B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES
ET DES LETTRES DE ŁÓDŹ

SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXIV, no. 2

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE: RECHERCHES SUR LES DÉFORMATIONS

Volume LXIV, no. 2

ŁÓDŹ 2014

ŁÓDZKIE TOWARZYSTWO NAUKOWE
PL-90-505 Łódź, ul. M. Curie-Skłodowskiej 11
tel. (42) 66-55-459, fax (42) 66 55 464
sprzedaż wydawnictw: tel. (42) 66 55 448, http://sklep.ltn.lodz.pl
e-mail: biuro@ltn.lodz.pl; http://www.ltn.lodz.pl/

**Wydano z pomocą finansową Ministerstwa Nauki
i Szkolnictwa Wyższego**

PL ISSN 0459–6854

Wydanie 1.
Nakład 200 egz.
Skład komputerowy: Zofia Fijarczyk
Druk i oprawa: 2K Łódź sp. z o.o.
Łódź, ul. Plocka 35/45
www.2k.com.pl, 2k@2k.com.pl

The journal appears in the bases *Copernicus* and *EBSCOhost*

INSTRUCTION AUX AUTEURS

1. La présente Série du Bulletin de la Société des Sciences et des Lettres de Łódź comprend des communications du domaine des mathématiques, de la physique ainsi que de leurs applications liées aux déformations au sense large.
2. Toute communications est présentée à la séance d'une Commission de la Société par un des members (avec deux opinions de spécialistes designés par la Rédaction). Elle doit lui être adressée directement par l'auteur.
3. L'article doit être écrit en anglais, français, allemand ou russe et débuté par un résumé en anglais ou en langue de la communication présentée. Dans tous les travaux écrits par des auteurs étrangers le titre et le résumé en polonais seront préparés par la rédaction. Il faut fournir le texte original qui ne peut contenir plus de 15 pages (plus 2 copies).
4. Comme des articles seront reproduits par un procédé photographique, les auteurs sont priés de les préparer avec soin. Le texte tapé sur un ordinateur de la classe IBM PC avec l'utilisation d'un imprimante de laser, est absolument indispensable. Il doit être tapé préférablement en *AMS-TEX* ou, exceptionnellement, en *Plain-TEX* ou *LATEX*. Après l'acceptation de texte les auteurs sont priés d'envoyer les disquettes (PC). Quelle que soient les dimensions des feuilles de papier utilisées, le texte ne doit pas dépasser un cadre de frappe de 12.3×18.7 cm (0.9 cm pour la page courante y compris). Les deux marges doivent être le la même largeur.
5. Le nom de l'auteur (avec de prénom complet), écrit en italique sera placé à la 1ère page, 5.6 cm au dessous du bord supérieur du cadre de frappe; le titre de l'acticle, en majuscules d'orateur 14 points, 7.1 cm au dessous de même bord.
6. Le texte doit être tapé avec les caractères Times 10 points typographiques et l'interligne de 14 points hors de formules longues. Les résumés, les rénvois, la bibliographie et l'adresse de l'auteurs doivent être tapés avec le petites caractères 8 points typographiques et l'interligne de 12 points. Ne laissez pas de "blancs" inutiles pour respecter la densité du texte. En commençant le texte ou une formule par l'alinéa il faut taper 6 mm ou 2 cm de la marge gauche, respectivement.
7. Les texte des thèorémes, propositions, lemmes et corollaries doivent être écrits en italique.
8. Les articles cités seront rangés dans l'ordre alphabétique et précédés de leurs numéros placés entre crochets. Après les références, l'auteur indiquera son adress complète.
9. Envoi par la poste: protégez le manuscript à l'aide de cartons.
10. Les auteurs recevront une copie de fascicule correspondant à titre gratuit.

Adresse de la Rédaction de la Série:
Département de la Physique d'etat solide
de l'Université de Łódź
Pomorska 149/153, PL-90-236 Łódź, Pologne

Name and surname of the authors

TITLE – INSTRUCTION FOR AUTHORS SUBMITTING THE PAPERS FOR BULLETIN

Summary

Abstract should be written in clear and concise way, and should present all the main points of the paper. In particular, new results obtained, new approaches or methods applied, scientific significance of the paper and conclusions should be emphasized.

1. General information

The paper for BULLETIN DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ should be written in LaTeX, preferably in LaTeX 2e, using the style (the file **bull.cls**).

2. How to prepare a manuscript

To prepare the LaTeX 2e source file of your paper, copy the template file **instr.tex** with **Fig1.eps**, give the title of the paper, the authors with their affiliations/addresses, and go on with the body of the paper using all other means and commands of the standard class/style 'bull.cls'.

2.1. Example of a figure

Figures (including graphs and images) should be carefully prepared and submitted in electronic form (as separate files) in Encapsulated PostScript (EPS) format.

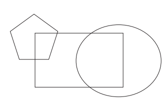

Fig. 1: The figure caption is located below the figure itself; it is automatically centered and should be typeset in small letters.

2.2. Example of a table

Tab. 1: The table caption is located above the table itself; it is automatically centered and should be typeset in small letters.

Description 1	Description 2	Description 3	Description 4
Row 1, Col 1	Row 1, Col 2	Row 1, Col 3	Row 1, Col 4
Row 2, Col 1	Row 2, Col 2	Row 2, Col 3	Row 2, Col 4

2.3. “Ghostwriting” and “guest authorship” are strictly forbiden

The printed version of an article is primary (comparing with the electronic version). Each contribution submitted is sent for evaluation to two independent referees before publishing.

3. How to submit a manuscript

Manuscripts have to be submitted in electronic form, preferably via e-mail as attachment files sent to the address **zofija@uni.lodz.pl**. If a whole manuscript exceeds 2 MB composed of more than one file, all parts of the manuscript, i.e. the text (including equations, tables, acknowledgements and references) and figures, should be ZIP-compressed to one file prior to transfer. If authors are unable to send their manuscript electronically, it should be provided on a disk (DOS format floppy or CD-ROM), containing the text and all electronic figures, and may be sent by regular mail to the address: **Department of Solid State Physics, University of Lodz, Bulletin de la Société des Sciences et des Lettres de Łódź, Pomorska 149/153, 90-236 Łódź, Poland**.

References

[1]

Affiliation/Address

TABLE DES MATIÈRES

Professor Zygmunt Charzyński (1914–2001) in 1957.
Photo by Olli Tammi.

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 11–17

In memory of
Professor Zygmunt Charzyński (1914–2001)

Zbigniew J. Jakubowski and Anna Łazińska

AN EXAMPLE OF A BIVALENT HOLOMORPHIC FUNCTION

Summary

In many problems of the geometric function theory for holomorphic functions the problem of the starlikeness or the convexity of images of circles or discs is considered under the assumption that the functions are univalent. In this paper we present some observations on the geometric properties of images of circles with the centre 0 under a selected bivalent function. This simple example allows us to see some differences between this kind of properties for univalent functions and for multivalent functions.

Keywords and phrases: holomorphic functions, bivalent functions, starlikness, convexity

1.

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and

$$f(z) := \frac{z}{a}\frac{z-a}{az-1}, \qquad z \in \Delta, \quad a \in (0,1). \tag{1}$$

Obviously, for a fixed $a \in (0,1)$ the function f is holomorphic in Δ, $f(0) = f(a) = 0$, $f'(0) = 1$ and

$$f(\Delta) = \left\{w \in \mathbb{C} : |w| < \frac{1}{a}\right\}.$$

Moreover, for $z_1, z_2 \in \Delta$, $z_1 \neq z_2$, the equality $f(z_1) = f(z_2)$ holds if and only if

$$z_2 = \frac{z_1 - a}{az_1 - 1}.$$

Thus the function f of the form (1) is a function bivalent in the disc Δ and the image of Δ under this mapping, as a planar set, is a disc, so it is a convex domain and it is starlike with respect to the point 0.

Let us fix $a \in (0,1)$. According to (1) we have

$$f'(z) = \frac{az^2 - 2z + a}{a(az-1)^2}, \qquad z \in \Delta, \tag{2}$$

so f' takes the value 0 in the disc Δ at the point

$$x_1 = \frac{1 - \sqrt{1-a^2}}{a} \in (0, a).$$

Directly from the definition one can examine that in the disc $K_1 = \{z \in \mathbb{C} : |z| < x_1\}$ the function f is univalent.

In the monograph [4] (pp. 108–110) A. W. Goodman gives the definition of a curve starlike with respect to a fixed point, which is not on the curve, and the definition of a convex curve. The author gives also the related analytic conditions.

If Γ_z is a regular arc with a parametrization $z = \gamma(t)$, $t \in \langle \alpha, \beta \rangle$, such that $\gamma'(z) \neq 0$, $t \in \langle \alpha, \beta \rangle$, F is a function holomorphic on the arc Γ_z and Γ_w denotes the image of Γ_z under the mapping F, then the arc Γ_w is called:

- *starlike with respect to* w_0 ($w_0 \notin \Gamma_w$), if the function $t \mapsto \arg(F(\gamma(t)) - w_0)$ is nondecreasing in the interval $\langle \alpha, \beta \rangle$;

- *convex*, if the argument of the tangent to Γ_w is a nondecreasing function of t in the interval $\langle \alpha, \beta \rangle$.

These concepts also apply to the corresponding regular curves, in particular circles. Let further C_r be a positively oriented circle with the centre at the point 0 and the radius $r > 0$. According to [4] (p.111), for $r > 0$ the image of the circle C_r under a holomorphic function F, such that $F(z) \neq 0$ for $|z| = r$, is starlike with respect to the the point 0 if and only if the known condition

$$\operatorname{Re} \frac{zF'(z)}{F(z)} \geq 0, \qquad z = re^{it}, \quad t \in \langle 0, 2\pi \rangle, \tag{3}$$

holds.

If $F'(z) \neq 0$, $|z| = r$, where $r > 0$, then the image of the circle C_r under the mapping F is convex if and only if

$$\operatorname{Re} \left(1 + \frac{zF''(z)}{F'(z)}\right) \geq 0, \qquad z = re^{it}, \quad t \in \langle 0, 2\pi \rangle. \tag{4}$$

A. W. Goodman emphasizes the importance of the assumption that the function F is univalent, when we deduce the relevant geometric properties of the image of the disc with the centre at the point 0 and the radius $r > 0$, applying conditions (3) and (4). It is associated with the definitions of starlike domains and convex domains on the plane and with the multi-sheeted images for multivalent functions.

For the function f of the form (1), by (2), we have

$$G(z) := \frac{zf'(z)}{f(z)} = \frac{az^2 - 2z + a}{(az-1)(z-a)}, \qquad z \in \Delta \setminus \{a\}, \tag{5}$$

and

$$H(z) := 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2z(1-a^2)}{(az^2-2z+a)(az-1)}, \qquad z \in \Delta \setminus \{x_1\}. \tag{6}$$

Using (5) we can check that for $r \in (0, x_1\rangle \cup (a, 1\rangle$ the condition (3) holds, i.e.

$$\operatorname{Re} G\left(re^{it}\right) \geq 0, \qquad t \in \langle 0, 2\pi\rangle,$$

so the images of the circles C_r, $r \in (0, x_1\rangle \cup (a, 1\rangle$, under the mapping f are starlike with respect to 0. In the case $r \in (x_1, a)$ the corresponding inequality does not hold on the whole circle C_r. For $r = a$ the point 0 belongs to the image of the circle, so we can not talk about starlikeness of this curve with respect to zero.

From (6) we conclude that

$$\operatorname{Re} H\left(re^{it}\right) \geq 0, \qquad t \in \langle 0, 2\pi\rangle,$$

for $r \in (0, x_0\rangle \cup (x_1, 1\rangle$, where

$$x_0 \in \left(\frac{x_1}{2}, x_1\right)$$

is the unique real root of the polynomial

$$W(z) = a^2z^3 - 3az^2 + (4-a^2)z - a.$$

Thus the images of the circles C_r, $r \in (0, x_0\rangle \cup (x_1, 1\rangle$, under the mapping f are convex in terms of the quoted definition. If $r \in (x_0, x_1)$, then the required inequality is not satisfied on the whole circle C_r, so the image of this circle is not a convex curve.

2.

Below we present sketches of the curves that are the images of the circles C_r for selected values $r \in (0, 1)$, when $a = 0,5$. Then we have $x_1 \cong 0,267949192$ and $x_0 \cong 0,1411109281$.

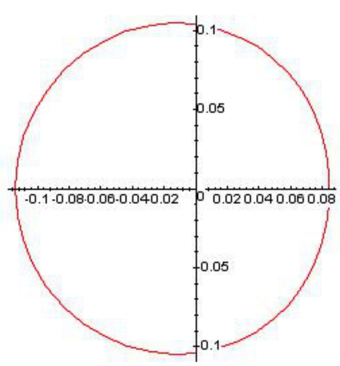

Fig. 1: $r = 0,1$.

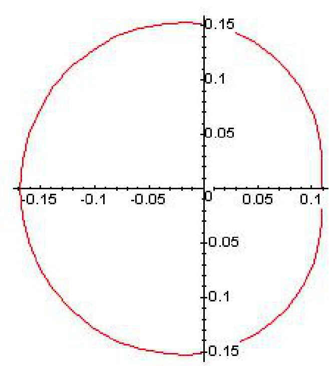

Fig. 2: $r = x_0 \cong 0,1411109281$.

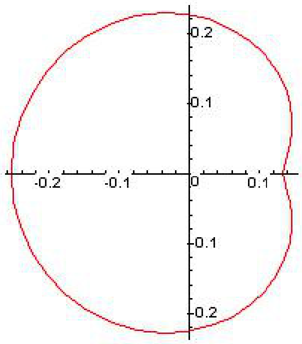

Fig. 3: $r = 0,2$.

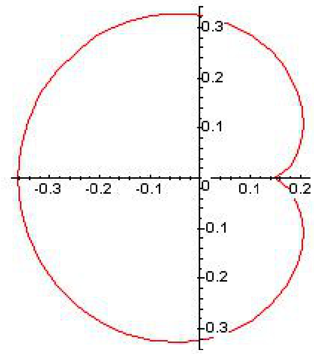

Fig. 4: $r = x_1 \cong 0,267949192$.

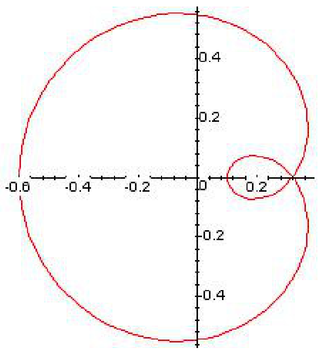

Fig. 5: $r = 0,4$.

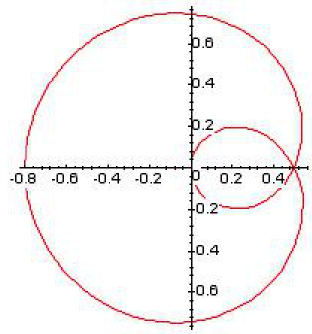

Fig. 6: $r = 0,5$.

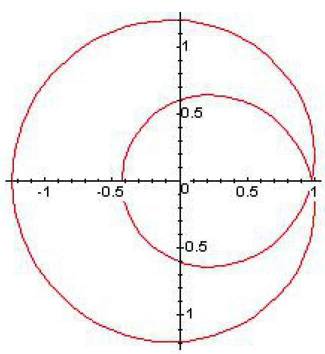

Fig. 7: $r = 0,7$.

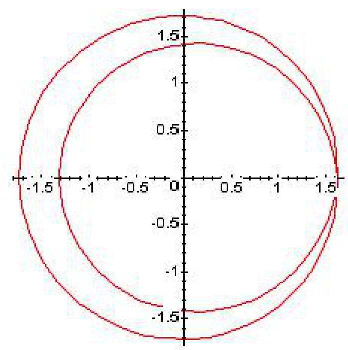

Fig. 8: $r = 0,9$.

3.

Note further that for the function f of the form (1) we have

$$f(z) = z + \sum_{n=2}^{+\infty} b_n z^n, \quad z \in \Delta,$$

where

(7) $$b_n = -\frac{1-a^2}{a^3} a^n, \qquad n = 2, 3, \dots .$$

Let $r \in (0, 1\rangle$ and denote

$$g_r(z) := \frac{1}{r} f(rz), \quad z \in \Delta.$$

Hence we have

$$g_r(z) = z + \sum_{n=2}^{+\infty} B_n z^n, \qquad z \in \Delta, \quad B_n = b_n r^{n-1}, \quad n = 2, 3, \dots .$$

Therefore

$$S_1(r) := \sum_{n=2}^{+\infty} n|B_n| = \frac{1-a^2}{a^2} \sum_{n=2}^{+\infty} n(ar)^{n-1} = \frac{1-a^2}{a} \, \frac{2r - ar^2}{(1-ar)^2}.$$

It is easy to check that $S_1(r) \le 1$ if and only if $r \in (0, x_1\rangle$. Hence we conclude that for $r \in (0, x_1\rangle$ the function g_r is univalent and starlike in the disc Δ (see [3]) and consequently the function f is univalent and starlike in any disc with the centre 0 and the radius $r \in (0, x_1\rangle$. It is seen that the applied coefficient condition significantly simplifies the substantial part of the earlier reasoning.

It is worth noting that the coefficients (7) of the expansion of f are negative. As we know (see [7]), in such a situation the condition

$$\sum_{n=2}^{+\infty} n|b_n| \le 1$$

is also necessary for the univalence and starlikeness of the function in Δ. However, for the function $f = g_1$ we have

$$S_1(1) = \frac{2 + a - a^2}{a(1-a)} > 1,$$

although

$$f(\Delta) = \left\{ w \in \mathbb{C} : |w| < \frac{1}{a} \right\}$$

is a planar starlike domain.

Similarly, considering the sum

$$S_2(r) := \sum_{n=2}^{+\infty} n^2|B_n| = \frac{1-a^2}{a^2} \sum_{n=2}^{+\infty} n^2(ar)^{n-1} = \frac{1-a^2}{a} \, \frac{a^2r^3 - 3ar^2 + 4r}{(1-ar)^3},$$

we can check that $S_2(r) \leq 1$ if and only if $r \in (0, x_0\rangle$, which means that for any $r \in (0, x_0\rangle$ the function f maps univalently the disc $\{z \in \mathbb{C} : |z| < r\}$ onto a convex domain (see [3]). Moreover, we have

$$S_2(1) = \frac{a^3 - 2a^2 + a + 4}{a(a^2 + a + 1)} > 1$$

for the negative coefficients, even though the image of the disc Δ is the disc with the centre 0 and the radius $\frac{1}{a}$. Clearly, this is associated with the bivalence of the function f (compare [7]).

4.

The starlikeness of multivalent functions was considered by many authors, e.g. [2], [5], [6]. In the papers [2] and [6] a function F holomorphic in the disc Δ, with $F(0) = 0$, is called *starlike with respect to the origin of order* p ($p \in \mathbb{N}$) if there exists $\rho \in (0, 1)$ such that for all $r \in (\rho, 1)$ we have $F(z) \neq 0$ for $|z| = r$ and

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > 0, \qquad \int_0^{2\pi} \frac{zF'(z)}{F(z)}\, dt = 2p\pi, \qquad \text{where} \quad z = re^{it}, \quad t \in \langle 0, 2\pi\rangle.$$

This means geometrically that for $r \in (\rho, 1)$ when z traverses the circle C_r in the anti-clockwise direction, then the vector joining the origin and the point $F(z)$ turns continuously through an angle $2p\pi$ in the same direction. Consequenttly, functions starlike of order p have exactly p zeros in the disc Δ and are p-valent in this disc (see [2]).

The functions starlike of order p in the sense of the above definition, but without the assumption $F(0) = 0$, J. A. Hummel (see [5]) called *annular p-valent starlike functions (with respect to the point 0)*. He also introduced a different concept of starlikeness.

A function F holomorphic in Δ J. A. Hummel called *geometrically starlike (of order p) with respect to 0*, if F has exactly p zeros in Δ and, given any $z_0 \in \Delta$, there exists a continuous path $\Gamma : z = \gamma(t)$, $t \in \langle 0, 1\rangle$, such that $F(\gamma(0)) = 0$, $\gamma(1) = z_0$, $|\gamma(t)| < 1$ for all $t \in \langle 0, 1\rangle$, $t \mapsto \arg F(\gamma(t))$ is a constant for all $t \in \langle 0, 1\rangle$ and $t \mapsto |F(\gamma(t))|$ is a strictly increasing function in the interval $\langle 0, 1\rangle$.

In [5] the author proved that each function annular starlike of order p is geometrically starlike of order p. Moreover, each function geometrically starlike of order p is p-valent in the disc Δ.

In view of the mentioned definitions, note that the function f of the form (1) is annular starlike of order 2 (with respect to 0) and in this case we have $\rho = a$. In consequence, it is also geomerically starlike of order 2.

When $p = 1$ the geometric starlikeness is the "classical" starlikeness. Some authors consider also the annular starlikeness of univalent functions with respect to a fixed point w_0 (e.g. [1], [8]).

References

[1] D. Bshouty, A. Lyzzaik, *On a conjecture of D. Styer regarding univalent geometric and annular starlike functions*, Proc. Amer. Math. Soc. **133** (2005), 1485–1490.

[2] A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1950), 204–223.

[3] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. **8**(3) (1957), 598–601.

[4] A. W. Goodman, *Univalent functions*, vol. 1, Mariner Publishing Company, Inc., Tampa, Florida 1983.

[5] J. A. Hummel, *Multivalent starlike functions*, J. Analyse Math. **18** (1967), 133–160.

[6] M. S. Robertson, *Multivalently star-like functions*, Duke Math. J. **20** (1953), 539–550.

[7] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1) (1975), 109–116.

[8] D. Styer, *Geometric and annular starlike functions*, Complex Variables **29**(1) (1996), 189–191.

Department of Nonlinear Analysis
Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22, PL-90-238 Łódź
Poland
e-mails: zjakub@math.uni.lodz.pl
lazinska@math.uni.lodz.pl

Presented by Zbigniew Jerzy Jakubowski and Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 4, 2014

PEWIEN PRZYKŁAD DWULISTNEJ FUNKCJI HOLOMORFICZNEJ

S t r e s z c z e n i e

W wielu zagadnieniach geometrycznej teorii funkcji holomorficznych problem gwiaździstości lub wypukłości obrazów okręgów, czy też kół, jest rozważany przy założeniu jednolistności odwzorowania. W pracy przedstawiono pewne uwagi o geometrycznych własnościach obrazów okręgów o środku w punkcie 0 przy odwzorowaniu wybraną funkcją dwulistną. Ten prosty przykład pozwala zaobserwować pewne różnice między tego typu własnościami funkcji jednolistnych i funkcji wielolistnych.

Słowa kluczowe: funkcje holomorficzne, funkcje dwulistne, gwiaździstość, wypukłość

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 19–27

In memory of
Professor Zygmunt Charzyński (1914–2001)

Ikkei Hotta and Andrzej Michalski

LOCALLY ONE-TO-ONE HARMONIC FUNCTIONS WITH STARLIKE ANALYTIC PART

Summary

Let L_H denote the set of all normalized locally one-to-one and sense-preserving harmonic functions in the unit disc Δ. It is well-known that every complex-valued harmonic function in the unit disc Δ can be uniquely represented as $f = h + \overline{g}$, where h and g are analytic in Δ. In particular the decomposition formula holds true for functions of the class L_H. For a fixed analytic function h, an interesting problem arises - to describe all functions g, such that f belongs to L_H.

The case when $f \in L_H$ and h is the identity mapping was considered in [3]. More general results are given in [4], where $f \in L_H$ and h is a convex analytic mapping. The focus of our present research is to characterize the set of all functions $f \in L_H$ having starlike analytic part h. In this paper, we provide coefficient, distortion and growth estimates of g. We also give growth and Jacobian estimates of f.

Keywords and phrases: harmonic mappings, starlike conformal mappings

1. Introduction

A complex-valued harmonic function f in the open unit disc $\Delta \subset \mathbb{C}$ can be uniquely represented as

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Δ with $g(0) = 0$. Hence, f is uniquely determined by coefficients of the following power series expansions

$$h(z)=\sum_{n=0}^{\infty}a_nz^n,\quad g(z)=\sum_{n=1}^{\infty}b_nz^n,\quad z\in\Delta, \tag{1.2}$$

where $a_n\in\mathbb{C}$, $n=0,1,2,\cdots$ and $b_n\in\mathbb{C}$, $n=1,2,3,\cdots$.

Such a function f, not identically constant, is said to be sense-preserving in Δ if and only if it satisfies the equation

$$g'=\omega h', \tag{1.3}$$

where ω is analytic in Δ with $|\omega(z)|<1$ for all $z\in\Delta$. The function ω is called the second complex dilatation of f, and it is closely related to the Jacobian of f defined as follows

$$J_f(z):=|h'(z)|^2-|g'(z)|^2,\quad z\in\Delta. \tag{1.4}$$

Recall that a necessary and sufficient condition for f to be locally one-to-one and sense-preserving in Δ is $J_f(z)>0$, $z\in\Delta$. This is an immediate consequence of Lewy's theorem (see [5]). Observe that if $J_f(z)>0$ then $|h'(z)|>0$ and hence $g'(z)/h'(z)$ is well defined for every $z\in\Delta$. Thus the dilatation ω of a locally univalent and sense-preserving function f in Δ can be expressed as

$$\omega(z)=\frac{g'(z)}{h'(z)},\quad z\in\Delta. \tag{1.5}$$

Let L_H denote the set of all locally one-to-one and sense-preserving harmonic functions f in Δ satisfying 1.1, such that $h(0)=0$ and $h'(0)=1$ ($g(0)=0$ by the uniqueness of 1.1). Note that the known family S_H introduced in [1] by Clunie and Sheil-Small is a subset of L_H, in fact, S_H consists of all one-to-one functions in L_H.

The main idea of our research is to characterize subclasses of L_H defined by some additional geometric conditions on h. The case when h is the identity mapping was studied in [3]. The paper [4] was devoted to the case when h is a convex analytic mapping. In this paper we consider functions $f\in L_H$ having starlike analytic part h.

Definition 1.1. Let $\alpha\in[0,1)$. We define the class $\check{L}_H^{\alpha}$ of all $f\in L_H$, such that $|b_1|=\alpha$ and h is a starlike analytic function, where b_1 is taken from the power series expansion 1.2 of f. Additionally, we define the class

$$\check{L}_H:=\bigcup_{\alpha\in[0,1)}\check{L}_H^{\alpha}.$$

Remark 1.2. Note that the estimate $|b_1|<1$ holds for all $f\in L_H$ (see [6], p. 79), in particular, this explains why we have taken $\alpha\in[0,1)$ in Definition 1.

Example 1.3. For a fixed $\zeta\in\Delta$ consider $f_\zeta:=k+\overline{g_\zeta}$ in Δ, where

$$k(z):=\frac{z}{(1-z)^2},\quad z\in\Delta$$

and

$$g_\zeta(z) := \left(\frac{1-\zeta}{1+\zeta}\right)^2 \log\frac{1+\zeta z}{1-z} + \frac{z}{(1-z)^2} - \left(\frac{1-\zeta}{1+\zeta}\right)\frac{2z}{1-z}, \quad z \in \Delta.$$

Straightforward calculation leads to the formula for the dilatation ω_ζ of f_ζ, i.e.

$$\omega_\zeta(z) := \frac{z+\zeta}{1+\zeta z}, \quad z \in \Delta,$$

which ensures that f_ζ is locally one-to-one and sense-preserving in Δ. Clearly, $k(0) = 0$, $k'(0) = 1$ and $g_\zeta(0) = 0$, hence $f_\zeta \in L_H$. Finally, k is well-known starlike function (Koebe function) and one can easily check that $g'_\zeta(0) = \zeta$, therefore $f_\zeta \in \check{L}_H^\alpha$ with $\alpha := |\zeta|$.

2. Results

At the beginning of this section we present a connection, discovered by us, between $\check{L}_H^\alpha$ and the class of normalized harmonic mappings with convex analytic part (introduced in [4]). It does not seem to be surprising in view of the classical result concerning convex and starlike analytic functions due to J. W. Alexander.

Theorem 2.1. *If $f \in \check{L}_H^\alpha$, then $F := H + \overline{G}$ belongs to S_H, H is a convex analytic function and $|G'(0)| = \alpha$, where H and G satisfy the conditions $h(z) = zH'(z)$ and $g(z) = zG'(z)$, $z \in \Delta$ with the normalization $H(0) = 0$ and $G(0) = 0$.*

Proof. Let $f \in \check{L}_H^\alpha$. By definition of $\check{L}_H^\alpha$, h is a normalized starlike function. Hence, by Alexander's theorem, the function H is a normalized convex function, namely,

$$H(0) = 0 \quad \text{and} \quad H'(0) = \lim_{z\to 0} h(z)/z = h'(0) = 1.$$

Moreover,

$$G'(0) = \lim_{z\to 0} g(z)/z = g'(0) \quad \text{and} \quad |G'(0)| = \alpha.$$

In particular, $|G'(0)| < |H'(0)|$. Now, observe that for all $z \in \Delta \setminus \{0\}$ the interval $[0, h(z)]$ is a subset of $h(\Delta)$ since h is a starlike function. Hence

$$\text{(2.1)}\ |g(z)| = \left|g \circ h^{-1}(h(z))\right| \le \int_0^{h(z)} \left|\frac{\mathrm{d}(\mathrm{g} \circ \mathrm{h}^{-1})}{\mathrm{d}\zeta}(\zeta)\right| |\mathrm{d}\zeta| < \int_0^{\mathrm{h}(z)} |\mathrm{d}\zeta| = |\mathrm{h}(z)|.$$

Again, by definition of $\check{L}_H^\alpha$, f is a sense-preserving harmonic function which together with 2.1 gives

$$|G'(z)| = \lim_{\zeta\to z}\left|\frac{g(\zeta)}{\zeta}\right| < \lim_{\zeta\to z}\left|\frac{h(\zeta)}{\zeta}\right| = |H'(z)|, \quad z \in \Delta \setminus \{0\}.$$

It shows that F is a locally univalent and sense-preserving harmonic function in Δ. Finally, appealing to [4, Corollary 2.3], we conclude that $F \in S_H$. □

For $f \in \check{L}_H^\alpha$, the classical theory of univalent functions says (see e.g. [7], [8])

$$|a_n| \le n, \quad n = 2, 3, 4, \cdots . \tag{2.2}$$

The following theorem gives an estimate of the coefficient b_n.

Theorem 2.2. *If $f \in \check{L}_H^\alpha$, then we have*

$$|b_2| \leq 2\alpha + \frac{1-\alpha^2}{2} \tag{2.3}$$

and

$$|b_n| \leq \alpha + \sqrt{(n-\alpha^2)(n-1)}, \quad n = 3, 4, 5, \cdots . \tag{2.4}$$

Proof. If $f \in \check{L}_H^\alpha$, then by Theorem 2.1 the function $F := H + \overline{G}$ belongs to S_H, H is a convex analytic function and $|G'(0)| = \alpha$, where H and G satisfy the conditions $h(z) = zH'(z)$ and $g(z) = zG'(z)$ for all $z \in \Delta$ with the normalization $H(0) = 0$ and $G(0) = 0$. Clearly G can be expanded in a power series, say

$$G(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in \Delta,$$

where $B_n \in \mathbb{C}$, $n = 1, 2, 3, \cdots$. Using this expansion together with the expansion 1.2 of g and the formula $g(z) = zG'(z)$, we obtain $b_n = nB_n$, $n = 1, 2, 3, \cdots$. Next, by applying [4, Theorem 3.1] we have

$$|b_n| \leq n \frac{\alpha + \sqrt{(n-\alpha^2)(n-1)}}{n}, \quad n = 2, 3, 4, \cdots,$$

which gives 2.4. To improve the estimate in the case $n = 2$, consider the function

$$F(z) := \frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)}, \quad z \in \Delta,$$

where ω is the dilatation of f. Since ω is analytic in Δ it has a power series expansion, say

$$\omega(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta,$$

where $c_n \in \mathbb{C}$, $n = 0, 1, 2, \cdots$ and $|c_0| = |\omega(0)| = |g'(0)| = |b_1| = \alpha$. Recall that $|\omega(z)| < 1$ for all $z \in \Delta$. Hence we can apply the Schwarz lemma to F and obtain $|F'(0)| \leq 1$, which yields

$$|c_1| = |\omega'(0)| \leq 1 - |c_0|^2. \tag{2.5}$$

On the other hand, the formula 1.5 gives

$$2b_2 = 2a_2 c_0 + c_1,$$

which together with 2.2 and 2.5 leads to the estimate

$$2|b_2| \leq 4|c_0| + 1 - |c_0|^2.$$

Since this is equivalent to 2.3, the proof is completed. □

We have the following immediate corollary from Theorem 2.2.

Corollary 2.3. *If* $f \in \check{L}_H$ *then* $|b_n| < n$, $n = 2, 3, 4, \cdots$.

Recall that by definition the analytic part h of $f \in \check{L}_H$ is starlike. Hence, it is known that

$$\frac{1-|z|}{(1+|z|)^3} \le |h'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in \Delta. \tag{2.6}$$

Our next result is the distortion estimate of the anti-analytic part g of $f \in \check{L}_H^\alpha$.

Theorem 2.4. *If* $f \in \check{L}_H^\alpha$, *then*

$$|g'(z)| \ge \begin{cases} \dfrac{(\alpha-|z|)(1-|z|)}{(1-\alpha|z|)(1+|z|)^3}, & |z| < \alpha, \\ 0, & |z| \ge \alpha, \end{cases} \tag{2.7}$$

where $z \in \Delta$ *and*

$$|g'(z)| \le \frac{(\alpha+|z|)(1+|z|)}{(1+\alpha|z|)(1-|z|)^3}, \quad z \in \Delta. \tag{2.8}$$

Proof. Let ω of the form 1.5 be the dilatation of $f \in \check{L}_H^\alpha$ and let $g'(0) = \alpha e^{\mathrm{i}\varphi}$. Then the function

$$\Omega(z) = \frac{e^{-\mathrm{i}\varphi}\omega(z) - \alpha}{1 - \alpha e^{-\mathrm{i}\varphi}\omega(z)}, \quad z \in \Delta$$

satisfies the assumptions of the Schwarz lemma, which gives

$$\left|e^{-\mathrm{i}\varphi}\omega(z) - \alpha\right| \le |z| \left|1 - \alpha e^{-\mathrm{i}\varphi}\omega(z)\right|, \quad z \in \Delta.$$

Equivalently we can write

$$\left|e^{-\mathrm{i}\varphi}\omega(z) - \frac{\alpha(1-|z|^2)}{1-\alpha^2|z|^2}\right| \le \frac{(1-\alpha^2)|z|}{1-\alpha^2|z|^2}, \quad z \in \Delta$$

and the equality holds only for the functions satisfying

$$\omega(z) = e^{i\varphi}\frac{e^{i\psi}z + \alpha}{1 + \alpha e^{i\psi}z}, \quad z \in \Delta,$$

where $\psi \in \mathbb{R}$. Hence, by the triangle inequality we have

$$\frac{\alpha - |z|}{1-\alpha|z|} \le |\omega(z)| \le \frac{\alpha+|z|}{1+\alpha|z|}, \quad z \in \Delta. \tag{2.9}$$

Finally, applying the estimate 2.9 together with 2.6 to the identity 1.3 we obtain 2.7 and 2.8, so the proof is completed. □

Corollary 2.5. *If* $f \in \check{L}_H$, *then*

$$|g'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in \Delta.$$

Using the distortion estimates we can easily deduce the following Jacobian estimates of f.

