cap W^{1,1}(0,T;Q),$$

$$\text{(4.4)} \qquad \beta \in W^{2,1}\left(0,T;L^2(\Gamma_3)\right) \cap C^1\left([0,T];H^1(\Gamma_3)\right),$$

$$\text{(4.5)} \qquad \beta_0 \in H^1(\Gamma_3).$$

Then, there exists a positive constant c *independent of discretization parameters* h *and* k *such that the following error estimate holds true for all* $v^h = \left(v_j^h\right)_{j=1}^{N} \subset K^h$:

$$\text{(4.6)} \qquad \begin{aligned} &\max_{1\leq n\leq N}\left\{\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}^2\right\} \\ &\leq c \max_{1\leq n\leq N}\inf_{v^h\in K^h}\left(\left\|u_n - v^h\right\|_V^2 + \left\|u_{n\nu} - v_\nu^h\right\|_V\right) + c\left(h^2 + k^2\right). \end{aligned}$$

Proof. First, let us obtain an error estimate on the displacement field. We rewrite variational inequality (2.16) at time $t = t_n$ for $v = u_n^{hk} \in K^h$ to obtain

$$\text{(4.7)} \qquad (F\varepsilon(u_n), \varepsilon\left(u_n - u_n^{hk}\right))_Q + j\left(\beta_n, u_n, u_n - u_n^{hk}\right) \leq \left(f_n, u_n - u_n^{hk}\right)_V.$$

Therefore, after some algebraic manipulations, we have

$$(4.8)\qquad \begin{cases} \left(F\varepsilon\left(u_n^{hk}\right),\varepsilon\left(u_n^{hk}-u_n\right)\right)_Q+j\left(\beta_n^{hk},u_n^{hk},u_n^{hk}-u_n\right) \\ \leq \left(F\varepsilon\left(u_n^{hk}\right),\varepsilon\left(v^h-u_n\right)\right)_Q+j\left(\beta_n^{hk},u_n^{hk},v^h-u_n\right) \\ +\left(f_n,u_n^{hk}-v^h\right)_V \quad \forall v_h\in K_h. \end{cases}$$

By adding the inequalities (4.7) and (4.8) we get

$$(4.9)\qquad \begin{cases} \left(F\varepsilon\left(u_n\right)-F\varepsilon\left(u_n^{hk}\right),\varepsilon\left(u_n-u_n^{hk}\right)\right)_Q+j\left(\beta_n,u_n,u_n-u_n^{hk}\right) \\ -j\left(\beta_n^{hk},u_n^{hk},u_n-u_n^{hk}\right) \\ \leq \left(F\varepsilon\left(u_n^{hk}\right),\varepsilon\left(v^h-u_n\right)\right)_Q+j\left(\beta_n^{hk},u_n^{hk},v^h-u_n\right) \\ -\left(f_n,v^h-u_n\right)_V \quad \forall v^h\in K^h. \end{cases}$$

Hence we deduce from (4.9) the following inequality

$$(4.10)\qquad \begin{cases} \left\langle F\varepsilon\left(u_n\right)-F\varepsilon\left(u_n^{hk}\right),\varepsilon\left(u_n-u_n^{hk}\right)\right\rangle_Q+j\left(\beta_n,u_n,u_n-u_n^{hk}\right) \\ -j\left(\beta_n^{hk},u_n^{hk},u_n-u_n^{hk}\right) \\ \leq \left(F\varepsilon\left(u_n^{hk}\right)-F\varepsilon\left(u_n\right),\varepsilon\left(v^h-u_n\right)\right)_Q+j\left(\beta_n^{hk},u_n^{hk},v^h-u_n\right) \\ -j\left(\beta_n,u_n,v^h-u_n\right)+L_n\left(v^h\right), \end{cases}$$

where

$$L_n\left(v^h\right)=\left(F\varepsilon\left(u_n\right),\varepsilon\left(v^h-u_n\right)\right)_Q+j\left(\beta_n,u_n,v^h-u_n\right)-\left(f_n,v^h-u_n\right)_V.$$

Next, denote

$$\sigma_n=F\varepsilon\left(u_n\right)$$

and using Green's formula we have

$$L_n\left(v^h\right)=\int_{\Gamma_3}\left(\sigma_{n\nu}-c_\nu\beta_n^2R_\nu\left(u_{n\nu}\right)+p\left(u_{n\nu}\right)\right)\left(v_\nu^h-u_{n\nu}\right)da.$$

