B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXV, no. 3

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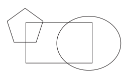

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References

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vol. LXV, no. 3

Série:
RECHERECHES SUR LES DÉFORMATIONS

Guest Editor of the Volume

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Professors Julian Ławrynowicz and Leszek Wojtczak
(accompanied by Professor Antoni Różalski, Pro-Rector and President of the Łódź Society of Sciences and Arts) during the ceremony of renewing of their doctorates (1964) after 50 years

Preface

Professor Julian Ławrynowicz, born in 1939 in Łódź, graduated in physics and mathematics at the Faculty of Mathematics, Physics and Chemistry of the University of Lódz in 1960. He gained the PhD in mathematical and physical sciences in 1964 and the degree of habilitated doctor in 1968. In 1976 Julian Ławrynowicz was appointed extraordinary professor, and in 1992 ordinary professor in mathematics. The scientific interest in the field of mathematics and physics includes complex analysis, Clifford algebras, fractals, field theory and some aspects of solid state physics. Professor Julian Ławrynowicz is recognized as a very active researcher with a rich scientific output and many contributions in the international conferences. In the period 1972–2002 he served in the Institute of Mathematics of the Polish Academy of Sciences as the Head of the Department of Complex Analysis and Differential Geometry.

Professor Leszek Wojtczak, born in 1939 in Gozdów, graduated in physics and mathematics at the Faculty of Mathematics, Physics and Chemistry of the University of Łódź in 1961. He gained the PhD in mathematical and physical sciences in 1964 and the degree of habilitated doctor in theoretical physics in 1969. In 1976 Leszek Wojtczak was appointed extraordinary professor, and in 1985 ordinary professor in physics. Employed in the University of Lódz since 1961 Professor Leszek Wojtczak organized research in the field of solid state physics, in particular on topical problems of surfaces and thin films, and at the same time actively participated in organizing the research group in theoretical electrochemistry of superficial layer properties. In 1974, he created the Department of Solid State Physics and headed it till 1999. Besides his research achievement confirmed by many publications in journals of high international standard, Professor Leszek Wojtczak took various administrative functions, was elected vice dean and dean of the Faculty of Mathematics, Physics and Chemistry, prorector and rector of the University of Łódź, as well as the first president of the Polish Universities Rectors Conference.

For the first time They met each other in 1954 as pupils participated in mathematics contest and since then their private and professional life interweave. Common interests, with physical problems formulated by Professor Leszek Wojtczak and the use of appropriate mathematical methods by Professor Julian Ławrynowicz resulted in several tens of joint publications. The important aspect of Professors collaboration is connected with the social scientific service in the Łódź Society of Sciences and Arts.

In 2014 we celebrated 50th anniversary of PhD degree received by Professors Julian Ławrynowicz and Leszek Wojtczak. To commemorate this jubilee, the Łódź

Society of Sciences and Arts offered a special issue of Bulletin de la Société des Sciences et des Lettres de Łódź (Série: Recherches sur les Déformations). The present collection of papers dedicated to Professors Julian Ławrynowicz and Leszek Wojtczak and submitted by their colleagues and coworkers, who spontaneously answered to the proposal of this edition, reflects their stimulating role in the development of different scientific subjects as well as in conducting the fruitful scientific cooperation.

The authors of the presented contributions and myself would like to congratulate Professors Julian Ławrynowicz and Leszek Wojtczak on this glorious jubilee and express our worm greetings.

Ilona Zasada

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Contribution to the jubilee volume, dedicated
to Professors J. Ławrynowicz and L. Wojtczak

Ryszard Taranko and Tomasz Kwapiński

ELECTRON DYNAMICS IN QUANTUM QUBIT INTERACTING WITH TWO SINGLE ELECTRON TRANSISTORS

Summary

We investigate theoretically the qubit in the form of the double quantum dot (QD) coupled electrostatically with two detectors composed of single electron transistors. The equation of motion method for the appropriate correlation functions with the special decoupling procedure for higher-order functions was used in the calculations of the QD occupancies and the current flowing in detectors. We have considered the qubit dynamics in the presence of different types of perturbations imposed on both detectors, i.e. for the constant and harmonic perturbations of the detectors QD energy levels. It was shown that the qubit oscillations of the qubit being already in the stationary state can be restored using the abrupt short-time perturbations acting on both detectors QDs. We have found that in the case of harmonically driven detectors QDs energy levels the qubit QD occupancy oscillates with the perturbation frequency for sufficiently long time after the perturbation has been applied. However, for shorter time we have observed overlap of these oscillations together with the damped oscillations of a free qubit.

Keywords and phrases: qubit, double quantum dot, single electron transistor, decoherence

1. Introduction

The progress in nanotechnology and the research on quantum computing have motivated interest in both theoretical and experimental studies of the electron dynamics in different quantum dot (QD) systems. The transient and steady-state electron transport through various configurations of QDs coupled with leads was investigated in the literature , e.g. [1–14]. The simplest quantum mechanical system which plays an important role in quantum computation is the double QD (DQD), the so-called qubit, in which a single excess electron occupies the ground state of either one dot or the other. In order to analyze the qubit dynamics (time-variations of the qubit QDs occupancies) we should perform the measurements using some external nanoscopic device. The qubit is usually placed in close proximity to the charge sensitive detector. The current flowing through such a charge meter depends on the occupancy of the nearby qubit QD. Detectors can be realized in the form of a quantum point contact (QPC) [15–24], a single QD placed between two leads (the so-called single electron transistor (SET)) [25–27], a DQD in a linear or vertical configuration between leads (the so-called double-dot detector) e.g. [22, 28–30], see also [31]. In most hitherto studies the detector electrons interact with the qubit electron localized on the qubit QD being in close proximity to the detector. In this case the second qubit QD is not coupled with the environment so the interaction between the qubit and the detector is strongly asymmetric. The environment proximity is responsible for the qubit decoherence processes (vanishing of the qubit electron oscillations). Asymmetrical qubit-detector configuration leads to nonequivalent occupations of the qubit QDs which strongly disturbs the qubit decoherence and is often non-physical.

In this paper we study the interaction between the qubit and the environment in the form of two SETs placed symmetrically on both sides of the qubit (see Fig. 1). In such a case the qubit electron interacts all time with the environment independently of the electron localization on the first or the second qubit QD – the qubit-detector interaction is fully symmetrical. Such a configuration of the qubit between two detectors (in the form of two SETs) allows us to check the accuracy of the approximations done in calculating of the required quantities. As the qubit-detectors interaction is described by the Coulomb electron repulsion, in order to solve the electron transport problem or calculate the QDs occupancies one is forced to assume some approximations. For the setup considered in this work, the asymptotic occupations of the qubit QDs should be equal to one-half and such a result should be obtained using reliable approximations. Our calculational procedures fulfil this requirement. In this paper we concentrate on the effect of the environment in the forms of two independent SETs on the qubit dynamics, and calculate the qubit QD occupancy and its dependence on the external perturbations. We analyze also the detector currents flowing in both SETs which are related with the qubit electron oscillations. In our calculations we use the equation of motion (EOM) approach for the appropriate correlation functions.

The outline of this paper is as follows. In Sec. 2 we present the model and derive the set of differential equations for the appropriate correlation functions describing the QD occupancies and the current flowing through the system. Section 3 is devoted to the presentation of the numerical results for the qubit charge oscillations and their reaction to some modifications of the qubit environment and finally we conclude in Section 4.

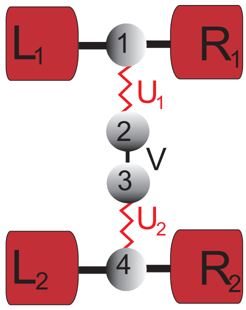

Fig. 1: The sketch of the qubit coupled electrostatically with the upper and bottom SETs. Both quantum dots (1 and 4) between the left and right electron reservoirs stand for the qubit charge detectors. Qubit is represented by two coupled quantum dots (2 and 3) occupied by a single electron. Straight (zig-zag) lines correspond to tunnel matrix elements (Coulomb interactions, U_1, U_2) between the appropriate states.

2. Hamiltonian and formalism

We consider the qubit in the form of the DQD coupled electrostatically with two SETs as depicted in Fig. 1. The Hamiltonian can be written as $H = H_{SET1} + H_{SET2} + H_{qubit} + H_{qubit-SETs}$, where

$$(1) \qquad H_{SETj} = \sum_{k,\alpha=L_j,R_j} \varepsilon_{\alpha k} c^+_{\alpha k} c_{\alpha k} + \varepsilon_j c^+_j c_j + \sum_{k,\alpha=L_j,R_j} V^{(j)}_{\alpha k} c^+_{\alpha k} c_l + h.c.\,,$$

$$(2) \qquad H_{qubit} = \sum_{i=2,3} \varepsilon_i c^+_i c_i + V_{23} c^+_2 c_3 + h.c.\,,$$

$$(3) \qquad H_{qubit-SETs} = U_1 c^+_1 c_1 c^+_2 c_2 + U_2 c^+_3 c_3 c^+_4 c_4\,,$$

where $j = 1, 2$ and $l = 1(4)$ for $j = 1(2)$. The operators $c_i(c^+_i)$ are the creation (annihilation) operators of electrons localized on $i-$th QD, $i = 1, 2, 3, 4$ and $c^+_{k\alpha}(c_{k\alpha})$ are the corresponding operators describing the electrons with $k-$wave vectors contained in the α-th lead ($\alpha = L_1, L_2, R_1, R_2$). The electron energy spectrum of the α-th lead is characterized by $\varepsilon_{\alpha k}$ and ε_i denotes the energy level of $i-$th QD. The interdot tunnel matrix element in the qubit is denoted by $V_{23} = V$ and $V^{(l)}_{\alpha k}$ describes the coupling between α-th lead and l-th QD. U_1 and U_2 stand for the corresponding Coulomb interactions between electrons localized on the qubit and SET QDs, respectively. All parameters, $\varepsilon_{\alpha k}$, ε_i, V, and $V^{(l)}_{\alpha k}$ can be time-dependent.