Theorem 2.6. *If $f \in \check{L}_H^\alpha$, then*

$$\frac{(1-\alpha^2)(1-|z|)^3}{(1+\alpha|z|)^2(1+|z|)^5} \le J_f(z) \le \begin{cases} \dfrac{(1-\alpha^2)(1+|z|)^3}{(1-\alpha|z|)^2(1-|z|)^5}, & |z| < \alpha, \\[2ex] \dfrac{(1+|z|)^2}{(1-|z|)^6}, & |z| \ge \alpha, \end{cases}$$

where $z \in \Delta$.

Proof. Observe that if $f \in \check{L}_H^\alpha$ then h' does not vanish in Δ and we can write the Jacobian of f given by 1.4 in the form

$$\mathbf{J}_f(z) = |h'(z)|^2 \left(1 - |\omega(z)|^2\right), \quad z \in \Delta,$$

where ω is the dilatation of f. By applying 2.6 and 2.9 to the above formula we obtain

$$\frac{(1-\alpha^2)(1-|z|^2)}{(1+\alpha|z|)^2} \frac{(1-|z|)^2}{(1+|z|)^6} \le J_f(z) \le \begin{cases} \dfrac{(1-\alpha^2)(1-|z|^2)}{(1-\alpha|z|)^2} \dfrac{(1+|z|)^2}{(1-|z|)^6}, & |z| < \alpha, \\[2ex] \dfrac{(1+|z|)^2}{(1-|z|)^6}, & |z| \ge \alpha \end{cases}$$

and the proof is completed. Note that these estimates can also be deduced from a more general result given in [2]. $\square$

The growth estimate of the analytic part h of $f \in \check{L}_H$ is known to be of the form

$$\frac{|z|}{(1+|z|)^2} \le |h(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \Delta. \tag{2.10}$$

In the following theorem we give the growth estimate of the anti-analytic part g.

Theorem 2.7. *If $f \in \check{L}_H^\alpha$, then*

$$|g(z)| \ge \begin{cases} \dfrac{|z|(\alpha-|z|)}{(1-\alpha|z|)(1+|z|)^2}, & |z| < \alpha, \\ 0, & |z| \ge \alpha, \end{cases} \tag{2.11}$$

where $z \in \Delta$ and

$$\begin{aligned} |g(z)| &\le \left(\frac{1-\alpha}{1+\alpha}\right)^2 \log\frac{1+\alpha|z|}{1-|z|} + \frac{|z|}{(1-|z|)^2} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{2|z|}{1-|z|} \\ &\le \frac{|z|(\alpha+|z|)}{(1+\alpha|z|)(1-|z|)^2}, \quad z \in \Delta. \end{aligned} \tag{2.12}$$

Proof. If $f \in \check{L}_H^\alpha$, then by Theorem 2.1 the function $F := H + \overline{G}$ belongs to S_H, H is a convex analytic function and $|G'(0)| = \alpha$, where H and G satisfy the conditions $h(z) = zH'(z)$ and $g(z) = zG'(z)$ for all $z \in \Delta$ with the normalization $H(0) = 0$

and $G(0) = 0$. Hence, by applying [4, Theorem 3.5] together with $g(z) = zG'(z)$ we have

$$|z|\frac{\alpha - |z|}{(1-\alpha|z|)(1+|z|)^2} \le |g(z)| \le |z|\frac{\alpha + |z|}{(1+\alpha|z|)(1-|z|)^2}, \quad z \in \Delta.$$

To prove the first inequality in 2.12 we estimate the integral of g' along $\gamma := [0, z]$ by applying 2.8, i.e.

$$\begin{aligned} |g(z)| &= \left|\int_\gamma g'(\zeta)\mathrm{d}\zeta\right| \le \int_\gamma |g'(\zeta)||\mathrm{d}\zeta| \le \int_0^{|z|} \frac{(\alpha+\rho)(1+\rho)}{(1+\alpha\rho)(1-\rho)^3}\mathrm{d}\rho \\ &= \left(\frac{1-\alpha}{1+\alpha}\right)^2 \log\frac{1+\alpha|z|}{1-|z|} + \frac{|z|}{(1-|z|)^2} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{2|z|}{1-|z|}, \quad z \in \Delta. \end{aligned}$$

Finally, observe that the function f_ζ defined in Example 1 with suitably chosen $\zeta \in \Delta$ shows that the first inequality in 2.12 is best possible, which completes the proof. □

Corollary 2.8. *If $f \in \check{L}_H$, then*

$$|g(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \Delta.$$

Now, we can deduce the growth estimate of f.

Theorem 2.9. *If $f \in \check{L}_H^\alpha$, then*

$$|f(z)| \ge \frac{(1-\alpha)|z|(1-|z|)}{(1+\alpha|z|)(1+|z|)^2}, \quad z \in \Delta. \tag{2.13}$$

and

$$\begin{aligned} |f(z)| &\le \left(\frac{1-\alpha}{1+\alpha}\right)^2 \log\frac{1+\alpha|z|}{1-|z|} + \frac{2|z|}{(1-|z|)^2} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{2|z|}{1-|z|} \\ &\le \frac{(1+\alpha)|z|(1+|z|)}{(1+\alpha|z|)(1-|z|)^2}, \quad z \in \Delta. \end{aligned} \tag{2.14}$$

Proof. If $f \in \check{L}_H^\alpha$, then by Theorem 2.1 the function $F := H + \overline{G}$ belongs to S_H, H is a convex analytic function and $|G'(0)| = \alpha$, where H and G satisfy the conditions $h(z) = zH'(z)$ and $g(z) = zG'(z)$ for all $z \in \Delta$ with the normalization $H(0) = 0$ and $G(0) = 0$. Hence, by applying inequality 2.10 and 2.9, which also holds true for the dilatation of F (see [4, the proof of Theorem 3.5]) we have

$$\begin{aligned} |f(z)| &= |h(z) + \overline{g}(z)| \ge |h(z)| - |g(z)| = |h(z)|\left(1 - \left|\frac{g(z)}{h(z)}\right|\right) \\ &= |h(z)|\left(1 - \left|\frac{G'(z)}{H'(z)}\right|\right) \ge \frac{|z|}{(1+|z|)^2}\left(1 - \frac{\alpha+|z|}{1+\alpha|z|}\right), \quad z \in \Delta. \end{aligned}$$

This proves the inequality 2.13. To prove 2.14 we use the triangle inequality $|f(z)| \leq |h(z)| + |g(z)|$, which together with 2.10 and 2.12 leads to 2.14 so the proof is completed. $\square$

Corollary 2.10. *If* $f \in \check{L}_H$ *then*

$$|f(z)| \leq \frac{2|z|}{(1-|z|)^2}, \quad z \in \Delta.$$

Remark 2.11. The estimates 2.4, 2.11 and 2.13 are probably not precise.

References

[1] J. G. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 3–25.

[2] D. Klimek-Smęt and A. Michalski, *Jacobian estimates for harmonic mappings generated by convex conformal mappings*, Bull. Soc. Sci. Lettres Łódź, Sér. Rech. Deform. vol. **43**, no. 1 (2013), 79–83.

[3] D. Klimek and A. Michalski, *Univalent anti-analytic perturbation of the identity in the unit disc*, Sci. Bull. Chełm **1** (2006), 67–78.

[4] D. Klimek and A. Michalski, *Univalent anti-analytic perturbation of convex conformal mapping in the unit disc*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **61** (2007), 39–49.

[5] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.

[6] P. L. Duren, *Harmonic Mappings in the Plane*, Cambridge Univ. Press, Cambridge 2004.

[7] P. L. Duren, *Univalent Functions*, Springer-Verlag, Berlin-New York 1983.

[8] A. W. Goodman, *Univalent Functions*, Mariner Publishing, Tampa 1983.

Department of Mathematics
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku
Tokyo 152-8551
Japan
e-mail: ikkeihotta@gmail.com

Department of Complex Analysis
The John Paul II Catholic
University of Lublin
Konstantynów 1H, PL-20-950 Lublin
Poland
e-mail: amichal@kul.lublin.pl

Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 20, 2014

LOKALNIE RÓŻNOWARTOŚCIOWE FUNKCJE HARMONICZNE Z GWIAŹDZISTĄ CZĘŚCIĄ ANALITYCZNĄ

S t r e s z c z e n i e

Niech L_H oznacza zbiór wszystkich unormowanych, lokalnie różnowartościowych i zachowujących orientację funkcji harmonicznych w kole jednostkowym Δ. Dobrze znany jest fakt, że każda funkcja harmoniczna określona w kole jednostkowym Δ o wartościach zespolonych może być jednoznacznie przedstawiona w postaci $f = h + \overline{g}$, gdzie h i g są analityczne w Δ. W szczególności powyższa reprezentacja jest słuszna dla funkcji z klasy L_H. Dla dowolnie ustalonej funkcji analitycznej h, jednym z ciekawych problemów jest opisanie wszystkich funkcji g takich, że f należy do L_H.

Przypadek, gdy $f \in L_H$ i h jest odwzorowaniem identycznościowym, był rozważany w [3]. Bardziej ogólne wyniki zostały przedstawione w [4], gdzie $f \in L_H$ i h jest wypukłą funkcją analityczną. Celem obecnie prowadzonych przez nas badań jest charakteryzacja zbioru funkcji $f \in L_H$, których część analityczna h jest gwiaździsta. W niniejszej pracy prezentujemy oszacowania współczynników, zniekształcenia oraz wzrostu funkcji g, a także oszacowania wzrostu i Jakobianu funkcji f.

Słowa kluczowe: odwzorowania harmoniczne, gwiaździste odwzorowania konforemne

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 29–40

In memory of
Professor Zygmunt Charzyński (1914–2001)

Oliwia Chojnacka and Adam Lecko

ON SOME DIFFERENTIAL SUBORDINATION OF HARMONIC MEAN

Summary

In this paper, we study some first order differential subordination related to the harmonic mean, in analogy to the differential subordination related to the arithmetic and geometric mean.

Keywords and phrases: differential subordination, arithmetic mean, geometric mean, harmonic mean, convex function

1. Introduction

Given $z_0 \in \mathbb{C}$ and $r > 0$, let

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\} \quad \text{and} \quad \overline{\mathbb{D}}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

Given $r > 0$, let $\mathbb{D}_r := \mathbb{D}(0, r)$, $\overline{\mathbb{D}}_r := \overline{\mathbb{D}}(0, r)$ and $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$. Let $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{T} := \mathbb{T}_1$.

Let D be a domain in $\mathbb{C}$ and $\mathcal{H}(D)$ be the class of all analytic functions $f : D \to \mathbb{C}$. Let $\mathcal{H} := \mathcal{H}(\mathbb{D})$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ of functions f normalized by $f(0) := 0$ and $f'(0) := 1$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions. Throughout the whole paper, we assume that D is a domain $\mathbb{C}$.

A function $f \in \mathcal{H}$ is said to be subordinate to a function $F \in \mathcal{H}$ if there exists $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = F \circ \omega$ in $\mathbb{D}$. We write then $f \prec F$.

When F is univalent, then

$$f \prec F \Leftrightarrow \left(f(0) = F(0) \wedge f(\mathbb{D}) \subset F(\mathbb{D})\right).$$

A function $f \in \mathcal{H}$ is said to be convex if it is univalent and $f(\mathbb{D})$ is a convex domain.

Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ and $h \in \mathcal{H}$ be univalent. We say that a function $p \in \mathcal{H}$ satisfies the first-order subordination if the function

$$\mathbb{D} \ni z \mapsto \psi\left(p(z), zp'(z); z\right)$$

is well defined and analytic, and

$$\psi\left(p(z), zp'(z); z\right) \prec h(z), \quad z \in \mathbb{D}. \tag{1.1}$$

The question when (1.1) yields $p \prec h$ is the basis in the theory of differential subordinations (see Lewandowski, Miller and Złotkiewicz [9], Miller, Mocanu [10], [11] and [12]). Further details and references can be found in the book of Miller and Mocanu [13].

Let $t \in [0, 1]$ and a, $b \in \mathbb{C}$. By

$$\mathrm{A}_t(a, b) := (1 - t)a + tb$$

we denote the arithmetic mean of a and b, and when $b + t(b - a) \neq 0$, by

$$\mathrm{H}_t(a, b) := \frac{ab}{b + t(a - b)}$$

we denote the harmonic mean of a and b.

Definition 1.1. Let $t \in [0, 1]$. For Ψ, $p \in \mathcal{H}$, let

$$P_{t;\Psi,p}(z) := p(z) + tzp'(z)\Psi(z), \quad z \in \mathbb{D}.$$

The example of (1.1) studied by many authors (see e.g. [13, pp. 39, 69-71]) is as follows:

$$P_{1;\Psi,p} \prec h.$$

Particularly, when $\Psi \equiv 1/\gamma$, $\gamma \in \mathbb{C} \setminus \{0\}$, the above subordination is of the form

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad z \in \mathbb{D}, \tag{1.2}$$

and was studied by Hallenbeck and Ruscheweyh [3]. Setting $\gamma := 1/\alpha$, $\alpha \in (0, 1]$, we see that (1.2) leads to

$$p(z) + \alpha zp'(z) \prec h(z), \quad z \in \mathbb{D}. \tag{1.3}$$

Assuming now that $\alpha \in [0, 1]$, the above yields

$$(1 - \alpha)p(z) + \alpha(p(z) + zp'(z)) \prec h(z), \quad z \in \mathbb{D},$$

or

$$\mathrm{A}_\alpha(p(z), p(z) + zp'(z)) \prec h(z), \quad z \in \mathbb{D}.$$

As an example, for $f \in \mathcal{A}$ setting

$$p(z) := \frac{f(z)}{z}, \quad z \in \mathbb{D} \setminus \{0\}, \quad p(0) := 1,$$

and

$$p(z) := f'(z), \quad z \in \mathbb{D},$$

into (1.3), we get, respectively,

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec h(z), \quad z \in \mathbb{D}.$$

and

$$(1-\alpha)f'(z) + \alpha\left(zf'(z)\right)' \prec h(z), \quad z \in \mathbb{D}.$$

The differential subordination related to the geometric mean was introduced in [5]. For further results in this direction see e.g. [6], [7], [8], [2] and [1]. We omit the details because the description requires some additional notation.

Definition 1.2. Let $t \in [0,1]$ and $\Psi \in \mathcal{H}$. By $\mathcal{H}(t;\Psi)$ we denote the subclass of $\mathcal{H}$ of all functions p such that

$$H_{t;\Psi,p}(z) := \begin{cases} \dfrac{P_{0;\Psi,p}(z)P_{1;\Psi,p}(z)}{P_{1-t;\Psi,p}(z)}, & z \in \mathbb{D} \setminus P^{-1}_{1-t;\Psi,p}(0), \\[2ex] \lim_{\mathbb{D}\ni\zeta\to z} \dfrac{P_{0;\Psi,p}(\zeta)P_{1;\Psi,p}(\zeta)}{P_{1-t;\Psi,p}(\zeta)}, & z \in P^{-1}_{1-t;\Psi,p}(0), \end{cases} \tag{1.4}$$

is an analytic function in $\mathbb{D}$.

When $p \equiv 0$, set

$$H_{t;\Psi,p} \equiv 0. \tag{1.5}$$

By $\mathcal{H}^0(t;\Psi)$ we denote the subclass of $\mathcal{H}(t;\Psi)$ of all functions p such that

$$P_{1-t,\Psi,p}(z) \neq 0, \quad z \in \mathbb{D}.$$

Remark 1.3. 1. Note that, when $p \in \mathcal{H}^0(t;\Psi)$, then

$$\begin{aligned} H_{t;\Psi,p}(z) &= \frac{P_{0;\Psi,p}(z)P_{1;\Psi,p}(z)}{P_{1-t;\Psi,p}(z)} \\ &= \frac{p(z)(p(z)+zp'(z)\Psi(z))}{p(z)+(1-t)zp'(z)\Psi(z)}, \quad z \in \mathbb{D}. \end{aligned} \tag{1.6}$$

2. Observe that

$$H_{0;\Psi,p}(z) = P_{0;\Psi,p}(z) = p(z), \quad z \in \mathbb{D}, \tag{1.7}$$

and

$$H_{1;\Psi,p}(z) = P_{1;\Psi,p}(z) = p(z) + zp'(z)\Psi(z), \quad z \in \mathbb{D}. \tag{1.8}$$

3. (a) Let $\Psi \in \mathcal{H}$ be arbitrary. By (1.7) and (1.8), for every $p \in \mathcal{H}$ the functions $H_{0;\Psi,p}$ and $H_{1;\Psi,p}$ are analytic in $\mathbb{D}$, so the classes $\mathcal{H}(0;\Psi)$ and $\mathcal{H}(1;\Psi)$ are non-empty.

(b) Fix $t \in (0,1)$. Assume that $\Psi \in \mathcal{H}$ is bounded, i.e., there is $M > 0$ such that

$$|\Psi(z)| \le M, \quad z \in \mathbb{D}. \tag{1.9}$$

Fix $c_0 \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. Take r such that

$$0 < r < \frac{|c_0|}{1 + (1-t)nM}. \tag{1.10}$$

Given $c \in \mathbb{T}_r$, set

$$p(z) := c_0 + cz^n, \quad z \in \mathbb{D}. \tag{1.11}$$

We have

$$\begin{aligned} P_{1-t;\Psi,p}(z) &= p(z) + (1-t)zp'(z)\Psi(z) \\ &= c_0 + c\,(1 + (1-t)n\Psi(z))\,z^n, \quad z \in \mathbb{D}. \end{aligned} \tag{1.12}$$

From (1.9) and (1.10) we obtain

$$\begin{aligned} |c\,(1 + (1-t)n\Psi(z))| &\le r\,(1 + (1-t)n|\Psi(z)|) \\ &\le r(1 + (1-t)nM) < |c_0|, \quad z \in \mathbb{D}. \end{aligned} \tag{1.13}$$

Hence and from (1.12) we get

$$\begin{aligned} |P_{1-t;\Psi,p}(z)| &= |c_0 + c\,(1 + (1-t)n\Psi(z))\,z^n| \\ &\ge |c_0| - |c\,(1 + (1-t)n\Psi(z))|\,|z|^n \\ &> |c_0| - |c_0| = 0, \quad z \in \mathbb{D}. \end{aligned}$$

Thus $P_{1-t;\Psi,p}(z) \neq 0$ for $z \in \mathbb{D}$, so $H_{t;\Psi,p}$ is given by (1.6) and it is analytic in $\mathbb{D}$. Hence $p \in \mathcal{H}(t;\Psi)$. In this way, we showed that for every $t \in [0,1)$ and every bounded $\Psi \in \mathcal{H}$ the class $\mathcal{H}(t;\Psi)$ is non-empty.

(c) Fix $t \in (0,1)$. Assume that $\Psi \in \mathcal{H}$ and

$$\operatorname{Re}\Psi(z) > 0, \quad z \in \mathbb{D}. \tag{1.14}$$

Given $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, set

$$p(z) := cz^n, \quad z \in \mathbb{D}.$$

Then

$$\begin{aligned} P_{1-t;\Psi,p}(z) &= p(z) + (1-t)zp'(z)\Psi(z) \\ &= cz^n\,(1 + (1-t)n\Psi(z))\,, \quad z \in \mathbb{D}. \end{aligned} \tag{1.15}$$

Since, in view of (1.14),

$$\Psi(z) \neq -\frac{1}{(1-t)n}, \quad z \in \mathbb{D},$$

so by (1.15), we have $P^{-1}_{1-t;\Psi,p}(0) = \{0\}$. Consequently, by (1.4) we have

$$H_{t;\Psi,p}(z) = cz^n \frac{1+n\Psi(z)}{1+(1-t)n\Psi(z)}, \quad z \in \mathbb{D},$$

and $H_{t;\Psi,p}$ is an analytic function in $\mathbb{D}$. Thus $p \in \mathcal{H}(t;\Psi)$. In this way, we showed that for every $t \in (0,1)$ and every $\Psi \in \mathcal{H}$ with the positive real part in $\mathbb{D}$, the class $\mathcal{H}(t;\Psi)$ is non-empty.

In this paper, we study the differential subordination related to harmonic mean. For $\beta \in [0,1]$, $\Psi \in \mathcal{H}$ and a univalent function $h \in \mathcal{H}$, we consider the differential subordination of the type

$$H_{\beta;\Psi,p} \prec h. \tag{1.16}$$

When $p \in \mathcal{H}^0(\beta;\Psi)$, then taking into account (1.6), the condition (1.16) is of the form

$$\frac{p(z)(p(z)+zp'(z)\Psi(z))}{p(z)+(1-\beta)zp'(z)\Psi(z)} \prec h(z), \quad z \in \mathbb{D}. \tag{1.17}$$

Equivalently,

$$\mathrm{H}_\beta\left(p(z), p(z)+zp'(z)\Psi(z)\right) \prec h(z), \quad z \in \mathbb{D}.$$

Assuming additionally that

$$p(z) \neq 0, \quad p(z)+zp'(z)\Psi(z) \neq 0, \quad z \in \mathbb{D},$$

i.e., that

$$p \in \mathcal{H}^0(\beta;\Phi) \cap \mathcal{H}^0(1;\Phi) \cap \mathcal{H}^0(0;\Phi),$$

the condition (1.17) can be written as

$$\frac{1}{\dfrac{1-\beta}{p(z)} + \dfrac{\beta}{p(z)+zp'(z)\Psi(z)}} \prec h(z), \quad z \in \mathbb{D},$$

or as

$$p(z)\frac{1+\dfrac{zp'(z)}{p(z)}\Psi(z)}{1+(1-\beta)\dfrac{zp'(z)}{p(z)}\Psi(z)} \prec h(z), \quad z \in \mathbb{D}.$$

The basis of this paper is Theorem 2.4, where the differential subordination of the form (1.16), with h being a convex function with a piecewise smooth boundary curve of $h(\mathbb{D})$ having a finite number of corners, namely, with h from the subclass $\mathcal{Q}$ of convex functions, is considered. For the proof, Lemma 2.1 on harmonic mean is applied. Moreover, the well known Lemma 2.2 (see Lemma 2.2d of [13, p. 24]) was used in case when a dominant function h is from the class $\mathcal{Q}$. The class $\mathcal{Q}$ is enough general for typical applications.

2. Main result

In this section we prove the main theorem of this paper, namely, Theorem 2.4.

Given $A \subset \mathbb{C}$, by $\overline{A}$ we denote the closure of A in $\mathbb{C}$. For $\mathbb{H}$ being an open half-plane in $\mathbb{C}$, let $\overline{\mathbb{H}}^0 := \overline{\mathbb{H}} \setminus \{0\}$.

Define

$$\hbar(z) := \frac{1}{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

For the proof of Theorem 2.4 we need the following lemma.

Lemma 2.1. *Let $\mathbb{H}$ be a half-plane in $\mathbb{C}$ such that $0 \notin \mathbb{H}$. Then*

$$\bigwedge_{a,b \in \overline{\mathbb{H}}} \bigwedge_{t \in [0,1]} \left(b + t(a-b) \neq 0 \Rightarrow \mathrm{H}_t(a,b) \in \overline{\mathbb{H}} \right).$$

Proof. Let $\mathbb{H}$ be a half-plane in $\mathbb{C}$ such that $0 \notin \mathbb{H}$. Let a, $b \in \overline{\mathbb{H}}$ and $t \in [0,1]$ be such that

$$b + t(a-b) \neq 0. \tag{2.1}$$

When $t = 0$, then by (2.1), $b \neq 0$ and

$$\mathrm{H}_0(a,b) = a \in \overline{\mathbb{H}}.$$

When $t = 1$, then by (2.1), $a \neq 0$ and then

$$\mathrm{H}_1(a,b) = b \in \overline{\mathbb{H}}.$$

Let $t \in (0,1)$. Since $0 \notin \mathbb{H}$, either $0 \in \partial\mathbb{H}$ or $0 \notin \overline{\mathbb{H}}$.

Assume that $ab = 0$, which is possible only when $0 \in \partial\mathbb{H}$. When $a = 0$, then by (2.1), $(1-t)b \neq 0$, so $b \neq 0$. Thus

$$\mathrm{H}_t(a,b) = \mathrm{H}_t(0,b) = 0 = a \in \overline{\mathbb{H}}.$$

When $b = 0$, by (2.1), $ta \neq 0$, so $a \neq 0$. Thus

$$\mathrm{H}_t(a,b) = \mathrm{H}_t(a,0) = 0 = b \in \overline{\mathbb{H}}.$$

Assume that

$$ab \neq 0. \tag{2.2}$$

Then

$$\mathrm{H}_t(a,b) = \frac{1}{(1-t)\dfrac{1}{a} + t\dfrac{1}{b}} = \hbar\left(\mathrm{A}_t\left(\hbar(a), \hbar(b)\right)\right). \tag{2.3}$$

If $0 \in \partial\mathbb{H}$, then $\hbar(\mathbb{H})$ is an open half-plane, say $\mathbb{E}$, with $0 \in \partial\mathbb{E}$. Moreover

$$\hbar\left(\overline{\mathbb{H}}^0\right) = \overline{\mathbb{E}}^0. \tag{2.4}$$

If $0 \notin \overline{\mathbb{H}}$, then

$$\hbar\left(\overline{\mathbb{H}}\right) = \overline{\mathbb{D}}(\xi, |\xi|) \setminus \{0\} \tag{2.5}$$

for some $\xi \in \mathbb{C} \setminus \{0\}$.

To end proof, we consider the case (2.4). The case (2.5) follows analogously. In view of (2.2) we have a, $b \in \overline{\mathbb{H}}^0$, so from (2.4) it follows that

$$\{\hbar(a), \hbar(b)\} \subset \hbar\left(\overline{\mathbb{H}}^0\right) = \overline{\mathbb{E}}^0. \tag{2.6}$$

Since by (2.1),

$$\mathrm{A}_t\left(\hbar(a), \hbar(b)\right) = \frac{1-t}{a} + \frac{t}{b} = \frac{b + t(a-b)}{ab} \neq 0,$$

so hence and from (2.6) we get

$$\mathrm{A}_t\left(\hbar(a), \hbar(b)\right) \in \overline{\mathbb{E}}^0. \tag{2.7}$$

In this way, by (2.3), (2.7) and (2.4) we obtain

$$\begin{aligned} \mathrm{H}_t(a,b) &= \hbar\left(\mathrm{A}_t\left(\hbar(a), \hbar(b)\right)\right) \in \hbar\left(\overline{\mathbb{E}}^0\right) \\ &= \hbar\left(\hbar\left(\overline{\mathbb{H}}^0\right)\right) = \overline{\mathbb{H}}^0, \end{aligned}$$

which ends the proof.

Now we introduce the subclass $\mathcal{Q}$ of convex functions h with some natural regularity on the boundaries of $h(\mathbb{D})$ (for details on corners of curves see e.g., [14, pp. 51–65]).

Definition 2.2. By $\mathcal{Q}$ we denote the class of convex functions h with the following properties:

(a) $h(\mathbb{D})$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity),

(b) $E(h)$ is the set of all points $\zeta \in \mathbb{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbb{D})$,

(c) $h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbb{T} \setminus E(h)$.

The lemma below is similar to Lemma 2.3d of [13, p. 24]. However, in [1] it was proved in details for a dominant being a convex function in the class $\mathcal{Q}$, so with a piecewise smooth boundary curve having a finite number of corners.

Lemma 2.2. *Let $h \in \mathcal{Q}$ and $p \in \mathcal{H}$ be a nonconstant function with $p(0) = h(0)$. If p is not subordinate to h, then there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T} \setminus E(h)$ such that*

$$p\left(\mathbb{D}_{|z_0|}\right) \subset h(\mathbb{D}) \tag{2.8}$$

and

$$p(z_0) = h(\zeta_0). \tag{2.9}$$

The lemma below is a modification of Lemma 2.2c [13, p. 22]. In this form it follows directly from Jack's Lemma [4].

Lemma 2.3. *Let $h \in \mathcal{H}$ be univalent and assume that $h'(\zeta_0) \neq 0$ at $\zeta_0 \in \mathbb{T}$ exists. Let $p \in \mathcal{H}$ be a nonconstant function with $p(0) = h(0)$. If $z_0 \in \mathbb{D} \setminus \{0\}$ is such that* (2.8) *and* (2.9) *hold, then*

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0)$$

for some $m \geq 1$.

The main theorem of this paper is the following.

Theorem 2.4. *Let $\beta \in [0, 1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$ and $\Psi \in \mathcal{H}$ be such that*

$$\operatorname{Re} \Psi(z) > 0, \quad z \in \mathbb{D}. \tag{2.10}$$

If $p \in \mathcal{H}(\beta; \Psi)$, $p(0) = h(0)$ and

$$H_{\beta;\Psi,p} \prec h, \tag{2.11}$$

then

$$p \prec h. \tag{2.12}$$

Proof. Since by (1.7) we have $H_{0;\Psi,p} \equiv p$, so for $\beta := 0$, the subordination (2.12) holds evidently.

Let $\beta \in (0, 1]$. Suppose that $p \equiv p(0) \in \mathcal{H}(\beta; \Psi)$. Then by (1.4) and (1.5), we have $H_{\beta;\Psi,p} \equiv p$, so (2.12) holds evidently again.

Let now $p \in \mathcal{H}(\beta; \Psi)$ be nonconstant and, on the contrary, suppose that p is not subordinate to h. By Lemma 2.2, there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T} \setminus E(h)$ such that (2.8) and (2.9) hold. Since $h \in \mathcal{Q}$, so $h'(\zeta_0) \neq 0$ exists. Hence, applying Lemma 2.3 we have

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) \tag{2.13}$$

for some $m \geq 1$. Let

$$a := p(z_0), \tag{2.14}$$

and

$$b := p(z_0) + z_0 p'(z_0)\Psi(z_0). \tag{2.15}$$

From (2.9), (2.14), (2.13) and (2.15) we have

$$a = h(\zeta_0), \quad b = h(\zeta_0) + m\zeta_0 h'(\zeta_0)\Psi(z_0). \tag{2.16}$$

Let $\mathbb{H}$ be an open half-plane supporting the convex domain $h(\mathbb{D})$ at $h(\zeta_0)$. Thus

$$a = h(\zeta_0) \in \partial\overline{\mathbb{H}} \tag{2.17}$$

and

$$h(\mathbb{D}) \cap \overline{\mathbb{H}} = \emptyset. \tag{2.18}$$

Since $\zeta_0 h'(\zeta_0) \neq 0$ is an outward normal to $\partial h(\mathbb{D})$ at $h(\zeta_0)$, so from (2.17), (2.10) with $z := z_0$, and (2.18) we obtain

$$b = h(\zeta_0) + m\zeta_0 h'(\zeta_0)\Psi(z_0) \in \overline{\mathbb{H}}. \tag{2.19}$$

Hence and in view of (2.17) we have

$$\{a, b\} \subset \overline{\mathbb{H}}. \tag{2.20}$$

Consequently,

$$b + \beta(a-b) \in \overline{\mathbb{H}}.$$

Moreover, by (2.14) and (2.15) we have

$$P_{0;\Psi,p}(z_0) = a, \quad P_{1;\Psi,p}(z_0) = b \tag{2.21}$$

and

$$\begin{aligned} P_{1-\beta;\Psi,p}(z_0) &= p(z_0) + (1-\beta) z_0 p'(z_0)\Psi(z_0) \\ &= \beta p(z_0) + (1-\beta)\left(p(z_0) + z_0 p'(z_0)\Psi(z_0)\right) \\ &= \beta a + (1-\beta) b = b + \beta(a-b). \end{aligned} \tag{2.22}$$

(A) Suppose that

$$b + \beta(a-b) \neq 0,$$

i.e., by (2.22), that

$$P_{1-\beta;\Psi,p}(z_0) \neq 0.$$

Thus $z_0 \in \mathbb{D} \setminus P^{-1}_{1-\beta;\Psi,p}(0)$ and by (2.21) and (2.22) we obtain

$$\begin{aligned} H_{\beta;\Psi,p}(z_0) &= \frac{P_{0;\Psi,p}(z_0) P_{1;\Psi,p}(z_0)}{P_{1-\beta;\Psi,p}(z_0)} \\ &= \frac{ab}{b + \beta(a-b)} = \mathrm{H}_\beta(a, b). \end{aligned}$$

Taking into account (2.20), by applying Lemma 2.1, we get

$$H_{\beta;\Psi,p}(z_0) = \mathrm{H}_\beta(a, b) \in \overline{\mathbb{H}}.$$

Hence and from (2.18) it follows that

$$H_{\beta;\Psi,p}(z_0) \notin h(\mathbb{D}),$$

which contradicts (2.11).

(B) Suppose that

$$b + \beta(a-b) = 0, \tag{2.23}$$

i.e., by (2.22), that

$$P_{1-\beta;\Psi,p}(z_0) = 0.$$

Thus $z_0 \in P^{-1}_{1-\beta;\Psi,p}(0)$. Since, from the assumption, the finite limit

$$H_{\beta;\Psi,p}(z_0) = \lim_{\mathbb{D} \ni \zeta \to z_0} \frac{P_{0;\Psi,p}(\zeta) P_{1;\Psi,p}(\zeta)}{P_{1-\beta;\Psi,p}(\zeta)}$$

exists, so

$$P_{0;\Psi,p}(z_0) P_{1;\Psi,p}(z_0) = 0,$$

i.e. by (2.21),

$$ab = 0.$$

Suppose that

$$a = 0. \tag{2.24}$$

Hence and from (2.23) we have

$$(1 - \beta)b = 0. \tag{2.25}$$

Note now that in view of (2.16) and (2.24),

$$b = m\zeta_0 h'(\zeta_0)\Psi(z_0).$$

But $h'(\zeta_0) \neq 0$ and by (2.10), $\Psi(z_0) \neq 0$. Thus $b \neq 0$. Hence it follows that (2.25) holds if and only if $\beta = 1$. Consequently, by (1.8), (2.15) and (2.19) we get

$$H_{1;\Psi,p}(z_0) = p(z_0) + z_0 p'(z_0)\Psi(z_0) = b \in \overline{\mathbb{H}}.$$

This together with (2.18) yield

$$H_{1;\Psi,p}(z_0) \notin h(\mathbb{D}),$$

which contradicts (2.11).

Suppose that

$$b = 0. \tag{2.26}$$

Hence and from (2.23) we have $\beta a = 0$. Since $\beta \neq 0$, so $a = 0$. Thus by (2.16), we see that (2.26) is equivalent to

$$\zeta_0 h'(\zeta_0)\Psi(z_0) = 0. \tag{2.27}$$

However, $h'(\zeta_0) \neq 0$ and by (2.10), $\Psi(z_0) \neq 0$, so (2.27) does not hold. This ends the proof of the theorem.

Corollary 2.5. *Let $\beta \in [0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$ and $\Psi \in \mathcal{H}$ be such that* (2.10) *hold. If*

$$p \in \mathcal{H}^0(\beta;\Psi), \quad p(0) = h(0)$$

and

$$\frac{p(z)(p(z) + zp'(z)\Psi(z))}{p(z) + (1-\beta)zp'(z)\Psi(z)} \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

For $\beta := 1$, we get

Corollary 2.6. *Let $h \in \mathcal{Q}$ and $\Psi \in \mathcal{H}$ be such that* (2.10) *hold. If $p \in \mathcal{H}$ and*

$$p(z) + zp'(z)\Psi(z) \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

References

[1] N. E. Cho, O. Chojnacka, and A. Lecko, *On differential subordination of geometric mean*, Bull. Soc. Sci. Lettres Łódź (submited).

[2] O. Crişan and S. Kanas, *Differential subordinations involving arithmetic and geometric means*, Appl. Math. Comp. **222** (2013), 123–131.