So taking into account (4.3) which implies that $\sigma_\nu\in C\left([0,T];L^2\left(\Gamma_3\right)\right)$ and using (2.14) we deduce the following estimate

$$\left|L_n\left(v^h\right)\right|\leq c\left\|v_\nu^h-u_{n\nu}\right\|_{L^2(\Gamma_3)}.$$

Now we turn to the other estimates. Indeed, using again (2.14) we have

$$\begin{aligned} & j\left(\beta_n^{hk},u_n^{hk},u_n-u_n^{hk}\right)-j\left(\beta_n,u_n,u_n-u_n^{hk}\right) \\ & \leq c'\left(\left\|\beta_n-\beta_n^{hk}\right\|_{L^2(\Gamma_3)}+\left\|u_n-u_n^{hk}\right\|_V\right)\left\|u_n^{hk}-u_n\right\|_V \end{aligned}$$

and

$$j\left(\beta_n^{hk}, u_n^{hk}, v^h - u_n\right) - j\left(\beta_n, u_n, v^h - u_n\right)$$

$$\leq c^{'}(\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)} + \left\|u_n - u_n^{hk}\right\|_V) \left\|v^h - u_n\right\|_V .$$

Moreover using the inequality (4.10) and applying Young's inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \forall a, b \in \mathbb{R},\, \delta > 0,$$

after some caluclations, it follows that

$$\left\|u_n - u_n^{hk}\right\|_V^2 \leq$$

$$c\left(\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}^2 + \left\|u_n - v^h\right\|_V^2 + \left\|u_{n\nu} - v_\nu^h\right\|_{L^2(\Gamma_3)}\right) \ \forall v^h \in K^h.$$

Then we get the estimate

$$\begin{aligned} &\left\|u_n - u_n^{hk}\right\|_V^2 \leq \\ &c\left(\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}^2 + \left\|v^h - u_n\right\|_V^2 + \left\|u_{n\nu} - v_\nu^h\right\|_{L^2(\Gamma_3)}\right) \ \forall v^h \in K^h. \end{aligned} \tag{4.11}$$

Now, keeping in mind (3.43) [20, page 64], (4.4) and (4.5) with the estimate

$$\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} \leq ch,$$

where β_0^h is the orthogonal projection of β_0 on B^h, we have

$$\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}^2 \leq c\sum_{j=1}^{n} k(\left\|u_j - u_j^{hk}\right\|_V^2 + \left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Gamma_3)}^2) + c\left(h^2 + k^2\right). \tag{4.12}$$

Combining (4.11) and (4.12) yields

$$\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}^2$$

$$\leq c \left(\begin{array}{c} \left\|u_n - v^h\right\|_V^2 + \left\|u_{n\nu} - v_\nu^h\right\|_{L^2(\Gamma_3)} + h^2 + k^2 \\ \\ +\sum_{j=1}^{n} k(\left\|u_j - u_j^{hk}\right\|_V^2 + \left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Gamma_3)}^2) \end{array} \right)$$

Applying discrete Gronwall's inequality and the arbitrariness of $v^h \in K_h$ leads the the estimate (4.6). □

We now have the following result of error estimate.

Theorem 4.6. *Let the assumptions of Theorem* 3.1 *and Lemma* 4.1 *hold. Moreover under the regularity conditions*

$$u \in C\left([0,T]\,;\left(H^2\left(\Omega\right)\right)^d\right) \ \textit{and}\ u_\nu \in C\left([0,T]\,;H^2\left(\Gamma_3\right)\right),$$

there exists a constant $c > 0$ independent of the discretization parameters h and k such that

$$\max_{1\le n\le N}\left(\left\|u_n - u_n^{hk}\right\|_V + \left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)}\right) \le c\,(h+k)\,. \tag{4.13}$$

Proof. Recall that we have the following approximation properties of the finite element space V^h (see [5]),

$$\begin{aligned}
&\max_{1\le n\le N}\inf_{v^h\in V^h}\left\|u_n - v^h\right\|_V \le ch\,\|u\|_{C\left([0,T];\left(H^2(\Omega)\right)^d\right)},\\
&\max_{1\le n\le N}\inf_{v^h\in V^h}\left\|u_\nu - v^h_\nu\right\|_V \le ch\,\|u\|_{C([0,T];(H^2(\Gamma_3)))},
\end{aligned} \tag{4.14}$$