In order to describe the qubit dynamics, the knowledge of the qubit QDs occupancies and the currents flowing through both SETs are required. We calculate them using the EOM method for appropriate correlation functions. In general, the current flowing e.g. from the $\alpha-$th lead can be written as (e.g. [1]):

$$(4) \qquad j_\alpha(t) = -ie\langle[H, N_\alpha]\rangle = 2e\texttt{Im}\sum_k V^{(l)}_{\alpha k}(t)\langle c^+_l(t) c_{\alpha k}(t)\rangle\,,$$

where $N_\alpha = \sum_k c^+_{\alpha k} c_{\alpha k}$ and the Heisenberg picture is used. Here $\langle \ldots \rangle$ denotes the quantum-statistical average and the index l identifies the QD coupled with the α-th lead. Using the exact representation for $c_{\alpha k}(t)$, e.g. [12]:

$$\begin{aligned} c_{\alpha k}(t) &= c_{\alpha k}(0) \exp\left(-i \int_0^t dt_1 \varepsilon_{\alpha k}(t_1)\right) \\ &\quad -i \int_0^t dt_1 V^{(l)*}_{\alpha k}(t_1) \exp\left(-\int_{t_1}^t dt_2 \varepsilon_{\alpha k}(t_2)\right) c_l(t_1)\,, \end{aligned} \tag{5}$$

the current $j_\alpha(t)$ can be written as follows:

$$j_\alpha(t) = 2e\mathtt{Im}\left(S^{(l)}_\alpha(t) - i\int_0^t dt_1 K^{(l)}_\alpha(t,t_1)\langle c^+_l(t) c_l(t_1)\rangle\right)\,. \tag{6}$$

In the above relation the integral kernel $K_\alpha(t,t_1)$ and $S^{(l)}_\alpha(t)$ functions are expressed by:

$$S^{(l)}_\alpha(t) = \sum_k V^{(l)}_{\alpha k}(t) \exp\left(-\int_0^t dt_1 \varepsilon_{\alpha k}(t_1)\right)\langle c^+_l(t) c_{\alpha k}(0)\rangle \tag{7}$$

$$K^{(l)}_\alpha(t,t_1) = |V^{(l)}_\alpha|^2 u_\alpha(t) u_\alpha(t_1) \exp\left(-\int_{t_1}^t dt' \Delta_\alpha(t')\right) D_\alpha(t-t_1)\,, \tag{8}$$

where $V^{(l)}_{\alpha k}(t) = V^{(l)}_{\alpha k} u_\alpha(t)$, $D_\alpha(t)$ denotes the Fourier transform of the α-th lead density of states and we have assumed $V^{(l)}_{\alpha k} = V^{(l)}_\alpha$. The function $u_\alpha(t)$ is responsible for the initial switching on the couplings between leads and SET QDs, i.e. $u_\alpha(t) = 0$ for $t < 0$ and $u_\alpha(t) \neq 0$ for $t \geq 0$. Formula 6 for the current is valid for the time-dependent α-th lead electron spectrum $\varepsilon_{\alpha k}(t) = \varepsilon^0_{\alpha k} + \Delta_\alpha(t)$ and can be used for describing the behaviour of the considered system in the case of the time-dependent bias voltage. In the following we consider the case for which the lead energy bandwidth is the largest energy in the system and the so-called wide–band limit (WBL), e.g. [37], is a good approximation in calculating the integral in Eq. 6. Now the formula for the current becomes local in time and the second term (with the time integral) is reduced to $i\frac{\Gamma^{(l)}_\alpha}{2}\langle n_l(t)\rangle$ where $\Gamma^{(l)}_\alpha = 2\pi\sum_k |V^{(l)}_{\alpha k}|^2 u^2(t)\delta(\varepsilon - \varepsilon^0_{\alpha k})$.

The current formula requires the knowledge of the QD occupancies, $\langle n_i(t)\rangle \equiv n_i(t)$, and the correlation functions $\langle c^+_j(t) c_{\alpha k}(0)\rangle$. For the QD occupancies within the EOM method and WBL approximation we obtain:

$$\frac{d}{dt} n_1(t) = 2e\mathtt{Im}\left\{S^{(1)}_{L_1}(t) + S^{(1)}_{R_1}(t) - i\frac{\Gamma^{(1)}_{L_1} + \Gamma^{(1)}_{R_1}}{2} n_1(t)\right\}\,, \tag{9}$$

$$\frac{d}{dt} n_4(t) = 2e\mathtt{Im}\left\{S^{(4)}_{L_2}(t) + S^{(4)}_{R_2}(t) - i\frac{\Gamma^{(4)}_{L_2} + \Gamma^{(4)}_{R_2}}{2} n_4(t)\right\}\,, \tag{10}$$

$$\frac{d}{dt} n_2(t) = 2e\mathtt{Im}\left\{V\langle c^+_2(t) c_3(t)\rangle\right\}\,, \tag{11}$$

where we assumed constant values for $V_{\alpha k}^{(l)}(t)$, i.e. $u_\alpha(t) = 1$ and $\Gamma_\alpha^{(l)}$ is the time independent. As usually in the EOM method, writing e.g. the equation for $\langle c_2^+(t)c_3(t)\rangle$ the higher-order functions appear. Note that only two kinds of functions are present in the subsequent equations of motion. The first one can be written schematically as $\langle f_n(c_i^+(t), c_j(t))\rangle$, where f_n are the products of the QD electron creation and annihilation operators. These functions, n_1, n_2, n_4, $\langle c_2^+c_3\rangle$, $\langle c_2^+c_3n_1\rangle$, $\langle c_2^+c_3n_4\rangle$, $\langle n_1n_2\rangle$, $\langle c_2^+c_3n_1n_4\rangle$, $\langle n_2n_4\rangle$, $\langle n_1n_2n_4\rangle$ and $\langle n_1n_4\rangle$ (here for brevity we have omitted the time-dependence of all operators) satisfy the closed set of differential equations. The functions of the second type correspond to the averages of a number of QD electron operators taken at a given time t and leads electron operators taken at the initial time $t = 0$. This class of functions is generated by the EOM method used for $\langle c_j^+(t)c_{\alpha k}(0)\rangle$ (see Eq. 7) for which we have (e.g. for $j = 1$ and $\alpha = L_1$):

$$
\begin{aligned}
\frac{d\langle c_1^+(t)c_{L_1k}(0)\rangle}{dt} &= \left(i\varepsilon_1 - \frac{\Gamma_{L_1}^{(1)} + \Gamma_{R_1}^{(1)}}{2}\right)\langle c_1^+(t)c_{L_1k}(0)\rangle \\
&+ iU_1\langle c_1^+(t)n_2(t)c_{L_1k}(0)\rangle \\
&+ i\sum_q \tilde{V}_{L_1q}^{(l)}(t)\langle n_2(t)c_{L_1q}^+(0)c_{L_1k}(0)\rangle\,,
\end{aligned}
\tag{12}
$$

where $\tilde{V}_{L_1q}^{(l)}(t) = V_{L_1q}^{(l)}\exp\left(-i\int_0^t dt_1\varepsilon_{L_1q}(t_1)\right)$. The function $\langle n_2(t)c_{L_1q}^+(0)c_{L_1k}(0)\rangle$ which appears in the above equation, generates the next higher-order functions and, unfortunately, this is never ending process. In order to terminate this infinite set of equations, one should assume a truncation procedure. We apply the following decouplings:

$$
\begin{aligned}
\langle f_n(c_i^+(t), c_j(t))c_{\alpha k}^+(0)c_{\beta q}(0)\rangle &\simeq \langle f_n(c_i^+(t), c_j(t))\rangle\langle c_{\alpha k}^+(0)c_{\beta q}(0)\rangle \\
&= \langle f_n(c_i^+(t), c_j(t))\rangle\delta_{\alpha\beta}\delta_{k,q}\langle n_{\alpha k}(0)\rangle\,,
\end{aligned}
\tag{13}
$$

$$
\langle f_n(c_i^+(t), c_j(t))c_{\alpha k}^+(0)c_{\beta q}^+(0)\rangle \simeq \langle f_n(c_i^+(t), c_j(t))\rangle\langle c_{\alpha k}^+(0)c_{\beta q}^+(0)\rangle = 0\,, \tag{14}
$$

$$
\langle f_n(c_i^+(t), c_j(t))c_{\alpha k}(0)c_{\beta q}(0)\rangle \simeq \langle f_n(c_i^+(t), c_j(t))\rangle\langle c_{\alpha k}(0)c_{\beta q}(0)\rangle = 0\,, \tag{15}
$$

where $\langle n_{\alpha k}(0)\rangle$ is the Fermi distribution function for α-th lead electrons taken at the initial time $t = 0$. Such approximation preserves the correlation between electrons localized on different QDs - compare the mean-field Hartree-Fock approximation, $\langle n_2(t)c_1^+(t)c_{\alpha k}(t)\rangle \to \langle n_2(t)\rangle\langle c_1^+(t)c_{\alpha k}(t)\rangle$, where such correlations are destroyed. The method of calculations and decouplings used here are similar to the well known Hubbard-I approximation although here we have the time-dependent problem cf. [4]. After doing such decouplings of the higher-order functions we obtain the closed set of equations for the second type functions: $\langle c_1^+c_{\alpha k}(0)\rangle$, $\langle c_1^+n_2c_{\alpha k}(0)\rangle$, $\langle c_1^+c_2^+c_3c_{\alpha k}(0)\rangle$, $\langle c_1^+n_4c_{\alpha k}(0)\rangle$, $\langle c_1^+c_2^+c_3n_4c_{\alpha k}(0)\rangle$, $\langle c_1^+c_2c_3^+n_4c_{\alpha k}(0)\rangle$, $\langle c_1^+n_2n_4c_{\alpha k}(0)\rangle$ (here, as before, we have omitted time-dependence of all QDs operators), where $\alpha = L_1, R_1$. The slightly modified functions appear also for $\alpha = L_2, R_2$. Finally, the set of 43 differential equations for the functions of the first and second types is constructed and solved numerically for every $k-$vector used in the corresponding sums $S_{\alpha}^{(l)}(t)$. The number of k-vectors taken in the calculations of $S_{\alpha}^{(l)}(t)$ usually extends from 501 to 3001 depending on the system parameters. In order to check the correctness of our calculations we considered a simple qubit-detector system (the qubit coupled only with one SET detector) and we compared our results with those obtained by other methods and we found good agreement between them. Thus we believe that the calculating approach used here can be successfully used also for more complicated QD systems, see e.g. [26, 30].

3. Numerical results and discussion

In our calculation we set $\hbar = e = k_B = 1$ and $\Gamma = 1$ as an energy unit is assumed ($\Gamma_{L_1} = \Gamma_{R_1} = \Gamma_{L_2} = \Gamma_{R_2} = \Gamma$). The current and time are expressed in the units of $2e\Gamma/\hbar$ and $\hbar/\Gamma$, respectively. We consider the DQD playing the role of the qubit (dots 2 and 3) containing one excess electron which interacts electrostatically with two SETs (see Fig. 1). The qubit electron, depending on which qubit QD is localized, interacts with electrons flowing through the upper or lower SET QD. The bias voltage is applied symmetrically to the left and right leads, e.g. for the upper SET we assume $\mu_{L_1/R_1} = \mu_0 \pm eV_{bias}/2$ where $\mu_0 = 0$ is the chemical potential of the unbiased leads. Note that for the same bias voltages of both SETs and for $U_1 = U_2$ the system is fully symmetrical and the asymptotic occupations of the qubit QDs should achieve the stationary value 0.5. This remark is very useful in testing the approximations done during our calculations. In our studies we analyze the qubit QD occupation (qubit oscillations) calculated for different parameters characterizing the system, i.e. the inter-dot tunnelling couplings, the qubit-SET interaction strength and the bias voltage in both SETs. In addition, we also present the current flowing in both SETs and show the modifications of the qubit oscillation, $n_2(t)$, in response to the abrupt changes of the SETs QDs energy levels. These studies allow us to find such experimental setups for which the qubit state is minimally destroyed during the external disturbances of the SETs QDs.