[3] D. J. Hallenbeck and S. Ruscheweyh, *Subordination by convex function*, Proc. Amer. Math. Soc. **52** (1975), 191–195.

[4] I. S. Jack, *Functions starlike and convex of order alpha*, J. London Math. Soc. **3** (1971), 469–474.

[5] S. Kanas, A. Lecko, and J. Stankiewicz, *Differential subordinations and geometric means*, Complex Variables **28** (1996), 201–209.

[6] Y. C. Kim and A. Lecko, *On differential subordination related to convex functions*, J. Math. Anal. Appl. **235** (1999), 130–141.

[7] A. Lecko, *On differential subordinations and inclusion relation between classes of analytic functions*, Complex Variables **40** (2000), 371–385.

[8] A. Lecko and M. Lecko, *Differential subordinations of arithmetic and geometric means of some functionals related to a sector*, Int. J. Math. Math. Sci. **2011** (2011), no. Article ID 205845, 1–19.

[9] ______, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56** (1976), 111–117.

[10] S. S. Miller and P. T. Mocanu, *Second order differential inequatities in the complex plane*, J. Math. Anal. Appl **65** (1978), no. 2, 289–305.

[11] ______, *Differential subordination and univalent functions*, Mich. Math. J. **28** (1981), 157–171.

[12] ______, *On some classes of first-order differential subordinations*, Mich. Math. J. **32** (1985), 185–195.

[13] ______, *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York-Basel 2000.

[14] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, Berlin-Heidelberg-New York 1992.

Department of Analysis and Differential Equations
University of Warmia and Mazury
Słoneczna 54, PL-10-710 Olsztyn
Poland
e-mail: alecko@matman.uwm.edu.pl

Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 24, 2014

O PEWNYM PODPORZĄDKOWANIU RÓŻNICZKOWYM ŚREDNIEJ HARMONICZNEJ

S t r e s z c z e n i e

Dla $t \in [0,1]$ i a, $b \in \mathbb{C}$ takich, że $b + t(b-a) \neq 0$, niech

$$\mathrm{H}_t(a,b) := \frac{ab}{b + t(a-b)}$$

oznacza średnią harmoniczną liczb a i b. W pracy tej badane jest różniczkowe podporządkowanie następującej postaci:

$$\mathrm{H}_\beta(p(z), p(z) + zp'(z)\Psi(z)) \prec h(z) \Rightarrow p(z) \prec h(z), \quad z \in \mathbb{D},$$

gdzie $\beta \in [0,1]$, zaś p, h i Ψ spełniają potrzebne założenia.

Słowa kluczowe: podporządkowanie różniczkowe, średnia arytmetyczna, średnia geometryczna, średnia harmoniczna, funkcja wypukła

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 41–56

In memory of
Professor Zygmunt Charzyński (1914–2001)

Tomasz Kapitaniak, Anna Karmazyn, Przemysław Perlikowski,
and Andrzej Stefański

SYNCHRONIZATION IN COUPLED MECHANICAL OSCILLATORS

Summary

The purpose of this paper is to present that coupled mechanical oscillator can be synchronised. Moreover, for identical pendulums as well as for non-identical pendulums synchronization can be observed. In particular, complete synchronization is obtained in coupled identical mechanical oscillators, and imperfect complete synchronization is obtained in coupled non-identical mechanical oscillators.

Keywords and phrases: complete synchronizastion, imperfect complete synchronizastion, synchronization in phase, synchronization in anti-phase, Lyapunov exponents, bifurcation diagram, Poincaré map

1. Introduction

Synchronization is a powerful basic concept in nature regulating a large variety of processes. Synchronous behaviour has attracted a large interest in physics, engineering, biology, ecology, sociology, communication and other fields of science and technology [1–3]. It is known that synchrony is rooted in human life from the metabolic processes in our cells to the highest cognitive tasks we perform as a group of individuals. Therefore, synchronization phenomena have been extensively studied and models robustly capturing the dynamical synchronization process have been proposed.

Historically, the analysis of synchronization in the evolution of dynamical systems has been a subject of investigation since the earlier days of physics [4]. It started when

Christian Huygens find that two very weakly coupled pendulum clocks (hanging at the same beam) become synchronized in phase [5–10]. In the last two decades it has been demonstrated that any set of chaotic systems can synchronize by linking them with mutual coupling or with a common signal or signals [11–14]. Synchronization has been to employ control theory as a control problem. Particularly this approach can be applied in robotics when two or more robot – manipulators have to operate synchronously in a dangerous environment [15, 16]. Pogromsky et al. [17] designed a controller for a synchronization problem comprising two pendulous suspended on an elastically supported rigid beam.

The phenomenon of synchronization in dynamical and, in particular, mechanical systems has been known for a long time. The idea of synchronization has been adopted for chaotic systems [11–14, 18–20]. It has been demonstrated that two or more chaotic systems can synchronize by linking them with mutual coupling or with a common signal or signals. In the case of linking a set of identical chaotic systems, which is connected with the same set of ODE's and values of the system parameters, complete synchronization (CS) can be obtained. The CS takes place when all trajectories converge to the same value and remain in step with each other during further evolution (i.e., $\lim t - > \infty ||\mathbf{x}(t) - \mathbf{y}(t)|| = 0$ for two arbitrarily chosen trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$). In such a situation all subsystems of the augmented system evolve on the same manifold on which one of these subsystems evolves (the phase space is reduced to the synchronization manifold). Linking homochaotic systems (i.e., systems given by the same set of ODE's but with different values of the system parameters) can lead to imperfect synchronization (i.e., $\lim_{t->\infty} ||\mathbf{x}(t) - \mathbf{y}(t)|| = \varepsilon$, where ε is a vector of small parameters) or to the phase synchronization. In such linked systems it can be also observed a significant change of the chaotic behaviour of one or more systems.

CS was the first discovered and is the simplest form of synchronization in chaotic systems [13, 19, 20]. In the early 1980s, Fujisaka and Yamada [19, 20] showed how two identical chaotic oscillators under variation of the coupling strength can attain a state of complete synchronization in which the motion of the coupled system takes place on an invariant subspace in the phase space, the synchronization manifold. It is also described in the literature as the identical or full synchronization [11, 22].

Synchronization of real systems can not be defined by the same relationship as in the case of identical oscillators. It is impossible to obtain two identical springs, dampers, resistors, etc., each of these elements have tolerances and differences in the structure of the material. Taking into account this fact in experimental settings must be applied other criteria for detection of CS. Introduced, so the concept of imperfect complete synchronization (ICS). Among the many works on this subject it is worth mentioning an article which presents the experimental results. Blakely et. al [21] presented in the theoretical part of the work a full review of criteria for the occurrence of ICS, while presenting the results of the experimental part of the modelled electrical oscillators. None of the criteria are not allowed for precise determination of the behaviour of coupled oscillators (ICS threshold). The transition to phase synchronization for two coupled oscillators has been firstly characterized with reference to the Rössler system [14].

Coupling through an elastic structure allows one to investigate how the dynamics of the particular pendulum is influenced by the dynamics of other subsystems and this is the main purpose of the research. Synchronization in coupled dynamical systems is associated with the emergence of collective coherent behaviour between identical or similar subsystems. In this paper attention is focused on the possibility of the occurrence of complete and phase synchronization of pendulums.

The majority of the oscillatory systems that we meet in everyday life experience some sort of irreversible energy loss whilst they are in motion, which is due, for instance, to frictional or viscous heat generation. We would therefore expect oscillations excited in such systems to eventually be damped away. A pendulum is a well known concept and remains very common in the research [23–29]. The pendulum's attraction and interest is associated with the familiar regularity of its swings, and as the consequence its relation to the fundamental natural force of gravity.

2. Model of the system

The dynamics of physical pendulums coupled through an elastic structure, as shown in Fig. 1a, is considered. The adopted model takes the form of one mass M_{b1} concentrated in the point connected to the light elastic beam. To the mass attached two identical physical pendulums (on each side of the beam). In the present case, the excitation on both sides of the elastic beam are the same $x_{z1} = x_{z2} = x_z = z\ \sin\Omega t$. We assumed that the mass M_{b1} was located in the middle of the beam.

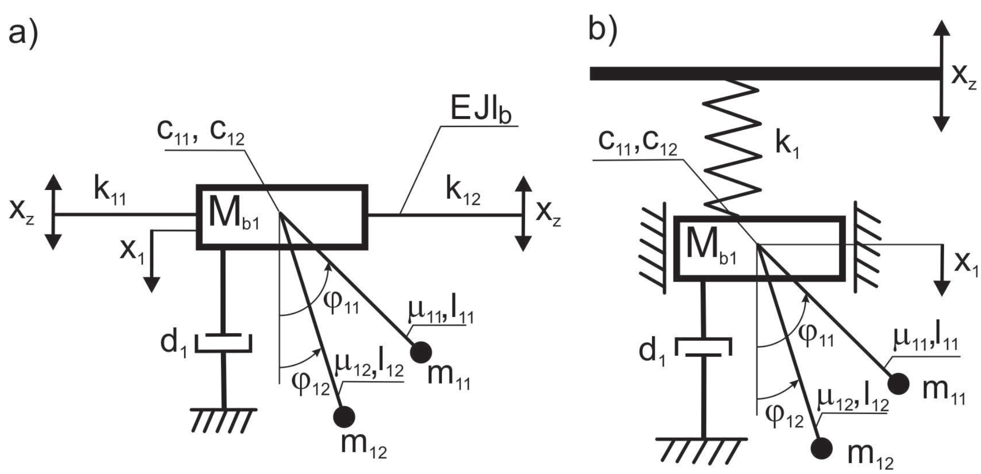

Fig. 1: a) System of one mass and two pendulums located on (coupled through) a light elastic structure, b) the elastically supported mass with two physical pendulums.

It should be pointed out that this work is focused on the analysis of a quite general model of the coupling trough the elastic structure (beam). This coupling is common in mechanical systems. The continuous beam of mass M_b was replaced by the massless beam. It is based on flexibility coefficients method [30]. The stiffness of the beam fulfils the relation $[k_{ji}] = [a_{ji}]^{-1}$, where $[a_{ji}]$ is the 2×2 dimensional matrix of flexibility coefficients and it is dependent on the quantity $E\,J\,l_b$ and the location of mass M_{b1}.

In addition, we assumed that the massless beam was displaced by the spring with a stiffness k_1. Mass M_{b1} are constrained to move only in vertical direction and thus are described by the coordinate x_1. The considered discrete model is shown in Fig. 1b.

The next assumption is that the beam is simply supported at both ends. Hence, we have the following boundary conditions:

$$z(0,\ t) = 0, \quad z(l_b,\ t) = 0, \quad \frac{d^2\, z(0,\ t)}{d\, x^2} = 0 \quad \text{and} \quad \frac{d^2\, z(l_b,\ t)}{d\, x^2} = 0.$$

The equations of motion of system presenting in Fig. 1b can be derived using Lagrange equations of the second type. The kinetic energy T, potential energy V and Rayleigh dissipation D are given respectively by

$$\begin{aligned}
T &= \frac{1}{2}\, M_{b1}\, \dot{x}_1^2 + \frac{1}{2} B_{11}\, \dot{\varphi}_{11}^2 + (m_{11} + \mu_{11})\, b_{11}\, \dot{x}_1\, \dot{\varphi}_{11}\, \sin\varphi_{11} \\
&\quad + \frac{1}{2} B_{12}\, \dot{\varphi}_{12}^2 + (m_{12} + \mu_{12})\, b_{12}\, \dot{x}_1\, \dot{\varphi}_{12}\, \sin\varphi_{12}, \\
V &= (m_{11} + \mu_{11})\, g\, b_{11}\, (1 - \cos\varphi_{11}) + (m_{12} + \mu_{12})\, g\, b_{12}\, (1 - \cos\varphi_{12}), \\
D &= \frac{1}{2}\, d_1\, \dot{x}_1^2 + \frac{1}{2}\, c_{11}\, \dot{\varphi}_{11}^2 + \frac{1}{2}\, c_{12}\, \dot{\varphi}_{12}^2,
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
B_{11,\,12} &= B_{S1,S2} + (m_{11,\,12} + \mu_{11,\,12})\, {b_{11,\,12}}^2, \\
M_1 &= M_{b1} + m_{11} + \mu_{11} + m_{12} + \mu_{12}.
\end{aligned} \tag{2}$$

M_{b1} is mass of the oscillator [kg], $m_{11,12}$ is mass concentrated at the point in the end of rod [kg], $\mu_{11,12}$ is mass of the rod [kg], $l_{11,12}$ is length of the rod [m], $B_{1\ S1,1\ S2}$ is the inertia moment of the mass [kg m^2] given by

$$B_{1\ S1,1\ S2} = m_{11,12}(l_{11,12} - b_{11,12})^2 + \frac{1}{12}\mu_{11,12} l_{11,12}^2 + \mu_{11,12}(b_{11,12} - \frac{1}{2} l_{11,12})^2$$

and

$$b_{11,12} = \frac{(m_{11,12} + \frac{1}{2}\mu_{11,12})l_{11,12}}{(m_{11,12} + \mu_{11,12})}$$

is distance from the center of mass to the center of rotation of the rod and ball $m_{11,12}$, $c_{11,12}$ is damping factor at the node [Nms], d_1 is viscous damping [Ns/m], g is acceleration due to the gravity [m/s^2]. Parameters of the beam: mass M_b [kg], length l_b [m], modulus of elasticity E [N/m^2] and the inertial momentum of cross-section I [m^4].

We obtain the following equation describing the dynamics of the 3DOF system (M_{b1}, m_{11} and m_{12}):

$$B_{11}\,\ddot{\varphi}_{11} + A_{11}\,\ddot{x}_1\,\sin\varphi_{11} + A_{11}\,g\,\sin\varphi_{11} + c_{11}\,\dot{\varphi}_{11} = 0,$$

$$B_{12}\,\ddot{\varphi}_{12} + A_{12}\,\ddot{x}_1\,\sin\varphi_{12} + A_{12}\,g\,\sin\varphi_{12} + c_{12}\,\dot{\varphi}_{12} = 0,$$

$$M_1\,\ddot{x}_1 + A_{11}\,(\ddot{\varphi}_{11}\,\sin\varphi_{i1} + \dot{\varphi}_{11}^2\,\cos\varphi_{11}) + A_{12}\,(\ddot{\varphi}_{12}\,\sin\varphi_{12} + \dot{\varphi}_{12}^2\,\cos\varphi_{12})$$
$$+ d_1\,\dot{x}_1 + k_1\,(x_1 - x_z) = 0,$$

where $A_{11,\,12} = (m_{11,\,12} + \mu_{11,\,12})\,b_{11,\,12}$, k_{11} is spring stiffness coefficients [N/m].

2.1. Identical pendulums

In the numerical simulations of the system described by Eq. (3) the following parameter values for the identical pendulums were used:

$M_{b1} = 0.2$ [kg]	$M_1 = 0.5$ [kg]	$k_1 = 600$ $[\frac{\mathrm{N}}{\mathrm{m}}]$
$m_{11,12} = 0.1$ [kg]	$A_{1,\,2} = 0.01875$ [kgm]	$c_{11,12} = 0.01$ [Nms]
$\mu_{11,12} = 0.05$ [kg]	$B_{11,12} = 0.002625$ $[\frac{\mathrm{kg}}{\mathrm{m}^2}]$	$d_1 = 1.5$ $[\frac{\mathrm{Ns}}{\mathrm{m}}]$
$l_{11,12} = 0.15$ [m]	$b_{11,12} = 0.125$ [m]	$z = 0.01$ [m]

and frequency Ω is a control parameter. Above values correspond to real parameters measured and estimated on the experimental rig. Damping coefficients were approximated with classical free vibrations probe.

Therefore, the mass of this single pendulum was divided symmetrically between both pendulums in Eq. (3). Remaining parameters were left unchanged. The pendulums are released from the initial conditions $\varphi_{11} = 3.0$, $\dot{\varphi}_{11} = 0.0$, $\varphi_{12} = 2.0$, $\dot{\varphi}_{12} = 0.0$, $x_1 = 0.05$, $\dot{x}_1 = 0.0$. As an external excitation, we have chosen a sinusoidal signal with amplitude $z = 0.01$ [m].

In Figs. 2a–c and their enlargements in Figs. 2a–c bifurcation diagrams of mass and pendulums positions versus driving frequency are demonstrated. We can see that nonlinear oscillations are activated near the principal resonance frequency $\omega = \sqrt{\frac{k_{11}}{M_1}}$.

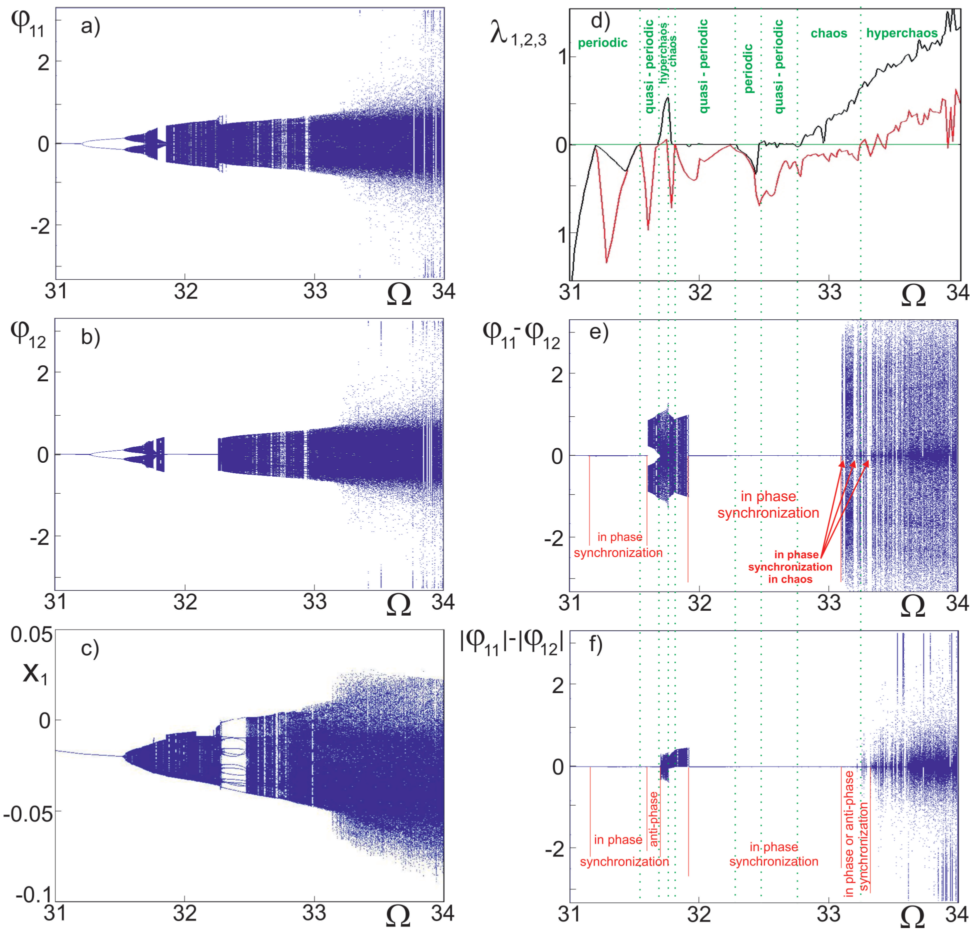

Fig. 2: Bifurcation diagrams for the case of identical pendulums Eq. (3): a) pendulum angle φ_{11}, b) mass displacement x_1 versus Ω. Diagram of Lyapunov exponents λ_1, λ_2, λ_3 (the green one is equal to zero) (d) and corresponding bifurcation diagrams presenting the occurrence of synchronization $\varphi_{11} - \varphi_{12}$ (e), $|\varphi_{11}| - |\varphi_{12}|$ (f) versus Ω.

In Figs. 2d–f (corresponding to Figs. 2a–c) course of largest Lyapunov exponents (Fig. 2d) and bifurcation diagrams depicting an occurrence of complete (Fig. 2e) and combined in phase and in anti-phase synchronization (Fig. 2f) are demonstrated. These diagrams clearly illustrate the sequence of bifurcations, dominant solutions and their correlation with the synchronization of pendulums. For the frequency $\Omega = 31.22\ [\frac{\text{rad}}{\text{s}}]$ we can see the tendency of pendulums to stabilize in the lower stationary position (Figs. 2a–b). The period doubling bifurcation at $\Omega = 31.22\ [\frac{\text{rad}}{\text{s}}]$ stimulate periodic, completely synchronized response of the pendulums within the interval $31.22 < \Omega < 31.64\ [\frac{\text{rad}}{\text{s}}]$ In this range all the values of Lyapunov exponents are negative ($\lambda_i < 0$). Further increase of driving frequency in the interval under

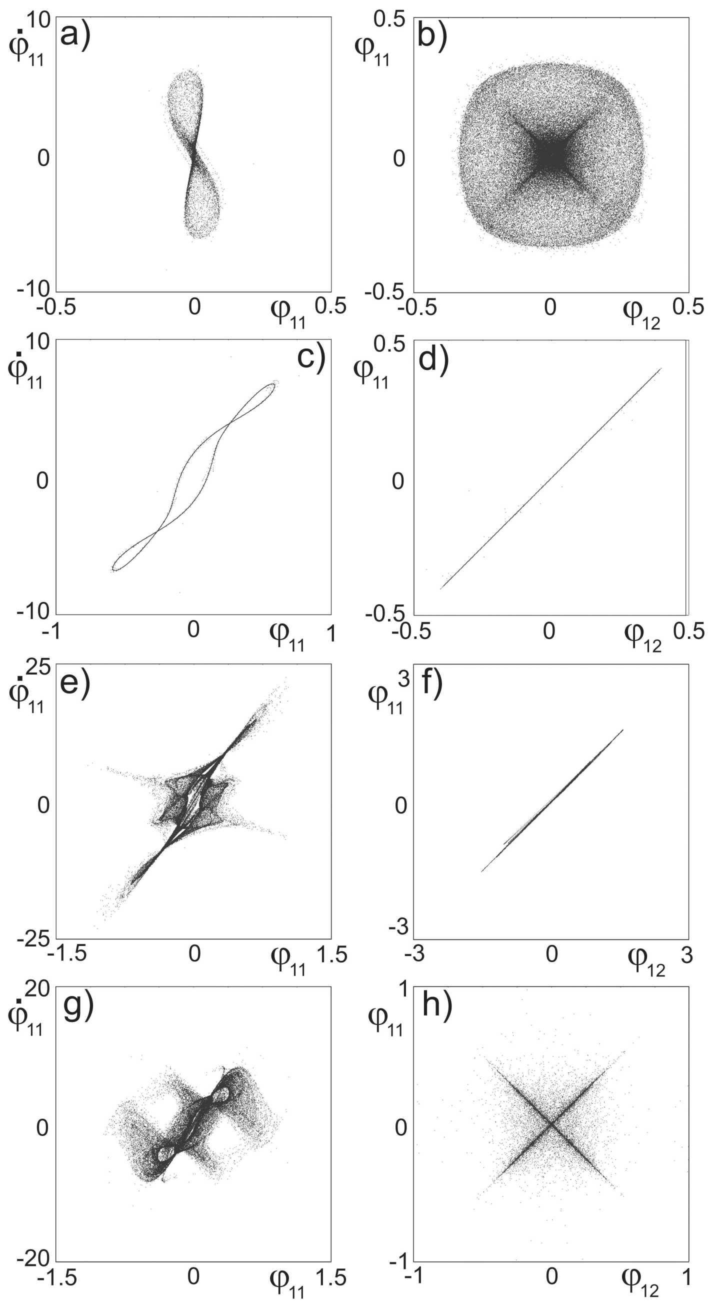

Fig. 3: Poincaré maps (left column) and synchronization diagrams (right column) for identical pendulums for $\Omega = 31.75$ $[\frac{\text{rad}}{\text{s}}]$ (a, b), for $\Omega = 32.20$ $[\frac{\text{rad}}{\text{s}}]$ (c, d), for $\Omega = 33.20$ $[\frac{\text{rad}}{\text{s}}]$ (e, f) and for $\Omega = 33.50$ $[\frac{\text{rad}}{\text{s}}]$ (g, h).

consideration (up to $\Omega = 34.00$ $[\frac{\text{rad}}{\text{s}}]$) results in alternately appearing intervals of periodic, quasi-periodic, chaotic (one positive Lyapunov exponent, see Fig. 2d) and finally hyperchaotic (two positive Lyapunov exponents, see Fig. 2d) states.

Comparison of synchronous intervals in Figs. 2e and 2f indicates coexistence of phase and anti-phase synchronization of pendulums. In Figs. 3 Poincaré maps demonstrating the pendulum dynamics (left column) and corresponding synchronization tendency (right column) are shown. We can see that pendulums desynchronization is typical for hyperchaos (Figs. 3a–b, 3g–h). On the other hand, the synchronous behaviour is possible in chaotic (Fig. 3e–f) and quasi-periodic (Fig. 3c–d) regime of pendulums oscillation. Such phenomena of chaotic and quasi-periodic synchronization is verified by comparison of Figs. 2d and 2f where in-phase or anti-phase synchronization range (Fig. 2f) is corresponding to the scope of one positive Lyapunov exponent (Fig. 2d). In-phase and anti-phase synchronous regimes in the system under consideration are demonstrated with time series in Fig. 4a and Fig. 4b, respectively.

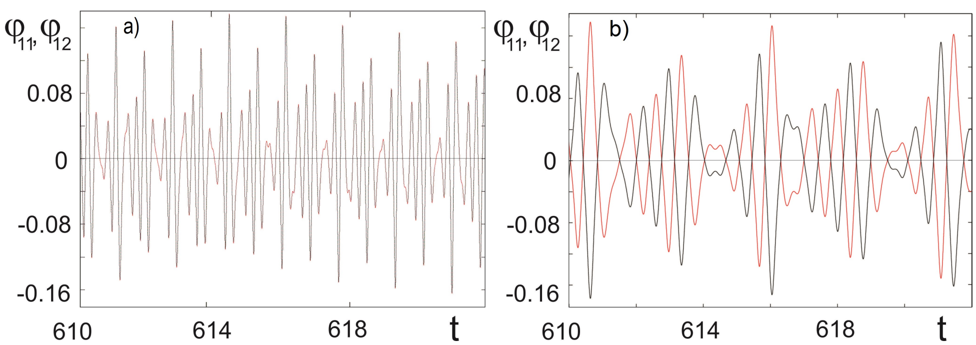

Fig. 4: Time diagram for identical pendulums φ_{11}, φ_{12} for a) $\Omega = 32.20$ $[\frac{\text{rad}}{\text{s}}]$ – pendulums in phase (complete synchronization) and b) $\Omega = 33.28$ $[\frac{\text{rad}}{\text{s}}]$ – pendulums in anti-phase (anti-phase synchronization).

3. Experimental studies

Coupled dynamical systems whose elements can be treated as mathematical or physical pendulum, are widely used technical and engineering devices. They may be parts of machines, such as components of cranes and port crane, where the pendulum plays the role of a crane arm. Also each suspension element, which during motion of the machine varies, can be considered as the pendulum, e.g., the engine suspension system for an aircraft wing. Furthermore, pendulums are increasingly used to eliminate the vibrating components of bridges, tall chimneys and towers. Oscillating systems of multiple degrees-of-freedom, comprising of pendulums, may exhibit the phenomenon of energy transfer between the degrees-of-freedom as a result of various types of couplings.

The flow of energy can be partial or complete, and it depends on the choice of parameters. It turns out that the total energy transfer occurs when the ratio of self-oscillation frequency is equal to the ratio of the integers, i.e., when this frequencies are commensurate. This phenomenon is called an internal resonance. If we have damping in the system, then the internal resonance of the vibration with one frequency or the vibration in the type of beats (with superimposed at least two frequencies) may occur.

In systems with many degrees-of-freedom with inertial coupling the parametric vibration may appear. Parametric vibration problem is quite complex, and solutions are only known for a specific form of the equations in the case of equations with one degree-of-freedom. The phenomenon of excited of auto-parametric vibration have been observed in the thirties of the last century by Gorelik and Witt [31] and many others [32–36].

Fig. 5: An experimental rig consisting of the beam with pendulums mounted on the electrodynamic shaker.

In our experiments, we have used the rig with one pair of pendulums shown in Fig. 6. The vertical oscillations can be seen here as a blurry contour of the rig frame. Before numerical simulations and experiments some basic measurements of the individual pendulums have been carried out.

In order to verify the numerical results in practice, the experimental study has been conducted. Real parameters of experimental rig approach nominally identical

Fig. 6: Experimental rig consisting of one oscillator node ($n = 1$).

values assumed in numerical simulations. Parameters of pendulums: lengths $l_{11} = l_{12} = 0.15$ [m], masses of the rods $\mu_{11} = \mu_{12} = 0.05$ [kg] with masses $m_{11} = m_{12} = 0.10$ [kg], mass at the end $M_{b1} = 0.2$ [kg]. Parameters of the beam: mass $m_b = 0.64$ [kg], length $l_b = 1.0$ [m], height $h = 0.002$ [m] and width $a = 0.03$ [m], modulus of elasticity $E = 0.74e^{11}$ [N/m^2], the inertial momentum of cross-section I [m^4] and $EI = 1.48$ [Nm^2]. The rig has been mounted on the shaker LDS V780 Low Force Shaker (basic data are as follows: sine force peak 5120 [N]; max random force (rms) 4230 [N]; max acceleration sine peak $g_n = 0.111$ [m]; system velocity sine peak 1.9 [m/s]; displacement pk-pk $g_n = 0.254$ [m]; moving element mass 4.7 [kg]). The shaker introduces kinematic periodic excitation $z \cos \Omega t$, where z and Ω are the amplitude and the frequency of the excitation, respectively. At initial moments of the lower pendulum have been assumed to be in the upper position $\varphi_{1i} = \pi \pm \pi/36$ for $i = 1,\ 2$. We fix the value of the excitation amplitude $z = 0.0082 \pm 0.004$ [m] and consider excitation frequency Ω as a control parameter. The rig was excited around its natural frequency Ω in the approximated range between 5 and 7 [Hz], i.e. approximately $30 - 40$ [rad/s].

The amplitude of external excitation $z = 0.0082$ [m] was calculated for $n = 30$ peaks from following formula

$$z = \frac{\sum_{i=1}^{n} z^i_{\max} - z^i_{\min}}{n}. \tag{3}$$

In designing the rig, we deliberately chosen identical masses and length of elements (the differences in masses and length are about 1% between the maximum and the minimum values). Our goal was to check if the theoretically predicted synchronization of the nominally identical pendulums can be observed experimentally. Dynamics of the system has been video recorded and the beam and pendulum's trajectories have been determined using image analysis software *Kinovea* [37].

In Fig. 6 experimental rig consisting of one mass-pendulums component is presented, whereas, in Figs. 7 a–b we see pictures documenting synchronous behaviour of the pendulums – in phase and in anti-phase, respectively, observed during the experiment.

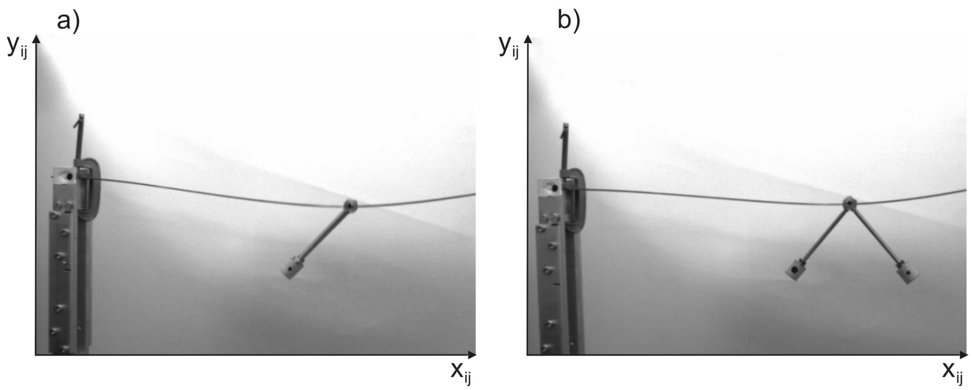

Fig. 7: Experiment: a) phase synchronization (practical): $\varphi_{11} = \varphi_{12}$, b) anti-phase synchronization: $\varphi_{11} = -\varphi_{12}$ for $\Omega = 36$ [rad/s].

In general, carried out experimental tests brought a variety of interesting dynamical behaviours of pendulums, e.g. chaotic rotation of the first pendulum while the second remains in the top position of equilibrium or anti-phase synchronization of the quasi-periodic behaviour of pendulums.

An exemplary analysis of experimental results is demonstrated in Figs. 8 and 9. The measurement equipment allows us to detect a position of the pendulum in an orthogonal coordinate system $(x_{ij},\ y_{ij})$ as shown in Fig. 8. Hence, angular displacement can be determined from obvious relation

$$\tan(\varphi_{ij}) = \frac{y_{ij}}{x_{ij}}. \tag{4}$$

In Fig. 9 a diagram illustrating the synchronization tendency of both pendulums (i.e., vertical position x_{11} and x_{12} of pendulums), corresponding to Fig. 8, is shown. Due to applied method of collecting the experimental data in orthogonal directions, trajectories shown in Fig. 8 and Fig. 9 seem to represent a case of irregular motion. However, we have to remember that collected signal also contain the vertical displacement of the beam with the mass. Thus, in fact pendulum's oscillations are periodic. This is one of the most interesting cases, where imperfect complete synchro-

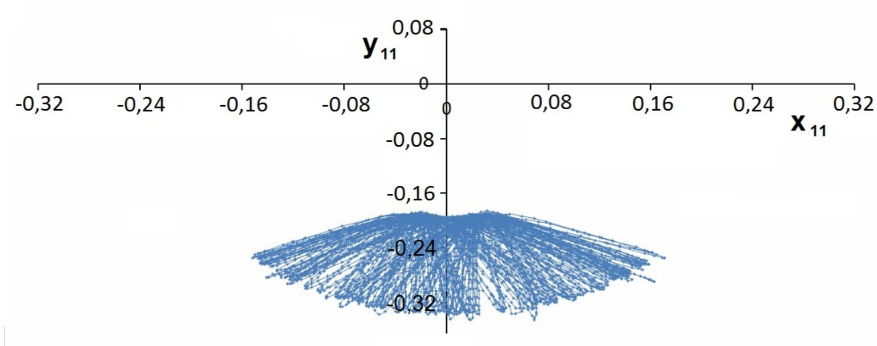

Fig. 8: Trajectory of the motion of pendulum described by angle φ_{11} at the plane $(x_{11},\ y_{11})$ for $\Omega = 36$ [rad/s].