Combining (4.6) and (4.14) we obtain the error estimate (4.13). □

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Presented by Marek Moneta at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 12, 2014

ANALIZA I APROKSYMACJA W JEDNOSTRONNYM PROBLEMIE PRZYLEGANIA Z ADHEZJĄ II

ISTNIENIE, JEDNOZNACZNOŚĆ WYNIKU I PODEJŚCIE NUMERYCZNE

S t r e s z c z e n i e

Artykuł poświęcony jest problemowi kwazi-statycznego, beztarciowego kontaktu pomiędzy nieliniowo elastycznym ciałem a podłożem. Kontakt jest modelowany za pomocą normalnych warunków odkształcenia związanych z jednostronnymi więzami Signoriniego.

Przyleganie pomiędzy stykającymi się powierzchniami zostało uwzględnione i wymodelowane za pomocą pola powierzchniowych oddziaływań, którego zmiana opisywana jest równaniami różniczkowymi 1. stopnia. W drugiej części pracy dowodzimy istnienia i jednoznaczności rozwiązania. Dowód oparty jest na zależnych od czasu nierównościach wariacyjnych, równaniach różniczkowych i twierdzeniu Banacha o punkcie stałym. Wykorzystujemy także w pełni nieciągłe podejście oparte na metodzie elementów skończonych w celu przybliżenia zmiennych przestrzennych oraz odwrotne Eulera w celu określenia nieciągłych pochodnych czasowych. Oszacowaliśmy błąd przybliżonego rozwiązania przy odpowiednich warunkach regularności.

Słowa kluczowe: elastyczny, normalne warunki odkształcenia, adhezja, pozbawiony tarcia, jednostronny, słabe rozwiązanie.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 1

pp. 109–119

In memory of
Professor Claude Surry

Krzysztof Pomorski, Mariusz Zubert, and Przemysław Prokopow

NUMERICAL SOLUTIONS OF NEARLY TIME-INDEPENDENT GINZBURG-LANDAU EQUATION FOR VARIOUS SUPERCONDUCTING STRUCTURES
III. ANALYTICAL SOLUTIONS AND RELAXATION METHOD

Summary
In this work the concept of modified unconventional Josephson junction (muJJ) is introduced. It is the structure made from deformation of uJJ by placement of nonsuperconducting rectangular bar on the top of superconducting bar and by shift of nonsuperconducting bar deeper in the superconductor. Some analytic solutions are obtained for such structure with use of linearized Ginzburg-Landau equation. The concept of muJJ is introduced for rectangular, cylindrical and spherical geometry. Basing on obtained analytical results the development of relaxation method is proposed. It can be further generalized nearly time independent Ginzburg-Landau equation.

Keywords and phrases: Schrödinger equation, linear Ginzburg-Landau equation, GL equation, modified unconventional Josephon junction and device, relaxation method

1. Description of superconducting structures by linearized Ginzburg-Landau equation

We study various cases of system of superconductor with non-superconductor. In one dimension we have the following GL equation

$$0 = \alpha\psi + \beta|\psi|^2\psi - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) \tag{1}$$

with α and β parameters.

If superconducting order parameter is small such equation can become linear. We thus have

$$0 = \alpha\psi - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x). \tag{2}$$

In superconductor we have $\alpha < 0$, while in normal state we can assume $\alpha > 0$ or $\alpha = 0$.

Therefore we can recognize linearized GL equation in superconductor Schrödinger equation quite easily with positive eigenenergy values α

$$|\alpha|\psi(x) = -\alpha\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x). \tag{3}$$

Such solutions of Schrödinger equation correspond to free particle solutions of Schrödinger equation since $-\alpha > 0$ in the superconductor. The solutions are of the form

$$\psi(x) = A\sin(kx) + B\cos(kx) + Cx + D, \tag{4}$$

where A, B, C, D, k are constants.