In Fig. 2 we plot the qubit QD occupation, $n_2(t)$, as a function of time and inter-dot tunnelling amplitude, V_{23}. It is assumed that for $t < 0$ all detectors elements are isolated and at $t = 0$ the couplings between them are switched on. Similarly, up to $t = 0$ the qubit electron is localized on the upper qubit QD, $n_3(t < 0) = 1$, and begins to oscillate at $t = 0$. The upper and bottom panels correspond to the large and small bias voltages, respectively. One can observe that the asymptotic values of $n_2(t)$ achieve a half filling independently of the inter-dot tunnelling amplitude and bias voltages, as expected. The decoupling procedure (performed in order to close the corresponding set of equation of motion for higher-order correlation functions) is then justified for the considered system. As one can see the qubit oscillations are evident and the period of these oscillations strongly depends on the qubit coupling V_{23}. For very weak couplings the steady-state value of $n_2(t)$ is achieved very fast in

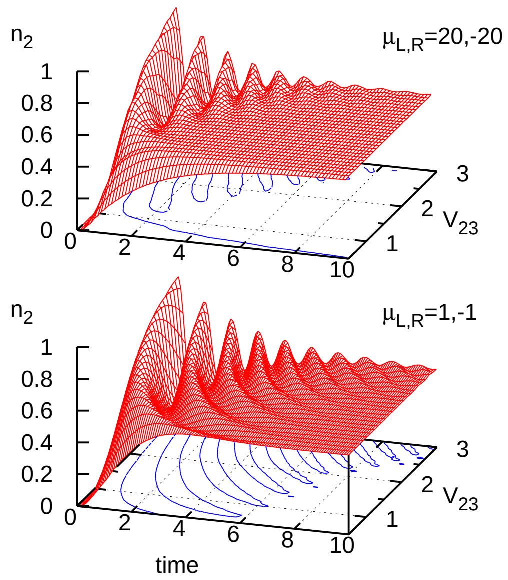

Fig. 2: The qubit QD occupancy $n_2(t)$ as a function of time and the qubit inter-dot coupling V_{23}. The upper (bottom) panel corresponds to $\mu_L = -\mu_R = 20$ ($\mu_L = -\mu_R = 1$). The other parameters are: $\varepsilon_{1,2,3,4} = 0$, $U_{12} = U_{34} = 5$, $\Gamma_{L_1/L_2/R_1,R_2} = 1$ and the initial conditions are: $n_1(t) = n_2(t) = n_4(t) = 0$ for $t < 0$ and $n_3(t < 0) = 1$.

comparison with the case of large V_{23} parameter for which high-amplitude oscillations are observed. Note, that these oscillations depend also on the bias voltage and much faster oscillations with smaller amplitudes appear for larger voltages (upper panel). On the other hand, weak decoherence takes place for smaller voltages ($n_2(t)$ oscillations survive longer in time).

Next we analyze the qubit dynamics in the presence of external time-dependent perturbations which can change the qubit decoherence process. In Fig. 3 we show the oscillations of the qubit QD occupancy and its reaction to the short-time disturbances of both SETs. At the specific time moments (here at $t = 60$ and $t = 90$) the SETs QDs energy levels are abruptly changed (by applying the bias voltage to the appropriate QDs) for a short time from $\varepsilon_1 = \varepsilon_4 = 0$ up to $\varepsilon_1 = 10$ and $\varepsilon_4 = -10$. Note that this perturbation was applied to the system after the qubit achieved its stationary value of both QD occupancies, $n_1 = n_2 = 0.5$. Such anti-phase impulses lead again to the qubit oscillations and appearance of transient current oscillations. In the lower panel of Fig. 3 we show also the changes of the SETs QDs occupations, $n_1(t)$ and $n_4(t)$, which were forced during such short-lived perturbation of the corresponding SETs QDs energy levels. These abrupt disturbances change the occu-

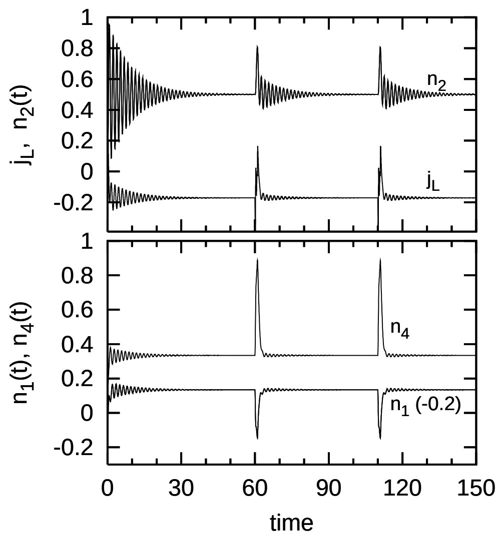

Fig. 3: The occupancies of both QD SETs, $n_1(t)$ and $n_4(t)$ (bottom panel), and the qubit occupancy $n_2(t)$ and the current flowing from the left lead $j_L \equiv j_{L_1}$ (upper panel) as a function of time for the perturbation which changes for a short time the values of the energy levels from $\varepsilon_{1,4} = 0$ to $\varepsilon_1 = -\varepsilon_4 = 10$ at $t = 60$ and $t = 90$. The other parameters are: $\mu_L = -\mu_R = 1$, $\varepsilon_{2,3} = 0$, $U_{12} = U_{34} = 2$, $V_{23} = 2$, $\Gamma_i = 1$.

pancies $n_1(t)$ and $n_4(t)$ (up to about 0.05 and 0.85, respectively) and are responsible for the revival of the qubit oscillations. In this case also the repeated current oscillations are observed. In other words, in order to evoke the qubit oscillations again, even for the qubit in the stationary state (full decoherence has been achieved), it is sufficient to disturb asymmetrically (for a short time) the SETs QDs energy levels, e.g. moving ε_1 and ε_4 in opposite directions on the energy scale.

The behaviour of the qubit oscillations is quite different for in-phase perturbations, e.g in the case when both ε_1 and ε_4 are changed in the same way, $\varepsilon_1 \to \varepsilon_1 + \Delta$, $\varepsilon_4 \to \varepsilon_4 + \Delta$. In Fig. 4 we analyze the reaction of the qubit oscillations to such perturbations. Now the SET QDs energy levels are abruptly changed up to new values (the same for both dots) for some interval of time. The upper panel shows the modifications of $n_2(t)$ when the perturbations act persistently from $t = 10$ up to $t = 25$ and from $t = 80$ up to $t = 100$. Note, that the first impulse influences the qubit dynamics before the stationary qubit state is achieved. It is interesting that the amplitude of the qubit oscillations is 'frozen' (is nearly constant) in this case. For $t = 25$ the perturbation ends and the qubit continues the damped oscillations as before the perturbation appeared. If, however, the in-phase perturbation acts on the stationary qubit state, this state is not changed at all, see the occupancy curve for $t > 80$, the upper panel. The explanation of such behaviour is relatively simple. As the QDs energy levels ε_1 and ε_4 are moved up to higher energies, their occupations

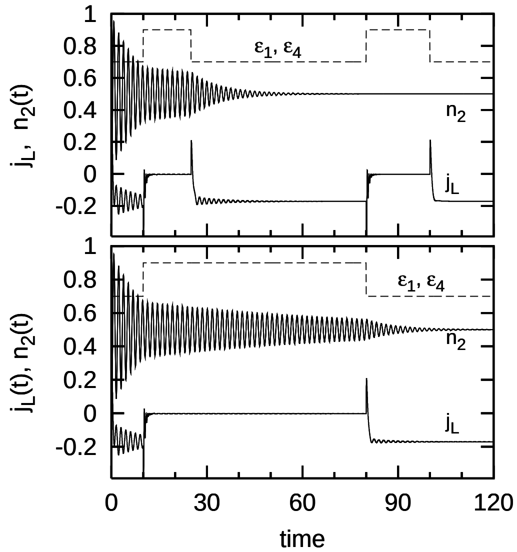

Fig. 4: The qubit QD occupancy, $n_2(t)$, and the current flowing from the left lead $j_L \equiv j_{L_1}$ as a function of time for the perturbation which changes the values of the SETs QDs energy levels from $\varepsilon_{1,4} = 0$ to $\varepsilon_1 = \varepsilon_4 = 10$ at $t = 10$ and $t = 80$ (upper panel, the perturbation duration is $\delta t = 15$ and 20, respectively) and at $t = 10$ (bottom panel, the perturbation duration is $\delta t = 70$). The broken lines show the time-positions of the QDs energy levels, ε_1 and ε_4 (not in scale). The other parameters are: $\mu_L = -\mu_R = 1$, $\varepsilon_{2,3} = 0$, $U_{12} = U_{34} = 2$, $V_{23} = 2$, $\Gamma_\alpha = 1$.

are considerably reduced but they are still equal to one another. Thus the system symmetry is not broken in this case and the motion of the qubit electron is nearly not disturbed. The qubit preserves its state as it was before the perturbation appeared.

The lower panel shows the qubit oscillations when the in-phase perturbation acts for sufficiently long time (here from $t = 10$ up to $t = 80$). In this case we observe similar behaviour of the qubit dynamics to that in the upper panel. Here very low occupations of the SET QDs slightly disturb the qubit state which leads to a relatively slow decrease of the oscillation amplitude. This result indicates that one can nearly freeze the qubit decoherence by applying the in-phase gate voltage perturbations. On both panels we also show the current flowing from the left lead, $j_L \equiv j_{L_1}$. As expected, the time-dependence of the current reflects the values of the SETs QDs occupancies. In the time interval when ε_1 and ε_4 are moved up on the energy scale (much higher than the chemical potentials of both leads i.e. beyond the voltage windows) the current drops to zero value except the transients observed just after the abrupt changes of ε_1 and ε_4 energy levels.

In Fig. 5 we show the effect of another type of perturbations acting on the qubit state. We assume that energy levels of both SETs QDs are driven harmonically

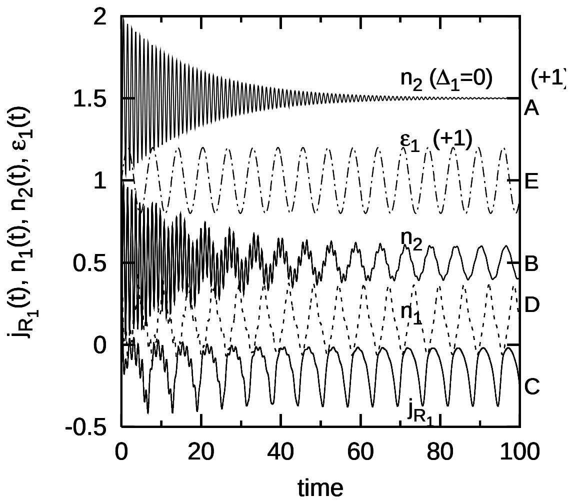

Fig. 5: The qubit QD occupancy, $n_2(t)$, the SET QD occupation, $n_1(t)$, and the current flowing from the left lead $j_L \equiv j_{L_1}$ as a function of time for the harmonic perturbation of $\varepsilon_1(t) = \Delta \sin(\omega t)$ ($\varepsilon_4(t) = -\varepsilon_1(t)$) for $\Delta = 2$, $\omega = 1$ and for $\mu_L = -\mu_R = 1$, $\varepsilon_{2,3} = 0$, $U_{12} = U_{34} = 2$, $V_{23} = 2$, $\Gamma_\alpha = 1$. Curve E for $\varepsilon_1(t)$ was divided by 10 and shifted by $+1$ for better visualization.

in time, i.e. $\varepsilon_i(t) = \Delta_i \sin(\omega_i t)$, $i = 1, 4$, $\Delta_1 = -\Delta_4$. The upper curve illustrates the occupancy of the qubit QD, $n_2(t)$, for the constant values of the SETs QDs energy levels, $\Delta_i = 0$. We observe decaying oscillations due to the interaction of the qubit electron with the environment represented by two SETs. The period of these oscillations is approximately equal to $T_{qubit} \simeq \frac{\pi}{V}$, which corresponds to the qubit decoupled from both SETs. Approximately at $t = 100$ the oscillations are washed out completely and the qubit is in the stationary state with the occupancy $n_2 = n_3 = 0.5$. If the anti-phase harmonic perturbations are applied to the SETs energy levels, $\varepsilon_1(t)$ and $\varepsilon_4(t)$, then the qubit oscillations are composed of two overlapping signals: the first one is related to the driving harmonic force which disturbs the positions of $\varepsilon_1(t)$ (curve E) and $\varepsilon_4(t)$ (not shown here) and the second signal is related to the dumped oscillations in the absence of the external perturbations (curve A). For sufficiently long time the period of the qubit oscillations is equal to the period of the external harmonic perturbations represented by oscillating ε_1 and ε_4 energy levels. Note that the minima of ε_1 coincidence with the minima of $n_2(t)$. Due to the repulsive interaction between the electrons localized on the first QD, $n_1(t)$, (the upper SET) and on the second qubit QD, $n_2(t)$, the occupancy of the qubit QD possesses local minima at maximal values of $n_1(t)$ - compare the curves D and B, respectively. In Fig. 5 we show also the time-dependent current flowing to the right lead, $j_{R_1}(t)$ (curve C). As one can see, the qubit dynamics can be detected only in the time interval which is compared with the decoherence time of the free qubit (see the curve A). For longer time the current oscillations (due to harmonic perturbations) are similar to the occupancy oscillations of a single QD driven harmonically and coupled with two leads, cf. [37, 38].