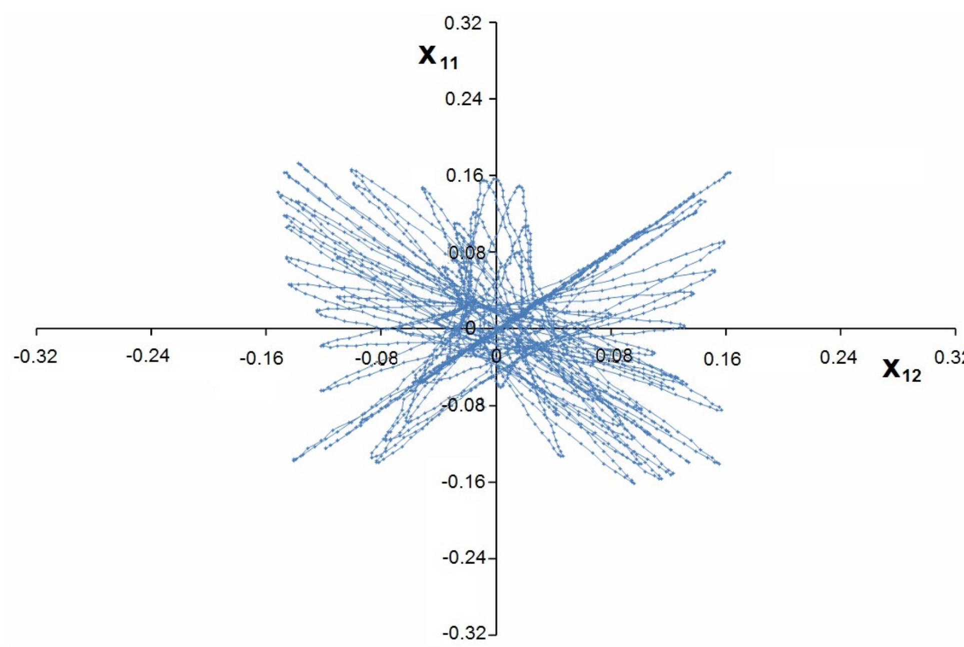

Fig. 9: Experiment: coexistence of phase synchronization (practical) – $\varphi_{11} = \varphi_{12}$ and anti-phase synchronization – $\varphi_{11} = -\varphi_{12}$ for $\Omega = 36$ [rad/s].

nization in phase (Fig. 7a) and in anti-phase (Fig. 7b) is observed during periodic motion of pendulums (Fig. 8). One can observe transition between the synchronization in phase and in anti-phase - Fig. 9, which was also obtained in the computer simulations. On the other hand, the synchronization of chaotic pendulums has not been observed experimentally.

4. Conclusions

To summarize, during the analysis, particular attention was paid to study the phenomenon of synchronization between pendulums suspended on the elastic beam. This phenomenon has been detected both in numerical simulations and during the experiment in the case of one 3DOFs oscillator (mass with two pendulums) located on the beam.

Especially noteworthy here is the chaotic synchronization of pendulums revealed in case of one oscillator with two pendulums (Eqs (3). This fact can be treated as an original and scientifically valuable result of this paper. Occurrence of synchronization is common for periodic or quasi-periodic system's response (Fig. 2d-f).

Numerical analysis of the system described by Eqs (3)a–c demonstrates that due to vertical direction of forcing, angular oscillations and rotations of pendulums are activated only in the direct neighbourhood of principal natural frequency of the system. Outside this range of excitation frequency we observe an extinction of angular oscillations of pendulums and system described by Eqs (3a–c) is reduced to 1DOF linear oscillator executing vertical vibration.

However, close to resonance high amplitude of the mass excites nonlinear response of pendulums and transition nonlinear dynamics takes place. The Ω-interval of chaotic synchronization is clearly depicted in Figs. 2d–f. This is complete synchronization in phase or in anti-phase, which are equivalent coexisting solutions. Chaotic synchronization state is characterized with one positive Lyapunov exponent (see Fig. 2d). Loss of stability of chaotic synchronization is caused by chaos-hyperchaos transition when second Lyapunov exponent becomes positive. Thus, this exponent play a rule of transversal or conditional Lyapunov exponent quantifying the stability of synchronous regime. Other reason of observed desynchronization can be possible coexistence of attractors, even in the Ω-range of regular motion, e.g., one pendulum oscillates while the second is stabilized in upper or lower position.

Numerical results for one mass with two pendulums have been verified experimentally qualitatively and, in general, quantitatively. Phenomena of in phase (Fig. 7a), in anti-phase (Fig. 7b) and transition between state in phase to anti-phase (Fig. 9) have been observed, registered and analysed during experimental study.

References

[1] I. I. Blekhman, *Synchronization in Science and Technology*, ASME Press, New York 1988.

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge 2001.

[3] L. M. Pecora, T. L. Carroll, G. A. Johnson, D. J. Mar, and J. F. Heagy, *Fundamentals of synchronization in chaotic systems, concepts, and applications*, Chaos **7** (1997), 520.

[4] G. D. Birkhoff, *Dynamical Systems*, AMS Colloquium Publications, Providence (1927), 305.

[5] C. Huygens, *The Pendulum Clock*, Iowa State University Press, Ames 1986.

[6] J. Pena-Ramirez, R. H. B. Fey, and H. Nijmeijer, *An experimental study on synchronization of nonlinear oscillators with Huygens' coupling*, Nonlinear Theory and its Applications, IEICE **3**, no. 2 (2012), 128.

[7] J. Pena-Ramirez, R. H. B. Fey, and H. Nijmeijer, *In-phase and anti-phase synchronization of oscillators with Huygens' coupling*, Cybernetics and Physics **1**, no. 1 (2012), 58–66.

[8] M. Kapitaniak, K. Czołczyński, P. Perlikowski, A. Stefański, and T. Kapitaniak, *Synchronization of clocks*, Physics Reports **517**, no. 1–2 (2012), 1.

[9] K. Czołczyński, P. Perlikowski, A. Stefański, and T. Kapitaniak, *Huygens' odd sympathy experiment revisited*, International Journal of Bifurcations and Chaos **21**, no. 7 (2011), 2047.

[10] R. Dilao, *Anti-phase and in-phase synchronization of nonlinear oscillators: The Huygens's clocks system*, Chaos: An Interdisciplinary Journal of Nonlinear Science **19**, no. 2 (2009), 023118.

[11] L. M. Pecora and T. S. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett. **64** (1990), 821.

[12] T. Kapitaniak, *Controlling Chaos*, Academic Press, London 1996.

[13] G. Chen, *Controlling Chaos and Bifurcations*, CRC Press, Boca Raton 1999.

[14] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, *The synchronization of chaotic systems*, Physics Reports **366** (2002), 1.

[15] A. Rodrigues-Angeles and H. Nijmeijer, *Coordination of two robot manipulators based on position measurements only*, International Journal of Control **74** (2001), 1311.

[16] Y.-H. Liu, Y. Xu, and M. Bergerman, *Coordination control of multiple manipulators with passive joints*, IEEE Transactions on Robotics and Automation **15** (1999), 258.

[17] A. Yu. Pogromsky, Y. Xu, V. N. Belykh, and H. Nijmeijer, *Controlled synchronization of pendula*, Proceedings of the 42nd IEEE Conference on Design and Control, Maui, Hawaii 2003, 4381.

[18] A. S. Pikovsky, *On the interaction of strange attractors*, Zeitschrift Physik B **55** (1984), 149.

[19] H. Fujisaka and T. Yamada, *Stability theory of synchronized motion in coupled oscillator systems*, Prog. Theor. Phys. **69** (1983), 32.

[20] H. Fujisaka and T. Yamada, *Stability theory of synchronized motion in coupled-oscillator systems. ii – the mapping approach*, Prog. Theor. Phys. **70** (1983).

[21] J. N. Blakely, D. J. Gauthier, G. Johnson, T. L. Carroll, and L. M. Pecora, *Experimental investigation of high-quality synchronization of coupled oscillators*, Chaos **10** (2000), 738.

[22] M. Rosenblum, A. S. Pikovsky, and J. Kurths, *From phase to lag synchronization in coupled chaotic oscillators*, Phys. Rev. Lett. **78** (1997), 4193.

[23] V. Vaziri, A. Najdecka, and M. Wiercigroch, *Experimental control for initiating and maintaining rotation of parametric pendulum*, The European Physical Journal, Special Topics **223**, no. 4 (2014), 795.

[24] J. Strzałko, J. Grabski, J. Wojewoda, M. Wiercigroch, and T. Kapitaniak, *Synchronous rotation of the set of double pendula: Experimental observations*, Chaos **22** no. 4 (2012), 047503.

[25] E. Pavlovskaia, B. W. Horton, M. Wiercigroch, S. Lenci, and G. Rega, *Approximate rotating solutions of pendulum under combined vertical and horizontal excitation*, International Journal of Bifurcation and Chaos **22**, no. 5 (2012), 1250100.

[26] A. S. de Paula, M. A. Savi, M. Wiercigroch, and E. Pavlovskaia, *Bifurcation control of pendulum dynamics*, International Journal of Bifurcation and Chaos **22**, no. 5 (2012), 1250111.

[27] M. Wiercigroch and G. Rega, *Dynamics, synchronization and control of parametric pendulums*, Springer Netherlands, IUTAM Symposium on Nonlinear Dynamics for Advanced Technologies and Engineering Design **32** (2013), 185.

[28] A. S. de Paula, M. A. Savi, M. Wiercigroch, and E. Pavlovskaia, *Chaos control methods applied to avoid bifurcations in pendulum dynamics*, Springer Netherlands, IUTAM Symposium on Nonlinear Dynamics for Advanced Technologies and Engineering Design **32** (2013), 387.

[29] J. M. T. Thompson, M. Wiercigroch, J. Sieber, and B. W. Horton, *Dynamics of the nearly parametric pendulum*, International Journal of Non-Linear Mechanics **46**, no. 2 (2011), 436.

[30] D. G. Fertis, *Advanced Mechanics of Structures*, CRC Press, Akron 1996.

[31] G. S. Gorelik and A. A. Witt, *Swing of an elastic pendulum as an example of two paramerticall bound linear vibration systems*, J. Tech. Physics **3**, no. 2–3 (1933), 294.

[32] X. Xu, E. Pavlovskaia, M. Wiercigroch, F. Romeo, and S. Lenci, *Dynamic interactions between parametric pendulum and electro-dynamical shaker*, ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik **87**, no. 2 (2007), 172.

[33] K. Czoł,czyński, P. Perlikowski, A. Stefański, and T. Kapitaniak, *Synchronization of slowly rotating pendulums*, Int. J. Bifurcation Chaos **22** (2012), 1250128.

[34] K. Czołczyński, P. Perlikowski, A. Stefański, and T. Kapitaniak, *Synchronization of pendula rotating in different directions*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 3658.

[35] D. Sado and M. Kot, *Nonlinear oscillations of a coupled autoparametrical system with ideal and nonideal sources of power*,Hindawi Publishing Corporation, Mathematical Problem in Engineering **2006** (2006).

[36] D. Sado and M. Kot, *Chaotic vibration of an autoparametrical system with ideal and nonideal sources of power*, Journal of Theoretical and Applied Mechanics **45**, no. 1 (2007), 119.

[37] www.kinovea.org

Division of Dynamics
Technical University of Łódź
Stefanowskiego 1/15, PL-90-924 Łódź,
Poland
e-mail: anna.karmazyn@dokt.p.lodz.pl

Presented by Jakub Rembieliński at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 4, 2014

SYNCHRONIZACJA SPRZĘŻONYCH OSCYLATORÓW MECHANICZNYCH

S t r e s z c z e n i e

Celem niniejszego artykułu jest przedstawienie synchronizacji w układzie sprzężonych oscylatorów mechanicznych. W przypadku identycznych wahadeł synchronizacja w fazie i w antyfazie może być zaobserwowana. Ponadto, pełną synchronizację uzyskano dla układu identycznych wahadeł połączonych z oscylatorem mechanicznym a niedoskonałą pełną synchronizację uzyskano w rzeczywistym układzie mechanicznym.

Słowa kluczowe: synchronizacja kompletna, niedoskonała synchronizacja kompletna, synchronizacja w fazie, synchronizacja w antyfazie, wykładniki Lyapunova, wykres bifurkacyjny, mapa Poincarégo

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 57–70

In memory of
Professor Zygmunt Charzyński (1914–2001)

Renata Długosz and Piotr Liczberski

NEW PROPERTIES OF BAVRIN'S HOLOMORPHIC FUNCTIONS ON BANACH SPACES

Summary

The paper concerns properties of complex valued functions which are holomorphic in the unit ball $\mathbb{B}$ of a complex Banach space X and fulfil some geometric conditions. Such functions were considered earlier by I. Bavrin [1] and the second author [9]. The above functions can be applied to research some families of locally biholomorphic mappings from $\mathbb{B}$ into X. For some families of such functions we present an uniqueness theorem, some embedding theorems and we solve a few extremal problems on these families.

Keywords and phrases: holomorphic functions in Banach spaces, Bavrins families, uniqueness theorems, embeding theorems, growth theorems, bounds for polynomials of functions development on series of homogeneous polynomials

Let X be a complex Banach space, $\mathbb{B} = \{x \in X : \|x\| < 1\}$ and $\mathcal{L}(X, Y)$ be the space of all linear continuous operators from X into a complex Banach space Y. A mapping $g : \mathbb{B} \to Y$ is said to be holomorphic ($g \in \mathcal{H}(\mathbb{B}, Y)$) if it is Frechet differentiable at each point $x_0 \in \mathbb{B}$, i.e., there exist a bounded linear mapping $Dg(x_0) : X \to Y$ and a mapping $R(x_0) : X \to Y$, $R(x_0)h = o(||h||)$, such that for small $\|h\|$

$$g(x_0 + h) - g(x_0) = Dg(x_0)h + R(x_0)h,\ \ h \in X.$$

T. J. Suffridge initiated in 1973 [14] the investigations a family $S^*_{\mathbb{B}}$ of biholomorphic mappings g of $\mathbb{B}$ onto starlike domains $g(\mathbb{B}) \subset X$, normalized by the conditions $g(0) = 0, Dg(0) = I$ (I is the identity in the space of linear operators from X into itself). In his characterization an important role plays the following result:

Theorem A. *A locally biholomorphic mapping $g : \mathbb{B} \to X$, normalized by the equalities $g(0) = 0, Dg(0) = I$, belongs to $S^*_{\mathbb{B}}$ if and only if there exists a mapping $\omega \in \mathcal{H}(\mathbb{B}, X), \omega(0) = 0$, such that*

$$g(x) = Dg(x)\omega(x), x \in \mathbb{B} \tag{1}$$

and for every $a \in \mathbb{B} \setminus \{0\}$ and every functional x^* of the complex dual space $X^*, \|x^*\| = 1, x^*(a) = \|a\|$, there holds the inequality

$$\operatorname{Re} x^*(\omega(a)) > 0. \tag{2}$$

Observe that $S^*_{\mathbb{B}}$ includes a proper subfamily $\widetilde{S^*_{\mathbb{B}}}$ of all mappings

$$g(x) = xf(x), x \in \mathbb{B}, \tag{3}$$

with

$$f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1) = \{f \in \mathcal{H}(\mathbb{B}, \mathbb{C}) : f(0) = 1\},$$

for which

$$\operatorname{Re} \frac{Lf(x)}{f(x)} > 0, x \in \mathbb{B}, \tag{4}$$

where $L : \mathcal{H}(\mathbb{B}, \mathbb{C}) \to \mathcal{H}(\mathbb{B}, \mathbb{C})$ is the operator defined as follows

$$Lf(x) = f(x) + Df(x)x, x \in \mathbb{B}. \tag{5}$$

The family of all functions $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ fulfilling (4) we will denote by $\mathcal{M}_{\mathbb{B}}$. It is known, see [1], that $f(x)Lf(x) \neq 0$ for $x \in \mathbb{B}$ and $f \in \mathcal{M}_{\mathbb{B}}$.

A confirmation of the relation $\widetilde{S^*_{\mathbb{B}}} \subset S^*_{\mathbb{B}}$ bases on Suffridge's theorem. Indeed, since g is a product of a function $f : \mathbb{B} \to \mathbb{C}$ and the identity function $id_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ we obtain by the differential-product rule, that

$$Dg(x)h = f(x)(D(id_B)(x)h) + (Df(x)h)x, \ x \in \mathbb{B}, h \in X.$$

Replacing h by

$$\omega(x) = x\frac{f(x)}{Lf(x)}$$

in this formula we have

$$\begin{aligned} Dg(x)\omega(x) &= f(x)x\frac{f(x)}{Lf(x)} + \left(Df(x)x\frac{f(x)}{Lf(x)}\right)x \\ &= f(x)x\frac{f(x)}{Lf(x)} + (Df(x)x)\,x\frac{f(x)}{Lf(x)} \\ &= (f(x) + Df(x)x)\,x\frac{f(x)}{Lf(x)} = g(x), \end{aligned}$$

hence the mapping g fulfils (1). On the other hand, if $a \in \mathbb{B}\setminus\{0\}$ and $x^* \in X^*, \|x^*\| = 1, x^*(a) = \|a\|$, then

$$\operatorname{Re} x^*(\omega(a)) = \operatorname{Re} x^*\left(a\frac{f(a)}{Lf(a)}\right) = ||a||\operatorname{Re}\frac{f(a)}{Lf(a)} > 0,$$

iff the condition (4) is fulfiled.

The equality $S_{\mathbb{B}}^{*}=\widetilde{S_{\mathbb{B}}^{*}}$ is false even in the case if $X=\mathbb{C}^{n}, n>1$, with the maximum norm, because the mapping

$$g(z)=\left(\frac{z_1}{(1-z_1)^2},...,\frac{z_n}{(1-z_n)^2}\right), z=(z_1,...,z_n)\in\mathbb{B},$$

belongs to $S_{\mathbb{B}}^{*}$ (see for instance [15, Ex.4]) but the form of $g(x)$ is different from the form (3).

Now we give another necessary and sufficient condition for the functions to belong to the family $\mathcal{M}_{\mathbb{B}}$. It has an integral form and is based on functions from the family

$$\mathcal{C}_{\mathbb{B}}=\{h\in\mathcal{H}(\mathbb{B},\mathbb{C},1):\operatorname{Re}h(x)>0, x\in\mathbb{B}.\}$$

Theorem 1. *A function $f\in\mathcal{H}(\mathbb{B},\mathbb{C},1)$ belongs to the family $\mathcal{M}_{\mathbb{B}}$ if and only if there exists a function $h\in\mathcal{C}_{\mathbb{B}}$ such that*

$$f(x)=\exp\left[\int_0^1\frac{1}{t}\left(h\left(tx\right)-1\right)dt\right],\ x\in\mathbb{B}.$$

Proof. Assume that $f\in\mathcal{M}_{\mathbb{B}}$. Since the above formula is true for $x=0$, let $x\in\mathbb{B}\setminus\{0\}$ be arbitrarily fixed. Then, by the definition of $\mathcal{M}_{\mathbb{B}}$, $f(x)\neq0$ and there exists an $h\in\mathcal{C}_{\mathbb{B}}$ such that

$$\frac{Df\left(x\right)x}{f\left(x\right)}=h\left(x\right)-1,\ x\in\mathbb{B}.$$

Replacing $x\in\mathbb{B}\setminus\{0\}$ in the above equality, by $\zeta\frac{x}{\|x\|}$ with $\zeta\in\Delta$, where $\Delta=\{\zeta\in\mathbb{C}:|\zeta|<1\}$, we have that the complex valued function

$$\frac{h\left(\zeta\frac{x}{\|x\|}\right)-1}{\zeta},\quad\zeta\in\Delta$$

is holomorphic. Indeed, it is holomorphic in $\Delta\setminus\{0\}$ and has a removable singularity at $\zeta=0$, because by the holomorphicity of h

$$\lim_{\zeta\leftarrow0}\frac{h\left(\zeta\frac{x}{\|x\|}\right)-1}{\zeta}=\lim_{\zeta\leftarrow0}\frac{Dh(0)\zeta\frac{x}{\|x\|}+o\left(\left\|\zeta\frac{x}{\|x\|}\right\|\right)}{\zeta}=Dh(0)\frac{x}{\|x\|}.$$

Moreover, we have also

$$\frac{Df\left(\zeta\frac{x}{\|x\|}\right)\frac{x}{\|x\|}}{f\left(\zeta\frac{x}{\|x\|}\right)}=\frac{h\left(\zeta\frac{x}{\|x\|}\right)-1}{\zeta},\quad\zeta\in\Delta,$$

which can be rewritten in the form

$$\frac{d}{d\zeta}\log f\left(\zeta\frac{x}{\|x\|}\right)=\frac{1}{\zeta}\left(h\left(\zeta\frac{x}{\|x\|}\right)-1\right),\ \zeta\in\Delta,$$

with such branch of the log that $\log1=0$. Integrating this equality on the segment $[0,\|x\|]$ we obtain

$$\log f(x) = \int_0^1 \frac{1}{t}\left(h(xt) - 1\right) dt.$$

This gives the required integral formula for f in $\mathbb{B} \setminus \{0\}$.

Now let us assume that a function $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ is defined as in the thesis. Then, there holds the relation

$$\frac{Lf(x)}{f(x)} = h(x), x \in \mathbb{B}.$$

Indeed, for $x \in \mathbb{B}$

$$\frac{Lf(x)}{f(x)} = 1 + \int_0^1 Dh(xt)\, x dt = 1 + \int_0^1 \frac{d}{dt} h(xt)\, dt = h(x),$$

hence the relation $f \in \mathcal{M}_{\mathbb{B}}$ follows from the fact that $h \in \mathcal{C}_{\mathbb{B}}$.

This completes the proof of the Theorem 1. □

Using the formula from Theorem 1 we will construct a function f_1 belonging to $\mathcal{M}_{\mathbb{B}}$. By f_1 let us denote the function defined by the formula from Theorem 1 with the function

$$h(x) = \frac{1 + \eta x^*(x)}{1 - \eta x^*(x)}, x \in \mathbb{B},$$

where $|\eta| = 1$ and $x^* \in X^*, \|x^*\| = 1$ are arbitrarily fixed. The required in Theorem 1 relation $h \in \mathcal{C}_{\mathbb{B}}$, follows from the fact that $h(x) = q(\eta x^*(x)), x \in \mathbb{B}$, where q is the linear fractional transformation $q(\zeta) = (1-\zeta)^{-1}(1+\zeta), \zeta \in \Delta$, which maps the disc Δ onto the right half plane $\Pi = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$. Since for every $x \in \mathbb{B}$ there hold the equalities

$$\begin{aligned} f_1(x) &= \exp\left[\int_0^1 \frac{1}{t}\left(\frac{1 + \eta x^*(tx)}{1 - \eta x^*(tx)} - 1\right) dt\right] = \exp\left[-2\log(1 - \eta x^*(x)\right] \\ &= \frac{1}{\left(1 - \eta x^*(x)\right)^2}, \end{aligned}$$

we obtain that

$$f_1(x) = \frac{1}{\left(1 - \eta x^*(x)\right)^2}, x \in \mathbb{B}.$$

Of course if we know the form of the function f_1, then the relation $f_1 \in \mathcal{M}_{\mathbb{B}}$ can be confirmed by proving that f_1 fulfils the condition (4).

Bavrin [1] actually defined the family $\mathcal{M}_{\mathbb{B}}$ as follows:

A function $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ belongs to $\mathcal{M}_{\mathbb{B}}$ if and only if the function

$$F_b(\zeta) = \zeta f(\zeta b), \ \zeta \in \Delta \tag{6}$$

belongs, for every $b \in \partial\mathbb{B}$, to the class S^*_Δ of functions $F : \Delta \to \mathbb{C}$ univalent, starlike and normalized as follows $F(0) = 0, F'(0) = 1$.

In the papers [1], [9] there are also considered families $\mathcal{N}_{\mathbb{B}}, Q_{\mathbb{B}}$ of functions $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ such that for every $b \in \partial\mathbb{B}$ the function F_b from (6) belongs to the class S^c_Δ, S_Δ of functions $F:\Delta\to\mathbb{C}$ normalized, univalent and convex and only normalized univalent, respectively. For the characterization of the families $S_\Delta, S^*_\Delta, S^c_\Delta$ and further S^{cc}_Δ we refer to the monograhs of P.Duren [4] and Ch.Pommerenke [12]. In [9] it was proved also that an $f\in\mathcal{H}(\mathbb{B},\mathbb{C},1)$ belongs to $\mathcal{N}_\mathbb{B}$ iff

$$\frac{LLf(x)}{Lf(x)}>0, x\in\mathbb{B}. \tag{7}$$

Now we will show that an example of a function of the family $\mathcal{N}_\mathbb{B}$ is the following function

$$f_2(x)=\frac{1}{1-\eta x^*(x)}, x\in\mathbb{B},$$

where $|\eta|=1, x^*\in X^*, \|x^*\|=1$, are arbitrarily fixed. Indeed. Since $f_2\in\mathcal{H}(\mathbb{B},\mathbb{C},1)$ and for $x\in\mathbb{B}$

$$Lf_2(x)=\frac{1}{(1-\eta x^*(x))^2},\ LLf_2(x)=\frac{1+\eta x^*(x)}{(1-\eta x^*(x))^3},$$

it suffices to observe, in view of (7), that

$$\operatorname{Re}\frac{LLf_2(x)}{Lf_2(x)}=\operatorname{Re}\frac{1+\eta x^*(x)}{1-\eta x^*(x)}>0, x\in\mathbb{B}.$$

Note that $\mathcal{M}_\mathbb{B}$ and $\mathcal{N}_\mathbb{B}$ there are in the following relationships

$$f\in\mathcal{N}_\mathbb{B}\Longrightarrow Lf\in\mathcal{M}_\mathbb{B},\quad f\in\mathcal{M}_\mathbb{B}\Longrightarrow L^{-1}f\in\mathcal{N}_\mathbb{B}, \tag{8}$$

where

$$L^{-1}f(x)=\int_0^1 f(tx)dt, x\in\mathbb{B}. \tag{9}$$

The second implication was given in the paper [9] and the first one follows directly from the conditions (4) and (7), defining the families $\mathcal{M}_\mathbb{B}$ and $\mathcal{N}_\mathbb{B}$, respectively.

Observe that L^{-1} is the inverse operator with respect to L. Indeed, by the definitions of L and L^{-1}, in view of the fact [9] that

$$D\left(\int_0^1 f(tx)dt\right)x=\int_0^1 Df(tx)txdt,$$

we obtain

$$\begin{aligned} LL^{-1}f(x) &= L\left(\int_0^1 f(tx)dt\right)=\int_0^1 \left(f(tx)dt+Df(tx)tx\right)dt \\ &= \int_0^1 \frac{d}{dt}\left(tf(tx)\right)dt=f(x), \end{aligned}$$

for every $f\in\mathcal{H}(\mathbb{B},\mathbb{C})$. On the other hand by the definitions of L and L^{-1} we have also

$$\begin{aligned} L^{-1}Lf(x) &= \int_0^1 Lf(tx)dt = \int_0^1 \left(f(tx) + Df(tx)tx\right)dt = \\ \int_0^1 \frac{d}{dt}\left(tf(tx)\right)dt &= f(x) \end{aligned}$$

for every $f \in \mathcal{H}(\mathbb{B}, \mathbb{C})$, hence $LL^{-1}f = L^{-1}Lf$.

Let us note that the relation $f_2 \in \mathcal{N}_{\mathbb{B}}$ follows also, in view of (8), from the facts that $f_2 = L^{-1}f_1$ and $f_1 \in \mathcal{M}_{\mathbb{B}}$.

In the paper [9] there are given some sufficient conditions for the functions f to belong to the families $\mathcal{M}_{\mathbb{B}}, \mathcal{N}_{\mathbb{B}}$. These conditions are formulated in form of some bounds for the norms $\|P_{f,k}\|$ of all k-homogeneous polynomials $P_{f,k}$ from the unique developmens

$$f(x) = 1 + \sum_{k=1}^{\infty} P_{f,k}(x), x \in \mathbb{B} \tag{10}$$

of functions $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$. Note that the above series converges uniformly on every ball $r\overline{\mathbb{B}}, r \in (0,1)$ [11, §5-7]. This and the definitions of Lf and $L^{-1}f$ imply for the developmens of $Lf(x), L^{-1}f(x)$ at $x \in \mathbb{B}$ the following useful formulas

$$Lf(x) = 1 + \sum_{k=1}^{\infty}(k+1)P_{f,k}(x),\ L^{-1}f(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k+1}P_{f,k}(x). \tag{11}$$

Now we can prove some necessary conditions for the relations $f \in Q_{\mathbb{B}}$, $f \in \mathcal{M}_{\mathbb{B}}, f \in \mathcal{N}_{\mathbb{B}}$.

Theorem 2. *For $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ of the form* (10) *there hold*

(*i*) *if $f \in Q_{\mathbb{B}}$ and $k \in \mathbb{N}$, then $\|P_{f,k}\| \leqslant k+1$,*

(*ii*) *if $f \in \mathcal{M}_{\mathbb{B}}$ and $k \in \mathbb{N}$, then $\|P_{f,k}\| \leqslant k+1$,*

(*iii*) *if $f \in \mathcal{N}_{\mathbb{B}}$ and $k \in \mathbb{N}$, then $\|P_{f,k}\| \leqslant 1$.*

All estimations are optimal.

Proof. Let $f \in Q_{\mathbb{B}}$ has the form (10) and $b \in \partial\mathbb{B}$ be arbitrarily fixed. Then by the definition of the family $Q_{\mathbb{B}}$ the function F_b from (6) belongs to the class S_{Δ}. On the other hand, in view of (10), its Taylor series has the form

$$F_b(\zeta) = \zeta + \sum_{k=1}^{\infty} P_{f,k}(b)\zeta^{k+1}, \zeta \in \Delta,$$

hence

$$|P_{f,k}(b)| \leqslant k+1, \quad k \in \mathbb{N},$$

by the de Branges theorem [3]. Thus

$$\|P_{f,k}\| = \sup_{x \in \partial\mathbb{B}} |P_{f,k}(x)| \leqslant k+1, k \in \mathbb{N}.$$

This gives the estimation (i). The estimation is sharp. Indeed, the function f_1 belongs to $Q_{\mathbb{B}}$, because $f_1 \in \mathcal{M}_{\mathbb{B}}$. Moreover, the homogeneous polynomials $P_{f_1,k}$ in its series (10) have the form

$$P_{f_1,k}(x) = (k+1)\eta^k \left(x^*(x)\right)^k, x \in \mathbb{B},$$

hence for every $k \in \mathbb{N}$

$$\|P_{f_1,k}\| = (k+1) \sup_{x \in \partial\mathbb{B}} \left|\left(x^*(x)\right)^k\right| = k+1.$$

The statement (ii) follows from the fact that $\mathcal{M}_{\mathbb{B}} \subset Q_{\mathbb{B}}$ [1] and the fact that the extremal function $f_1 \in Q_{\mathbb{B}}$ belongs also to $\mathcal{M}_{\mathbb{B}}$.

Now let us assume that f belongs to $\mathcal{N}_{\mathbb{B}}$ and has the form (10). Then, in view of (8), $Lf \in \mathcal{M}_{\mathbb{B}}$ and by (11) the estimations (iii) follow from the assertion (ii). The estimations (iii) are sharp, because the function f_2 belongs to $\mathcal{N}_{\mathbb{B}}$ and realises the equality in the inequalities (iii). □

Now we will present a theorem which brings a corollary connected with the uniqueness of the extremal functions in the estimations from the parts (ii) and (iii) of Theorem 2.

Theorem 3. *Let $m_2 \in \mathbb{N}$ divides $m_1 \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ has the form* (10). *Then*

(i) *if $f \in \mathcal{M}_{\mathbb{B}}$, then there holds the following inequality*

$$m_1 + 1 - A_{f,m_1} \leqslant \frac{m_1+1}{m_2+1}\left(\frac{m_1}{m_2}\right)^2 \left(m_2 + 1 - A_{f,m_2}\right), \tag{12}$$

(ii) *if $f \in \mathcal{N}_{\mathbb{B}}$, then there holds the following inequality*

$$1 - A_{f,m_1} \leq \left(\frac{m_1}{m_2}\right)^2 \left(1 - A_{f,m_2}\right), \tag{13}$$

where

$$A_{f,m_j} = \sup_{x \in \mathbb{B}} Re\, P_{f,m_j}(x), j = 1, 2.$$

The both inequalities are optimal.

Proof. We start with the proof of the inequality (12). Applying a result of Ruscheweyh [13] to the function F_b defined in (6) by $f \in \mathcal{M}_{\mathbb{B}}$ and arbitrarily fixed $b \in \partial\mathbb{B}$, we obtain

$$\operatorname{Re}\left[m_1 + 1 - P_{f,m_1}(b)\right] \leq \frac{m_1+1}{m_2+1}\left(\frac{m_1}{m_2}\right)^2 \operatorname{Re}\left[m_2 + 1 - P_{f,m_2}(b)\right].$$

This, in view of the arbitrariness of $b \in \partial\mathbb{B}$, implies

$$\inf_{b \in \mathbb{B}}\left[m_1 + 1 - \operatorname{Re} P_{f,m_1}(b)\right] \leq \frac{m_1+1}{m_2+1}\left(\frac{m_1}{m_2}\right)^2 \inf_{b \in \mathbb{B}}\left[m_2 + 1 - \operatorname{Re} P_{f,m_2}(b)\right],$$

which is equivalent to the inequality (12).

Finally let us observe that the form of the homogeneous polynomials of the function $f_1 \in \mathcal{M}_{\mathbb{B}}$, with $\eta = 1$, (see the proof of Theorem 2) shows that in the inequality (12) also holds the equality.

The inequality (13) follows from the inequality (12), by the first one of the relations (8) and the equality (11). The equality in (13) holds for the function $f_2 \in \mathcal{N}_{\mathbb{B}}$, with $\eta = 1$. □

Below is a kind of uniqueness theorem for functions $f \in \mathcal{M}_{\mathbb{B}}, f \in \mathcal{N}_{\mathbb{B}}$.

Corollary 1. *Let* $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ *has the form* (10).

(i) *If* $f \in \mathcal{M}_{\mathbb{B}}$ *and* $A_{f,1} = 2$, *then* $A_{f,m} = m + 1, m \in \mathbb{N}$;

(ii) *if* $f \in \mathcal{N}_{\mathbb{B}}$ *and* $A_{f,1} = 1$, *then* $A_{f,m} = 1, m \in \mathbb{N}$.

Proof. We start with the proof of the statement (i). Let us put $m_2 = 1$ and $m_1 = m$ for $f \in \mathcal{M}_{\mathbb{B}}$ in Theorem 3. Then the inequality (12) reduces to the following

$$m + 1 - A_{f,m} \leqslant \frac{(m+1)m^2}{2} \left(2 - A_{f,1}\right).$$

Since, in view of the assumption the right hand side vanish, we have that $m + 1 \leqslant A_{f,m}$. This implies the thesis (i), because $A_{f,m} \leqslant \|P_{f,m}\|$ and by part (ii) of Theorem 2, $\|P_{f,m}\| \leqslant m + 1$.

The proof of the part (ii) of the Corollary 1 runs similarly. □

Now we give some estimations of the growth of $|f(x)|$ and $|Lf(x)|$ for $f \in \mathcal{M}_{\mathbb{B}}$ and $f \in \mathcal{N}_{\mathbb{B}}$, similar to these for $f \in Q_{\mathbb{B}}$ from [1].

Theorem 4. *For every arbitrarily fixed* $r \in [0, 1)$ *there hold the relations:*

(i) *If* $f \in \mathcal{M}_{\mathbb{B}}$, *then*

$$\frac{1}{(1+r)^2} \leqslant |f(x)| \leqslant \frac{1}{(1-r)^2}, \|x\| \leqslant r,$$

$$\frac{1-r}{(1+r)^3} \leqslant |Lf(x)| \leqslant \frac{1+r}{(1-r)^3}, \|x\| \leqslant r.$$

(ii) *If* $f \in \mathcal{N}_{\mathbb{B}}$, *then*

$$\frac{1}{1+r} \leqslant |f(x)| \leqslant \frac{1}{1-r}, \|x\| \leqslant r,$$

$$\frac{1}{(1+r)^2} \leqslant |Lf(x)| \leqslant \frac{1}{(1-r)^2}, \|x\| \leqslant r.$$

All of the above estimations are sharp.