The following relation is being fulfilled

$$\frac{\hbar^2}{2m}k^2 = |\alpha|.$$

On another hand in non-superconducting medium we have $\alpha > 0$. This makes linear GL equation to be of the form as:

$$-|\alpha|\psi(x) = -\alpha\psi(x) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) \tag{5}$$

and have solutions of Schrödinger equation corresponding to bounded-states where total energy of the system which is the sum of kinetic and potential term is smaller than zero. It has the following solution of the form

$$\psi(x) = Ae^{\lambda x} + Ce^{-\lambda x} + Dx + E, \tag{6}$$

where A, B, C, D, E, λ are constants. Here

$$\frac{\hbar^2}{2m}\lambda^2 = |\alpha|.$$

We should remember that linear equation with given boundary conditions have infinite number of solutions. Our aim is at first to find solutions for linear Ginzburg-Landau equation. Later we come back to non-linear version of GL equation. It guarantees that we have only one unique solution.

1.1. Boundary conditions for GL equation

If we deal with superconductor-nonsuperconductor system the canonical momentum with respect to interface needs to be zero since no electric current can escape the superconductor. We have

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x\right)\psi(x) = 0$$

at the superconductor-normal metal interface. At superconductor-normal metal interface the boundary conditions are

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \frac{2e}{c}A_x\right)\psi(x) = \frac{1}{b}\psi(x),$$

where b is constant depending on the type of material. For non-conducting metal b tends to infinity and for well conducting metal b is of medium value.

It is quite straightforward to generalize the given consideration to 2 or 3 dimensions. One should remember that superconductor is the source of Cooper pairs flowing to non-superconducting area.

2. Various types of solutions of GL equation

2.1. Case of Sc-N and Sc-N-Sc system

Let us assume that $\alpha > 0$ inside normal metal. Having Sc-N interface at $x = 0$ and for infinite superconductor inside normal strip we have have $\psi(x) = Ae^{-\lambda x}$. In such case

$$-\lambda = \frac{1}{b}.$$

Let us consider Sc and Sc-N-Sc system. We treat the system to be one dimensional. It is quite obvious to assign the symmetry of solutions. In N area we have $\psi(x) = A\cosh(\lambda) + A\sinh(\lambda)$. Symmetry imposes $\psi(x) = A\cosh(\lambda)$. From boundary conditions we have

$$\tanh(\lambda d) = \frac{\lambda}{b}.$$

2.2. Case of N-Sc-N system

Let us refer to situation depicted in Fig. 1. In such one dimensional system we obtain linear solution of GL equation given as

$$\psi(x, y) = A\cos(ky)$$

what implies

$$\psi(x, y) = A\cos\left(\left(\frac{2m|\alpha|}{\hbar}\right)^{\frac{1}{2}} y\right).$$

Additional boundary condition implies

$$\tan(kd) = \frac{b}{k}.$$

3. Simple considerations on analytic solution of GL equation for uJJ structures

Let us consider the structure as depicted in Fig. 2 described by numbers L_1,L_2, L_3 and L_4 and constants b_1, b_2, b_3, b_4, b_5. It mimics the uJJ structure, but it is

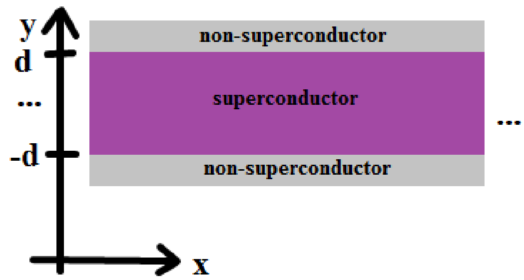

Fig. 1: Case of superconductor embedded from two sides by non-superconductor.

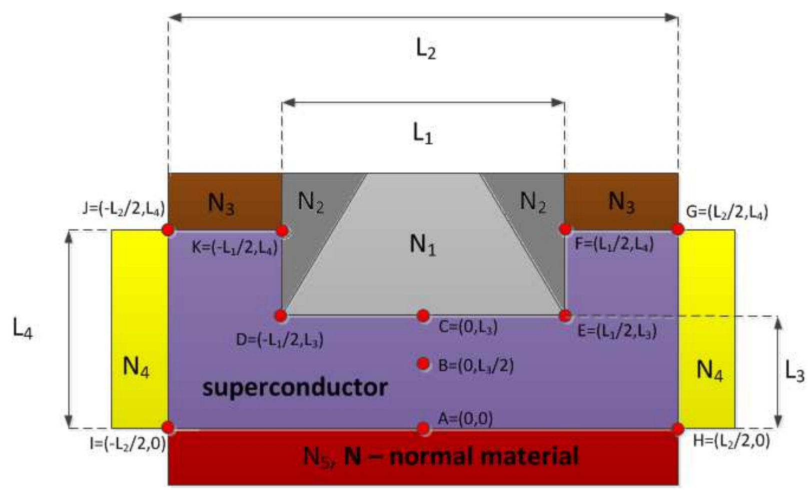

Fig. 2: Schematic view of modified uJJ in rectangular geometry.

the deformed version of it, since non-superconducting strip is placed deeper in the superconductor than in the case of uJJ when it is placed directly on superconductor. Nevertheless in the technological process of synthesis of uJJ the diffusion of evaporated material can penetrate into superconductor to certain thickness.