4. Conclusions

We have considered the coherent oscillations of the qubit electron in the system in which the qubit (double QD) is coupled electrostatically with two SETs playing the role of detectors of the qubit state. The system is fully symmetrical in opposition to most setups investigated in the literature in which the qubit interacts asymmetrically with one detector. The qubit QD occupancies and the current flowing in SETs were calculated using the equation of motion method for the appropriate correlation functions for which a special decoupling procedure for higher-order functions was applied. We have focused our attention on the qubit dynamics in the presence of different types of perturbations imposed on both detectors i.e. short and long-time impulses or harmonic perturbations of the QD energy levels.

It was shown that even for the qubit in the stationary state (occupancy oscillations are washed out) the oscillation can be again restored using the abrupt short anti-phase perturbations acting on both SETs QDs (the energy levels are driven in the opposite directions on the energy scale). On the other hand, if the in-phase perturbation changes simultaneously the SETs QDs energy levels even during longer time, the qubit decoherence process can be almost stopped and the damping of the qubit oscillations is considerably reduced. However, if this perturbation of the energy levels appears when the qubit is in the stationary state, the qubit does not respond to such disturbance at all. We have also considered the influence of the oscillating SETs QDs energy levels on the qubit dynamics. With sufficiently long time from the moment when the perturbation began to disturb the system, the qubit QD occupancy oscillates with the same frequency as the perturbation, however, for smaller time we observe additional oscillations with the damped amplitude and frequency of the free qubit.

References

[1] E. Taranko, M. Wiertel, and R. Taranko, J. Appl. Phys. **111** (2012), 023711.

[2] H. Pan and Y. Zhao, J. Phys.: Condens. Matter **21** (2009), 265501.

[3] B. Dong, I. Djuric, H. L. Cui, and X. L. Lei, J. Phys.: Condens. Matter **16** (2004), 4303.

[4] Y. X. Deng, X. H. Jan, and N. S. Tang, Phys. E **41** (2009), 353.

[5] Z. T. Jiang, J. Yang, Y. Wang, X. F. Wei, and Q. Z. Han, J. Phys.: Condens. Matt. **20** (2008), 445216.

[6] J. Y. Luo, S. K. Wang, X. L. Me, X. Q. Li, and Y. J. Yan, J. Appl. Phys. **108** (2010), 083720.

[7] Y. Han, W. Gong, H. Wu, and G. Wei, Physica B **404** (2009), 201.

[8] F. F. Fanchini, L. K. Castelano, and A. O. Caldeira, New Journal of Phys. **12** (2010), 073009.

[9] M. L. Ladron de Guevara and P. A. Orellana, Phys. Rev. B **73** (2006), 205303.

[10] W. Xie, H. Pan, W. Onu, W. Zhang, and S. Duan, arXiv 0801.3515 (2008).

[11] J. Łuczak and B. R. Bułka, Phys. Rev. B **90** (2014), 165427.

[12] T. Kwapiński and R. Taranko, Physica E **63** (2014), 241.

[13] T. Kwapiński and R. Taranko, J. Phys.: Condens. Matt. **23** (2011), 405301.

[14] R. Taranko and P. Parafiniuk, Physica E **43** (2010), 302.

[15] B. Liu, J. Song, and H. S. Song, Int. J. Theor. Phys. **51** (2012), 2930.

[16] J. Liu, Z. T. Jiang, and B. Shao, Phys. Rev. B **79** (2009), 115323.

[17] H. B. Sun and H. M. Wiseman, J.Phys.: Condens. Matter **21** (2009), 125301.

[18] M. T. Lee and W. M. Zhang, Phys. Rev. B **74** (2006), 085325.

[19] Y. Ye, Y. Cao, X. Q. Li, and S. Gurvitz, Phys. Rev. B **84** (2011), 245311.

[20] M. T. Lee and W. M. Zhang, J. Chem. Phys. **129** (2008), 224106.

[21] S. K. Wang, J. Jin, and X. Q. Li, Phys. Rev. B **75** (2007), 155304.

[22] H. J. Jiao, X. Q. Lin, and Y. J. Luo, Phys. Rev. B **75** (2007), 155333.

[23] J. P. Zhang, S. H. Ouyang, C. H. Lam, and J. Q. You, J. Phys.: Condens. Matt. **20** (2008), 395206.

[24] S. H. Ouyang, C. H. Lam, and J. Q. You, J. Phys.: Condens. Matt. **18** (2006), 115511.

[25] R. Taranko and P. Parafiniuk, Physica E **40** (2008), 2765.

[26] S. A. Gurvitz and G. P. Berman, Phys. Rev. B **72** (2005), 073303.

[27] H. J. Jiao, F. Li, S. K. Wang, and X. Q. Li, Phys. Rev. B **79** (2009), 075320.

[28] C. Kreisbeck, F. J. Kaiser, and S. Kohler, Phys. Rev. B **81** (2010), 125404.

[29] R. Taranko and T. Kwapiński, Physica E **48** (2013), 157; **70** (2015), 217.

[30] T. Gilad and S. A. Gurvitz, Phys. Rev. Lett. **97** (2006) 116806.

[31] V. H. Nguyen, J.Phys.: Condens. Matter **21** (2009), 273201.

[32] C. Nietner, G. Schaller, C. Poltl, and T. Brandes, Phys. Rev. B **85** (2012), 245431.

[33] G. Shinkai, T. Hayashi, T. Ota, and T. Fujisawa, Phys. Rev. Lett. **103** (2009), 056801.

[34] T. Tanamoto and S. Fujita, Phys. Rev. B **72** (2005), 085335.

[35] T. Tanamoto and X. Hu, Phys. Rev. B **69** (2004), 115301.

[36] J. Li and G. Johansson, Phys. Rev. B **75** (2007), 085312.

[37] A.-P. Jauho, N. S. Wingreen, and Y. Meir, Phys. Rev. B **50** (1994), 5528.

[38] T. Kwapiński and R. Taranko, Physica E **18** (2003), 402.

[39] S. A. Gurvitz, and D. Mozyrsky, Phys. Rev. B **77** (2008), 075325.

[40] X. Q. Li, W. K. Zhang, P. Cui, J. Shao, Z. Ma, and YiJing Yan, Phys. Rev. B **69** (2004), 085315.

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DYNAMIKA ELEKTRONU W KWANTOWYM QUBICIE SPRZĘŻONYM Z DWOMA JEDNOELEKTRONOWYMI TRANZYSTORAMI

S t r e s z c z e n i e

Wykonano badania teoretyczne dynamiki qubitu (w postaci podwójnej kropki kwantowej) sprzężonego elektrostatycznie z dwoma detektorami będącymi jednoelektronowymi tranzystorami. Obliczając stopień zapełnienia kropek kwantowych i prądy płynące przez detektory użyto metody równań ruchu dla odpowiednich funkcji korelacyjnych połączonej ze specjalną procedurą rozszczepienia funkcji wyższego rzędu. Zbadano dynamikę qubitu w obecności różnego rodzaju zaburzeń działających na oba detektory, tj. dla stałych oraz harmonicznych zmian wartości poziomów energetycznych detektorów. Pokazano, że oscylacje qubitu będącego już w stanie stacjonarnym mogą być ponownie wzbudzone przy użyciu nagłych, krótkich zaburzeń poziomów energetycznych detektorów. W przypadku zaburzeń harmonicznych zapełnienie kropek kwantowych qubitu oscyluje z częstością zaburzającą dla dostatecznie długiego czasu od momentu włączenia zaburzenia. Natomiast dla krótszych czasów obserwujemy nałożenie się tych oscylacji na tłumione oscylacje swobodnego qubitu.

Słowa kluczowe: qubit, kropka kwantowa podwójna, tranzystor jednoelektronowy, dekoherencja

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Władysław Wilczyński

MIXED PARTIAL DENSITY TOPOLOGY

Summary

The paper deals with the density-type topology in the plane generated by the lower density operator Φ_{xy} which is defined (in a sense) similarly to the mixed partial derivative of a function of two variables. This topology is different from ordinary and strong density topologies in the plane as well as from the product of two linear density topologies.

Keywords and phrases: density topologies in the plane, approximately continuous functions

1.

Let $\mathcal{L}^1$ ($\mathcal{L}^2$) be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ (resp. $\mathbb{R}^2$), $\mathcal{B}^1$ ($\mathcal{B}^2$) – the σ-algebra of Borel subsets of $\mathbb{R}$ (resp. $\mathbb{R}^2$), $\mathcal{I}^1$ ($\mathcal{I}^2$) – the σ-ideal of null sets in $\mathbb{R}$ (resp. $\mathbb{R}^2$) and λ_1 (λ_2) – the linear (planar) Lebesgue measure. If $A \subset \mathbb{R}^2$, then, as usual, $A_x = \{y : (x,y) \in A\}$ for $x \in \mathbb{R}$ and $A^y = \{x : (x,y) \in A\}$ for $y \in \mathbb{R}$. We shall say that the sets $A_1, A_2 \in \mathcal{L}^1$ (resp. $\mathcal{L}^2$) are equivalent ($A_1 \sim A_2$) if and only if $A_1 \triangle A_2 \in \mathcal{I}^1$ (resp. $\mathcal{I}^2$).

Recall that x_0 is a point of density of a set $A \in \mathcal{L}^1$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda_1(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

Let $\Phi(A) = \{x \in \mathbb{R} : x \text{ is a point of density of } A\}$ for $A \in \mathcal{L}^1$. The operator $\Phi : \mathcal{L}^1 \to 2^{\mathbb{R}}$ has the following properties (see [O], chapter 22):

1) for each $A \in \mathcal{L}^1$, $A \sim \Phi(A)$ (the Lebesgue Density Theorem),
2) for each $A_1, A_2 \in \mathcal{L}^1$ if $A_1 \sim A_2$, then $\Phi(A_1) = \Phi(A_2)$,

3) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,
4) for each $A_1, A_2 \in \mathcal{L}^1$ $\Phi(A_1 \cap A_2) = \Phi(A_1) \cap \Phi(A_2)$.

The operator fulfilling properties 1)-4) is called the lower density operator.

The family $\mathcal{T}_d = \{A \in \mathcal{L}^1 : A \subset \Phi(A)\}$ is the density topology, which is stronger than the natural topology on the real line. Observe also, that in fact $\Phi : \mathcal{L}^1 \to \mathcal{L}^1$ by virtue of the LDT.

2.

Our aim is to introduce a new operator $\Phi_{xy} : \mathcal{L}^2 \to \mathcal{L}^2$ having all properties similar to that of Φ and to study properties of the topology $\mathcal{T}_{xy} = \{A \in \mathcal{L}^2 : A \subset \Phi_{xy}(A)\}$ generated by this operator.

Let $\Phi_v(B) = \{(x,y) : y \in \Phi(B_x)\}$ and $\Phi_h(B) = \{(x,y) : x \in \Phi(B^y)\}$ for $B \in \mathcal{B}^2$. Since $B_x \in \mathcal{B}^1$ for each $B \in \mathcal{B}^2$ and also $B^y \in \mathcal{B}^1$ for each $B \in \mathcal{B}^2$, the operators Φ_v (of "vertical" density points) and Φ_h (of "horizontal" density points) are well defined. Moreover, $\Phi_v(B)$ and $\Phi_h(B)$ are Borel sets for $B \in \mathcal{B}^2$ (see [M], th. 1).