Proof. The estimations (i) follow from similar bounds for $f \in Q_{\mathbb{B}}$, [1], because $\mathcal{M}_{\mathbb{B}} \subset Q_{\mathbb{B}}$ and the extremal function $f_1 \in Q_{\mathbb{B}}$ is also in $\mathcal{M}_{\mathbb{B}}$.

The estimations of $|Lf(x)|$ in the statement (ii) follow from the estimations of $|f(x)|$ in (i), by the first of the relations (8).

Now we prove the estimations (ii) of $|f(x)|$. Of course, the estimations hold for $x = 0$. Let us fix an $x \in \mathbb{B} \setminus \{0\}$. Then the function $F_b, b = \frac{x}{\|x\|}$, defined in (6), belongs to the family S^c_Δ of functions convex univalent and normalized in the unit disc Δ, hence by the Löwner estimations [10] of $|F(\zeta)|$ for $F \in S^c_\Delta$ in every point $\zeta \in \Delta$, we have

$$\frac{r}{1+r} \leqslant \left|\zeta f(\zeta \frac{x}{\|x\|})\right| \leqslant \frac{r}{1-r}, |\zeta| \leqslant r \in [0,1).$$

Now, it suffices to put $\zeta = \|x\|$ in the above.

Finally, let us note that sharpness of the estimations $|f(x)|$ in (ii) confirm the fact that for the points $x = r$ and the function $f_2 \in \mathcal{N}_\mathbb{B}$, with adequate $\eta, |\eta| = 1$, we obtain here the equalities. □

Now we will complete the information about the double inclusion $\mathcal{N}_\mathbb{B} \subset \mathcal{M}_\mathbb{B} \subset Q_\mathbb{B}$ given in [1] and [9]. We give a first embedding theorem for Bavrin's families.

Theorem 5. *There hold the proper inclusions* $\mathcal{N}_\mathbb{B} \subsetneq \mathcal{M}_\mathbb{B} \subsetneq Q_\mathbb{B}$.

Proof. The relation $\mathcal{M}_\mathbb{B} \setminus \mathcal{N}_\mathbb{B} \neq \emptyset$ follows from the fact that $f_1 \in \mathcal{M}_\mathbb{B} \setminus \mathcal{N}_\mathbb{B}$. Since $f_1 \in \mathcal{M}_\mathbb{B}$, it suffices to show that $f_1 \notin \mathcal{N}_\mathbb{B}$. This is a direct consequence of the contraditions between the estimations from the part (iii) of Theorem 2 and the equality $\|P_{f_1,k}\| = k+1$.

Now we will show that $Q_\mathbb{B} \setminus \mathcal{M}_\mathbb{B} \neq \emptyset$. To do it, let us consider the function

$$f_3(x) = L^{-1}\left[\frac{(1+ix^*(x))}{(1-ix^*(x))\,(1-x^*(x))^2}\right], x \in \mathbb{B},$$

where $x^* \in X^*$, $\|x^*\| = 1$, is arbitrarily fixed. Then, in view of (9), for a point $b \in \partial\mathbb{B}$ such that $x^*(b) = 1$, the function F_b from (6) with $f = f_3$, has the form

$$\begin{aligned} F_b(\zeta) &= \zeta\int_0^1 \frac{1+ix^*(t\zeta b)}{(1-ix^*(t\zeta b))\,(1-x^*(t\zeta b))^2}dt \\ &= \zeta\int_0^1 \frac{1+it\zeta}{(1-it\zeta)\,(1-t\zeta)^2}dt, \zeta \in \Delta. \end{aligned}$$

This gives the relation $F_b \notin S^*_\Delta$ [2] and consequently, $f_3 \notin \mathcal{M}_\mathbb{B}$. The relation $f_3 \in Q_\mathbb{B}$ we prove further, after the proof of Theorem 6. □

The relation $\mathcal{N}_\mathbb{B} \subsetneq \mathcal{M}_\mathbb{B} \subsetneq Q_\mathbb{B}$ says that the family $\mathcal{M}_\mathbb{B}$ plays a special role: it separates the families $\mathcal{N}_\mathbb{B}$ and $Q_\mathbb{B}$. Therefore, it is very natural to ask whether by a condition similar to (3) and (7), we can define another family $\mathcal{R}_\mathbb{B}$ which separates $\mathcal{M}_\mathbb{B}, Q_\mathbb{B}$. We define the family $\mathcal{R}_\mathbb{B}$ as follows: a function $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ belongs to $\mathcal{R}_\mathbb{B}$, iff there exists a function $\varphi \in \mathcal{N}_\mathbb{B}$ such that

$$\text{Re}\,\frac{Lf(x)}{L\varphi(x)} > 0, x \in \mathbb{B}. \tag{14}$$

Let us note that the family $\mathcal{R}_{\mathbb{B}}$ can be defined similarly as the families $\mathcal{N}_{\mathbb{B}}, \mathcal{M}_{\mathbb{B}}, Q_{\mathbb{B}}$ in the papers [1], [9]. A Function $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ belongs to $\mathcal{R}_{\mathbb{B}}$ iff for every $b \in \partial\mathbb{B}$ the function F_b from (6) belongs to the class S_{Δ}^{cc} of functions $F : \Delta \to \mathbb{C}$ close-to-convex and normalized by $F(0) = 0, F'(0) = 1$. Directly from such interpretations of the families $Q_{\mathbb{B}}, \mathcal{M}_{\mathbb{B}}, \mathcal{R}_{\mathbb{B}}$ there follow the inclusions $\mathcal{M}_{\mathbb{B}} \subset \mathcal{R}_{\mathbb{B}} \subset Q_{\mathbb{B}}$.

A more precise result we present in the next embedding theorem.

Theorem 6. *There hold the proper inclusions*

$$\mathcal{M}_{\mathbb{B}} \subsetneq \mathcal{R}_{\mathbb{B}} \subsetneq Q_{\mathbb{B}}.$$

Proof. We start with the proof of the first one of the above inclusions. Let $f \in \mathcal{M}_{\mathbb{B}}$, then by (8) $\varphi = L^{-1}f \in \mathcal{N}_{\mathbb{B}}$ and

$$\operatorname{Re} \frac{Lf(x)}{L\varphi(x)} = \operatorname{Re} \frac{Lf(x)}{f(x)} > 0, x \in \mathbb{B}.$$

This gives the inclusion $\mathcal{M}_{\mathbb{B}} \subset \mathcal{R}_{\mathbb{B}}$. To prove the relation $\mathcal{R}_{\mathbb{B}} \setminus \mathcal{M}_{\mathbb{B}} \neq \emptyset$ we consider the function f_3. Since the function

$$\varphi(x) = \frac{1}{1 - ix^*(x)}, x \in \mathbb{B},$$

belongs to $\mathcal{N}_{\mathbb{B}}$ ($\varphi = f_{2,}$ with $\eta = i$) and

$$\operatorname{Re} \frac{Lf_3(x)}{L\varphi(x)} = \operatorname{Re} \frac{1 + ix^*(x)}{1 - ix^*(x)} > 0, x \in \mathbb{B},$$

we have that $f_3 \in \mathcal{R}_{\mathbb{B}}$. The relation $f_3 \notin \mathcal{M}_{\mathbb{B}}$ was proved during the proof of the Theorem 5.

The relation $\mathbb{Q}_{\mathbb{B}} \setminus \mathcal{R}_{\mathbb{B}} \neq \emptyset$ follows from the above interpretation of the families $Q_{\mathbb{B}}, \mathcal{R}_{\mathbb{B}}$ and the fact that there holds the proper inclusion $S_{\Delta}^{cc} \subsetneq S_{\Delta}$ (see [7]). $\square$

At this place we can return to the mentioned relation $f_3 \in Q_{\mathbb{B}}$, from the proof of Theorem 5. It follows from the inclusion $\mathcal{R}_{\mathbb{B}} \subset Q_{\mathbb{B}}$ and the relation $f_3 \in \mathcal{R}_{\mathbb{B}}$, proved above.

Now we give some properties of the functions from the family $\mathcal{R}_{\mathbb{B}}$.

Theorem 7. *If $f \in \mathcal{R}_{\mathbb{B}}$ has the form* (10), *then the following sharp estimations hold*

$$\|P_{f,k}\| \leqslant k + 1, k \in \mathbb{N}.$$

Proof. Since $\mathcal{R}_{\mathbb{B}} \subset Q_{\mathbb{B}}$, the above estimations follow from the part (i) of Theorem 2, because the extremal function $f_1 \in Q_{\mathbb{B}}$ belongs also to the family $\mathcal{R}_{\mathbb{B}}$. $\square$

Theorem 8. *For every $f \in \mathcal{R}_{\mathbb{B}}$ and $r \in [0, 1)$ there hold the following sharp estimations:*

$$\frac{1}{(1+r)^2} \leqslant |f(x)| \leqslant \frac{1}{(1-r)^2}, \|x\| \leqslant r$$
$$\frac{1-r}{(1+r)^3} \leqslant |Lf(x)| \leqslant \frac{1+r}{(1-r)^3}, \|x\| \leqslant r.$$

Proof. Since $\mathcal{R}_{\mathbb{B}} \subset Q_{\mathbb{B}}$, the above estimations follow from the estimations in the family $Q_{\mathbb{B}}$, given in [1], because the extremal function $f_1 \in Q_{\mathbb{B}}$, with adequates η, belongs also to the family $\mathcal{R}_{\mathbb{B}}$. $\square$

From the proper inclusions

$$\mathcal{N}_{\mathbb{B}} \subsetneq \mathcal{M}_{\mathbb{B}} \subsetneq Q_{\mathbb{B}},\ \mathcal{M}_{\mathbb{B}} \subsetneq \mathcal{R}_{\mathbb{B}} \subsetneq Q_{\mathbb{B}}$$

there follows that does not hold any of the four inclusions $\mathcal{M}_{\mathbb{B}} \subset \mathcal{N}_{\mathbb{B}}$, $Q_{\mathbb{B}} \subset \mathcal{M}_{\mathbb{B}}, \mathcal{R}_{\mathbb{B}} \subset \mathcal{M}_{\mathbb{B}}$ and $Q_{\mathbb{B}} \subset \mathcal{R}_{\mathbb{B}}$. However, we can ask whether for an pair of families $\mathcal{X}_{\mathbb{B}}, \mathcal{Y}_{\mathbb{B}}$ of the above kind there exists $r \in (0,1]$ such that $\mathcal{X}_{\mathbb{B}} \subset \mathcal{Y}^r_{\mathbb{B}}$, where $\mathcal{Y}^r_{\mathbb{B}}$ denotes the family of all $f \in \mathcal{H}(\mathbb{B}, \mathbb{C}, 1)$ which fulfils for $x \in r\mathbb{B}$ the condition defining $\mathcal{Y}_{\mathbb{B}}$. Let us denote

$$r_{\mathcal{Y}}(\mathcal{X}) = r_{\mathcal{Y}_{\mathbb{B}}}(\mathcal{X}_{\mathbb{B}}) = \sup\left\{r \in (0,1] : f \in \mathcal{X}_{\mathbb{B}} \Rightarrow f \in \mathcal{Y}^r_{\mathbb{B}}\right\}.$$

Further we solve the problem to find all numbers $r_{\mathcal{N}}(\mathcal{M})$, $r_{\mathcal{M}}(Q)$, $r_{\mathcal{M}}(\mathcal{R})$, $r_{\mathcal{R}}(Q)$. Our answer for this question will be preceded by the following lemma.

Lemma 1. *For every function $h \in \mathcal{C}_{\mathbb{B}}$ and $r \in [0,1)$ there hold the following sharp estimations:*

$$\frac{1-r}{1+r} \le Re\, h(x) \le \frac{1+r}{1-r}, \|x\| \le r,$$
$$\left|\frac{Dh(x)x}{h(x)}\right| \le \frac{2r}{1-r^2}, \|x\| \le r.$$

Proof. Let $h \in \mathcal{C}_{\mathbb{B}}$ be arbitrarily fixed and let

$$q(x) = \frac{h(x)-1}{h(x)+1}, x \in \mathbb{B}.$$

Then $q \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ and

$$|q(x)|^2 = \frac{|h(x)|^2 - 2\mathrm{Re}\, h(x) + 1}{|h(x)|^2 + 2\mathrm{Re}\, h(x) + 1} < 1, x \in \mathbb{B}.$$

Therefore, for function q there holds the following inequality [9]

$$|Dq(x)x| \leqslant \frac{\|x\|}{1-\|x\|^2}\left(1 - |q(x)|^2\right), x \in \mathbb{B},$$

hence by the relation

$$\frac{Dh(x)x}{h(x)} = \frac{2}{1-(q(x))^2} Dq(x)x,$$

we have also

$$\left|\frac{Dh(x)x}{h(x)}\right| \leqslant \frac{2\left(1-|q(x)|^2\right)}{\left|1-(q(x))^2\right|}\frac{\|x\|}{1-\|x\|^2}, x \in \mathbb{B},$$

which implies the second inequality of the Lemma 1.

Now let us observe that our function q has the required normalization $q(0) = 0$ in a version of the Schwarz Lemma for $q \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ (for Schwarz Lemma in Banach spaces see for instance [5, Subsection 6.1.5]). Thus $|q(x)| \leqslant \|x\|$ for every $x \in \mathbb{B}$. This and the equality

$$\operatorname{Re} h(x) = \frac{1-|q(x)|^2}{1-2\operatorname{Re} q(x) + |q(x)|^2}, x \in \mathbb{B}$$

implies the first inequality of the Lemma 1.

To confirm of the sharpness of the proved estimations, let us consider the function $h \in \mathcal{C}_{\mathbb{B}}$ which was used to construct the function $f_1 \in \mathcal{M}_{\mathbb{B}}$. If we choose an adequate $\eta \in \mathbb{C}, |\eta| = 1$ and a functional $x^* \in X^*$, $\|x^*\| = 1$, such that $x^*(a) = \|a\|$ at a point $a \in \mathbb{B}\diagdown\{0\}$, then we obtain equalities in the inequalities from the Lemma 1. □

Theorem 9. *There holds the following equalities:*

(*i*) $r_{\mathcal{N}}(\mathcal{M}) = 2-\sqrt{3}$;

(*ii*) $r_{\mathcal{M}}(\mathcal{R}) = 4\sqrt{2}-5$;

(*iii*) $r_{\mathcal{M}}(Q) = \tanh\frac{\pi}{4}$;

(*iv*) $r_{\mathcal{R}}(Q)$ *is the unique root* $r \in (\tanh\frac{\pi}{4}, 1)$ *of the equation*

$$2\mathit{arccot}\left(\frac{1-r^2}{1+r^2}x(r)\right) + \log\left(1+x^2(r)\right) - 2\log\frac{2r}{1-r^2} = 0,$$

where $x(r)$, *for* $r \in (\tanh\frac{\pi}{4}, 1)$, *means the unique root of the polynomial equation*

$$x^3 - \frac{1+r^2}{1-r^2}x^2 + \left(\frac{1+r^2}{1-r^2}\right)x - \frac{1+r^2}{1-r^2} = 0.$$

Any of the above constants $r_{\mathcal{Y}}(\mathcal{X})$ *can not be improved.*

Proof. We start with the proof of the equality (*i*). Let $f \in \mathcal{M}_{\mathbb{B}}$. Then there exists a function $h \in \mathcal{C}_{\mathbb{B}}$ such that

$$Lf(x) = f(x)h(x), x \in \mathbb{B}.$$

Since for $x \in \mathbb{B}$

$$\operatorname{Re}\frac{LLf(x)}{Lf(x)} = \operatorname{Re}\left(h(x) + \frac{Dh(x)x}{h(x)}\right) \geq \operatorname{Re} h(x) - \left|\frac{Dh(x)x}{h(x)}\right|,$$

we have by the Lemma 1 that

$$\operatorname{Re}\frac{LLf(x)}{Lf(x)} \geq \frac{r^2-4r+1}{1-r^2}, \|x\| \leq r \in [0,1).$$

Thus

$$\operatorname{Re}\frac{LLf(x)}{Lf(x)} > 0, \|x\| \leq r \in [0, 2-\sqrt{3})$$

and $f \in \mathcal{N}_{\mathbb{B}}^{r}$ with $r = 2 - \sqrt{3}$.

The maximality of the radius $r = r_{\mathcal{N}}(\mathcal{M}) = 2 - \sqrt{3}$ confirms the function $f_1 \in \mathcal{M}_{\mathbb{B}}$. Indeed, for f_1 and a point $a \in \partial r\mathbb{B}$ such that $x^*(\|a\|^{-1} a) = 1$ for the functional x^* used in (6), we have

$$\operatorname{Re} \frac{LLf_1(a)}{Lf_1(a)} = 0, \operatorname{Re} \frac{LLf_1(x)}{Lf_1(x)} > 0, \|x\| \leq r \in [0, 2 - \sqrt{3}).$$

This completes the prof of the thesis (i).

The equalities $(ii), (iii), (iv)$ we will prove on another way. To do it let $f \in \mathcal{R}$ and $x \in \mathbb{B} \diagdown \{0\}$ be arbitrarily fixed. Then the functions $F_b(\zeta)$ from (6), with $b = ||x||^{-1}x$ belongs to the family S_{Δ}^{cc}. Therefore, by a Lewandowski's result [8] about the radius of starlikeness in the family S_{Δ}^{cc},

$$\operatorname{Re} \frac{Lf(\zeta b)}{f(\zeta b)} = \operatorname{Re} \frac{F_b'(\zeta)}{F_b(\zeta)} > 0, |\zeta| \leqslant r \in [0, 4\sqrt{2} - 5).$$

Choosing in the above $\zeta = ||x||$, we have the relation

$$\operatorname{Re} \frac{Lf(x)}{f(x)} > 0, \|x\| < r, r \in (0, 4\sqrt{2} - 5],$$

which implies that $f \in \mathcal{M}_{\mathbb{B}}^{r}$, with $r = 4\sqrt{2} - 5$.

The proofs of the equalities (iii) and (iv) run similarly as the proof of the equality (ii). In these cases is enough to use a Grunsky's result [6] about the radius of starlikeness in the family S_{Δ} and a result due to Krzyż [7] about the radius of close-to-convexity in the family S_{Δ}, respectively. The constants $r_{\mathcal{M}}(Q)$, $r_{\mathcal{R}}(Q)$ are best possible, because the above radii of starlikeness within the family S_{Δ} and of close-to-convexity within the family S_{Δ} are best possible. $\square$

References

[1] I. I. Bavrin, *On a class of univalent mappings in Banach spaces*, Acad. Sci. Georgian SSR **97** (1980), 285–288 (in Russian).

[2] I. E. Bazilevich, *On a case of integrability in quadrates of the Loewner-Kufarev equation*, Mat. Sb. NS **37979** (1955), 471–476.

[3] L. De Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[4] P. Duren, *Univalent Functions*, Springer-Verlag, New York 1983.

[5] I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker, INC, New York-Basel 2003.

[6] H. Grunsky, *Zwei Bemerkungen zur konformen Abbildung (Two remarks of conformal mappings)*, Jahresber. Deutsch. Math. Verein. **43** (1933), 140–143, (in German).

[7] J. Krzyż, *The radius of close-to-convexivity within the family of univalent functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 201–204.

[8] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes II*, Ann. Univ. Mariae Curie-Skłodowska **14** (1960), 19–46.

[9] P. Liczberski, *On Bavrin's families of holomorphic maps of Banach spaces into the complex plane*, Folia Sci. Univ. Techn. Resoviensis, Mat. **1** (1988), 77–85.

[10] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einhetskreises, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden (Investigations of the distortion of conforme mappings of the unit disc, which are realised by fuctions with nonvanishing differential)*, Ber. über d.Verh. d. Königl. Sächs. Ges. d. Wissensch. **69** (1917), (in German).

[11] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*, Springer-Verlag, New York 1969.

[12] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Goetingen 1975.

[13] S. Ruscheweyh, *Nicht lineare Extremalproblem für holomorphe Stieltjesintegrale (Non linear extremal problem for holomorphic Stieltjes integral)*, Math. Zeitschrift **142** (1975), (in German).

[14] T. J. Suffrige, *Starlike and convex maps in Banach spaces*, Pacif. J. Math. **46** (1973), 575–589.

[15] T. J. Suffrige, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Math. **599** (1975), 146–159.

Institute of Mathematics
Łódź University of Technology
Wólczańska 215, PL-90-924 Łódź
Poland
e-mails: renata.dlugosz@p.lodz.pl
piliczb@p.lodz.pl

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 20, 2014

NOWE WŁASNOŚCI HOLOMORFICZNYCH BAVRINA W PRZESTRZENIACH BANACHA

S t r e s z c z e n i e

Praca dotyczy własności funkcji zespolonych, które są holomorficzne w kuli jednostkowej $\mathbb{B}$ zespolonej przestrzeni Banacha $\mathbb{X}$ oraz spełniają pewne warunki geometryczne. Tego typu funkcje były rozważane wcześniej przez I. Bavrina oraz drugiego autora. Powyższe funkcje można wykorzystać do badania pewnych rodzin odwzorowań biholomorficznych kuli $\mathbb{B}$ na obszary w $\mathbb{X}$. W pracy podane są, dla takich rodzin funkcji, twierdzenia o jednoznaczności, twierdzenia o włożeniu oraz rozwiązane zostały pewne problemy ekstremalne.

Słowa kluczowe: funkcje holomorficzne w przestrzeniach Banacha, rodziny Bavrinowskie, twierdzenia o jednoznaczności, twierdzenia o włożeniu, oszacowania w rozwinięciach funkcji w szereg wielomianów jednorodnych

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 71–80

In memory of
Professor Zygmunt Charzyński (1914–2001)

Marek Bienias, Szymon Głąb, and Władysław Wilczyński

CARDINALITY OF SETS OF ρ-UPPER AND ρ-LOWER CONTINUOUS FUNCTIONS

Summary

We prove that the cardinality of the set of all 1-upper continuous functions $f : (0, 1) \to \mathbb{R}$ equals $2^{\mathfrak{c}}$. In particular, there is a non-Borel 1-upper continuous function. We also prove that there are $2^{\mathfrak{c}}$ ρ-lower continuous functions for $\rho \in (0, \frac{1}{2})$.

Keywords and phrases: Lebesgue density, path continuity, continuous functions

1. Introduction

S. Kowalczyk and K. Nowakowska in [4] introduced the notion of ρ-upper continuous functions, where $\rho \in (0, 1)$. The notion of ρ-upper continuity is an example of the so called path continuity, which was widely described in [1]. They prove that each function of that class is Lebesgue measurable and has the Denjoy property. They also show that for any $\rho \in (0, \frac{1}{2})$, there are ρ-continuous functions which are not of Baire class one. Similar class, of the so called $[\lambda, \rho]$-continuous functions, was studied by K. Nowakowska in [7]. In [5] and [6], S. Kowalczyk and K. Nowakowska studied the so-called maximal additive and multiplicative classes for $[\lambda, \rho]$-continuous and ρ-upper continuous functions. A. Karasińska and E. Wagner-Bojakowska (cf. [2]) showed that there exists a function which is 1-upper continuous (i.e. ρ-upper continuous for each $\rho \in [0, 1)$) and is not approximately continuous. Moreover, they showed that there is a function which is 1-upper continuous but is not of Baire class one.

In this paper we prove that there are $2^{\mathfrak{c}}$ functions which are 1-upper continuous and $2^{\mathfrak{c}}$ functions which are ρ-lower continuous, for $\rho \in (0, \frac{1}{2})$. In particular, there are non-Borel 1-upper continuous and ρ-lower continuous functions. We also show that the class of all ρ-upper continuous functions for $\rho \in (0,1)$ is not closed under point-wise addition, and therefore it does not form a linear subspace of $\mathbb{R}^{\mathbb{R}}$.

We use standard set-theoretic notation – for any undefined notion we refer the reader to A. Kechris's monograph [3]. Let $\mathbb{N} = \{1,2,3,...\}$ stands for the set of all natural numbers and let m stands for Lebesgue measure on the real line. Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. The numbers

$$\underline{d}^+(E,x) = \liminf_{t \to 0^+} \frac{m(E \cap [x, x+t])}{t}$$

and

$$\overline{d}^+(E,x) = \limsup_{t \to 0^+} \frac{m(E \cap [x, x+t])}{t}$$

are called respectively the right lower density of E at x and right upper density of E at x. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E,x) = \overline{d}^+(E,x) \quad \text{and} \quad \underline{d}^-(E,x) = \overline{d}^-(E,x),$$

then we call these numbers the right density and left density of E at x, respectively. The numbers

$$\underline{d}(E,x) = \liminf_{t,k \to 0^+} \frac{m(E \cap [x-t, x+k])}{t+k}$$

and

$$\overline{d}(E,x) = \limsup_{t,k \to 0^+} \frac{m(E \cap [x-t, x+k])}{t+k}$$

are called the upper and lower density of E at x, respectively. Note that

$$\underline{d}(E,x) = \min\{\underline{d}^-(E,x), \underline{d}^+(E,x)\}$$

and

$$\overline{d}(E,x) = \max\{\overline{d}^-(E,x), \overline{d}^+(E,x)\}.$$

If $\underline{d}(E,x) = \overline{d}(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$. If $d(E,x) = 1$, then we say that x is a density point of E.

Let us recall the notion of ρ-upper and ρ-lower continuity.

Definition 1. Let $\rho \in (0,1)$ and let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval. We say that f is

(*i*) ρ-upper continuous at $x \in I$ provided there exists a measurable set $E \subseteq I$ with $x \in E$, such that $\overline{d}(E,x) > \rho$ and $f|_E$ is continuous at x;

(*ii*) ρ-lower continuous at $x \in I$ provided there exists a measurable set $E \subseteq I$ with $x \in E$, such that $\underline{d}(E,x) > \rho$ and $f|_E$ is continuous at x;

If f is ρ-upper (ρ-lower resp.) continuous at every point of I, we say that f is ρ-upper (ρ-lower resp.) continuous.

We will denote the class of all ρ-upper (ρ-lower resp.) continuous functions defined on a unit interval $(0,1)$ by $\mathcal{UC}_\rho$ ($\mathcal{LC}_\rho$ resp.). We say that f is 1-upper continuous if it is ρ-upper continuous for every $\rho \in [0,1)$.

For any nonempty set A we will denote the family of all finite sequences of elements of A by $A^{<\mathbb{N}}$. For any finite sequence $s = (s_1, ..., s_n) \in A^{<\mathbb{N}}$ and $a \in A$ by $s\hat{\ }a$ we denote a concatenation of s and a, i.e. $s\hat{\ }a = (s_1, ..., s_n, a)$. By $|s|$ we denote the length of s. If $\alpha \in A^{\mathbb{N}}$, then let $\alpha|n = (\alpha(1), ..., \alpha(n))$ and $\alpha|0 = \emptyset$. Moreover, by $2^{<\mathbb{N}}$ (resp. $2^{\mathbb{N}}$) we mean the set $\{0,1\}^{<\mathbb{N}}$ (resp. $\{0,1\}^{\mathbb{N}}$). For $n \in \mathbb{N}$ we denote $2^n = \{s \in 2^{<\mathbb{N}} : \ |s| = n\}$ and $2^0 = \{\emptyset\}$.

2. Cardinality of the set $\mathcal{UC}_\rho$

Note that the definition of upper 1-continuous functions and approximately continuous functions are similar but not identical. We have that f is 1-upper continuous at x if there is a measurable set E such that $\overline{d}(E,x) = 1$ and $f|_{E\cup\{x\}}$ is continuous, and we say that f is approximately continuous at x if there is a measurable set E such that $d(E,x) = 1$ and $f|_{E\cup\{x\}}$ is continuous. This slight difference in the definition has a huge consequence. Since an approximately continuous function is of Baire class one, there are $\mathfrak{c}$ approximately continuous functions. In this section we show that there are $2^{\mathfrak{c}}$ functions which are 1-upper continuous.

The main idea is the following. We may define sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$x_{n+1} < y_{n+1} < u_{n+1} < w_{n+1} < x_n \quad \text{for each} \quad n \in \mathbb{N},\ (x_n)_{n\in\mathbb{N}}$$

tends to 0 and

$$\overline{d}(\bigcup_{n\in\mathbb{N}} [x_n, y_n], 0) = \overline{d}(\bigcup_{n\in\mathbb{N}} [u_n, w_n], 0) = 1.$$

Then, we define $f : \mathbb{R} \to \mathbb{R}$ in the following way:

- put $f(x) = 1$ if $x \in [x_n, y_n]$ or $-x \in [x_n, y_n]$ or $x > x_1$ or $x < -x_1$
- put $f(x) = 0$ if $x \in [u_n, w_n]$ or $-x \in [u_n, w_n]$
- on $\mathbb{R} \setminus \{0\}$ define f to be (locally) affine.

The question is how to define f at 0? One can put $f(0) = 1$ or $f(0) = 0$. In both cases f is 1-upper continuous at 0, and consequently f is 1-upper continuous on its domain. Our plan is to make a similar construction of a function f for which the set A where we can freely put 0 or 1 is large, i.e. of cardinality $\mathfrak{c}$. Since there are $2^{\mathfrak{c}}$ functions from A to $\{0,1\}$ our construction will show that we may define f in $2^{\mathfrak{c}}$ ways to get a 1-upper continuous function.

Theorem 2. *The set $\mathcal{UC}_1$ has cardinality $2^{\mathfrak{c}}$. In particular there is a non-Borel 1-upper continuous function.*

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers from the interval $(0,\frac{1}{3})$ that it is convergent to 0. One can construct a sequence $\{I_s : s \in 2^{<\mathbb{N}}\}$ of closed subintervals of $[0,1]$ such that

1. $I_\emptyset = [0,1]$;
2. $\forall s \in 2^{<\mathbb{N}}\ \min I_{s\hat{}0} = \min I_s, \max I_{s\hat{}1} = \max I_s$;
3. $\forall s \in 2^{<\mathbb{N}}\ \forall i \in \{0,1\}\ |I_{s\hat{}i}| = q_{|s|+1}|I_s|$.

Let

$$\mathcal{C} = \bigcup_{\alpha\in 2^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} I_{\alpha|n}.$$

One can easily check, that $\mathcal{C}$ is a Cantor-like set (i.e. perfect, zero-dimensional and of measure zero).

The above construction is similar to the classical geometric construction of the ternary Cantor. Starting from the unit interval $I_\emptyset = [0,1]$, in the first step we remove the open middle subinterval leaving two subintervals: I_0, I_1 of lengths q_1. In the second step the open middle subintervals of the remaining intervals I_0, I_1 are removed, leaving four segments: $I_{00}, I_{01}, I_{10}, I_{11}$ of lengths $q_2|I_0| = q_2 \cdot q_1$ and so on.

Let $A, B \subseteq [0,1]$ be the unions of the closures of intervals removed in the odd and even steps respectively, i.e.

$$A = \bigcup_{k\in\mathbb{N}} \bigcup_{s\in 2^{2k-2}} \overline{I_s \setminus (I_{s\hat{}0} \cup I_{s\hat{}1})}$$
$$B = \bigcup_{k\in\mathbb{N}} \bigcup_{s\in 2^{2k-1}} \overline{I_s \setminus (I_{s\hat{}0} \cup I_{s\hat{}1})}$$

Observe, that $A \cup B = [0,1]\setminus\mathcal{C}'$, where $\mathcal{C}' \subseteq \mathcal{C}$ is created by removing the boundary of the set $[0,1]\setminus\mathcal{C}$ from $\mathcal{C}$. In particular $\mathcal{C}'$ has cardinality $\mathfrak{c}$. Let $g : A\cup B \to \{0,1\}$ be a characteristic function of A. To illustrate the idea, the function g is created by putting 1 over the closure of the intervals removed in the odd step and putting 0 elsewhere. Let $\mathcal{C}'' = \mathcal{C}' \setminus \{0,1\}$, let $F \subseteq \mathcal{C}''$ and define a function $g_F : (0,1) \to \{0,1\}$ by the formula

$$g_F(x) = \begin{cases} 1, \text{ when } x \in F \\ g(x), \text{ when } x \in A\cup B \\ 0, \text{ otherwise.} \end{cases}$$

We will show that g_F is 1-upper continuous. Let $x \in (0,1)$ and let $E = g_F^{-1}(\{g_F(x)\})$. We have the following possibilities:

1. $x \in A$, then $\overline{d}^+(E,x) = 1$ or $\overline{d}^-(E,x) = 1$. Hence, $\overline{d}(E,x) = 1$ and g_F is 1-upper continuous at x.
2. $x \in B$, then $\overline{d}^+(E,x) = 1$ or $\overline{d}^-(E,x) = 1$. Hence, $\overline{d}(E,x) = 1$ and g_F is 1-upper continuous at x.

3. $x \in F$, then there are sequences: $(n_k)_{k\in\mathbb{N}}$ of odd numbers, $(J_{n_k})_{k\in\mathbb{N}}$ of intervals, $(s_{n_k})_{k\in\mathbb{N}}$ of finite $0-1$ sequences, such that for every $k \in \mathbb{N}$:
   a) $|s_{n_k}| = n_k - 1$;
   b) $\{x\} = \bigcap_{k\in\mathbb{N}} I_{s_{n_k}}$;
   c) J_{n_k} is a connected component of A;
   d) $\frac{|J_{n_k}|}{|I_{s_{n_k}}|} = 1 - 2q_{n_k}$;
   e) both sequences $(\min J_{n_k})_{k\in\mathbb{N}}, (\max J_{n_k})_{k\in\mathbb{N}}$ converge to x from the right.

   The above sequences can be chosen in the following way: there is a sequence $(J_{n_k})_{k\in\mathbb{N}}$ of intervals removed in the odd step (i.e. connected components of A) that is convergent (in the sense of (e)) to x from the right and such that the interval J_{n_k} was removed exactly from $I_{s_{n_k}}$. By c) and the definition of g, we have that $J_{n_k} \subseteq E$. Hence, by b), d) and e) we obtain that $\overline{d}^+(E,x) = 1$ and g_F is 1-upper continuous at x.
4. $x \in \mathcal{C}'' \setminus F$, then by a similar reasoning as in the case when $x \in F$, the function g_F is 1-upper continuous at x.

Since there are exactly $2^{\mathfrak{c}}$ subsets of $\mathcal{C}''$, the set $\mathcal{UC}_1$ has cardinality $2^{\mathfrak{c}}$. □

3. Cardinality of the set $\mathcal{LC}_\rho$

Let $\rho \in (0, \frac{1}{2})$. This section is devoted to the proof that the cardinality of the set $\mathcal{LC}_\rho$ is $2^{\mathfrak{c}}$. The idea of a construction of $2^{\mathfrak{c}}$ ρ-lower continuous functions is similar to the one from the previous section and uses a geometric construction of the ternary Cantor set.

Lemma 3. *Let*

$$b_n = \frac{1}{2n-1} \quad \textit{and} \quad a_n = \frac{1}{2n}, \quad \textit{for} \quad n \in \mathbb{N}.$$

Let

$$A = \bigcup_{n\in\mathbb{N}} [a_n, b_n), \quad \textit{then} \quad d^+(A,0) = \frac{1}{2}.$$

The above fact is probably folklore, but for the reader's convenience we state the proof.