Let us concentrate at first on linear GL equation. We have

$$\alpha\psi - \frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi = 0 \tag{7}$$

and quite obviously its solutions are the solutions of Schrodinger equation.

Therefore for the symmetric system around OY axis we can write:

$$\psi_{sc}(x,y) = [A_1 - A_2\cos(k_1(x-x_0))\cos(k_2(y-y_0))] + A_3 y \tag{8}$$

and set $x_0 = 0$. Therefore we obtain the equation

$$\psi_{sc}(x,y) = [A_1 - A_2 \cos(k_1 x)\cos(k_2(y-y_0))] + A_3 y.$$

We identify the following variables: $A_1, A_2, A_3, k_1, k_2, x_0, y_0$.

From first two equations we obtain the equations:

$$\frac{\hbar^2}{2m}(k_1^2 + k_2^2) = \alpha, \tag{9}$$

$$\frac{d\psi_{sc}}{dx} = [+A_2 k_1 \sin(k_1(x-x_0))\cos(k_2(y-y_0))] \tag{10}$$

$$\frac{d\psi_{sc}}{dy} = [+A_2 k_2 \cos(k_1(x-x_0))\sin(k_2(y-y_0))] + A_3. \tag{11}$$

Let us consider IH and DE line and consider the boundary conditions

$$\left(\frac{d}{dy}\psi(IH,y) = \frac{1}{b_5}\psi(IH,y)\right),$$

or

$$\left(\frac{d}{dy}\psi(DE,y) = \frac{1}{b_1}\psi(DE,y)\right).$$

For IH, where $x \in (-L_2/2, L_2/2)$ line we have:

$$\begin{aligned}&\{[A_1 - A_2\cos(k_1(x-x_0)\cos(k_2(-y_0))))]\}\frac{1}{b_5} = \\ &[+A_2 k_2 \cos(k_1(x-x_0))\sin(k_2(-y_0)) + A_3]\end{aligned} \tag{12}$$

what brings

$$\frac{A_1}{b_5} - A_3 = 0 \quad \text{so} \quad \frac{A_1}{b_5} = A_3 \tag{13}$$

and

$$0 = \cos(k_1(x-x_0))A_2\left[\frac{1}{b_5}\cos(k_2 y_0) - k_2\sin(k_2 y_0)\right] \tag{14}$$

makes finally

$$\tan(k_2 y_0) = \left(\frac{1}{b_5 k_2}\right). \tag{15}$$

Let us consider the boundary condition along DE line

$$\begin{aligned}&([A_1 - A_2\cos(k_1(x))\cos(k_2(L_3-y_0))] + A_3 L_3)\frac{1}{b_1} = \\ &[+A_2 k_2 \cos(k_1 x)\sin(k_2(L_3-y_0)) + A_3].\end{aligned} \tag{16}$$

It can be rearranged to be of the form

$$\begin{aligned}0 \quad = \quad &\left(\frac{A_1 + A_3 L_3}{b_1}\right) - A_3 = \\ &A_2\cos(k_1 x)\left(\frac{1}{b_1}\cos(k_2(L_3-y_0)\right) + k_2\sin(k_2(L_3-y_0)))\end{aligned} \tag{17}$$

what brings

$$\tan(k_2(L_3 - y_0)) = -\left(\frac{1}{b_1 k_2}\right) \tag{18}$$

and

$$0 = \left(\frac{A_1 + A_3 L_3}{b_1}\right) - A_3. \tag{19}$$

Equation (13) and (7) brings the relation between b_1 and b_5 to be as $b_1 - L_3 = b_5$. Third condition along FG line brings the following relation to be valid