Definition 1. We shall say that (x_0, y_0) is a mixed (x, y) partial density point of a set $A \in \mathcal{L}^2$ if and only if $(x_0, y_0) \in \Phi_h(\Phi_v(B))$ for some $B \in \mathcal{B}^2$ such that $A \sim B$. We shall write $(x_0, y_0) \in \Phi_{xy}(A)$.

To prove that the operator Φ_{xy} is uniquely defined we shall need the following proposition:

Proposition 1. *If $B_1, B_2 \in \mathcal{B}^2$ and $B_1 \sim B_2$, then $\Phi_v(B_1) \sim \Phi_v(B_2)$.*

Proof. From Fubini theorem it follows that $(B_1)_x \sim (B_2)_x$ for a.e. $x \in \mathbb{R}$, so $\Phi((B_1)_x) = \Phi((B_2)_x)$ for a.e. $x \in \mathbb{R}$ by virtue of 2) and finally $\Phi_v(B_1) \sim \Phi_v(B_2)$ again using Fubini theorem. □

Proposition 2. *If $B_1, B_2 \in \mathcal{B}^2$ and $B_1 \sim B_2$, then $(\Phi_v(B_1))^y \sim (\Phi_v(B_2))^y$ for each $y \in \mathbb{R}$.*

Proof. It follows immediately from the proof of Proposition 1. □

Proposition 3. *If $B_1, B_2 \in \mathcal{B}^2$ and $B_1 \sim B_2$, then $\Phi_h(\Phi_v(B_1)) = \Phi_h(\Phi_v(B_2))$.*

Proof. By virtue of Proposition 2 we have $\Phi((\Phi_v(B_1))^y) = \Phi((\Phi_v(B_2))^y)$ for each $y \in \mathbb{R}$, so $\Phi_h(\Phi_v(B_1)) = \Phi_h(\Phi_v(B_2))$. □

Theorem 1. *The operator $\Phi_{xy} : \mathcal{L}^2 \to 2^{\mathbb{R}^2}$ has the following properties:*

0) *for each $A \in \mathcal{L}^2$ $\Phi_{xy}(A) \in \mathcal{B}^2$,*
1) *for each $A \in \mathcal{L}^2$ $A \sim \Phi_{xy}(A)$,*
2) *for each $A_1, A_2 \in \mathcal{L}^2$ if $A_1 \sim A_2$, then $\Phi_{xy}(A_1) = \Phi_{xy}(A_2)$,*

3) $\Phi_{xy}(\emptyset) = \emptyset,\ \Phi_{xy}(\mathbb{R}^2) = \mathbb{R}^2$,
0) *for each* $A_1, A_2 \in \mathcal{L}^2 \quad \Phi_{xy}(A_1 \cap A_2) = \Phi_{xy}(A_1) \cap \Phi_{xy}(A_2)$.

Proof. 0) follows from the theorem of Mauldin ([M], th. 1).

1) Take arbitrary Borel set $B \sim A$. Then $B \sim \Phi_{xy}(B)$ (see [S], p. 298) and so $A \sim \Phi_{xy}(A)$,

2) If $A_1 \sim A_2$, $A_1 \sim B_1$ and $A_2 \sim B_2$, where $B_1, B_2 \in \mathcal{B}^2$, then also $B_1 \sim B_2$ and by virtue of Proposition 3 we have $\Phi_{xy}(B_1) = \Phi_{xy}(B_2)$,

3) is obvious.

4) If $A_1 \sim B_1$, $A_2 \sim B_2$, $B_1, B_2 \in \mathcal{B}^2$, then we have $\Phi_v(B_1 \cap B_2) = \{(x,y) : y \in \Phi((B_1 \cap B_2)_x)\} = \{(x,y) : y \in \Phi((B_1)_x \cap (B_2)_x)\} = \{(x,y) : y \in \Phi((B_1)_x) \cap \Phi((B_2)_x)\} = \Phi_v(B_1) \cap \Phi_v(B_2)$ and similarly for Φ_h. □

Theorem 2. *The family* $\mathcal{T}_{xy} = \{A \in \mathcal{L}^2 : A \subset \Phi_{xy}(A)\}$ *is a topology stronger than a natural topology on the plane.*

Proof. Since the operator Φ_{xy} is a lower density operator and the Lebesgue measure on the plane fulfills the countable chain condition, the proof that $\mathcal{T}_{xy}$ is a topology is exactly the same as in [O], chapter 22. Observe that $(\mathbb{R} \setminus Q) \times \mathbb{R} \in \mathcal{T}_{xy}$ and is not open in the natural topology, simultaneously each open set (in the natural topology) consists only of mixed partial density points, so is in $\mathcal{T}_{xy}$. □

Theorem 3. *The topology* $\mathcal{T}_{xy}$ *has following properties:*
a) *if* A *is compact in* $\mathcal{T}_{xy}$ *then card*$(A) < \aleph_0$.
b) *each segment* $I = [a,b] \times \{c\}$ *is connected in* $\mathcal{T}_{xy}$,
each segment I, *which is not horizontal, is not connected.*
c) $\mathcal{T}_{xy}$ *is Hausdorff but not regular.*

Proof. Ad a) The proof does not differ from that for $\mathcal{T}_d$ (see [W], p. 685).

Ad b) Suppose that $I = U_1 \cup U_2$, where $U_1, U_2 \in \mathcal{T}_{xy|I}$, which means that

$$U_1 = I \cap G_1,\ U_2 = I \cap G_2,\ G_1, G_2 \in \mathcal{T}_{xy}$$

and

$$U_1 \neq \emptyset,\ U_2 \neq \emptyset,\ U_1 \cap U_2 = \emptyset.$$

Observe that if $x \in [a,b]$, then $(x,c) \in U_1$ if and only if $x \in \Phi((\Phi_v(G_1))^c)$, so $U_1^c = \Phi((\Phi_v(G_1))^c)$ and similarly $U_2^c = \Phi((\Phi_v(G_2))^c)$. Hence $U_1^c, U_2^c \in \mathcal{T}_d$, $U_1^c \cup U_2^c = [a,b]$, $U_1^c \neq \emptyset$, $U_2^c \neq \emptyset$, $U_1^c \cap U_2^c = \emptyset$ – a contradiction, because $[a,b]$ is connected in $\mathcal{T}_d$ (see [GNN]).

If $I \subset \mathbb{R}^2$ is not horizontal segment, then let (x_0, y_0) be the end-point of I. Observe that if $G_1 = (\mathbb{R}^2 \setminus I) \cup \{(x_0, y_0)\}$, $G_2 = \mathbb{R}^2 \setminus \{x_0, y_0)\}$, then $G_1, G_2 \in \mathcal{T}_{xy}$. Hence $U_1 = G_1 \cap I$ and $U_2 = G_2 \cap I$ belong to $\mathcal{T}_{xy|I}$, $U_1 \cap U_2 = \emptyset$ and $I = U_1 \cup U_2$, so I is the union of two non-empty sets open in $\mathcal{T}_{xy|I}$.

Ad c) $\mathcal{T}_{xy}$ is Hausdorff since it is stronger than the natural topology in $\mathbb{R}^2$. To prove that $\mathcal{T}_{xy}$ is not regular we shall show that $(0,0)$ cannot be separated from $A = (\mathbb{R} \times \{0\}) \setminus \{(0,0)\}$. Observe first that A is closed in $\mathcal{T}_{xy}$, because $\lambda_2(A) = 0$. Suppose that $U_1, U_2 \in \mathcal{T}_{xy}$, $(0,0) \in U_1$ and $A \subset U_2$. The set $(\Phi_v(U_2))^0$ is of full measure (full measure in each interval), because $\mathbb{R} \setminus \{0\} \subset (\Phi_v(U_2))^0$. Simultaneously $\lambda_1((\Phi_v(U_1))^0) > 0$, because $0 \in \Phi((\Phi_v(U_1))^0)$. Hence $(\Phi_v(U_1))^0 \cap (\Phi_v(U_2))^0 \neq \emptyset$. If ξ belongs to both sets, then $(U_1)_\xi \cap (U_2)_\xi \neq \emptyset$, because $0 \in \Phi((U_1)_\xi) \cap \Phi((U_2)_\xi)$. Finally $U_1 \cap U_2 \neq \emptyset$. □

3.

Definition 2. We shall say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{T}_{xy}$-approximately path continuous at (x_0, y_0) if and only if there exists a set $A \subset \mathbb{R}^2$ such that $(x_0, y_0) \in \Phi_{xy}(A)$ and $f_{|A \cup \{(x_0,y_0)\}}$ is continuous at (x_0, y_0), which means

$$f(x_0, y_0) = \underset{\substack{(x,y) \to (x_0,y_0) \\ (x,y) \in A}}{\to} \lim f(x, y).$$

Definiton 3. We shall say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{T}_{xy}$-approximately continuous at (x_0, y_0) if and only if for each $\epsilon > 0$ we have

$$(x_0, y_0) \in \Phi_{xy}(f^{-1}(f(x_0, y_0) - \epsilon, f(x_0, y_0) + \epsilon)).$$

Theorem 4. *The $\mathcal{T}_{xy}$-approximate path continuity is equivalent to $\mathcal{T}_{xy}$-approximate continuity.*

Proof. The fact that $\mathcal{T}_{xy}$-approximate path continuity implies $\mathcal{T}_{xy}$-approximate continuity at (x_0, y_0) is clear. Suppose now that a function f is $\mathcal{T}_{xy}$-approximately continuous at (x_0, y_0) Then for each $n \in \mathbb{N}$ we have

$$(x_0, y_0) \in A_n = \Phi_{xy}\left(f^{-1}\left(f(x_0, y_0) - \frac{1}{n}, f(x_0, y_0) + \frac{1}{n}\right)\right).$$

A sequence $\{A_n\}_{n \in \mathbb{N}}$ is a descending sequence of measurable sets. Let $\{B_n\}_{n \in \mathbb{N}}$ be a descending sequence of Borel sets such that $B_n \sim A_n$ for each n. According to the definition of Φ_{xy} we have $x_0 \in \Phi(\Phi_v(B_n)^{y_0})$ for $n \in \mathbb{N}$ and the sequence $\{\Phi(\Phi_v(B_n)^{y_0}\}_{n \in \mathbb{N}}$ is also descending. Hence by virtue of the condition (J_2) (see [T], p. 29) there exists a decreasing sequence $\{h_n\}_{n \in \mathbb{N}}$ convergent to 0 such that $x_0 \in \Phi(B)$, where

$$B = \bigcup_{n=1}^{\infty} (\Phi(\Phi_v(B_n)^{y_0}) \setminus (x_0 - h_n, x_0 + h_n)).$$

Put

$$A = \bigcup_{n=1}^{\infty} (A_n \setminus ((x_0 - h_n, x_0 + h_n) \times \mathbb{R})).$$

Then $(x_0, y_0) \in \Phi_{xy}(A)$ and $f(x_0, y_0) = \underset{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in A}}{\to} \lim f(x, y)$. □

Observe now that $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{T}_{xy}$-approximately continuous if and only if $f^{-1}(G) \in \mathcal{T}_{xy}$ for each G open in the natural topology in $\mathbb{R}$.

Theorem 5. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{T}_{xy}$-approximately continuous, then f is of the third Baire class.*

Proof. We can suppose that f is bounded, $f : \mathbb{R}^2 \to [-1, 1]$. Put

$$f_{n,m}(x, y) = \frac{n}{2} \cdot \frac{m}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \int_{y-\frac{1}{n}}^{y+\frac{1}{n}} f(\xi, \eta) d\xi d\eta.$$

It is easy to see that $f_{n,m}$ is continuous for $n, m \in \mathbb{N}$.