Proof. For $h \in (0,1)$ let

$$\varphi(h) = \frac{m(A \cap (0,h))}{h}.$$

Let $n \in \mathbb{N}$ and observe that

$$\varphi(\frac{1}{2n-1}) > \frac{1}{2}$$

and

$$\varphi(\frac{1}{2n}) < \frac{1}{2}.$$

By the simple properties of the function φ, we have that

$$d^+(A,0) = \lim_{n\to\infty} \varphi(\frac{1}{n}) = \frac{1}{2}.$$

□

Lemma 4. *For every open interval $(a,b) \subseteq (0,1)$ and every $\varepsilon > 0$ there exists a set $E \subseteq (a,b)$ such that for all $h \in (0, b-a)$*

$$\left| \frac{m(E \cap (a, a+h))}{h} - \frac{1}{2} \right| < \varepsilon.$$

Proof. Let $(a,b) \subseteq (0,1), \varepsilon > 0$ and let $A \subseteq (0,1)$ be as in Lemma 3. There exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{m(A \cap (0, \frac{1}{n_0}))}{\frac{1}{n_0}} - \frac{1}{2} \right| < \varepsilon.$$

In particular for every $n \geq n_0$

$$\left| \frac{m(A \cap (0, \frac{1}{n}))}{\frac{1}{n}} - \frac{1}{2} \right| < \varepsilon.$$

Let us put

$$A_{n_0} = A \cap (0, \frac{1}{n_0}), \quad B_{n_0} = (0, \frac{1}{n_0}) \setminus A_{n_0}.$$

The idea is to fit the set A_{n_0} into the first half of (a,b) and to fit the set $-B_{n_0}$ into the second one, i.e. let

$$\tilde{E} = (\alpha \cdot A_{n_0} + a) \cup (b - \alpha \cdot B_{n_0}), \quad \text{where} \quad \alpha = \frac{(b-a)n_0}{2}.$$

For the further applications we have to modify the set $\tilde{E}$. Let $\{..., x_{-1}, x_0, x_1, x_2, ...\}$ be an increasing sequence of all endpoints of the intervals that form the set $\tilde{E}$, where

$$x_0 = \frac{a+b}{2}.$$

For every $i \in \mathbb{Z}$ one can choose a small enough $h_i > 0$ such that for the set

$$F = \bigcup_{i\in\mathbb{Z}} (x_i - h_i, x_i + h_i)$$

we have

- $d^+(F,a) = 0$;
- $d^-(F,b) = 0$.

It is easy to see that $E = \tilde{E} \setminus F$ has the desired properties. □

Remark 5. *In the sequel we will be using the following construction: for an interval $(a,b) \subseteq (0,1)$ and $\varepsilon > 0$, let $E \subseteq (a,b)$ be as in Lemma 4 and define a function $f : (a,b) \to [0,1]$ by the formula*

$$f(x) = \begin{cases} 1, \textit{ when } x \in E \\ 0, \textit{ when } x \in (a,b) \setminus (E \cup F) \\ \textit{locally affine, when } x \in F \end{cases}$$

where F is as in the proof of Lemma 4 and the locally affine mappings over F ensure that f is continuous.

Theorem 6. *Let $\rho \in (0, \frac{1}{2})$. The set $\mathcal{LC}_\rho$ has cardinality $2^{\mathfrak{c}}$. In particular there is a non-Borel ρ-lower continuous function.*

Proof. Consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, where

$$\varepsilon_n = \frac{1}{10^n} \quad \text{for} \quad n \in \mathbb{N}.$$

One can construct a sequence $\{I_s : s \in 2^{<\mathbb{N}}\}$ of closed subintervals of $[0,1]$ such that

1. $I_\emptyset = [0,1]$;
2. $\forall s \in 2^{<\mathbb{N}}$ $\min I_{s\hat{}0} = \min I_s, \max I_{s\hat{}1} = \max I_s$;
3. $\forall s \in 2^{<\mathbb{N}}$ $\forall i \in \{0,1\}$ $|I_{s\hat{}i}| = \frac{1}{3}|I_s|$.

Let

$$\mathcal{C} = \bigcup_{\alpha\in 2^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} I_{\alpha|n},$$

then $\mathcal{C}$ is the ternary Cantor set.

The idea of the proof is to define a function $f : (0,1) \setminus \mathcal{C} \to [0,1]$ by putting functions as in Remark 5 for $\varepsilon_n > 0$ inside all of the intervals removed in the n^{th} step. To formalize this concept: for every $n \in \mathbb{N}$ and $s \in 2^{n-1}$ let

$$J_s = I_s \setminus (I_{s\hat{}0} \cup I_{s\hat{}1}) \quad \text{and} \quad f_s : J_s \to [0,1]$$

be as in Remark 5 for $\varepsilon_{|s|+1} = \varepsilon_n$. Let us put

$$f = \bigcup_{s\in 2^{<\mathbb{N}}} f_s.$$

We will show that for any $D \subseteq \mathcal{C} \setminus \{0,1\}$, the function $f_D : (0,1) \to [0,1]$ defined by

$$f_D(x) = \begin{cases} 1, \text{ when } x \in D \\ f(x), \text{ when } x \in (0,1) \setminus \mathcal{C} \\ 0, \text{ otherwise,} \end{cases}$$

is ρ-lower continuous. Let $D \subseteq \mathcal{C} \setminus \{0,1\}$ and fix a point $x \in (0,1)$ and let $B = f_D^{-1}(\{f_D(x)\})$. We have the following possibilities:

1. $x \in (0,1) \setminus \mathcal{C}$, then f_D is continuous at x. Hence, it is ρ-lower continuous.

2. $x \in \mathcal{C} \setminus \{0,1\}$ and x is not an endpoint of any interval J_s. We will show that $\underline{d}^+(B,x) > \rho$. Let $h > 0$ and consider the right neighborhood of x of the form $(x, x+h)$. We have two possibilities:

   a) $x + h \in \mathcal{C}$, then for
   $$\mathcal{J} = \{J_s : J_s \subseteq (x, x+h)\}$$
   and
   $$\mathcal{S} = \{s \in 2^{<\mathbb{N}} : J_s \in \mathcal{J}\}$$
   we have that
   $$h = m((x, x+h)) = \sum_{J \in \mathcal{J}} m(J).$$
   Moreover, for $N_h = \min\{|s|+1 : s \in \mathcal{S}\}$, N_h tends to ∞ whenever $h \to 0$. By Lemma 4 the following estimation holds
   $$(\frac{1}{2} - \varepsilon_{N_h})h < m(B \cap (x, x+h)) < (\frac{1}{2} + \varepsilon_{N_h})h.$$

   (b) or $x+h \notin \mathcal{C}$, then there is a finite $0-1$ sequence s_h such that $x+h \in J_{s_h}$. Let
   $$\mathcal{J} = \{J_s : J_s \subseteq (x, x+h)\} \quad \text{and} \quad \mathcal{S} = \{s \in 2^{<\mathbb{N}} : J_s \in \mathcal{J}\}.$$
   We have that
   $$h = m((x, x+h) \cap (\bigcup \mathcal{J} \cup J_{s_h})).$$
   Moreover, let
   $$N_h = \min\{|s|+1 : s \in \mathcal{S}\} \quad \text{and} \quad N = \min\{N_h, |s_h|+1\}.$$
   Observe that $N \to \infty$ whenever $h \to 0$. Let $d = \min J_{s_h}$. Since
   $$m(B \cap (x, x+h)) = m(B \cap \bigcup \mathcal{J}) + m(B \cap (d, x+h)),$$
   by Lemma 4 the following inequalities hold:
   $$m(B \cap (x, x+h)) < (\frac{1}{2} + \varepsilon_{N_h})(d-x) + (\frac{1}{2} + \varepsilon_{|s_h|+1})(x+h-d) < (\frac{1}{2} + \varepsilon_N)h$$
   and
   $$m(B \cap (x, x+h)) > (\frac{1}{2} - \varepsilon_{N_h})(d-x) + (\frac{1}{2} - \varepsilon_{|s_h|+1})(x+h-d) > (\frac{1}{2} - \varepsilon_N)h.$$

   By the above calculations, for a small enough $h > 0$, the number
   $$\frac{m(B \cap (x, x+h))}{h}$$
   can be as close to $\frac{1}{2}$ as we want. In particular $\underline{d}^+(B,x) > \rho$. By a similar argument (taking the intervals from the left neighborhood of x) we may prove that $\underline{d}^-(B,x) > \rho$.

3. $x \in \mathcal{C} \setminus \{0,1\}$ and x is an endpoint of some interval J_s. In this case $\underline{d}^+(B,x) > \rho$ and $\underline{d}^-(B,x) > \rho$ as well, where one of these inequalities follows from Lemma 4 and the other one can be proved by a similar argument as in the previous case.

Summarizing, $\underline{d}(B,x) > \rho$ and f_D is ρ-lower continuous at x. The fact that there are $2^{\mathfrak{c}}$ subsets of $\mathcal{C} \setminus \{0,1\}$ completes the proof. □

We end the paper with two open questions:

1. Does there exist a linear space $X \subseteq \mathbb{R}^{\mathbb{R}}$ of dimension $2^{\mathfrak{c}}$ such that any $f \in X \setminus \{0\}$ is 1-upper continuous? In other words: is the set $\mathcal{UC}_1$ $2^{\mathfrak{c}}$-lineable?
2. What is the cardinality of the set of $[\lambda, \rho]$-continuous functions? Is it $2^{\mathfrak{c}}$ for some positive λ and ρ?

Acknowledgement

The first and the second author have been supported by the NCN grant DEC-2012/07/D/ST1/02087.

References

[1] A. M. Bruckner, R. J. O'Malley, and B. S. Thomson, *Path derivatives: a unified view of certain generalized derivatives*, Trans. Amer. Math. Soc. **283**, no. 1 (1984), 97–125.

[2] A. Karasińska and E. Wagner-Bojakowska, *Some remarks on ρ-upper continuous functions*, Tatra Mt. Math. Publ. **46** (2010), 85–89.

[3] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York 1998.

[4] S. Kowalczyk and K. Nowakowska, *A note on ρ-upper continuous functions*, Tatra Mt. Math. Publ. **44** (2009), 153–158.

[5] S. Kowalczyk and K. Nowakowska, *Maximal classes for the family of* $[\lambda, \rho]$*-continuous functions*, Real Anal. Exchange **36**, no. 2 (2010/11), 307–324.

[6] S. Kowalczyk and K. Nowakowska, *Maximal classes for ρ-upper continuous functions*, J. Appl. Anal. **19**, no. 1 (2013), 69–89.

[7] K. Nowakowska, *On the family of* $[\lambda, \rho]$*-continuous functions*, Tatra Mt. Math. Publ. **44** (2009), 129–138.

Institute of Mathematics
Łódź University of Technology
Wólczańska 215, 93-005 Łódź
Poland
e-mail: marek.bienias88@gmail.com
szymon.glab@p.lodz.pl

Department of Real Functions, Faculty of Mathematics
University of Łódź
Banacha 22, 90-238, Łódź
Poland
e-mail: wwil@uni.lodz.pl

Presented by Władysław Wilczyński at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 20, 2014

MOCE ZBIORÓW FUNKCJI ρ-GÓRNIE I ρ-DOLNIE CIĄGŁYCH

S t r e s z c z e n i e

W pracy pokazujemy, że istnieje $2^{\mathfrak{c}}$ funkcji 1-górnie ciągłych. W szczególności dowodzi to istnienie nieborelowskiej funkcji, która jest 1-górnie ciągła. W drugiej części pracy pokazujemy, że dla każdego $\rho \in (0, \frac{1}{2})$, zbiór funkcji ρ-dolnie ciągłych jest mocy $2^{\mathfrak{c}}$.

Słowa kluczowe: gęstość Lebesgue'a, ciągłość ścieżkowa, ciągłość funkcji

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 81–93

In memory of
Professor Zygmunt Charzyński (1914–2001)

Filip Strobin

AN APPLICATION OF A FIXED POINT THEOREM FOR MULTIFUNCTIONS IN A PROBLEM OF CONNECTEDNESS OF ATTRACTORS OF FAMILIES OF IFSs

Summary

Let X be a Banach space and $\mathcal{F},\mathcal{G}$ be iterated function systems (IFSs) on X. We apply a standard fixed point theorem for multifunctions to study the size of the set

$$C_{\mathcal{F},\mathcal{G}} := \{w \in X : \text{the attractor of the IFS } \mathcal{F}\cup\mathcal{G}_w \text{ is connected}\},$$

where $\mathcal{G}_w := \{g_w : g\in\mathcal{G}\}$ and $g_w(x) := g(x) + w$, $x \in X$.

The paper is a continuation of recent studies on similar (but more restrictive) problems undertaken by Mihail, Miculescu, Swaczyna and the author.

Keywords and phrases: fractals, iterated function systems, fixed points, Baire category, multifunctions

1. Introduction and basic notation

If (X,d) is a metric space then by $\mathcal{K}(X)$ we denote the space of all nonempty and compact subsets of X, endowed with the standard Hausdorff-Pompeiu metric H:

$$H(K,D) := \max\left\{\sup_{x\in K}\left(\inf_{y\in D} d(x,y)\right), \sup_{y\in D}\left(\inf_{x\in K} d(x,y)\right)\right\}$$

Every finite family of continuous selfmaps of a metric space X will be called *a function system.*

Every function system $\mathcal{F}$ generates the mapping $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by setting

$$\mathcal{F}(K) := \bigcup_{f\in\mathcal{F}} f(K)$$

We say that a function system $\mathcal{F}$ *generates a fractal*, if there is a unique set $A_{\mathcal{F}} \in \mathcal{K}(X)$ such that

$$\mathcal{F}(A_{\mathcal{F}}) = \bigcup_{f\in\mathcal{F}} f(A_{\mathcal{F}}) = A_{\mathcal{F}}$$

and, moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $(\mathcal{F}^{(k)}(K))_{k\in\mathbb{N}}$ converges to $A_{\mathcal{F}}$ with respect to the Hausdorff metric.

Sets $A_{\mathcal{F}}$ obtained in this way are called *attractors* or *fractals in the sense of Hutchinson and Barnsley*, and function systems which generate them – *iterated function systems* (IFSs in short).

A classical result due to Hutchninson [H] (cf. also [B]) says that

Theorem 1.1. *Every function system $\mathcal{F}$ on a complete metric space consisting of Banach contractions (i.e., Lipschitzian mappings with the Lipschitz constants less then 1) generates a fractal.*

This result can be proved for example by showing that the mapping $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a Banach contraction with respect to the Hausdorff metric. Hence (knowing that completeness of X implies completeness of $\mathcal{K}(X)$) the assumptions of the Banach fixed point theorem are satisfied, so $\mathcal{F}$ generates a fractal.

Theorem 1.1 was a starting point for a wide and important part of dynamical systems theory. In particular, mathematicians were interested in the question: which function systems generate fractals? For example it turned out that it is enough to assume that each $f \in \mathcal{F}$ is a *Matkowski contraction* [Mat], which means that there is a nondecreasing function (called a *comparison function*) $\varphi : [0, \infty) \to [0, \infty)$ such that the sequence of iterates $(\varphi^{(k)}(t))_{k\in\mathbb{N}}$ converges to 0 for $t \geq 0$, and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \quad x, y \in X$$

Clearly, each Banach contraction is a Matkowski contraction (simply take $\varphi(t) := Lip(f)t$, where $Lip(f)$ is the Lipschitz constant of f), but the converse is not true.

The "Matkowski" version of Theorem 1.1 can be proved in a similar way as the original one - we only have to observe that the mapping $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a Matkowski contraction, and use the Matkowski fixed point theorem [Mat], which is one of the strongest strengthenings of the Banach fixed point theorem.

Another direction of studies was concentrated on the topological structure of fractals generated by IFSs. The topic of our paper fits to this part of investigations.

Let X be a Banach space and $f, g : X \to X$ be Matkowski contractions. For every $w \in X$, let $g_w(x) := g(x) + w$, $x \in X$, and let A_w be the attractor of the IFS $\{f, g_w\}$. In the papers [MM1], [MM2] and [SS] the following problem was considered: what is the size of the set

$$C_{f,g} := \{w \in X : A_w \text{ is connected}\}$$

It was proved that (under some conditions) it is closed and is a countable union of compact sets. In consequence, if X is infinite dimensional, then it is nowhere dense.

Our paper is a further discussion of this topic.

Our main contribution (stated in Sections 2 and 3) is application of a classical fixed point theorem for contractive multifunctions in these studies. We not only reprove most results from [SS] in a much more general setting, but also state a result which do not have analogy there.

Also, in the last section, we apply the Kuratowski-Ulam theorem and consider the size of some further sets.

2. Auxiliary results concerning multifunctions

If $F : X \to \mathcal{P}(X)$ is a multifunction (by $\mathcal{P}(X)$ we denote the family of all subsets of X), then we say that $x \in X$ is a *fixed point of* F, if $x \in F(x)$. The following result seems to be folklore (a more general result can be found in [DMRZ]) but we give a sketch of a proof for the sake of completeness.

Lemma 2.1. *Assume that (X, d) is a complete metric space and $F : X \to \mathcal{K}(X)$ be such that the Lipschitz constant $Lip(F) < 1$. Then the set $Fix(F)$ of fixed points of F is nonempty and compact.*

Proof. Consider the mapping $\tilde{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\tilde{F}(K) := \bigcup_{x\in K} F(x)$. By continuity of F, the function $\tilde{F}$ is well defined in the sense that $\tilde{F}(K) \in \mathcal{K}(X)$ for every $K \in \mathcal{K}(X)$. It is also standard to check that $\tilde{F}$ is a Banach contraction (in fact, $Lip(\tilde{F}) = Lip(F)$). Hence by the Banach fixed point theorem, there is a compact set $A \subset X$ such that $\tilde{F}(A) = A$ and each sequence of iterates $(\tilde{F}^{(k)}(K))_{k\in\mathbb{N}}$ converges to A. Now if x is a fixed point of F, then we have $x \in F(x)$, hence $x \in \tilde{F}(\{x\}) \subset \tilde{F}^{(2)}(\{x\})$. In this way we can show that $x \in \tilde{F}^{(k)}(\{x\})$ for every $k \in \mathbb{N}$. Hence, by the fact that $\tilde{F}^{(k)}(\{x\}) \to A$, we get $x \in A$. All in all, $Fix(F) \subset A$. It is also standard to check that $Fix(F)$ is closed. $\square$

If $\mathcal{F}, \mathcal{G}$ are function systems on a metric space X, then put

$$D(\mathcal{F}, \mathcal{G}) := \max\left\{ \sup_{g\in\mathcal{G}, x\in X} \left(\inf_{f\in\mathcal{F}} d(f(x), g(x)) \right), \sup_{f\in\mathcal{F}, x\in X} \left(\inf_{g\in\mathcal{G}} d(f(x), g(x)) \right) \right\}.$$

Directly from the definition we can see that

$$D(\mathcal{F}, \mathcal{G}) = \sup\{H(\mathcal{F}(K), \mathcal{G}(K)) : K \in \mathcal{K}(X)\} \tag{1}$$

If $\mathcal{F}$ is a function system consisting of Matkowski contractions, then for each $f \in \mathcal{F}$ there is a comparison function φ_f which witness to the fact that f is a Matkowski contraction (recall that $\psi : [0, \infty) \to [0, \infty)$ is a comparison function if it is non-decreasing and $\psi^{(k)}(t) \to 0$ for $t \geq 0$). Since $\mathcal{F}$ is finite, the function $\varphi(t) := \max\{\varphi_f(t) : f \in \mathcal{F}\}$ is a comparison function for all $f \in \mathcal{F}$. This common function will be called *a comparison function for* $\mathcal{F}$. It is standard to check that in this case φ is a comparison function for the mapping $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ generated by $\mathcal{F}$.

If $\mathcal{F}$ is a function system consisting of Matkowski contractions on a complete metric space, then by $A_\mathcal{F}$ we will denote the attractor of $\mathcal{F}$.

Lemma 2.2. *Let $\mathcal{F},\mathcal{G}$ be function systems consisting of Matkowski contractions, φ be a comparison function for $\mathcal{F}\cup\mathcal{G}$ and $t>0$ be such that $D(\mathcal{F},\mathcal{G})\leq t-\varphi(t)$. Then $H(A_\mathcal{F},A_\mathcal{G})\leq t$.*

Proof. Let $K\in\mathcal{K}(X)$. Since $\mathcal{F}^{(k)}(K)\to A_\mathcal{F}$ and $\mathcal{G}^{(k)}(K)\to A_\mathcal{G}$, it is enough to observe that for every $k\in\mathbb{N}$,

$$H(\mathcal{F}^{(k)}(K),\mathcal{G}^{(k)}(K))\leq t \tag{2}$$

For $k=1$ it follows from assumptions and (1). Now assume that (2) holds for some $k\in\mathbb{N}$. Then by (1) we have

$$\begin{aligned}H(\mathcal{F}^{(k+1)}(K),\mathcal{G}^{(k+1)}(K))&\leq H(\mathcal{F}^{(k+1)}(K),\mathcal{F}(\mathcal{G}^{(k)}(K)))\\&\quad+H(\mathcal{F}(\mathcal{G}^{(k)}(K)),\mathcal{G}^{(k+1)}(K))\\&\leq \varphi(H(\mathcal{F}^{(k)}(K),\mathcal{G}^{(k)}(K)))+D(\mathcal{F},\mathcal{G})\leq\varphi(t)+t-\varphi(t)=t\end{aligned}$$

□

Now we define several multifunction maps which will be useful for further studies. Let X be a Banach space and fix function systems $\mathcal{F},\mathcal{G}$ consisting of Matkowski contractions defined on X, such that at least $\mathcal{G}$ is nonempty. If $w\in X$, then put

$$\mathcal{G}_w:=\{g_w:g\in\mathcal{G}\},$$

where

$$g_w(x):=g(x)+w,\quad x\in X.$$

Finally, let $D\in\mathcal{K}(X)$ and $n=1,2,\ldots$. For every $w\in X$, define

$$\begin{aligned}F_1(w)&:=A_{\mathcal{F}\cup\mathcal{G}_w}\\F_2(w)&:=\mathcal{F}(A_{\mathcal{F}\cup\mathcal{G}_w})-\mathcal{G}(A_{\mathcal{F}\cup\mathcal{G}_w})\\F_3(w)&:=D-\mathcal{G}(A_{\mathcal{G}_w})\\F_4(w)&:=D-\mathcal{G}(\mathcal{G}_w^{(n)}(D))\end{aligned}$$

Clearly, each F_i is a multifunction from X to $\mathcal{K}(X)$. The next result indicates important for our studies contractive properties of them. If $\mathcal{F}$ is a function system, then set

$$L(\mathcal{F}):=\max\{Lip(f):f\in\mathcal{F}\}$$

Let us remark that (in general) $L(\mathcal{F})$ may be not equal to the Lipschitz constant of the mapping $\mathcal{F}:\mathcal{K}(X)\to\mathcal{K}(X)$ generated by $\mathcal{F}$; however, it is greater or equal to this constant.

Lemma 2.3. *In the above setting:*

(i) *F_1 is uniformly continuous;*

(ii) *If $L(\mathcal{F} \cup \mathcal{G}) < 1$, then the Lipschitz constant*

$$Lip(F_1) \leq \frac{1}{1 - L(\mathcal{F} \cup \mathcal{G})};$$

(iii) *If $L(\mathcal{F} \cup \mathcal{G}) < 1$, then*

$$Lip(F_2) \leq \frac{L(\mathcal{F}) + L(\mathcal{G})}{1 - L(\mathcal{F} \cup \mathcal{G})};$$

(iv) *If $L(\mathcal{G}) < 1$, then*

$$Lip(F_3) \leq \frac{L(\mathcal{G})}{1 - L(\mathcal{G})};$$

(v) $Lip(F_4) \leq \sum_{k=1}^{n} L(\mathcal{G})^k$.

Proof. Ad(i). If φ is a comparison function for $\mathcal{F} \cup \mathcal{G}$, then it is a comparison function for $\mathcal{F} \cup \mathcal{G}_w$ for all $w \in X$. Also it is easy to see that for every $w, s \in X$,

$$D(\mathcal{F} \cup \mathcal{G}_w, \mathcal{F} \cup \mathcal{G}_s) \leq ||w - s||.$$

Hence uniform continuity follows from Lemma 2.2 and an easy fact that if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.

Ad(ii). If $\alpha := L(\mathcal{F} \cup \mathcal{G}) < 1$, then $\varphi(t) = \alpha t$ is a comparison function for $\mathcal{F} \cup \mathcal{G}$. Now take $w, s \in X$ and let $t \geq 0$ be such that

$$D(\mathcal{F} \cup \mathcal{G}_w, \mathcal{F} \cup \mathcal{G}_s) = (1 - \alpha)t.$$

Then by Lemma 2.2 we have that

$$\begin{aligned} H(F_1(w), F_1(s)) \leq t &= \frac{1}{1-\alpha}(1-\alpha)t \\ &= \frac{1}{1-\alpha} D(\mathcal{F} \cup \mathcal{G}_w, \mathcal{F} \cup \mathcal{G}_s) \leq \frac{1}{1-\alpha}||w - s||. \end{aligned}$$

Ad(iii). For every $w \in X$, set $A_w := A_{\mathcal{F} \cup \mathcal{G}_w}$. By point (i) and standard properties of the Hausdorff metric, we have:

$$\begin{aligned} H(F_2(w), F_2(s)) &= H(\mathcal{F}(A_w) - \mathcal{G}(A_w), \mathcal{F}(A_s) - \mathcal{G}(A_s)) \\ &\leq H(\mathcal{F}(A_w), \mathcal{F}(A_s)) + H(\mathcal{G}(A_w), \mathcal{G}(A_s)) \\ &\leq L(\mathcal{F})H(A_w, A_s) + L(\mathcal{G})H(A_w, A_s) \\ &\leq \frac{L(\mathcal{F}) + L(\mathcal{G})}{1 - L(\mathcal{F} \cup \mathcal{G})}||w - s||. \end{aligned}$$

Ad(iv). By point (i) we have:

$$\begin{aligned} H(F_3(w), F_3(s)) &= H(D - \mathcal{G}(A_{\mathcal{G}_w}), D - \mathcal{G}(A_{\mathcal{G}_s})) \leq H(\mathcal{G}(A_{\mathcal{G}_w}), \mathcal{G}(A_{\mathcal{G}_s})) \\ &\leq L(\mathcal{G})H(A_{\mathcal{G}_w}, A_{\mathcal{G}_s}) \leq \frac{L(\mathcal{G})}{1 - L(\mathcal{G})}||w - s|| \end{aligned}$$

Ad(v). We have

$$\begin{aligned} H(F_4(w),F_4(s)) &= H(D-\mathcal{G}(\mathcal{G}_w^{(n)}(D)),D-\mathcal{G}(\mathcal{G}_s^{(n)}(D))) \\ &\le H(\mathcal{G}(\mathcal{G}_w^{(n)}(D)),\mathcal{G}(\mathcal{G}_s^{(n)}(D))) \le L(\mathcal{G})H(\mathcal{G}_w^{(n)}(D),\mathcal{G}_s^{(n)}(D)) \\ &= L(\mathcal{G})H(\mathcal{G}(\mathcal{G}_w^{(n-1)}(D))+w,\mathcal{G}(\mathcal{G}_s^{(n-1)}(D))+s) \\ &\le L(\mathcal{G})||w-s||+L(\mathcal{G})H(\mathcal{G}_w^{(n-1)}(D),\mathcal{G}_s^{(n-1)}(D)) \\ &\le ... \le (L(\mathcal{G})+...+L(\mathcal{G})^n)\,||w-s||. \end{aligned}$$

□

3. Main results

In this section we assume that X is a Banach space and $\mathcal{F},\mathcal{G}$ are nonempty families of Matkowski contractions. Define

$$C_{\mathcal{F},\mathcal{G}} := \{w\in X : A_{\mathcal{F}\cup G_w} \text{ is connected}\}$$

(recall that $A_{\mathcal{F}\cup\mathcal{G}_w}$ is the attractor of the IFS $\mathcal{F}\cup\mathcal{G}_w$). Clearly, if

$$\mathcal{F}=\{f\} \quad \text{and} \quad \mathcal{G}=\{g\}$$

for some Matkowski contractions f,g (i.e., $\mathcal{F}$ and $\mathcal{G}$ are singletons), then

$$C_{\mathcal{F},\mathcal{G}} = C_{f,g}.$$

The first result shows that $C_{\mathcal{F},\mathcal{G}}$ is always closed. It is a strengthening of [SS, Theorem 2.2] (obtained in a bit simpler way):

Theorem 3.1. *The set $C_{\mathcal{F},\mathcal{G}}$ is closed.*

Proof. Observe that $C_{\mathcal{F},\mathcal{G}}$ is the preimage of the closed (in $\mathcal{K}(X)$) set $\{K\in\mathcal{K}(X) : K \text{ is connected}\}$ by a continuous (by Corollary 2.3(i)) function $w\to A_{\mathcal{F}\cup\mathcal{G}_w}$. The result follows. □

The next result is a generalization of [SS, Theorem 2.7(i)] for the case when $Lip(g)<\frac{1}{2}$. The proof will be quite similar, but it will use some additional reasonings; also, we will indicate the use of Lemma 2.1.
We say that $f:X\to X$ is *compact*, if the closure $\overline{f(D)}$ is compact for every bounded set $D\subset X$.

Theorem 3.2. *Assume that each $f\in\mathcal{F}$ is compact and $L(\mathcal{G})<\frac{1}{2}$. Then $C_{\mathcal{F},\mathcal{G}}$ is a countable union of compact sets. In particular, if X is infinite dimensional, then $C_{\mathcal{F},\mathcal{G}}$ is nowhere dense.*

To prove this result we need the following Lemma which is a direct generalization of [SS, Lemma 2.10].

Lemma 3.3. *Assume that $\mathcal{P},\mathcal{R}$ are function systems consisting of Matkowski contractions. Then*

$$\mathcal{R}(A_{\mathcal{P}\cup\mathcal{R}}) = \bigcup_{k=1}^{\infty} \mathcal{R}^{(k)}(\mathcal{P}(A_{\mathcal{P}\cup\mathcal{R}})) \cup A_{\mathcal{R}}$$

Proof. Put $A := A_{\mathcal{P}\cup\mathcal{R}}$. At first observe that

$$A = \mathcal{P}(A) \cup \mathcal{R}(A).$$

Hence

$$\mathcal{R}(A) = \mathcal{R}(\mathcal{P}(A)) \cup \mathcal{R}^{(2)}(A),$$

and in particular,

$$A = \mathcal{P}(A) \cup \mathcal{R}(A) = \mathcal{P}(A) \cup \mathcal{R}(\mathcal{P}(A)) \cup \mathcal{R}^{(2)}(A)$$

which gives us

$$\mathcal{R}(A) = \mathcal{R}(\mathcal{P}(A)) \cup \mathcal{R}^{(2)}(\mathcal{P}(A)) \cup \mathcal{R}^{(3)}(A).$$

Proceeding in this way we can show that for every $k \geq 1$,

$$\mathcal{R}(A) = \bigcup_{i=1}^{k} \mathcal{R}^{(i)}(\mathcal{P}(A)) \cup \mathcal{R}^{(k+1)}(A)$$

This together with a fact that $\mathcal{R}^{(k)}(A) \to A_{\mathcal{R}}$ easily imply the thesis. □

Proof. (of Theorem 3.2) Since $C_{\mathcal{F},\mathcal{G}}$ is closed, it is enough to show that it can be covered by countable union of compact sets.

Let $A_w := A_{\mathcal{F}\cup\mathcal{G}_w}$. For every $k \in \mathbb{N}$, put

$$D_k := \overline{\mathcal{F}\left(\bigcup_{w\in B(0,k)} A_w\right)},$$

where $B(0,k)$ is the closed ball in X centered in 0 with the radius k. Then each D_k is compact. Indeed, since the mapping $w \to A_w$ is uniformly continuous (this is exactly thesis of Lemma 2.3(i)), the set $\bigcup_{w\in B(0,k)} A_w$ is bounded, so D_k is compact since each $f \in \mathcal{F}$ is compact.

By Lemma 3.3 we have that

$$\begin{aligned}
C_{\mathcal{F},\mathcal{G}} &\subset \{w \in X : \mathcal{F}(A_w) \cap \mathcal{G}_w(A_w) \neq \emptyset\} \\
&= \left\{w \in X : \mathcal{F}(A_w) \cap \left(A_{\mathcal{G}_w} \cup \bigcup_{n=1}^{\infty} \mathcal{G}_w^{(n)}(\mathcal{F}(A_w))\right) \neq \emptyset\right\} \\
&= \{w \in X : \mathcal{F}(A_w) \cap A_{\mathcal{G}_w} \neq \emptyset\} \\
&\cup \bigcup_{n=1}^{\infty} \left\{w \in X : \mathcal{F}(A_w) \cap \mathcal{G}_w^{(n)}(\mathcal{F}(A_w)) \neq \emptyset\right\} \\
&= \bigcup_{k=1}^{\infty} \{w \in B(0,k) : \mathcal{F}(A_w) \cap A_{\mathcal{G}_w} \neq \emptyset\}
\end{aligned}$$

$$\cup \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ w \in B(0,k) : \mathcal{F}(A_w) \cap \mathcal{G}_w^{(n)}(\mathcal{F}(A_w)) \neq \emptyset \right\}$$
$$\subset \bigcup_{k=1}^{\infty} \{ w \in X : D_k \cap A_{\mathcal{G}_w} \neq \emptyset \} \cup \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ w \in X : D_k \cap \mathcal{G}_w^{(n)}(D_k) \neq \emptyset \right\}.$$

Hence it remains to check that for any $n \in \mathbb{N}$ and compact set D, sets

$$M_{\mathcal{G},n,D} := \{ w \in X : D \cap \mathcal{G}_w^{(n)}(D) \neq \emptyset \} \quad \text{and} \quad N_{\mathcal{G},D} := \{ w \in X : D \cap A_{\mathcal{G}_w} \neq \emptyset \}$$

are compact.

We first prove that each set $N_{\mathcal{G},D}$ is compact. Since

$$A_{\mathcal{G}_w} = \mathcal{G}_w(A_{\mathcal{G}_w}) = \mathcal{G}(A_{\mathcal{G}_w}) + w,$$

we have that $N_{\mathcal{G},D}$ is exactly the set of fixed points of the multifunction $F : X \to \mathcal{K}(X)$ defined by $F(w) := D - \mathcal{G}(A_{\mathcal{G}_w})$. By the fact that $L(\mathcal{G}) < 1/2$ and by Lemma 2.3(iv), we have that

$$Lip(F) \leq \frac{L(\mathcal{G})}{1 - L(\mathcal{G})} < 1.$$

Then Lemma 2.1 implies that $N_{\mathcal{G},D}$ is compact.

Now we prove compactness of the set $M_{\mathcal{G},n,D}$. Observe that $M_{\mathcal{G},n,D}$ is exactly the set of fixed points of multifunction $P : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$P(w) := D - \mathcal{G}(\mathcal{G}_w^{(n-1)}(D))$$

(we assume here that $\mathcal{G}_w^{(0)}(D) = D$). Hence, in view of Lemma 2.1 it is enough to show that $Lip(P) < 1$.

If $n = 1$, then

$$H(P(w), P(s)) = H(D - \mathcal{G}(D), D - \mathcal{G}(D)) = 0$$

and it is ok. If $n > 1$, then by Lemma 2.3(v) and our assumptions we have

$$Lip(P) \leq L(\mathcal{G}) + ... + L(\mathcal{G})^{n-1} \leq \frac{1}{2} + ... + \frac{1}{2^{n-1}} < 1$$

Finally assume that X is infinite dimensional. Then each compact set is nowhere dense, so by the Baire category theorem, $C_{\mathcal{F},\mathcal{G}}$ has empty interior. Since it is also closed, it is nowhere dense. □

The next result gives another sufficient condition for compactness of $C_{\mathcal{F},\mathcal{G}}$. Note that in [SS] there is no analogous result.