$$\tan(k_2(L_4 - y_0)) = -\left(\frac{1}{b_3 k_2}\right) \tag{20}$$

and

$$0 = \left(\frac{A_1 + A_3 L_4}{b_3}\right) - A_3. \tag{21}$$

For GH line we have

$$\begin{aligned}\frac{1}{b_4}(A_1 - A_2\cos(k_1(L_2/2))\cos(k_2(y - y_0)) + A_3 y) = \\ [+A_2 k_1 \sin(k_1(L_2/2))\cos(k_2(y - y_0))],\end{aligned} \tag{22}$$

which can be rewritten to be in the form

$$\begin{aligned}&\text{const} = \frac{1}{b_4}(A_1) = \\ &\cos\left(k_2(y - y_0)A_2\left(\cos\left(k_1(L_2/2)\frac{1}{b_4} + k_1\sin(k_1(L_2/2)\right)\right)\right) - (A_3 y)\frac{1}{b_4}.\end{aligned} \tag{23}$$

Solutions of such equation for infinite number of points ($y \in (0, L_4)$) exists only if $b_4 \to +\infty$. In such case we have:

$$k_1 \sin(k_1(L_2/2)) = 0. \tag{24}$$

Since k_1 is non zero then

$$k_1 = (\pi + 2n\pi)/L_2. \tag{25}$$

Knowledge of k_1 implies knowledge of k_2 from equation 3.

Quite similar situation occurs for EF line. We have

$$\begin{aligned}\frac{1}{b_2}(A_1 - A_2\cos(k_1(L_1/2))\cos(k_2(y - y_0)) + A_3 y) = \\ [+A_2 k_1 \sin(k_1(L_1/2))\cos(k_2(y - y_0))],\end{aligned} \tag{26}$$

which can be rewritten to be in the form

$$\begin{aligned}&\text{const} = \frac{1}{b_2}(A_1) = \\ &\cos(k_2(y - y_0)A_2(\cos(k_1(L_1/2)\frac{1}{b_2} + k_1\sin(k_1(L_1/2)))) - (A_3 y)\frac{1}{b_2}.\end{aligned} \tag{27}$$

Such equation has only the solution if $b_2 \to +\infty$ for $(y \in (L_3, L_4))$. Similarly as before we obtain the condition

$$k_1 \sin(k_1(L_1/2)) = 0 \tag{28}$$

which implies

$$k_1 = (\pi + 2n_1\pi)/L_1. \tag{29}$$

Equation (23) and (19) brings the following condition on L_1 and L_2

$$(1 + 2n)/L_2 = (1 + 2n_1)/L_1. \tag{30}$$

3.1. Summary of analytic solutions for linearized GL

First we apply equation (24) and use

$$n = 0, \quad n_1 = 1, \quad \frac{1}{3} = \frac{L_2}{L_1}.$$

Material N1 and N4 needs to be insulators to maintain structure of solution Equation (2) with two superconducting peaks. N3 and N5 materials can also be insulators. In such case equation (12) is prescription for y_0 coefficient. Equation (13) is prescription for ration between A_1 and A_2 coefficients. It is quite straightforward to generalized the obtained result to 3 dimensional case. Due to periodicity of cosine function one generalize the obtained results to structures depicted in Fig. 4, 5, 6. The next step in considerations are structures depicted in Fig. 3. Here SCOP is the cosine function factorized by Bessel function.

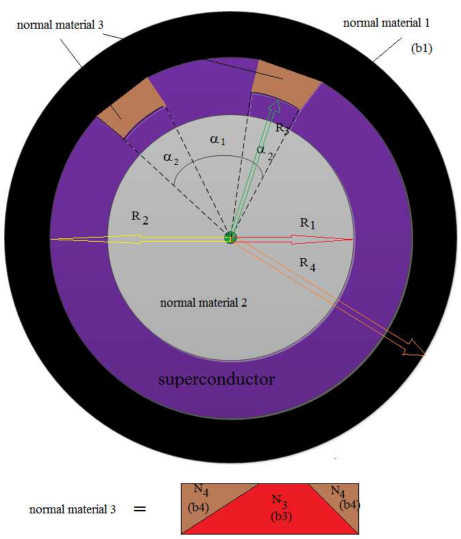

Fig. 3: Geometrical parametrization of modified spherical and cylindrical uJJ junction. The picture can be used as the definition of cylindrical mujj.