If

$$f_n(x, y) = \limsup_{m\to\infty} f_{n,m}(x, y),$$

then obviously f_n is of the second Baire class for $n \in \mathbb{N}$. We shall show that $f = \lim_{n\to\infty} f_n$, where the limit is pointwise, which yields that f is of the third Baire class.

Fix $(x, y) \in \mathbb{R}^2$. Let $k \in \mathbb{N}$ and let

$$E_k = f^{-1}\left(\left(f(x, y) - \frac{1}{k}, f(x, y) + \frac{1}{k}\right)\right).$$

Obviously $E \in \mathcal{L}^2$, moreover $(x, y) \in \Phi_{xy}(E_k)$. To simplify the denotation put $A_k = (\Phi_v(E_k))^y$. We have $x \in \Phi(A_k)$ and for each $\xi \in A_k$ $y \in \Phi((E_k)_\xi)$.

Again to simplify the denotation put

$$K_{n,m} = \left[x - \frac{1}{n}, x + \frac{1}{n}\right] \times \left[y - \frac{1}{m}, y + \frac{1}{m}\right].$$

We have

$$\begin{aligned} f_{n,m}(x, y) &= \frac{n}{2} \cdot \frac{m}{2} \cdot \iint_{K_{n,m}\cap E_k} f(\xi, \eta) d\xi d\eta + \iint_{K_{n,m}\setminus E_k} f(\xi, \eta) d\xi d\eta) \\ &\le \left(f(x, y) + \frac{1}{k}\right) + \frac{n}{2} \cdot \frac{m}{2} \lambda_2(K_{n,m} \setminus E_k) \end{aligned}$$

and similarly

$$\frac{n}{2} \cdot \frac{m}{2} \lambda_2(K_{n,m} \setminus E_k) + (f(x, y) - \frac{1}{k}) \le f_{n,m}(x, y).$$

We have

$$K_{n,m} \setminus E_k \subset (([x - \frac{1}{n}, x + \frac{1}{n}] \setminus A_k) \times [y - \frac{1}{m}, y + \frac{1}{m}]) \cup ((A_k \times [y - \frac{1}{m}, y + \frac{1}{m}]) \setminus E_k).$$

There exists $n_0 \in \mathbb{N}$ such that

$$\frac{\lambda_1(A_k \cap [x - \frac{1}{n}, x + \frac{1}{n}])}{\frac{2}{n}} > 1 - \frac{1}{k}$$

for $n \geq n_0$, so

$$\lambda_2\left(\left(\left[x-\frac{1}{n}, x+\frac{1}{n}\right] \setminus A_k\right) \times \left[y-\frac{1}{m}, y+\frac{1}{m}\right]\right) \leq \frac{2}{n} \cdot \frac{2}{m} \cdot \frac{2}{k}.$$

From the fact that $y \in \Phi((E_k)_\xi)$ for each $\xi \in A_\xi$ using the dominated convergence theorem we obtain

$$\lim_{m\to\infty} \frac{\lambda_2\left(((A_k \cap [x-\frac{1}{n}, x+\frac{1}{n}]) \times [y-\frac{1}{m}, y+\frac{1}{m}]) \cap E_k\right)}{\lambda_1(A_k)\cdot\frac{2}{m}} = 1,$$

hence for $n \geq n_0$ and sufficiently big $m \in \mathbb{N}$ we have

$$\lambda_2\left(\left(A_k \times \left[y-\frac{1}{m}, y+\frac{1}{m}\right]\right) \setminus E_k\right) \leq \lambda_1\left(A_k \cap \left[x-\frac{1}{n}, x+\frac{1}{n}\right]\right)\frac{2}{m}\cdot\frac{1}{k} \leq \frac{2}{n}\cdot\frac{2}{m}\cdot\frac{1}{k}.$$

Hence

$$\lambda_2(K_{n,m} \setminus E_k) \leq \frac{2}{n}\cdot\frac{2}{m}\cdot\frac{2}{k}.$$

Finally for $n \geq n_0$ and sufficiently big $m \in \mathbb{N}$

$$-\frac{2}{k} + f(x,y) - \frac{1}{k} \leq f_{nm}(x,y) \leq f(x,y) + \frac{1}{k} + \frac{2}{k}$$

and so

$$-\frac{3}{k} + f(x,y) \leq f_n(x,y) \leq f(x,y) + \frac{3}{k} \qquad \text{for } n \geq n_0.$$

It means that

$$f(x,y) = \lim_{n\to\infty} f_n(x,y)$$

and the theorem is proved. □

Problem. Whether $\mathcal{T}_{xy}$-approximately continuous function must be of the second Baire class?

Theorem 6. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is separately approximately continuous, then it is $\mathcal{T}_{xy}$-approximately continuous.*

Proof. Fix $(x_0, y_0) \in \mathbb{R}^2$ and take $\epsilon > 0$. There exists a set $A \subset \mathbb{R}$, $A \in \mathcal{L}^1$ such that

$$x_0 \in \Phi(A) \quad \text{and} \quad |f(x,y_0) - f(x_0,y_0)| < \frac{\epsilon}{2}$$

for $x \in A$, because $f(\cdot, y_0)$ is approximately countinuous at x_0.

For each $x \in A$ there exists a set $B(x) \in \mathcal{L}^1$ such that

$$x \in \Phi(B(x)) \quad \text{and} \quad |f(x,y) - f(x,y_0)| < \frac{\epsilon}{2}$$

for $y \in B(x)$. Hence

$$\bigcup_{x\in A}(\{x\} \times B(x)) \subset f^{-1}((f(x_0,y_0)-\epsilon, f(x_0,y_0)+\epsilon))$$

and the last set is measurable by virtue of [D], th. 1. We see at once that

$$(x_0,y_0) \in \Phi_{xy}(f^{-1}((f(x_0,y_0)-\epsilon, f(x_0,y_0)+\epsilon))).$$

From the arbitrariness of ϵ it follows that f is $\mathcal{T}_{xy}$-approximately continuous at (x_0, y_0). $\square$

Remark. If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined in the following way:

$f(x,y) = 0$ if $y \le x^2$,

$f(x,y) = 1$ if $y \ge 2x^2$, $(x,y) \ne (0,0)$,

f is linear in y and continuous on each closed interval joining (x, x^2) and $(x, 2x^2)$, then f is $\mathcal{T}_{xy}$-approximately continuous but not separately approximately continuous at $(0,0)$.

4.

Now we shall compare our topology $\mathcal{T}_{xy}$ with other density topologies in the plane (see [GNN] and [GW]).

Recall that (x_0, y_0) is a density point (or ordinary density point) of the set $A \in \mathcal{L}^2$ if and only if

$$\lim_{h \to 0^+} \frac{\lambda_2([A \cap (x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]))}{4h^2} = 1.$$

If $\Phi_0(A) = \{(x,y) \in \mathbb{R}^2 : (x,y)$ is an ordinary density point of $A\}$ for $A \in \mathcal{L}^2$, then Φ_0 is a lower density operator and the family

$$\mathcal{T}_0 = \{A \in \mathcal{L}^2 : A \subset \Phi_0(A)\}$$

is a topology in the plane called the (ordinary) density topology.

A point (x_0, y_0) is a strong density point of the set $A \in \mathcal{L}^2$ if and only if

$$\underset{\substack{h \to 0^+ \\ k \to 0^+}}{\to} \lim \frac{\lambda_2(A \cap ([x_0 - h, x + h] \times [y_0 - k, y_0 + k]))}{4hk} = 1.$$

If $\Phi_s(A) = \{(x,y) \in \mathbb{R}^2 : (x,y)$ is a strong density point of $A\}$ for $A \in \mathcal{L}^2$, then Φ_s is a lower density operator and the family

$$\mathcal{T}_s = \{A \in \mathcal{L}^2 : A \subset \Phi_s(A)\}$$

is a topology in the plane called the strong density topology.

Also it is known that the (ordinary) density topology in the plane is strictly stronger than the strong density topology, which in turn is strictly stronger than the natural topology in the plane.

Observe also that the product topology $\mathcal{T}_d \times \mathcal{T}_d$ is strictly stronger than the natural topology and strictly weaker than the strong density topology.

Theorem 7. *$\mathcal{T}_{xy} \setminus \mathcal{T}_0 \ne \emptyset$, $\mathcal{T}_0 \setminus \mathcal{T}_{xy} \ne \emptyset$.*

Proof. Let $A = \{(x,y) : |y| < x^2\} \cup \{(0,0)\}$. We have $A \in \mathcal{T}_{xy} \setminus \mathcal{T}_0$. Let now $B = \{(x,y) : |y| > x^2\} \cup \{(0,0)\}$. We have $B \in \mathcal{T}_0 \setminus \mathcal{T}_{xy}$. $\square$

Theorem 8. *$\mathcal{T}_{xy} \setminus \mathcal{T}_s \ne \emptyset$, $\mathcal{T}_s \setminus \mathcal{T}_{xy} \ne \emptyset$.*

Proof. If A is the set from the proof of the previous theorem, then $A \in \mathcal{T}_{xy} \setminus \mathcal{T}_s$.

The construction of the set B which belongs to $\mathcal{T}_s$ but not to $\mathcal{T}_{xy}$ is a little bit more complicated.

Let P_n for $n \in \mathbb{N}$ be an interval set, i.e. the set it the form

$$P_n = \bigcup_k (a_{n,k}, b_{n,k}), \quad 0 < a_{n,k} < b_{n,k} < a_{n,k-1}$$

such that:

$$P_n \subset (0,1) \quad \text{for} \quad n \in \mathbb{N}, \quad \frac{\lambda_1(P_n \cap (0,h))}{h} > 1 - \frac{1}{n}$$

for each $n \in \mathbb{N}$ and each $h \in (0,1)$ and

$$\liminf_{h \to 0^+} \frac{\lambda_1(P_n \cap [0,h])}{h} < 1.$$

Put

$$B = (\mathbb{R}^2 \setminus ([0,1] \times [0,1])) \cup (\bigcup_{n=1}^{\infty} ((\frac{1}{n+1}, \frac{1}{n}) \times P_n)) \cup \{(0,0)\}.$$

We see that if $h < \frac{1}{n}$, k is arbitrary in $(0,1)$, then

$$\frac{\lambda_2(B \cap ([-h,h] \times [-k,k]))}{4hk} > 1 - \frac{1}{n},$$

so $(0,0)$ is a strong density point of B. If $(x,y) \in B$ and $(x,y) \neq (0,0)$, then (x,y) is an interior point of B, so also a strong density point of B. Hence $B \in \mathcal{T}_s$.

Simultaneously $0 \notin \Phi(B_x)$ for $x \in (0,1)$, so $(0,0) \notin \Phi_{xy}(B)$ and $B \notin \mathcal{T}_{xy}$. □

Theorem 9. $\mathcal{T}_d \times \mathcal{T}_d \subsetneqq \mathcal{T}_{xy}$.

Proof. If $A \subset \mathcal{T}_d \times \mathcal{T}_d$ and $(x_0, y_0) \in A$, then there exists sets $B_1, B_2 \in \mathcal{T}_d$ such that $(x_0, y_0) \in B_1 \times B_2 \subset A$. It is not difficult to observe that $(x_0, y_0) \in \Phi_{xy}(A)$, so $A \in \mathcal{T}_{xy}$.

The set A from the proof of Theorem 7 belongs to $\mathcal{T}_{xy}$ but not to $\mathcal{T}_d \times \mathcal{T}_d$. □

References

[1] R. O. Davies, *Separate approximate continuity implies measurability*, Proc. Camb. Phil. Soc. **73** (1973), 461–465.