Theorem 3.4. *Assume that*

$$L(\mathcal{F}) + L(\mathcal{G}) + L(\mathcal{F} \cup \mathcal{G}) < 1 \tag{3}$$

Then $C_{\mathcal{F},\mathcal{G}}$ is compact. In particular if X is infinite dimensional then it is nowhere dense.

Before we give the proof let us remark that if

$$L(\mathcal{F} \cup \mathcal{G}) < \frac{1}{3},$$

then (3) is satisfied. However, if

$$L(\mathcal{F}) < \frac{1}{2} \quad (\text{or} \quad L(\mathcal{G}) < \frac{1}{2}),$$

then there is $\delta > 0$ such that if $L(\mathcal{G}) < \delta$ (or $L(\mathcal{F}) < \delta$), then (3) is also satisfied.

Proof. For every $w \in X$, set

$$A_w := A_{\mathcal{F} \cup \mathcal{G}_w}.$$

Observe that

$$C_{\mathcal{F},\mathcal{G}} \subset \{w \in X : \mathcal{F}(A_w) \cap (\mathcal{G}(A_w) + w) \neq \emptyset\} = \{w \in X : w \in \mathcal{F}(A_w) - \mathcal{G}(A_w)\}$$

Hence it is enough to show that the latter set is compact. But this set is a set of fixed points of the multivalued mapping $P : X \to \mathcal{K}(X)$ defined by

$$P(w) := \mathcal{F}(A_w) - \mathcal{G}(A_w).$$

In view of Lemma 2.1, it is enough to observe that

$$Lip(P) < 1.$$

By our assumptions and Lemma 2.3(iii) we have that

$$Lip(P) \leq \frac{L(\mathcal{F}) + L(\mathcal{G})}{1 - L(\mathcal{F} \cup \mathcal{G})} < 1.$$

□

Remark 3.5. Mihail and Miculescu proved in [MM2] that if X is a Hilbert space and f is a noncompact, bounded linear operator such that $||f|| < 1$, then there is a bounded linear operator g with $||g|| < 1$ and such that $C_{f,g}$ is not a countable union of compact sets. In view of Theorem 3.4 we see that the norm of such g cannot be as small as we want (observe that $Lip(g) = ||g||$).

The last result is a (kind of) generalization of [SS, Theorem 2.12].

Theorem 3.6. *Assume that that each $f \in \mathcal{F} \cup \mathcal{G}$ is compact. Then $C_{\mathcal{F},\mathcal{G}}$ is a countable union of compact sets. In particular if X is infinite dimensional then it is nowhere dense.*

Proof. Again it is enough to show that $C_{\mathcal{F},\mathcal{G}}$ can be covered by countable union of compact sets. If $w \in X$, then let $A_w := A_{\mathcal{F} \cup \mathcal{G}_w}$.
For every $k \in \mathbb{N}$, put

$$D_k := \overline{\mathcal{F}\left(\bigcup_{w \in B(0,k)} A_w\right)} \quad \text{and} \quad E_k := \overline{\mathcal{G}\left(\bigcup_{w \in B(0,k)} A_w\right)}.$$

Then each D_k's and E_k's are compact (see the proof of Theorem 3.2). Then we have

$$\begin{aligned} C_{\mathcal{F},\mathcal{G}} &\subset \{w \in X : \mathcal{F}(A_w) \cap \mathcal{G}_w(A_w) \neq \emptyset\} \\ &\subset \bigcup_{k=1}^{\infty} \{w \in B(0,k) : \mathcal{F}(A_w) \cap (\mathcal{G}(A_w) + w) \neq \emptyset\} \\ &\subset \bigcup_{k=1}^{\infty} \{w \in X : D_k \cap (E_k + w) \neq \emptyset\} = \bigcup_{k=1}^{\infty} (D_k - E_k) \end{aligned}$$

Since sets $D_k - E_k$ are compact, the result follows. □

Let us point out that we can look at sets $D_k - E_k$ from the above proof as sets of fixed point of constant multivalued functions $w \to D_k - E_k$.

The contractive assumptions in Theorem 3.2 and (especially) in Theorem 3.4 are rather strict and there may be suspicion that the sets $C_{\mathcal{F},\mathcal{G}}$ are singletons (or even empty) in those cases. The next two examples show that it is not true. By c_0 we denote the space of all sequences which converges to 0, endowed with the maximium norm.

Example 3.7. Let $X = c_0$. Define

$$f_1((x_i)_{i\in\mathbb{N}}) := \left(\frac{2}{7}x_1, \frac{1}{2}x_2, \frac{1}{2}x_3, ...\right) \text{ and } f_2((x_i)_{i\in\mathbb{N}}) := \left(\frac{2}{7}x_1 + 1, \frac{1}{2}x_2, \frac{1}{2}x_3, ...\right)$$

$$f_3((x_i)_{i\in\mathbb{N}}) := \left(\frac{2}{7}x_1 + 2, \frac{1}{2}x_2, \frac{1}{2}x_3, ...\right) \text{ and } g((x_i)_{i\in\mathbb{N}}) := \left(\frac{2}{7}x_1, \frac{1}{2}x_2, \frac{1}{2}x_3, ...\right)$$

and set

$$\mathcal{F} := \{f_1, f_2, f_3\} \quad \text{and} \quad \mathcal{G} := \{g\}.$$

Then

$$Lip(\mathcal{F}) = Lip(\mathcal{G}) = \frac{2}{7} < \frac{1}{3},$$

so assumptions of Theorem 3.4 are satisfied. It is easy to see that for every $w = (w_1, 0, 0, ...)$ such that that

$$w_1 \in \left[\frac{5}{2}, \frac{10}{3}\right],$$

we have that

$$A_{\mathcal{F}\cup\mathcal{G}_w} = \left[0, \frac{7}{5}w_1\right] \times \{0\} \times \{0\} \times ...$$

Hence

$$\left[\frac{5}{2}, \frac{10}{3}\right] \times \{0\} \times \{0\} \times ... \subset C_{\mathcal{F},\mathcal{G}}.$$

On the other hand f_1, f_2, f_3, g are not compact, so we cannot use Theorem 3.6 here.

Example 3.8. Again, let $X = c_0$. Define

$$f((x_i)_{i\in\mathbb{N}}) := \left(\frac{2}{3}x_1, 0, 0, ...\right) \quad \text{and} \quad g((x_i)_{i\in\mathbb{N}}) := \left(\frac{1}{3}|x_1|, \frac{1}{3}x_2, \frac{1}{2}x_3, ...\right)$$

and set $\mathcal{F} := \{f\}$ and $\mathcal{G} := \{g\}$. Since f is compact and

$$Lip(g) = \frac{1}{3} < \frac{1}{2},$$

assumptions of Theorem 3.2 are satisfied. It is easy to see that for every $w = (w_1, 0, 0, ...)$ with $w_1 \geq 0$, we have that

$$A_{\mathcal{F}\cup\mathcal{G}_w} = \left[0, \frac{3}{2}w_1\right] \times \{0\} \times \{0\} \times ...$$

Hence $[0, \infty) \times \{0\} \times \{0\} \times ... \subset C_{\mathcal{F},\mathcal{G}}$. On the other hand, g is not compact, so we cannot use Theorem 3.6. Moreover, g is not not affine, so we cannot use [SS, Theorem 2.7(i)] here.

4. Application of the Kuratowski-Ulam theorem

Recall the following classical Kuratowski-Ulam theorem and its converse.

If X, Y are sets, $A \subset X \times Y$ and $x \in X$, then we denote by $(A)_x$ the set $\{y \in Y : (x, y) \in A\}$, usually called the *$x$-section of A*.

Theorem 4.1. [Sr, Theorem 3.5.16] *Let X and Y be topological Baire spaces where Y is second countable. Suppose $A \subset X \times Y$ has the Baire property. Then A is of the first category if and only if, the set $\{x \in X : (A)_x$ is of the first category$\}$ is residual.*

As a direct application of this result we have the following:

Theorem 4.2. *Let X be an infinite dimensional, separable Banach space and $\mathcal{F}, \mathcal{G}$ be function systems consisting of Matkowski contractions and such that one of the following conditions hold:*

(i) *Each $f \in \mathcal{F}$ is compact and $Lip(\mathcal{G}) < \frac{1}{2}$.*

(ii) *Each $h \in \mathcal{F} \cup \mathcal{G}$ is compact;*

(iii) *$Lip(\mathcal{F}) + Lip(\mathcal{G}) + Lip(\mathcal{F} \cup \mathcal{G}) < 1$.*

Then the set

$$W_{\mathcal{F},\mathcal{G}} := \{(s, w) \in X \times X : A_{\mathcal{F}_s\cup\mathcal{G}_w} \textit{is connected}\}$$

is closed and nowhere dense.

Proof. The fact that $W_{\mathcal{F},\mathcal{G}}$ is closed can be proved in a similar way as in the case of set $C_{\mathcal{F},\mathcal{G}}$ - it is the preimage of the closed set $\{K \in \mathcal{K}(X) : K \text{ is connected}\}$ over the continuous function $X \times X \ni (w, s) \to A_{\mathcal{F}_s\cup\mathcal{G}_w} \in \mathcal{K}(X)$.

Observe that for every $s \in X$, $(W_{\mathcal{F},\mathcal{G}})_s = C_{\mathcal{F}_s,\mathcal{G}}$. Hence by Theorems 3.2, 3.4 and 3.6, each $(W_{\mathcal{F},\mathcal{G}})_s$ is nowhere dense, hence of the first category. By the Kuratowski-Ulam theorem, $W_{\mathcal{F},\mathcal{G}}$ is of the first category. Then it has empty interior, so it is closed with empty interior, so it is closed and nowhere dense. $\square$

The following example shows that $W_{\mathcal{F},\mathcal{G}}$ may contain an isometric copy of X, even in the case when all conditions (i)-(iii) are satisfied. In particular, we cannot strengthen Theorem 4.2 in the manner of Theorems 3.2, 3.4 and 3.6.

Example 4.3. Let f be any Matkowski contraction on X and $\mathcal{F} = \mathcal{G} := \{f\}$. Then for every $w \in X$, the set $A_{\mathcal{F}_w \cup \mathcal{G}_w}$ is a singleton (simply the fixed point of $f + w$). Hence $\{(w,w) : w \in X\} \subset W_{\mathcal{F},\mathcal{G}}$ and the set $\{(w,w) : w \in X\}$ is an isometric copy of X (provided we consider the maximum norm on the product $X \times X$).

Acknowledgement

The first author has been supported by the NCN Grant FUGA No. 2013/08/S/ST1/ /00541.

References

[B] M. F. Barnsley, *Fractals everywhere.* Academic Press Professional, Boston, MA, 1993.

[DMRZ] F. S. de Blasi, J. Myjak, S. Reich, and A. J. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued Var. Anal. **17** (2009), no. 1, 97-112.

[H] J. Hutchinson, *Fractals and self-similarity.* Indiana Univ. Math. J. **30** (1981), no 5, 713-747.

[Mat] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math. **127** (1975) 68 pp.

[MM1] A. Mihail, R. Miculescu, *On a family of IFSs whose attractors are not connected.* J. Math. Anal. Appl. 376 (2011), 187–192.

[MM2] A. Michalil, R. Miculescu *A Characterization of Compact Operators Via the Non-Connectedness of the Attractors of a Family of IFSs*, Complex Anal. Oper. Theory, DOI 10.1007/s11785-012-02

[Sr] S.M. Srivastava, *A course of Borel sets*, Springer, New York 1998.

[SS] F. Strobin, J. Swaczyna *Connectedness of attractors of certain family of IFSs*, submitted.

Institute of Mathematics
Jan Kochanowski University in Kielce
Kielce, Poland
and
Institute of Mathematics
Łódź University of Technology
Łódź, Poland
e-mail: filip.strobin@p.lodz.pl

Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 4, 2014

ZASTOSOWANIE TWIERDZENIA O PUNKCIE STAŁYM DLA MULTIFUNKCJI W PROBLEMIE SPÓJNOŚCI ATRAKTORÓW RODZIN IFSÓW

S t r e s z c z e n i e

Niech X będzie przestrzenią Banacha, a $\mathcal{F}$, $\mathcal{G}$ niech będą iterowanymi układami odwzorowań na X. W artykule wykorzystujemy klasyczne twierdzenie o punkcie stałym dla multifunkcji do badania wielkości zbioru

$$C_{\mathcal{F},\mathcal{G}} := \{w \in X : \text{atraktor generowany przez układ} \quad \mathcal{F} \cup \mathcal{G}_w \quad \text{jest spójny}\},$$

gdzie $\mathcal{G}_w := \{g_w : g \in \mathcal{G}\}$, a $g_w(x) := g(x) + w$ dla $x \in X$. Artykuł jest kontynuacją badań prowadzonych przez Mihaila i Miculescu oraz Swaczynę i autora.

Słowa kluczowe: fraktale, układy funkcji iterowanych, punkty stałe, kategorie Baire'a, wielofunkcje

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 95–100

In memory of
Professor Zygmunt Charzyński (1914–2001)

Anna Łazińska

SOME REMARKS ON THE ORDER OF STARLIKENESS IN NEIGHBOURHOODS OF HARMONIC FUNCTIONS

Summary

In 1984 J. Clunie and T. Sheil-Small published their studies of some geometric properties of harmonic univalent functions that are sense-preserving in the unit disc. Several authors examined the influence of some coefficient conditions on geometric properties of these functions. J. M. Jahangiri, H. Silverman and Evelyn M. Silvia published (2007) a theorem concerning the starlikeness of order β of harmonic functions in δ-neighbourhoods ([6], Th. 2.5). In the paper we would like to present a counterexample to this statement and to give the revised version of the theorem.

Keywords and phrases: harmonic functions, neighbourhood of a function, order of starlikeness, coefficient conditions

1.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. As we know, each complex function harmonic in Δ may be written in the form $f = h + \overline{g}$, where h, g are functions holomorphic in Δ.

Denote by S_H the class of harmonic functions $f = h + \overline{g}$ univalent and sense-preserving in Δ and such that $h(0) = h'(0) - 1 = 0$, $g(0) = 0$ ([3], see also [4], p. 78). In consequence, a function $f \in S_H$ is of the form

$$(1) \qquad f = h + \overline{g}, \quad h(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{+\infty} b_n z^n, \quad z \in \Delta,$$

where $|b_1| < 1$.

Let $\alpha \in \langle 0, 1)$. We say that a function $f \in S_H$ is starlike of order α in Δ

(see [10], [5]) if

$$\frac{\partial}{\partial t}\left(\arg f(re^{it})\right) \geq \alpha, \qquad r \in (0,1), \quad t \in \langle 0, 2\pi).$$

The subclass of S_H consisting of functions starlike of order α we denote by $S_H(\alpha)$, $\alpha \in \langle 0, 1)$.

In 1999 J. M. Jahangiri proved two following theorems.

Theorem A ([5], Th. 1). *Let f be of the form* (1) *and $\alpha \in \langle 0,1)$. If*

$$\sum_{n=2}^{+\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{+\infty} \frac{n+\alpha}{1-\alpha}|b_n| \leq 1, \tag{2}$$

then $f \in S_H(\alpha)$.

Theorem B ([5], Th. 2). *Let f be of the form* (1) *where $a_n \leq 0$, $n = 2, 3, \ldots$, $b_n \geq 0$, $n = 1, 2, \ldots$ and $\alpha \in \langle 0, 1)$. Then $f \in S_H(\alpha)$ if and only if the condition* (2) *holds.*

Geometric properties of the functions in question are systematically studied e.g. in [1–3, 5, 7, 8, 11, 12].

2.

For a function f of the form (1) and for $\delta > 0$, the δ-neighbourhood of f is defined as the class of all functions F of the form

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \quad z \in \Delta,$$

such that

$$\sum_{n=2}^{+\infty} n|a_n - A_n| + \sum_{n=1}^{+\infty} n|b_n - B_n| \leq \delta. \tag{3}$$

The δ-neighbourhood of f we denote by $N_\delta(f)$ ($\delta > 0$).

The δ-neighbourhoods for holomorphic functions were introduced by S. Ruscheweyh [9] and for harmonic functions – by Y. Avci and E. Złotkiewicz [2].

In the paper [6] the authors published, among others, the following statement.

Theorem C ([6], Th. 2.5). *Let $\alpha \in (0,1)$ and let a function f of the form* (1) *satisfy the condition* (2)*. Then $N_\delta(f)$ consists of functions that are starlike of order $\beta < \alpha$, when*

$$\delta = \frac{2(\alpha - \beta)}{(2-\alpha)(2-\beta)}.$$

We would like to present counterexamples to this statement.

Example.

a) Take

$$\alpha_0 = \frac{3}{4} \quad \text{and} \quad \beta_0 = \frac{1}{2}.$$

According to the statement from paper [6] in this case we obtain $\delta_0 = \dfrac{4}{15}$.

Consider the functions

$$f_1(z) = z - \frac{1}{7}\overline{z}, \quad z \in \Delta,$$

$$F_1(z) = z - \frac{43}{105}\overline{z}, \quad z \in \Delta.$$

The function f_1 satisfies the condition (2) with $\alpha_0 = \dfrac{3}{4}$ (we have $\dfrac{1+\frac{3}{4}}{1-\frac{3}{4}}\left|-\dfrac{1}{7}\right| = 1$). According to (3) we see that $F_1 \in N_{\frac{4}{15}}(f_0)$, because

$$\left|-\frac{1}{7} - \left(-\frac{43}{105}\right)\right| = \frac{28}{105} = \frac{4}{15}.$$

Now we show that the function F_1 is not starlike of order $\beta_0 = \dfrac{1}{2}$.

Using the known equalities (see e.g. [2], [5], [12]) we obtain

$$\frac{\partial}{dt}\left(\arg F_1(re^{it})\right) = \mathrm{Re}\frac{1 + \frac{43}{105}e^{-2it}}{1 - \frac{43}{105}e^{-2it}}, \quad r \in (0,1), \quad t \in \langle 0, 2\pi).$$

Setting, for example, $t = \frac{\pi}{2}$ we get

$$\mathrm{Re}\frac{1 - \frac{43}{105}}{1 + \frac{43}{105}} = \frac{62}{148} = \frac{31}{74} < \frac{1}{2} = \beta_0.$$

What is more, we can check that

$$\mathrm{Re}\frac{1 + \frac{43}{105}e^{-2it}}{1 - \frac{43}{105}e^{-2it}} \geq \frac{1}{2} \iff \left|\frac{43}{105}e^{-2it} - \frac{1}{3}\right| \leq \frac{2}{3}, \quad t \in \langle 0, 2\pi).$$

Hence the function F_1 is not starlike of order $\frac{1}{2}$ in Δ.

b) Let $0 < \beta < \alpha < 1$ and $\delta = \dfrac{2(\alpha-\beta)}{(2-\alpha)(2-\beta)}$ (like in Theorem C). We consider the functions

$$f_2(z) = z + \frac{1-\alpha}{1+\alpha}\overline{z}, \quad z \in \Delta,$$

$$F_2(z) = z + \left(\frac{1-\alpha}{1+\alpha} + \frac{2(\alpha-\beta)}{(2-\alpha)(2-\beta)}\right)\overline{z}, \quad z \in \Delta,$$

so we have $b_1 = \dfrac{1-\alpha}{1+\alpha} > 0$, $B_1 = \dfrac{1-\alpha}{1+\alpha} + \delta > 0$, $a_n = A_n = b_n = B_n = 0$, $n = 2, 3, \ldots$.

It is easily seen that f_2 satisfies the condition (2) and $F_2 \in N_\delta(f_2)$. Moreover, we obtain

$$\sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|A_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|B_n| = \frac{1+\beta}{1-\beta}\left(\frac{1-\alpha}{1+\alpha} + \frac{2(\alpha-\beta)}{(2-\alpha)(2-\beta)}\right) =$$

$$= 1 + \frac{6(\alpha-\beta)(\alpha+\beta-1)}{(2-\alpha)(1+\alpha)(2-\beta)(1-\beta)}.$$

If $\alpha + \beta > 1$, then this expression is greater than 1. Applying Theorem B we conclude that in this case the function F_2 is not starlike of order β in Δ.

3.

We will present the revised version of the mentioned theorem.

Theorem 1. *Let $\alpha \in (0,1)$ and let a function f of the form* (1) *satisfy the condition* (2)*. Then $N_\delta(f)$ consists of functions that are starlike of order $\beta < \alpha$ $(\beta \geq 0)$, when*

$$\delta = \frac{\alpha-\beta}{(2-\alpha)(1+\beta)}.$$

Proof. Let a function f of the form (1) satisfy the condition (2) with a fixed $\alpha \in (0,1)$ and let $\beta \in \langle 0, \alpha)$. Then, as it was shown in [6], we have

$$\sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|b_n| \leq \frac{(2-\beta)(1-\alpha)}{(1-\beta)(2-\alpha)}. \tag{4}$$

Let

$$\delta = \frac{\alpha-\beta}{(2-\alpha)(1+\beta)}.$$

Assume that a function F of the form

$$F(z) = z + \sum_{n=2}^{\infty}(a_n + c_n)z^n + \overline{\sum_{n=1}^{\infty}(b_n + d_n)z^n}, \quad z \in \Delta,$$

belongs to $N_\delta(f)$. By (3), it means that

$$|d_1| + \sum_{n=2}^{+\infty} n\left(|c_n| + |d_n|\right) \leq \delta. \tag{5}$$

From (4) we have

$$\sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|a_n + c_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|b_n + d_n| \leq \tag{6}$$

$$\leq \quad \frac{(2-\beta)(1-\alpha)}{(1-\beta)(2-\alpha)} + \sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|c_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|d_n|.$$

Observe that

$$\frac{n-\beta}{n(1-\beta)} = \frac{1}{1-\beta}\left(1-\frac{\beta}{n}\right) \le \frac{1}{1-\beta}, \quad n = 2,3,\dots,$$

$$\frac{n+\beta}{n(1-\beta)} = \frac{1}{1-\beta}\left(1+\frac{\beta}{n}\right) \le \frac{1+\beta}{1-\beta}, \quad n = 1,2,\dots,$$

and, of course,

$$\frac{1}{1-\beta} \le \frac{1+\beta}{1-\beta} \qquad (0 \le \beta < \alpha < 1).$$

Therefore, by (5), we obtain

$$\begin{aligned} \sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|c_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|d_n| &= \\ = \sum_{n=2}^{+\infty} \frac{n-\beta}{n(1-\beta)} n|c_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{n(1-\beta)} n|d_n| &\le \\ \le \frac{1+\beta}{1-\beta}\left(\sum_{n=2}^{+\infty} n|c_n| + \sum_{n=1}^{+\infty} n|d_n|\right) &\le \frac{1+\beta}{1-\beta}\,\delta. \end{aligned} \tag{7}$$

From (4), (6) and (7) we have

$$\sum_{n=2}^{+\infty} \frac{n-\beta}{1-\beta}|a_n+c_n| + \sum_{n=1}^{+\infty} \frac{n+\beta}{1-\beta}|b_n+d_n| \le \frac{(2-\beta)(1-\alpha)}{(1-\beta)(2-\alpha)} + \frac{1+\beta}{1-\beta}\,\delta =$$

$$= \frac{(2-\beta)(1-\alpha)}{(1-\beta)(2-\alpha)} + \frac{1+\beta}{1-\beta}\,\frac{\alpha-\beta}{(2-\alpha)(1+\beta)} = \frac{2-2\alpha-\beta+\alpha\beta+\alpha-\beta}{(1-\beta)(2-\alpha)} = 1.$$

Hence, by Theorem A, the function $F \in N_\delta(f)$ is starlike of order β.

References

[1] G. Adamczyk and A. Łazińska, *On some generalization of coefficient conditions for complex harmonic mappings*, Demonstratio Math. **37**, no. 2 (2004), 317–326.

[2] Y. Avci and E. Złotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae Curie-Skłodowska, Sec. A **44**, no. 1 (1990), 1–7.

[3] J. Clunie and T. Sheil-Small, *Harmonic univalent mappings*, Ann. Acad. Sci. Fenn., Ser. A. I. Math. **9** (1984), 3–25.

[4] P. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge 2004.

[5] J. M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Analysis and Appl. **235** (1999), 470–477.

[6] J. M. Jahangiri, H. Silverman, and E. M. Silvia, *Construction of planar harmonic functions*, Int. J. Math. Math. Sci., Article ID 70192, (2007), 1–11.

[7] Z. J. Jakubowski and A. Łazińska,*On some classes of harmonic functions with conditions imposed on coeficients and their arguments*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. **13** (2006), 95–111.

[8] Z. J. Jakubowski, A. Łazińska, and A. Sibelska, *On some properties of complex harmonic mappings with a two-parameter coefficient condition*, Math. Balkanica (New Series) **18** (2004), 313–319.

[9] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81**, no. 4 (1981), 521–527.

[10] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. **42**, no. 2 (1990), 237–248.

[11] H. Silverman, *Harmonic univalent mappings with negative coefficients*, J. Math. Anal. Appl. **220** (1998), 283–289.

[12] H. Silverman and E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zeland J. Math. **28** (1999), 275–284.

Department of Nonlinear Analysis
Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22, PL-90-238 Łódź
Poland
e-mail: lazinska@math.uni.lodz.pl

Presented by Zbigniew Jerzy Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 4, 2014

PEWNE UWAGI O RZĘDZIE GWIAŹDZISTOŚCI W OTOCZENIACH FUNKCJI HARMONICZNYCH

S t r e s z c z e n i e

J. Clunie i T. Sheil-Small w 1984 r. opublikowali pracę dotyczącą pewnych geometrycznych własności jednolistnych funkcji harmonicznych, które zachowują orientację w kole jednostkowym. Wielu autorów badało wpływ warunków współczynnikowych na geometryczne własności tych funkcji. W 2007 r. J. M. Jahangiri, H. Silverman i Evelyn M. Silvia opublikowali twierdzenie dotyczące gwiaździstości rzędu β funkcji harmonicznych w δ-otoczeniach ([6], Th. 2.5). W pracy przedstawimy kontrprzykład do tego stwierdzenia i podamy poprawioną wersję twierdzenia.

Słowa kluczowe: funkcje harmoniczne, otoczenie funkcji, rząd gwiaździstości, warunki współczynnikowe

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 101–109

In memory of prof. Claude Surry
our Dear Friend and Collegue

Małgorzata Nowak-Kępczyk

AN ALGEBRA GOVERNING REDUCTION OF QUATERNARY STRUCTURES TO TERNARY STRUCTURES I

REDUCTIONS OF QUATERNARY STRUCTURES TO TERNARY STRUCTURES

Summary

Reduction of quaternary structures to ternary structures is studied from the point of view of the governing algebra.

Keywords and phrases: noncommutative Galois extensions, finite dimensional algebras, associative rings and algebras, matrix rings

1. Programme of the paper

Reduction of quaternary structures to ternary structures is considered in analogy to the reduction of ternary structures to binary structures analyzed in [1]. Formally, we have to identify two complex $6{\times}6$-matrices in the ternary case and two complex $12{\times}12$-matrices in the quaternary case including the related Cauchy-Riemann-like ∂ end $\bar{\partial}$ operators and Galois extensions. The idea of replacing a ternary structure by an equivalent binary structure with the help of basic region [1] in a fractal representation [3] is shown in Fig. 1.

In order to discuss the reduction of ternary structures to binary structures and quaternary structures to ternary structures we need to consider five real or complex structure matrices, which might also represent matrix algebras over $\mathbb{R}$ or $\mathbb{C}$. The coefficients of these matrices might as well be real or complex matrices themselves.

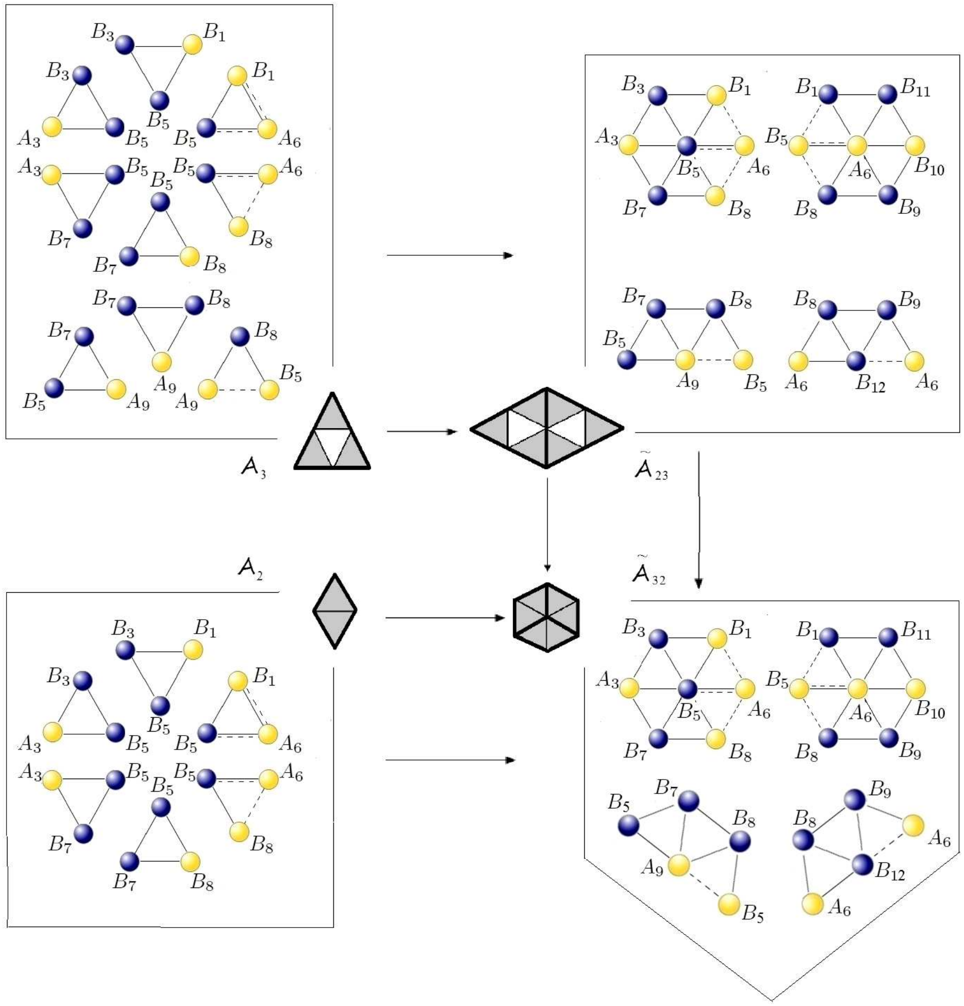

Fig. 1: The idea of replacing a ternary structure by an equivalent binary structure with the help of basic region [1] in a fractal representation [3]

The binary structure matrix has the form

$$\begin{matrix} \boxed{a} & \boxed{b} \\ \boxed{-b} & \boxed{a} \end{matrix}, \tag{1}$$

the ternary structure matrices have the form

$$(2)\ \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \quad \text{or} \quad (3)\ \begin{bmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{bmatrix},$$

and quaternary structure matrices have the form

$$(4)\ \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix} \quad \text{or} \quad (5)\ \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & -d & a & b \\ d & -c & -b & a \end{bmatrix}.$$

depending on weather we are investigating reduction of ternary or quaternary structure [4]. In analogy to (2) and (3) when, instead of reduction of quaternary structures, we turn our attention to quintenary structures, we have to replace structure matrices (4) and (5) with (6) and (7), respectively.

$$(6)\ \begin{bmatrix} a & b & c & d \\ -d & a & b & c \\ -c & -d & a & b \\ -b & -c & -d & a \end{bmatrix} \quad \text{or} \quad (7)\ \begin{bmatrix} a & b & c & d \\ b & a & -d & c \\ c & d & a & b \\ -d & c & b & a \end{bmatrix}.$$

2. Reduction of the ternary structures to binary structures without application of the ternary structure matrix

In the situation described in the title the construction relies upon the binary structure matrix (1). We have

$$\left[\begin{array}{cc|cc|cc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 \\ \hline x_5 & x_6 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_2 & x_1 & -x_4 & x_3 \\ \hline x_3 & x_4 & x_5 & x_6 & x_1 & x_2 \\ -x_4 & x_3 & -x_6 & x_5 & -x_2 & x_1 \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|ccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ y_4 & y_1 & y_2 & y_6 & y_4 & y_5 \\ y_3 & y_4 & y_1 & y_5 & y_6 & y_4 \\ \hline -y_4 & -y_5 & -y_6 & y_1 & y_2 & y_3 \\ -y_6 & -y_4 & -y_5 & y_4 & y_1 & y_2 \\ -y_5 & -y_6 & -y_4 & y_3 & y_4 & y_1 \end{array}\right]$$

3. Reduction of the ternary structures to binary structures with application of the ternary structure matrix

We turn our attention to the case where the ternary structure matrix (2) is applied. The resulting identification relation reads:

$$
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
-x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 \\
x_5 & x_6 & x_1 & x_2 & x_3 & x_4 \\
-x_6 & x_5 & -x_2 & x_1 & -x_4 & x_3 \\
x_3 & x_4 & x_5 & x_6 & x_1 & x_2 \\
-x_4 & x_3 & -x_6 & x_5 & -x_2 & x_1
\end{pmatrix}
\Updownarrow
\begin{pmatrix}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
y_3 & y_1 & y_2 & y_6 & y_4 & y_5 \\
y_2 & y_3 & y_1 & y_5 & y_6 & y_4 \\
-y_4 & -y_5 & -y_6 & y_1 & y_2 & y_3 \\
-y_6 & -y_4 & -y_5 & y_3 & y_1 & y_2 \\
-y_5 & -y_6 & -y_4 & y_2 & y_3 & y_1
\end{pmatrix}
$$

4. Reduction of the quaternary structures to ternary structures without application of the binary and quaternary structure matrices

The construction relies upon the ternary structure matrix (2). According to our programme the identified 12×12 complex matrices read

$$
\left[\begin{array}{ccc|ccc|ccc|ccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\
-x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} \\
-x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} \\
\hline
x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
-x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 \\
-x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 \\
\hline
x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
-x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 \\
-x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 \\
\hline
x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 \\
-x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 \\
-x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1
\end{array}\right]
$$

$$\Updownarrow$$

$$
\left[\begin{array}{cccc|cccc|cccc}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\
y_4 & y_1 & y_2 & y_3 & y_8 & y_5 & y_6 & y_7 & y_{12} & y_9 & y_{10} & y_{11} \\
y_3 & y_4 & y_1 & y_2 & y_7 & y_8 & y_5 & y_6 & y_{11} & y_{12} & y_9 & y_{10} \\
y_2 & y_3 & y_4 & y_1 & y_6 & y_7 & y_8 & y_5 & y_{10} & y_{11} & y_{12} & y_9 \\
\hline
-y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
-y_{12} & -y_9 & -y_{10} & -y_{11} & y_4 & y_1 & y_2 & y_3 & y_8 & y_5 & y_6 & y_7 \\
-y_{11} & -y_{12} & -y_9 & -y_{10} & y_3 & y_4 & y_1 & y_2 & y_7 & y_8 & y_5 & y_6 \\
-y_{10} & -y_{11} & -y_{12} & -y_9 & y_2 & y_3 & y_4 & y_1 & y_6 & y_7 & y_8 & y_5 \\
\hline
-y_5 & -y_6 & -y_7 & -y_8 & -y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 \\
-y_8 & -y_5 & -y_6 & -y_7 & -y_{12} & -y_9 & -y_{10} & -y_{11} & y_4 & y_1 & y_2 & y_3 \\
-y_7 & -y_8 & -y_5 & -y_6 & -y_{11} & -y_{12} & -y_9 & -y_{10} & y_3 & y_4 & y_1 & y_2 \\
-y_6 & -y_7 & -y_8 & -y_5 & -y_{10} & -y_{11} & -y_{12} & -y_9 & y_2 & y_3 & y_4 & y_1
\end{array}\right]
$$