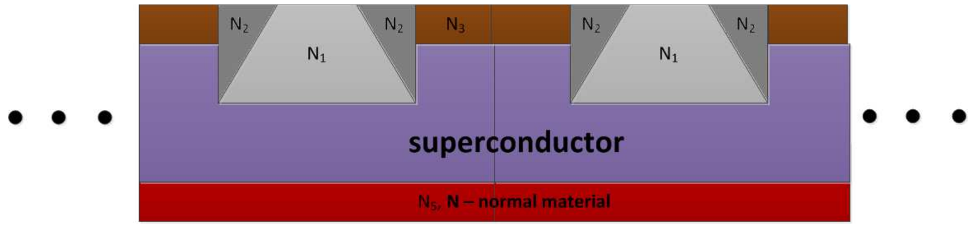

Fig. 4: Schematic view of infinite series of asymmetric modified uJJs in infinite array.

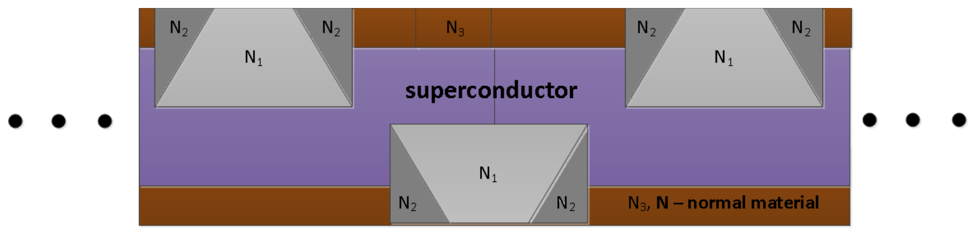

Fig. 5: Schematic view of infinite series of antisymmetric modified uJJs in infinite array.

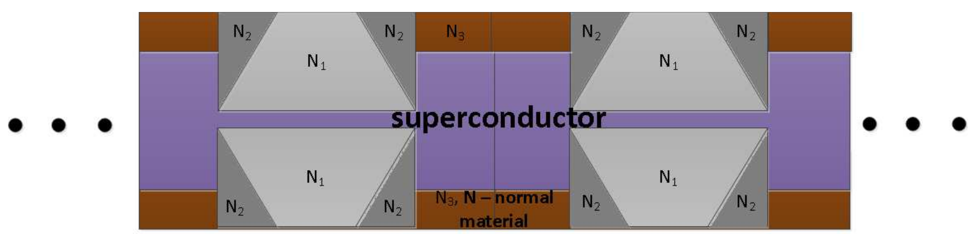

Fig. 6: Schematic view of infinite series of symmetric modified uJJs in infinite array.

3.2. Usage of non-linear term in GL equations

Previous solutions of ψ_0 do not give any knowledge on A_1, A_2 and A_3 factors except ration $\frac{A_1}{A_3}$. We only know that $(|A_1|^2+|A_2|^2+|A_3|^2)$ is small number. We have also assumed that $\beta|\psi_0|^2\psi_0$ is small. In the region inside superconductor not always is fully true. For two given points we could use full form of GL equation. With certain approximation we can write

$$\alpha\psi_0 - \frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\psi_0 + \beta|\psi_0|^2\psi_0 = 0, \tag{31}$$

which results in algebraic equation

$$|\psi_0|^2\psi_0 = 0. \tag{32}$$

We can have two equations

$$|\psi_0(x_{\min}, y_{\min})|^2\psi_0(x_{\min}, y_{\min}) = 0, |\psi_0(x_{\max}, y_{\max})|^2\psi_0(x_{\max}, y_{\max}) = 0, \tag{33}$$

where $x_{\min}, y_{\min}, x_{\max}, y_{\max}$ stands for minima and maxima of $\psi(x, y)$.

Thus we obtain

$$([A_1 - A_2 \cos(k_1 x_{\min}) \cos(k_2(y_{\min} - y_0))] + A_3 y_{\min})^3 = 0$$

and

$$([A_1 - A_2 \cos(k_1 x_{\max}) \cos(k_2(y_{\max} - y_0))] + A_3 y_{\max})^3 = 0.$$

We need to use formula:

$$\begin{aligned} & (a + (-b + c))^3 = a^3 + (-b + c)^3 + 3a^2(-b + c) + 3(-b + c)^2 a \\ = \;& a^3 + c^3 - b^3 + 3b^2 c - 3bc^2 + 3a^2(-b + c) + 3(-b + c)^2 a. \end{aligned}$$