[2] C. Goffman, C. J. Neugebauer, and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[3] C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121.

[4] J. Jędrzejewski, *On limit numbers of real functions*, Fund. Math. **83** (1973/74), 269–281.

[5] R. D. Mauldin, *One-to-one selections-marriage theorems*, Amer. J. Math. **104**, no. 4 (1982), 823–828.

[6] J. C. Oxtoby, *Measure and category*, Springer Verlag, New York-Heidelberg-Berlin 1980.

[7] S. Saks, *Theory of the integral*, Monografie Matematyczne, Warszawa-Lwów 1937.
[8] B. S. Thomson, *Theory of real functions*, Lecture Notes in Math., Springer-Verlag.
[9] W. Wilczyński, *Density topologies*, Chapter XV in Handbook of measure theory, Elsevier 2002, ed. E. Pap.

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MIESZANE CZĄSTKOWE TOPOLOGIE GĘSTOŚCI

S t r e s z c z e n i e

Praca zawiera konstrukcję topologii na płaszczyźnie typu topologii gęstości generowaną przez operator dolnej gęstości zdefiniowany (w pewnym sensie) podobnie, jak mieszana pochodna cząstkowa funkcji dwóch zmiennych. Pokazano, że topologia ta różni się od topologii zwykłej i silnej gęstości na płaszczyźne oraz od produktu dwóch topologii gęstości na prostej.

Słowa kluczowe: topologie gęstości na płaszczyźnie, funkcje aproksymatywnie ciągłe

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Szymon Brzostowski, Tadeusz Krasiński, and Justyna Walewska

NON-DEGENERATE JUMP OF MILNOR NUMBERS OF SURFACE SINGULARITIES

Summary

The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations (f_s). We give a formula for the jump in some class of surface singularities in the case where deformations are non-degenerate.

Keywords and phrases: Milnor number, deformation of singularity, non-degenerate singularity, Newton polyhedron

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *(isolated) singularity*, i.e. let f_0 be a germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \to \mathbb{C}$ of f_0 holomorphic in an open neighborhood U of the point $0 \in \mathbb{C}^n$ such that:

- $\hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$.

In the sequel we will identify germs of functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of n variables will be denoted by $\mathcal{O}_n$.

A *deformation of the singularity* f_0 is a germ of a holomorphic function $f = f(s,z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that:

- $f(0,z) = f_0(z)$,
- $f(s,0) = 0$,

The deformation $f(s,z)$ of the singularity f_0 will also be treated as a family (f_s) of germs, putting $f_s(z) := f(s,z)$. Since f_0 is an isolated singularity, f_s has also isolated singularities near the origin, for sufficiently small s [GLS07, Theorem 2.6 in Chap. I].

Remark 1. Notice that in the deformation (f_s) there can occur in particular *smooth* germs, that is germs satisfying $\nabla f_s(0) \neq 0$. In this context, the symbol ∇f_s will always denote $\nabla_z f_s(z)$.

By the above assumptions it follows that, for every sufficiently small s, one can define a (finite) number μ_s as the Milnor number of f_s, namely

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n/(\nabla f_s) = \mu\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right),$$

where the symbol

$$\mu\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

denotes intersection multiplicity of the ideal

$$\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_n \quad \text{in} \quad \mathcal{O}_n.$$

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities [GLS07, Theorem 2.57 in Chap. II], there exists an open neighborhood S of the point $0 \in \mathbb{C}$ such that

- $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
- $\mu_0 \geqslant \mu_s$ for $s \in S$.

The (constant) difference $\mu_0 - \mu_s$ for $s \in S \setminus \{0\}$ will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest nonzero value among all the jumps of deformations of the singularity f_0 (such a value exists because one can always consider a deformation of f_0 built of smooth germs and then for it it is $\mu_s = 0$; cf. Remark 1) will be called *the jump (of the Milnor number) of the singularity* f_0 and denoted by $\lambda(f_0)$.

The first general result concerning the jump was S. Gusein-Zade's [GZ93], who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities it holds $\lambda(f_0) = 1$. In [BK14] the authors proved that $\lambda(f_0)$ is not a topological invariant of f_0 but it is an invariant of the stable equivalence. The computation of $\lambda(f_0)$ for a specific reducible singularity (or for a class of reducible singularities) is not an easy task. It is related to the problem of adjacency of classes of singularities. Only for a few classess of singularities we know the exact value of $\lambda(f_0)$. For plane curve singularities ($n = 2$) we have (see [AGZV85] for terminology):

- for the one-modal family of singularities in the X_9 class, that is singularities of the form
$$f_0^a(x, y) := x^4 + y^4 + ax^2y^2, \quad a \in \mathbb{C}, \quad a^2 \neq 4,$$
we have $\lambda(f_0^a) = 2$ [BK14],
- for the two-modal family of singularities in the $W_{1,0}$ class, that is singularities of the form
$$f_0^{a,b}(x, y) := x^4 + y^6 + (a + by)x^2y^3, \quad a, b \in \mathbb{C}, \quad a^2 \neq 4,$$
we have
$$\lambda(f_0^{a,b}) = \begin{cases} 1, & \text{if } a = 0 \text{ [BK14]} \\ \geqslant 2, & \text{for generic } a, b \text{ [GZ93]}, \end{cases}$$
- for specific homogenous singularities $f_0^d(x, y) := x^d + y^d$, $d \geqslant 2$, we have $\lambda(f_0^d) = \left[\frac{d}{2}\right]$ [BKW14],
- for homogeneous singularities of degree d with generic coefficients f_0 we have $\lambda(f_0) < \left[\frac{d}{2}\right]$ [BKW14]

In the present paper we consider a weaker problem: *compute the jump* $\lambda^{\mathrm{nd}}(f_0)$ *of* f_0 *over all non-degenerate deformations of* f_0 (i.e. the f_s in the deformations (f_s) of f_0 are non-degenerate singularities). Clearly, we always have $\lambda(f_0) \leqslant \lambda^{\mathrm{nd}}(f_0)$. Up to now, this problem has been studied only for plane curve singularities

- A. Bodin [Bod07] gave a formula for $\lambda^{\mathrm{nd}}(f_0)$ for f_0 convenient with its Newton polygon reduced to one segment,
- J. Walewska in [Wal13] generalized Bodin's results to the non-convenient case,
- the authors [BKW14] calculated all possible Milnor numbers of all non-degenerate deformations of homogenous singularities,
- J. Walewska [Wal10] proved that the *second non-degenerate jump of* f_0 satisfying Bodin's assumptions is equal to 1.

In this paper we want to pass to surface singularities ($n = 3$). We give a formula (more precisely: a simple algorithm) for $\lambda^{\mathrm{nd}}(f_0)$ in the case where f_0 is non-degenerate, convenient and has its Newton diagram reduced to one triangle, (see Figure 1) i.e. f_0 of the form
$$f_0(x, y, z) = ax^p + by^q + cz^r + \dots \quad (p, q, r \geqslant 2, \ \ abc \neq 0).$$

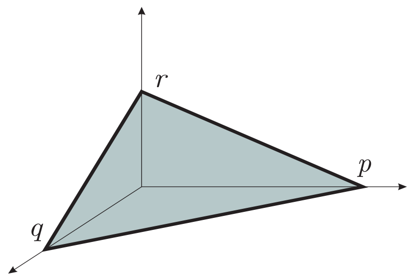

Fig. 1: The Newton diagram of $f_0(x,y,z) = ax^p + by^q + cz^r + \dots$

Moreover, for simplicity reasons, we will only consider the case of p, q, r being pairwise coprime integers. The general case of arbitrary p, q, r will be the topic of a next paper.

2. Non-degenerate singularities

In this Section we recall the notion of non-degenerate singularities. We restrict ourselves to surface singularities. All notions can easily be generalized to higher dimensions. Let

$$f_0(x,y,z) := \sum_{i,j,k\in\mathbb{N}} a_{ijk}x^iy^jz^k,$$

be a singularity. Let

$$\mathrm{supp(f_0) := \{(i,j,k) \in \mathbb{N}^3 : a_{ijk} \neq 0\}}$$

be the *support of* f_0. The *Newton polyhedron* $\Gamma_+(f_0)$ *of* f_0 is the convex hull of the set

$$\bigcup_{(i,j,k)\in\mathrm{supp(f_0)}} (i,j,k) + \mathbb{R}^3_+,$$

where $\mathbb{R}^3_+$ is the closed octant of $\mathbb{R}^3$ consisting of points with nonnegative coordinates. The boundary (in $\mathbb{R}^3$) of $\Gamma_+(f_0)$ is an unbounded polyhedron with a finite number of 2-dimensional faces, which are (not necessarily compact) polygons. The singularity f_0 is called *convenient* if $\Gamma_+(f_0)$ has some points in common with all three coordinate axes in $\mathbb{R}^3$. The set of compact faces (of all dimensions) of $\Gamma_+(f_0)$ constitutes the *Newton diagram of* f_0 and is denoted by $\Gamma(f_0)$. For each face $S \in \Gamma(f_0)$ we define a weighted homogeneous polynomial

$$(f_0)_S := \sum_{(i,j,k)\in S} a_{ijk}x^iy^jz^k.$$

We call the singularity f_0 *non-degenerate on* $S \in \Gamma(f_0)$ if the system of equations

$$\frac{\partial(f_0)_S}{\partial x}(x,y,z) = 0, \quad \frac{\partial(f_0)_S}{\partial y}(x,y,z) = 0, \quad \frac{\partial(f_0)_S}{\partial z}(x,y,z) = 0$$

has no solutions in $(\mathbb{C}^*)^3$; f_0 is *non-degenerate* (*in the Kouchnirenko sense*) if f_0 is non-degenerate on every face $S \in \Gamma(f_0)$.

Assume now that f_0 is convenient. We introduce the following notation:

- $\Gamma_-(f_0)$ – the compact polyhedron bounded by $\Gamma(f_0)$ and the three coordinate planes (labeled in a self-explanatory way as OXY, OXZ, OYZ); in other words, $\Gamma_-(f_0) := \overline{\mathbb{R}^3_+ \setminus \Gamma_+(f_0)}$,
- V – the volume of $\Gamma_-(f_0)$,
- P_1, P_2, P_3 – the areas of the two-dimensional faces of $\Gamma_-(f_0)$ lying in the planes OXY, OXZ, OYZ, respectively; e.g. P_1 is the area of the set $\Gamma_-(f_0) \cap \mathrm{OXY}$,
- W_1, W_2, W_3 – the lengths of the edges (= one-dimensional faces) of $\Gamma_-(f_0)$ lying in the axes OX, OY, OZ, respectively (see Figure 2).

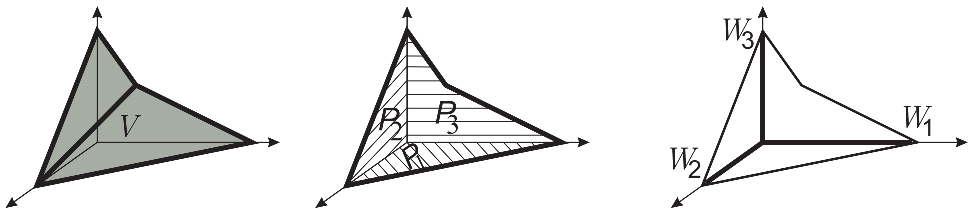

Fig. 2: Geometric meaning of volume V, areas P_i and lengths W_j.