5. Reduction of the quaternary structures to ternary structures with application of the binary structure matrix but without application of the quaternary structure matrix

The construction relies upon the binary structure matrix (1) and the ternary structure matrix (3). According to our programme the identified complex 12×12-matrices read

$$\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12}\\
-x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11}\\
-x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10}\\
-x_4 & -x_5 & -x_6 & x_1 & x_2 & x_3 & -x_{10} & -x_{11} & -x_{12} & x_7 & x_8 & x_9\\
x_6 & -x_4 & -x_5 & -x_3 & x_1 & x_2 & x_{12} & -x_{10} & -x_{11} & -x_9 & x_7 & x_8\\
x_5 & x_6 & -x_4 & -x_2 & -x_3 & x_1 & x_{11} & x_{12} & -x_{10} & -x_8 & -x_9 & x_7\\
-x_7 & -x_8 & -x_9 & -x_{10} & -x_{11} & -x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6\\
x_9 & -x_7 & -x_8 & x_{12} & -x_{10} & -x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5\\
x_8 & x_9 & -x_7 & x_{11} & x_{12} & -x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4\\
x_{10} & x_{11} & x_{12} & -x_7 & -x_8 & -x_9 & -x_4 & -x_5 & -x_6 & x_1 & x_2 & x_3\\
-x_{12} & x_{10} & x_{11} & x_9 & -x_7 & -x_8 & x_6 & -x_4 & -x_5 & -x_3 & x_1 & x_2\\
-x_{11} & -x_{12} & x_{10} & x_8 & x_9 & -x_7 & x_5 & x_6 & -x_4 & -x_2 & -x_3 & x_1
\end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12}\\
-y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & -y_8 & y_7 & -y_{10} & y_9 & -y_{12} & y_{11}\\
-y_3 & -y_4 & y_1 & y_2 & -y_7 & -y_8 & y_5 & y_6 & -y_{11} & -y_{12} & y_9 & y_{10}\\
y_4 & -y_3 & -y_2 & y_1 & y_8 & -y_7 & -y_6 & y_5 & y_{12} & -y_{11} & -y_{10} & y_9\\
-y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8\\
y_{10} & -y_9 & y_{12} & -y_{11} & -y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & -y_8 & y_7\\
y_{11} & y_{12} & -y_9 & -y_{10} & -y_3 & -y_4 & y_1 & y_2 & -y_7 & -y_8 & y_5 & y_6\\
-y_{12} & y_{11} & y_{10} & -y_9 & y_4 & -y_3 & -y_2 & y_1 & y_8 & -y_7 & -y_6 & y_5\\
-y_5 & -y_6 & -y_7 & -y_8 & -y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4\\
y_6 & -y_5 & y_8 & -y_7 & y_{10} & -y_9 & y_{12} & -y_{11} & -y_2 & y_1 & -y_4 & y_3\\
y_7 & y_8 & -y_5 & -y_6 & y_{11} & y_{12} & -y_9 & -y_{10} & -y_3 & -y_4 & y_1 & y_2\\
-y_8 & y_7 & y_6 & -y_5 & -y_{12} & y_{11} & y_{10} & -y_9 & y_4 & -y_3 & -y_2 & y_1
\end{pmatrix}$$

6. Reduction of the quaternary structures to ternary structures without application of the binary structure but with application of the quaternary structure matrix

The construction relies upon the ternary structure martix (3) and the quaternary structure matrix (4). According to our programme the identified complex 12×12-matrices read

$$
\begin{array}{ccc|ccc|ccc|ccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\
-x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} \\
-x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} \\
\hline
x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
-x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 \\
-x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 \\
\hline
x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
-x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 \\
-x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 \\
\hline
x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_1 & x_2 & x_3 \\
-x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} & -x_3 & x_1 & x_2 \\
-x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} & -x_2 & -x_3 & x_1
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{cccc|cccc|cccc}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\
y_4 & y_1 & y_2 & y_3 & y_8 & y_5 & y_6 & y_7 & y_{12} & y_9 & y_{10} & y_{11} \\
y_3 & y_4 & y_1 & y_2 & y_7 & y_8 & y_5 & y_6 & y_{11} & y_{12} & y_9 & y_{10} \\
y_2 & y_3 & y_4 & y_1 & y_6 & y_7 & y_8 & y_5 & y_{10} & y_{11} & y_{12} & y_9 \\
\hline
-y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
-y_{12} & -y_9 & -y_{10} & -y_{11} & y_4 & y_1 & y_2 & y_3 & y_8 & y_5 & y_6 & y_7 \\
-y_{11} & -y_{12} & -y_9 & -y_{10} & y_3 & y_4 & y_1 & y_2 & y_7 & y_8 & y_5 & y_6 \\
-y_{10} & -y_{11} & -y_{12} & -y_9 & y_2 & y_3 & y_4 & y_1 & y_6 & y_7 & y_8 & y_5 \\
\hline
-y_5 & -y_6 & -y_7 & -y_8 & -y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 \\
-y_8 & -y_5 & -y_6 & -y_7 & -y_{12} & -y_9 & -y_{10} & -y_{11} & y_4 & y_1 & y_2 & y_3 \\
-y_7 & -y_8 & -y_5 & -y_6 & -y_{11} & -y_{12} & -y_9 & -y_{10} & y_3 & y_4 & y_1 & y_2 \\
-y_6 & -y_7 & -y_8 & -y_5 & -y_{10} & -y_{11} & -y_{12} & -y_9 & y_2 & y_3 & y_4 & y_1
\end{array}
$$

7. Reduction of the quaternary structures to ternary structures with application of the binary and quaternary structure matrix

The construction relies upon the ternary structure martix (3) and the quaternary structure matrix (4). According to our programme the identified complex 12×12-matrices read

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \\
-x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 & -x_9 & x_7 & x_8 & -x_{12} & x_{10} & x_{11} \\
-x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 & -x_8 & -x_9 & x_7 & -x_{11} & -x_{12} & x_{10} \\
\hline
-x_4 & -x_5 & -x_6 & x_1 & x_2 & x_3 & -x_{10} & -x_{11} & -x_{12} & x_7 & x_8 & x_9 \\
x_6 & -x_4 & -x_5 & -x_3 & x_1 & x_2 & x_{12} & -x_{10} & -x_{11} & -x_9 & x_7 & x_8 \\
x_5 & x_6 & -x_4 & -x_2 & -x_3 & x_1 & x_{11} & x_{12} & -x_{10} & -x_8 & -x_9 & x_7 \\
\hline
-x_7 & -x_8 & -x_9 & -x_{10} & -x_{11} & -x_{12} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
x_9 & -x_7 & -x_8 & x_{12} & -x_{10} & -x_{11} & -x_3 & x_1 & x_2 & -x_6 & x_4 & x_5 \\
x_8 & x_9 & -x_7 & x_{11} & x_{12} & -x_{10} & -x_2 & -x_3 & x_1 & -x_5 & -x_6 & x_4 \\
\hline
x_{10} & x_{11} & x_{12} & -x_7 & -x_8 & -x_9 & -x_4 & -x_5 & -x_6 & x_1 & x_2 & x_3 \\
-x_{12} & x_{10} & x_{11} & x_9 & -x_7 & -x_8 & x_6 & -x_4 & -x_5 & -x_3 & x_1 & x_2 \\
-x_{11} & -x_{12} & x_{10} & x_8 & x_9 & -x_7 & x_5 & x_6 & -x_4 & -x_2 & -x_3 & x_1
\end{array}\right]$$

$$\Updownarrow$$

$$\left[\begin{array}{cccc|cccc|cccc}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} \\
-y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & -y_8 & y_7 & -y_{10} & y_9 & -y_{12} & y_{11} \\
-y_3 & -y_4 & y_1 & y_2 & -y_7 & -y_8 & y_5 & y_6 & -y_{11} & -y_{12} & y_9 & y_{10} \\
y_4 & -y_3 & -y_2 & y_1 & y_8 & -y_7 & -y_6 & y_5 & y_{12} & -y_{11} & -y_{10} & y_9 \\
\hline
-y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
y_{10} & -y_9 & y_{12} & -y_{11} & -y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & -y_8 & y_7 \\
y_{11} & y_{12} & -y_9 & -y_{10} & -y_3 & -y_4 & y_1 & y_2 & -y_7 & -y_8 & y_5 & y_6 \\
-y_{12} & y_{11} & y_{10} & -y_9 & y_4 & -y_3 & -y_2 & y_1 & y_8 & -y_7 & -y_6 & y_5 \\
\hline
-y_5 & -y_6 & -y_7 & -y_8 & -y_9 & -y_{10} & -y_{11} & -y_{12} & y_1 & y_2 & y_3 & y_4 \\
y_6 & -y_5 & y_8 & -y_7 & y_{10} & -y_9 & y_{12} & -y_{11} & -y_2 & y_1 & -y_4 & y_3 \\
y_7 & y_8 & -y_5 & -y_6 & y_{11} & y_{12} & -y_9 & -y_{10} & -y_3 & -y_4 & y_1 & y_2 \\
-y_8 & y_7 & y_6 & -y_5 & -y_{12} & y_{11} & y_{10} & -y_9 & y_4 & -y_3 & -y_2 & y_1
\end{array}\right]$$

8. Conclusions

The identifications discussed in Sections 1-7 show different possibilities of reducing ternary to binary and quaternary to ternary structures. The constructions shown in Sections 1-7 give us hints to find multiplication schemes for generators of the algebras involved.

The resulting ∂ and $\bar{\partial}$ Cauchy-Riemann operators and Galois extensions will be discussed in the next part of this paper as well as the multiplication schemes governing the resulting algebra.

References

[1] F. L. Castillo Alvarado, J. Ławrynowicz, Małgorzata Nowak-Kępczyk, *Fractal modelling of alloy thin films. Temperatures characteristic for local phase transitions*, in: Applied Complex and Quaternionic Approximation. Ed. by Ralitza K. Kovacheva, J. Ławrynowicz, and S. Marchiafava, Ediz. Univ. La Sapienza, Roma 2009, pp. 207–236.

[2] J. Ławrynowicz, K. Nôno, and O. Suzuki, *Binary and ternary Clifford analysis vs noncommutative Galois extensions II. The correspondence between the binary and ternary field operators*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 3 (2012), 109–118.

[3] J. Ławrynowicz, O. Suzuki, A. Niemczynowicz, and M. Nowak-Kępczyk, *Some algebraical and geometrical aspects of simple orthorhombic Ising-Onsager-like lattices*, to appear.

[4] I. R. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge studies in advanced mathematics **50**, Cambridge Univ. Press, Cambridge 1995.

The State School of Higher Education in Chełm
54 Pocztowa Street, PL-22-100 Chełm
Poland
e-mail: gosianmk@gmail.com

Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 20, 2014

REDUKCJA STRUKTUR TYPU KWATERNARNEGO DO STRUKTUR TYPU TERNARNEGO I

S t r e s z c z e n i e

Rozważane są redukcje struktur typu kwaternarnego do struktur typu ternarnego z punktu widzenia odpowiadającej im algebry.

Słowa kluczowe: nieprzemienne przedłużenia Galois, algebry skończenie wymiarowe, łączne pierścienie i algebry, pierścienie macierzy

PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2014 Vol. LXIV

Recherches sur les déformations no. 2

pp. 111–119

In memory of prof. Claude Surry
our Dear Friend and Collegue

Dominika Klimek-Smęt and Andrzej Michalski

HARMONIC MAPPINGS GENERATED BY CONVEX CONFORMAL MAPPINGS

Summary

Let S_H be the class of normalized univalent and sense-preserving harmonic mappings in the unit disc Δ introduced by Clunie and Sheil-Small. It is well-known that every $f \in S_H$ can be uniquely represented as $f = h + \overline{g}$, where h and g are analytic in Δ. We introduce a subclass of S_H consisting of all functions $f \in S_H$, such that $h + \varepsilon g$ is a convex function for some $\varepsilon \in \overline{\Delta}$, where $\overline{\Delta}$ is the closed unit disc. In fact, the condition defining the subclass provides an effective method for producing univalent sense-preserving harmonic functions using convex conformal mappings. Hence we find it very interesting to examine this subclass. We establish sharp coefficient and growth estimates, which are the main results of this paper.

Keywords and phrases: harmonic mappings, convex conformal mappings

1. Introduction

Let Δ stands for the open unit disc in the complex plane $\mathbb{C}$. Every complex-valued harmonic function f in Δ has unique representation of the form

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Δ with $g(0) = 0$. Moreover, every f satisfying (1.1) is uniquely determined by coefficients of the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \tag{1.2}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$.

A complex-valued harmonic function f, not identically constant, satisfying (1.1) is said to be sense-preserving in Δ if, and only if it satisfies the equation

$$g' = \omega h', \tag{1.3}$$

where ω is analytic in Δ with $|\omega(z)| < 1$, $z \in \Delta$. The function ω is called the second complex dilatation of f. This and much more information about harmonic functions can be found in e.g. [2].

Clunie and Sheil-Small introduced in [1] the family S_H of all univalent and sense-preserving harmonic functions f satisfying (1.1) in Δ, such that $h(0) = 0$ and $h'(0) = 1$. By S_H^0 they denoted the class of all $f \in S_H$ such that $g'(0) = 0$. In the same paper they gave the following sufficient condition for f to be in S_H.

Theorem 1.1. [[1], Theorem 5.17] *If f is a harmonic locally univalent and sense-preserving function in Δ satisfying* (1.1) *and there exists*

$$\epsilon \in \overline{\Delta} := \{z \in \mathbb{C} : |z| \leq 1\}$$

such that $h + \epsilon g$ is convex, then the function f is univalent, sense-preserving and close-to-convex.

We have observed in [4], that the Theorem 1.1 provides a method of generating univalent and sense-preserving harmonic functions in Δ by use of a convex conformal mapping in Δ and an analytic function of Δ into Δ. This observation enabled us to construct some interesting examples of harmonic mappings (see [4] and [5]) and gave us motivation to study the class of all harmonic mappings belonging to S_H possible to obtain in the mentioned way. Moreover, such a class contains all of harmonic mappings considered by us in [6] and [7].

To be more precise, let us define the class $\tilde{S}_H$ of all $f \in S_H$ satisfying (1.1) such that $h + \epsilon g$ is convex for some $\epsilon \in \overline{\Delta}$. We also define $\tilde{S}_H^0$ of all $f \in \tilde{S}_H$ with the property $g'(0) = 0$. The main results of this paper are solutions to the most common extremal problems in $\tilde{S}_H^0$ and $\tilde{S}_H$, such as coefficient and growth estimates. These estimates are known to hold for convex mappings of the class S_H, so that we also give an example of a function which belongs to $\tilde{S}_H^0$ but is not convex.

In fact, it is more convenient to study $\tilde{S}_H^0$ instead of $\tilde{S}_H$ and we mostly deal with this class.

2. Basic properties and examples

We start with some invariance property of the class $\tilde{S}_H$.

Proposition 2.1. *Let $\lambda \in \Delta$. If $f \in \tilde{S}_H$ satisfies* (1.1) *and*

$$f_\lambda(z) := \frac{f(z) + \lambda\overline{f(z)}}{1 + \lambda g'(0)}, \quad z \in \Delta, \tag{2.1}$$

then $f_\lambda \in \tilde{S}_H$.

Proof. Let $f \in \tilde{S}_H$ satisfies (1.1). Then, by definition of the class $\tilde{S}_H$, there exists $\epsilon \in \overline{\Delta}$ such that $h + \epsilon g$ is a convex conformal mapping. Consider

$$\delta := (\epsilon - \lambda)(1 - \epsilon\bar{\lambda})^{-1}.$$

Obviously, $\delta \in \overline{\Delta}$ since $\lambda \in \Delta$ and $\epsilon \in \overline{\Delta}$. Denoting

$$f_\lambda = h_\lambda + \overline{g_\lambda}$$

we derive from (2.1)

$$h_\lambda = \frac{h + \lambda g}{1 + \lambda g'(0)} \quad \text{and} \quad g_\lambda = \frac{g + \overline{\lambda} h}{1 + \lambda g'(0)}.$$

Next, simple calculation gives

$$h_\lambda + \delta g_\lambda = \frac{1 - |\lambda|^2}{[1 + \lambda g'(0)](1 - \epsilon\overline{\lambda})}(h + \epsilon g).$$

Hence, convexity of $h + \epsilon g$ implies convexity of $h_\lambda + \delta g_\lambda$. Obviously, f_λ is univalent, sense-preserving and suitably normalized so it belongs to the class $\tilde{S}_H$. □

The property above is called the affine invariance of the class $\tilde{S}_H$. In particular, it constitutes the corespondace between $\tilde{S}_H$ and $\tilde{S}_H^0$. Next, we prove that the property called linear invariance also holds for the class $\tilde{S}_H$.

Proposition 2.2. *Let ϕ be a conformal disc automorphism. If $f \in \tilde{S}_H$ satisfies* (1.1) *and*

$$f_\phi(z) := \frac{f(\phi(z)) - f(\phi(0))}{\phi'(0)h'(\phi(0))}, \quad z \in \Delta, \tag{2.2}$$

then $f_\phi \in \tilde{S}_H$.

Proof. Let $f \in \tilde{S}_H$ satisfies (1.1). Then, by definition of the class $\tilde{S}_H$, there exists $\epsilon \in \overline{\Delta}$ such that $h + \epsilon g$ is a convex conformal mapping. Denoting

$$f_\phi = h_\phi + \overline{g_\phi}$$

we derive from (2.2)

$$h_\phi = \frac{h(\phi) - h(\phi(0))}{\phi'(0)h'(\phi(0))} \quad \text{and} \quad g_\phi = \frac{g(\phi) - g(\phi(0))}{\phi'(0)h'(\phi(0))}.$$

Next, simple calculation gives

$$h_\phi + \epsilon g_\phi = \frac{1}{\phi'(0)h'(\phi(0))}(h + \epsilon g)(\phi) - \frac{(h + \epsilon g)(\phi(0))}{\phi'(0)h'(\phi(0))}.$$

Hence, convexity of $h + \epsilon g$ implies convexity of $h_\phi + \epsilon g_\phi$. Obviously, f_ϕ is univalent, sense-preserving and suitably normalized so it belongs to the class $\tilde{S}_H$. □

Now we construct one of the most important functions in our considerations.

Example 2.3. *Let f satisfies* (1.1). *Suppose that*

$$(h - g)(z) = \frac{z}{1 - z} \quad \text{and} \quad g'(z) = zh'(z), \quad z \in \Delta,$$

which imply

$$h'(z) = \frac{1}{(1-z)^3} \quad \textit{and} \quad g'(z) = \frac{z}{(1-z)^3}, \quad z \in \Delta.$$

Now, by integration, we obtain

$$h(z) = \frac{2z - z^2}{2(1-z)^2} \quad \textit{and} \quad g(z) = \frac{z^2}{2(1-z)^2}, \quad z \in \Delta.$$

Obviously, f is locally univalent, sense-preserving harmonic function in Δ and the function $\Delta \ni z \mapsto z(1-z)^{-1}$ is a well-known convex conformal mapping (*see e.g.* [3])*. Hence, in view of Theorem* 1.1*, the function f is univalent. Moreover, f is suitably normalized so that $f \in \tilde{S}_H^0$. Observe that we can express h and g in the form*

$$h(z) = \frac{1}{2}\left[\frac{z}{(1-z)^2} + \frac{z}{1-z}\right] \quad \textit{and} \quad g(z) = \frac{1}{2}\left[\frac{z}{(1-z)^2} - \frac{z}{1-z}\right], \quad z \in \Delta,$$

which easily yield

$$h(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n \quad \textit{and} \quad g(z) = \sum_{n=1}^{\infty} \frac{n-1}{2} z^n, \quad z \in \Delta.$$

Figure 1 *shows the graphical image of the function f.*

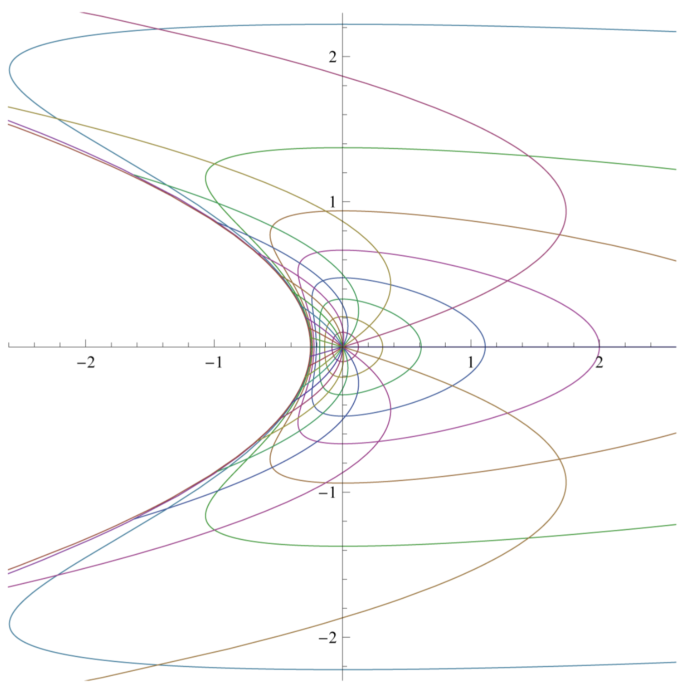

Fig. 1: Image of the function f constructed in Example 2.3

Before we present the main results, we give a reason why the class $\tilde{S}_H^0$ is more convenient to study then $\tilde{S}_H$.

Theorem 2.4. *The family $\tilde{S}_H^0$ is normal and compact. The family $\tilde{S}_H$ is normal but not compact.*

Proof. Both $\tilde{S}_H^0$ and $\tilde{S}_H$ are normal as subclasses of the normal family S_H. Since the sequence

$$\{0,1,2,\ldots\} \ni n \mapsto f_n(z) := z + n(n+1)^{-1}\overline{z}, \quad z \in \Delta$$

converges locally uniformly in Δ to the function $f(z) := 2Re\{z\}$, which is not univalent, the class $\tilde{S}_H$ is not compact. The compactness of $\tilde{S}_H^0$ follows from the compactness of the class S_H^0 and the compactness of the class of convex conformal mappings with usual normalization. $\square$

3. Main results

As the first result in this section we give coefficient estimate for the class $\tilde{S}_H^0$.

Theorem 3.1. *If $f \in \tilde{S}_H^0$ has the expansion* (1.2) *then*

$$|a_n| \le \frac{n+1}{2} \quad \textit{and} \quad |b_n| \le \frac{n-1}{2}, \quad n = 2,3,4,\ldots. \tag{3.1}$$

The inequalities (3.1) *cannot be improved.*

Proof. Let $f \in \tilde{S}_H^0$ satisfies (1.1) with the expansion (1.2). Then, by definition of the class $\tilde{S}_H^0$, there exists $\epsilon \in \overline{\Delta}$ such that $h + \epsilon g$ is a normalized convex conformal mapping. Hence we have

$$|a_n| - |\epsilon b_n| \le |a_n + \epsilon b_n| \le 1, \quad n = 2,3,4,\ldots. \tag{3.2}$$

On the other hand, for all $\alpha \in \overline{\Delta}$ the function $h + \alpha g$ is a normalized close-to-convex conformal mapping as it was shown in the proof of [1, Theorem 5.17]. Hence, there exists α, $|\alpha| = 1$ such that

$$|a_n| + |b_n| = |a_n + \alpha b_n| \le n, \quad n = 2,3,4,\ldots. \tag{3.3}$$

Combining (3.2) and (3.3) we obtain

$$|a_n| \le \frac{1 + n - (1 - |\epsilon|)|b_n|}{2} \le \frac{n+1}{2}, \quad n = 2,3,4,\ldots,$$

which proves the first inequality of the formula (3.1).

Now, we estimate $|b_n|$, $n = 2,3,4,\ldots$. If $\epsilon = 0$ then, by applying [7, Theorem 3.1 and Theorem 3.3], we obtain

$$|b_2| \le \frac{1}{2}, \quad \text{and} \quad |b_n| \le \sqrt{\frac{n-1}{n}} \le \frac{n-1}{2}, \quad n = 3,4,5,\ldots.$$

If $\epsilon \neq 0$ then, by [1, Lemma 5.11] and the maximum principle for harmonic functions, there exist $p, q \in \mathbb{R}$ such that

$$Re\left\{[e^{ip}(h+\epsilon g)'(z)](e^{iq}-e^{-iq}z^2)\right\} > 0, \quad z \in \Delta.$$

Let ω be the dilatation of f, i.e. ω satisfies (1.3) and denote

$$\begin{aligned} F_1(z) &:= [e^{ip}h'(z) + e^{ip}\epsilon g'(z)](e^{iq}-e^{-iq}z^2), \quad z \in \Delta, \\ F_2(z) &:= \frac{e^{iq}}{1-(e^{-iq}z)^2}, \quad z \in \Delta, \\ F_3(z) &:= \frac{\omega(z)}{1+\epsilon\omega(z)}, \quad z \in \Delta. \end{aligned}$$

Then we can express g' in the form

$$g'(z) = e^{-i(p+2q)}F_1(z)F_2(z)F_3(z), \quad z \in \Delta. \tag{3.4}$$

Denote also

$$\psi_1(z) := \frac{1+z}{1-z}, \quad \psi_2(z) := \frac{1}{1-z^2}, \quad \psi_3(z) := \frac{z}{1-z}, \quad z \in \Delta.$$

Now, observe that F_1 is analytic in Δ, has positive real part and $|F_1(0)| = 1$, so the function

$$F_4(z) := F_1(z) - F_1(0) + 1, \quad z \in \Delta$$

satisfies assumptions of Caratheodory's lemma (see [3, p. 41]) and hence the coefficients of F_4 are dominated in modulus by those of ψ_1. Since F_1 and F_4 differ by a constant, the coefficients of F_1 also are dominated by corresponding coefficients of ψ_1. The function F_2 is a rotation of ψ_2, so the modulus of corresponding coefficients of F_2 and ψ_2 is the same. Note that the dilatation ω of $f \in \tilde{S}_H^0$, by definition is an analytic function and satisfies the condition $|\omega(z)| < 1$, $z \in \Delta$. Additionally, by the normalization of $f \in \tilde{S}_H^0$, it has the property $\omega(0) = 0$. Hence, the dilatation ω satisfies the assumptions of the Schwarz lemma, which together with $\epsilon \in \overline{\Delta}$ guarantee that F_3 is subordinate to ψ_4, where

$$\psi_4(z) := -\epsilon^{-1}\psi_3(-\epsilon z), \quad z \in \Delta.$$

Obviously, ψ_4 is a normalized convex conformal mapping as a rotation and dilation of the well-known convex function ψ_3. Hence, by applying [3, Theorem 6.4] due to Rogosinski, we obtain that the coefficients of F_3 are dominated in modulus by corresponding coefficients of ψ_3. Finally, in view of the formula (3.4) and since $|e^{-i(p+2q)}| = 1$, we conclude that the coefficients of g' are dominated in modulus by those of the function ψ, where

$$\psi(z) := \psi_1(z)\psi_2(z)\psi_3(z) = \frac{z}{(1-z)^3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}z^n, \quad z \in \Delta.$$

This means that the following inequality holds

$$n|b_n| \leq \frac{(n-1)n}{2} \quad n = 2, 3, 4, \ldots,$$

which proves the second inequality of the formula (3.1). To complete the proof, observe that the case of equality in (3.1) occurs for the function constructed in Example 2.3. □

It is well-known that the estimate $|b_1| < 1$ holds for $f \in S_H$ with the expansion (1.2). It is not surprising that we have the same estimate for the class $\tilde{S}_H$.

Corollary 3.2. *If $f \in \tilde{S}_H$ has the expansion* (1.2) *then*

$$|a_n| < n \quad \textit{and} \quad |b_n| < n, \quad n = 2, 3, 4, \dots. \tag{3.5}$$

The inequalities (3.5) *cannot be improved.*

Proof. Observe that for each function $f \in \tilde{S}_H$ Proposition 2.1 provides the relation $f = f_0 + \overline{b_1 f_0}$ with some function $f_0 \in \tilde{S}_H^0$, where b_1 is taken from the expansion (1.2) of the function f. Thus, applying the inequalities (3.1) of Theorem 3.1 to the function f_0 and using the estimate $|b_1| < 1$ we obtain

$$|a_n| \leq \frac{n+1}{2} + \frac{n-1}{2}|b_1| < n \quad n = 2, 3, 4, \dots$$

and

$$|b_n| \leq \frac{n-1}{2} + \frac{n+1}{2}|b_1| < n, \quad n = 2, 3, 4, \dots.$$

To complete the proof, let f be the function constructed in Example 2.3 and observe that for each $0 < \lambda < 1$ the function $f + \lambda\overline{f}$ belongs to the class $\tilde{S}_H$, furthermore its coefficients can be arbitrary close to the bounds given in (3.5). □

Theorem 3.3. *If $f \in \tilde{S}_H^0$ then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad z \in \Delta. \tag{3.6}$$

The inequalities (3.6) *cannot be improved.*

Proof. As it was shown in Proposition 2.1 and Proposition 2.2 the class $\tilde{S}_H$ is affine and linear invariant. Clearly it is also subclass of S_H. Using Corollary 3.2 we easily deduce that $\sup\{|a_2|\} = 2$, where supremum is taken over all functions of the class $\tilde{S}_H$ with the expansion (1.2). Hence, we can refer to [9, Theorem 1], which yields

$$\frac{1}{4}\left[1 - \left(\frac{1-|z|}{1+|z|}\right)^2\right] \leq |f(z)| \leq \frac{1}{4}\left[\left(\frac{1+|z|}{1-|z|}\right)^2 - 1\right] \quad z \in \Delta.$$

Simple calculation shows that the above inequalities are equivalent to (3.6). To complete the proof, observe that the case of equality in (3.6) occurs for the function constructed in Example 2.3. □

Obviously, Theorem 3.3 and the maximum principle for harmonic functions give the following covering theorem.

Corollary 3.4. *If $f \in \tilde{S}_H^0$ then*

$$\left\{ z \in \mathbb{C} : \ |z| < \frac{1}{4} \right\} \subset f(\Delta). \tag{3.7}$$

The constant $\frac{1}{4}$ in (3.7) cannot be improved.

Our final remark is concerned with the function given in Example 2.3. Applying Corollary 3.4 we immediately deduce that the mentioned function is not convex, since the best radius in the corresponding covering theorem for convex harmonic mappings of S_H^0 is known to be $\frac{1}{2}$ (see e.g. [2, p. 50, Theorem 1]).

References

[1] J. G. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 3–25.

[2] P. L. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Math. 156, Cambridge Univ. Press, Cambridge 2004.

[3] P. L. Duren, *Univalent functions*, Grundlehren Math. Wiss. 259, Springer-Verlag, Berlin-New York, 1983.

[4] D. Klimek-Smęt and A. Michalski A., *Univalent harmonic functions generated by conformal mappings onto regular polygons*, Bull. Soc. Sci. Lettres Lodz **59**, Ser. Rech. Deform. **58** (2009), 33–44.

[5] D. Klimek and A. Michalski, *Univalent harmonic mappings with dilatation onto regular polygons by hypergeometric functions*, Sci. Bull. Chełm **2** (2006), 83–90.

[6] D. Klimek and A. Michalski, *Univalent anti-analytic perturbation of the identity in the unit disc*, Sci. Bull. Chełm **1** (2006), 67–76.

[7] D. Klimek-Smęt and A. Michalski, *Univalent anti-analytic perturbation of convex conformal mapping in the unit disc*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A **61** (2007), 39–49.

[8] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.

[9] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. **42** (1990), 237–248.

Department of Applied Mathematics
Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 5
PL-20-031 Lublin
Poland

Department of Complex Analysis
The John Paul II Catholic
University of Lublin
Konstantynów 1H, PL-20-950 Lublin
Poland
e-mail: amichal@kul.lublin.pl

Presented by Leon Mikołajczyk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 4, 2014

ODWZOROWANIA HARMONICZNE GENEROWANE PRZEZ WYPUKŁE ODWZOROWANIA KONFOREMNE

S t r e s z c z e n i e

Niech S_H oznacza klasę wszystkich unormowanych, różnowartościowych i zachowujących orientację odwzorowań harmonicznych w kole jednostkowym Δ wprowadzoną przez Cluniego i Sheil-Smalla. Dobrze znany jest fakt, że każda funkcja harmoniczna $f \in S_H$ może być jednoznacznie przedstawiona w postaci $f = h + \overline{g}$, gdzie h i g są analityczne w Δ. Wprowadzamy podklasę klasy S_H składającą się ze wszystkich funkcji $f \in S_H$ takich, że $h + \varepsilon g$ jest odwzorowaniem wypukłym dla pewnego $\varepsilon \in \overline{\Delta}$, gdzie $\overline{\Delta}$ jest domkniętym kołem jednostkowym. Okazuje się, że powyższy warunek pozwala efektywnie skonstruować różnowartościowe i zachowujące orientację odwzorowania harmoniczne przy pomocy wypukłych odwzorowań konforemnych, co skłoniło nas do zbadania tak zdefiniowanej podklasy klasy S_H. Dokładne oszacowania współczynników i wzrostu funkcji to główne wyniki prezentowane w niniejszej pracy.

Słowa kluczowe: odwzorowania harmoniczne generowane przez wypukłe odwzorowania konforemne

Rapporteurs – Referees

Richard A. Carhart (Chicago)
Fray de Landa Castillo Alvarado (México, D.F.)
Stancho Dimiev (Sofia)
Pierre Dolbeault (Paris)
Paweł Domański (Poznań)
Mohamed Saladin El Nashie (London)
Jerzy Grzybowski (Poznań)
Ryszard Jajte (Łódź)
Grzegorz Karwasz (Toruń)
Leopold Koczan (Lublin)
Dominique Lambert (Namur)
Andrzej Łuczak (Łódź)
Cecylia Malinowska-Adamska (Łódź)
Stefano Marchiafava (Roma)
Leon Mikołajczyk (Łódź)
Yuval Ne'eman (Haifa)
Kiyoharu Nôno (Fukuoka)
Adam Paszkiewicz (Łódź)
Sergey Plaksa (Kyiv)
Krzysztof Podlaski (Łódź)
Yaroslav G. Prytula (Kyiv)
Henryk Puszkarski (Poznań)
Jakub Rembieliński (Łódź)
Carlos Rentería Marcos (México, D.F.)
Lino F. Reséndis Ocampo (México, D.F.)
Stanisław Romanowski (Łódź)
Monica Roşiu (Craiova)
Jerzy Rutkowski (Łódź)
Ken-Ichi Sakan (Osaka)
Hideo Shimada (Sapporo)
Dawid Shoikhet (Karmiel, Israel)
Józef Siciak (Kraków)
Francesco Succi (Roma)
Osamu Suzuki (Tokyo)
Józef Szudy (Toruń)
Luis Manuel Tovar Sánchez (México, D.F.)
Massimo Vaccaro (Salerno)
Anna Urbaniak-Kucharczyk (Łódź)
Grzegorz Wiatrowski (Łódź)
Hassan Zahouani (Font Romeu)
Lawrence Zalcman (Ramat-Gan)
Yuri Zeliskiĭ (Kyiv)
Natalia Zoriĭ (Kyiv)

CONTENU DU VOLUME LXIV, no. 3