4. Perturbation solution of GL equation and enhancement of relaxation method

Let us assume that $\psi_0(x, y, z)$ is the solution of linear GL equation. In case of nonlinear GL equation we can introduce scalar field f so

$$\psi(x, y, z) = \psi_0(x, y, z) + f(x, y, z). \tag{34}$$

We assume that $f << \psi_0$ for any point. Let us express non-linear GL equations in terms of linear GL equation. We have

$$\alpha\psi + \beta|\psi|^2\psi - \frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi = 0. \tag{35}$$

Using algebraic property we have

$$(\psi_0 + f)^3 = \psi_0^3 + f^3 + 3\psi_0 f^2 + 3f^2\psi_0 \approx \psi_0^3 + 3f\psi_0^2, \tag{36}$$

since

$$\begin{aligned} & (a + b)^3 = (a^2 ab + b^2)(a + b) \\ = \;& a^3 + 2a^2 b + b^2 a + a^2 b + 2ab^2 + b^3 = a^3 + b^3 + 3a^2 b + 3b^2. \end{aligned} \tag{37}$$

More involving formulas appear if we consider the presence of vector potential components $A_x(x, y, z)$, $A_y(x, y, z)$, $A_z(x, y, z)$.

$$\begin{aligned} 0 \;=\; & \alpha\psi_0 + \beta|\psi_0|^2\psi_0 - (\frac{\hbar}{2m}2m\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\psi_0 + \\ & -\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)f + \alpha f + 3f^2\psi_0 + f^3 + 3\psi_0^2 f. \end{aligned} \tag{38}$$

The last equation is equivalent to the equation

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)f+\alpha f+3f^2\psi_0+f^3+3\psi_0^2 f=0. \tag{39}$$

The same boundary conditions applies to ψ_0 and f function.

What is more we can enhance the relaxation method using the knowledge on solutions of linearized GL equation. In very straightforward way we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)f+\alpha f+3f^2\psi_0+f^3+3\psi_0^2 f=\eta\frac{df}{di} \tag{40}$$

where, df is the change of wave-function f during each iteration and η is some constant.

Once we know the solutions of ψ as ψ_0+f we can turn on weak vector potential $A=(A_x,A_y,A_z)$. Turning weak vector potential would be the perturbation to the known solution ψ and would simply add phase factor. Thus we would obtain the distribution of the electric current density. But to have it we need to know the distribution of A potential. The following equation needs to be solved

$$\nabla^2 A_{x(,y,z)}=-cA_{x(,y,z)}|\psi(x,y,z)_0|^2, \tag{41}$$

where c is some constant. It can be achieved by the relaxation method applied to A field. In such way electric current density j, which is proportional to $|\psi|^2 A$ can be determined.

Obtained results allows to reduce one or two dimensional system of infinite array of modified rectangular uJJs to the case of superconducting interacting quantum dots and are presented in [1].

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[1] K. Pomorski, Doctoral thesis: *Mathematical description of unconventional Josephson junctions*, (2014).

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NUMERYCZNE ROZWIĄZANIA PRAWIE NIEZALEŻNYCH OD CZASU RÓWNAŃ GINZBURGA-LANDAUA DLA RÓŻNYCH NADPRZEWODZĄCYCH STRUKTUR

III. ROZWIĄZANIA ANALITYCZNE I METODA RELAKSACYJNA

S t r e s z c z e n i e

W pracy zostaje wprowadzona koncepcja zmodyfikowanego niekonwencjonalnego złącza Josephsona. Zostają zaprezentowane analityczne rozwiązania zlinearyzowanego równania Ginzburga-Landaua dla różnych geometrii realizujących koncepcje zmodfyfikowanego niekonwencjonalnego złącza Josephsona. Powstaje ono przez nałożenie nienadprzewodzącego paska na pasek nadprzewodzący, a następnie przesuniecie paska nienadprzewodzącego w głąb nadprzewodnika. Prezentujemy również rozwinięcie algorytmu relaksacyjnego rozwiązywania równań Ginzburga-Landaua dla przypadków niezależnych od czasu dla różnych geometrii nadprzewodzących struktur.

Słowa kluczowe: równanie Schrödingera, zlinearyzowane równanie Ginzburga-Lanadaua, zmodyfikowane niekonwencjonalne złącza Josephona, metoda relaksacyjna

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