We define the *Newton number* $\nu(f_0)$ of f_0 by

$$(\circ) \qquad \nu(f_0) := 3!V - 2!(P_1 + P_2 + P_3) + 1!(W_1 + W_2 + W_3) - 1.$$

The importance of $\nu(f_0)$ has its source in the celebrated *Kouchnirenko theorem*:

Theorem [Kou76] If f_0 is a convenient singularity, then

1. $\mu(f_0) \geqslant \nu(f_0)$,
2. if f_0 is non-degenerate then $\mu(f_0) = \nu(f_0)$.

Remark 2. The Kouchnirenko theorem is true in any dimension [Kou76].

3. Non-degenerate jump of Milnor numbers of singularities

Let $f_0 \in \mathcal{O}_3$ be a singularity. A deformation (f_s) of f_0 is called *non-degenerate* if f_s is non-degenerate for $s \neq 0$. The set of all non-degenerate deformations of the singularity f_0 will be denoted by $\mathcal{D}^{\mathrm{nd}}(f_0)$. *Non-degenerate jump* $\lambda^{\mathrm{nd}}(f_0)$ *of the singularity* f_0 is the minimal of non-zero jumps over all non-degenerate deformations of f_0, which means

$$\lambda^{\mathrm{nd}}(f_0) := \min_{(f_s) \in \mathcal{D}^{\mathrm{nd}}_0(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{\mathrm{nd}}(f_0)$ we denote all the non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$. Obviously

Proposition 3.1. *For each singularity f_0 we have the inequality*

$$\lambda(f_0) \leq \lambda^{\mathrm{nd}}(f_0).$$

In investigations concerning $\lambda^{\mathrm{nd}}(f_0)$ we may restrict our attention to non-degenerate f_0 because the non-degenerate jump for degenerate singularities can be found using the proposition below (cf. [Bod07, Lemma 5]). Let f_0^{nd} denote any non-degenerate singularity for which $\Gamma(f_0) = \Gamma(f_0^{\mathrm{nd}})$. Such singularities always exist.

Proposition 3.2. *If f_0 is degenerate then*

$$\lambda^{\mathrm{nd}}(f_0) = \begin{cases} \mu(f_0) - \mu(f_0^{\mathrm{nd}}), & \textit{if } \mu(f_0) - \mu(f_0^{\mathrm{nd}}) > 0 \\ \lambda^{\mathrm{nd}}(f_0^{\mathrm{nd}}), & \textit{if } \mu(f_0) - \mu(f_0^{\mathrm{nd}}) = 0 \end{cases}.$$

Proof. This follows from the fact that a generic small perturbation of coefficients of these monomials of f_0 which correspond to points belonging to $\bigcup \Gamma(f_0)$ (which are finite in number) give us non-degenerate singularities with the same Newton polyhedron as f_0. $\square$

Remark 3. By the Płoski theorem ([Pło90, Lemma 2.2], [Pło99, Theorem 1.1]), for degenerate plane curve singularities ($n = 2$) the second possibility in Proposition 3.2 is excluded.

A crucial rôle in the search for the formula for $\lambda^{\mathrm{nd}}(f_0)$ will be played by the monotonicity of the Newton number with respect to the Newton polyhedron. Namely, J. Gwoździewicz [Gwo08] and M. Furuya [Fur04] proved:

Theorem 3.3. (Monotonicity Theorem) *Let $f_0, \tilde{f}_0 \in \mathcal{O}_n$ be two convenient singularities such that $\Gamma_+(f_0) \subset \Gamma_+(\tilde{f}_0)$. Then $\nu(f_0) \geqslant \nu(\tilde{f}_0)$.*

By this theorem the problem of calculation of $\lambda^{\mathrm{nd}}(f_0)$ can be reduced to a purely combinatorial one. Namely, we define specific deformations of a convenient and non-degenerate singularity $f_0 \in \mathcal{O}_n$. Denote by J the set of integer points $\boldsymbol{i} = (i_1, \ldots, i_n) \neq 0$ lying in the closed domain bounded by coordinate hyperplanes $\{z_i = 0\}$ and the Newton diagram; in other words $J := \Gamma_-(f_0) \cap \mathbb{Z}^n$. Obviously, J is a finite set. For $\boldsymbol{i} = (i_1, \ldots, i_n) \in J$ we define the deformation $(f_s^{\boldsymbol{i}})_{s \in \mathbb{C}}$ of f_0 by the formula

$$f_s^{\boldsymbol{i}}(z_1, \ldots, z_n) := f_0(z_1, \ldots, z_n) + s z_1^{i_1} \ldots z_n^{i_n}.$$

Proposition 3.4. *For every $\boldsymbol{i} \in J$ the deformation $(f_s^{\boldsymbol{i}})$ of f_0 is convenient and non-degenerate for all sufficiently small $|s|$.*

Proof. See [Kou76] or [Oka79, Appendix]. □

Combining the Monotonicity Theorem with the above proposition we reach the conclusion that in order to find $\lambda^{\mathrm{nd}}(f_0)$ it is enough to consider only the non-degenerate deformations of the type $(f_s^{\boldsymbol{i}})$.

Theorem 3.5. *If f_0 is a convenient and non-degenerate singularity, then*

$$\lambda^{\mathrm{nd}}(f_0) = \min_{\boldsymbol{i}\in J_0} \lambda((f_s^{\boldsymbol{i}}))$$

where $J_0 \subset J$ is the set of these $\boldsymbol{i} \in J$ for which $\lambda^{\mathrm{nd}}((f_s^{\boldsymbol{i}})) > 0$.

Proof. By the Kouchnirenko theorem it suffices to consider non-degenerate deformations of f_0 of the form

$$(*) \qquad f_s(z_1, \ldots, z_n) = f_0(z_1, \ldots, z_n) + \sum_{\boldsymbol{i}\in J} a_{\boldsymbol{i}}(s) z^{\boldsymbol{i}},$$

where $a_{\boldsymbol{i}}(s)$ are holomorphic at $0 \in \mathbb{C}$ and $a_{\boldsymbol{i}}(0) = 0$. Then by the Monotonicity Theorem we may restrict the scope of deformations (3) to deformations with only one term added i.e. the deformations $(f_s^{\boldsymbol{i}})$ for $\boldsymbol{i} \in J_0$. □

Corollary 3.6. *If f_0 and $\tilde{f}_0$ are non-degenerate and convenient singularities and $\Gamma(f_0) = \Gamma(\tilde{f}_0)$ then $\lambda^{\mathrm{nd}}(f_0) = \lambda^{\mathrm{nd}}(\tilde{f}_0)$.*

4. An algorithm for $\lambda^{\mathrm{nd}}(f_0)$ in the case of one face Newton diagram of surface singularities

In this Section we give a simple algorithm for calculating $\lambda^{\mathrm{nd}}(f_0)$ provided that $f_0 \in \mathcal{O}_3$ is a convenient and non-degenerate singularity with one two-dimensional face of its Newton diagram. Let p, q, r be the first (i.e. nearest to the origin) points of $\Gamma_+(f_0)$ lying on the axes OX, OY and OZ, respectively. Then by Corollary 3.6 we may assume that

$$f_0(x, y, z) = x^p + y^q + z^r, \qquad p, q, r \geqslant 2.$$

By formula ($\circ$) we have $\mu(f_0) = (p-1)(q-1)(r-1)$. Moreover, without loss of generality we may also assume that

$$(\dagger) \qquad p \geqslant q \geqslant r.$$

Additionally, we demand that p, q, r are pairwise coprime

$$(**) \qquad \mathrm{GCD}(p, q) = \mathrm{GCD}(p, r) = \mathrm{GCD}(q, r) = 1.$$

By Theorem 3.5 we have to compare the jumps of deformations $(f_s^{\boldsymbol{i}})_{s\in\mathbb{C}}$, where $\boldsymbol{i} \in J$, i.e. $\boldsymbol{i}$ are integer points lying in the octant of $\mathbb{R}^3$ under the triangle with vertices $(p, 0, 0)$, $(0, q, 0)$, $(0, 0, r)$ (see Figure 1).

I. First we consider points in J lying on the axes. Using formula ($\circ$) and assumption ($\dagger$) we easily check that the axes-jump is realized by the deformation $(f_s^{(p-1,0,0)})$, i.e.

$$f_s^{(p-1,0,0)}(x,y,z) = x^p + y^q + z^r + sx^{p-1},$$

and the jump is equal to $(q-1)(r-1)$.

II. Now we consider points in J lying in coordinate planes. By the results of Bodin [Bod07] and Walewska [Wal10] we easily check that the minimal jumps on respective planes are realized by

   i. the deformation $(f^{(b_1,q-a_1,0)})$, where $a_1, b_1 \in \mathbb{Z}$ are such that $a_1p - b_1q = 1$ and $0 < a_1 < q$, $b_1 > 0$; this delivers the OXY-jump equal to $(r-1)$,
   ii. the deformation $(f_s^{(0,b_2,r-a_2)})$, where $a_2, b_2 \in \mathbb{Z}$ are such that $a_2q - b_2r = 1$ and $0 < a_2 < r$, $b_2 > 0$; this delivers the OYZ-jump equal to $(p-1)$,
   iii. the deformation $(f_s^{(b_3,0,p-a_3)})$, where $a_3, b_3 \in \mathbb{Z}$ are such that $a_3p - b_3r = 1$ and $0 < a_3 < p$, $b_3 > 0$; this delivers the OXZ-jump equal to $(q-1)$.

The above considerations imply that the jump realized by the points lying either in coordinate planes or on axes is equal to $(r-1)$.

III. Let us pass to the deformations $(f_s^{\boldsymbol{i}})$ for which the point $\boldsymbol{i}$ lies in the interior of the tetrahedron with vertices $(0,0,0)$, $(p,0,0)$, $(0,q,0)$, $(0,0,r)$. Any such point (α,β,γ) satisfies the conditions:

   (a) $0 < \alpha < p$, $0 < \beta < q$, $0 < \gamma < r$,
   (b) $\dfrac{\alpha}{p} + \dfrac{\beta}{q} + \dfrac{\gamma}{r} < 1$ or equivalently $\alpha qr + \beta pr + \gamma pq < pqr$.

   Moreover, the jump of the deformation $(f_s^{(\alpha,\beta,\gamma)})$ is equal to 6 times the volume of the tetrahedron with vertices $(p,0,0)$, $(0,q,0)$, $(0,0,r)$, (α,β,γ) i.e.

$$pqr - \alpha qr - \beta pr - \gamma pq.$$

Thus, we have reduced our original problem to a number theoretic one.

Problem 1. *Given pairwise coprime integers $p > q > r$ greater than 1. Find positive integers α, β, γ satisfying 0a and 0b for which the expression $pqr - \alpha qr - \beta pr - \gamma pq$ attains its positive minimum.*

In order to solve it, first notice that $\mathrm{GCD}(qr, pr, pq) = 1$. Consequently, there are integers a, b, c such that

$$(\ddagger) \qquad aqr + bpr + cpq = 1.$$

They can be obtained by the Euclid algorithm using the well-known associativity law: *for any integers u, v, w we have* $\mathrm{GCD}(u,v,w) = \mathrm{GCD}(\mathrm{GCD}(u,v),w)$. Notice that in any identity of the type (‡) it holds $abc \neq 0$. If we write $a = a'p + a''$, $0 \leqslant a'' < p$, then, by abuse of notation, we obtain yet another identity $aqr + bpr + cpq = 1$, but now $0 < a < p$. Next, we write $b = b'q - b''$, $0 < b'' < q$, and we use it to obtain a similar identity $aqr - bpr + cpq = 1$ in which $0 < a < p$ and $0 < b < q$. Notice that then $0 < |c| < r$. In fact, $|cpq| = |1 - aqr + bpr| \leqslant 1 + r|bp - aq| \leqslant 1 + r(pq - p - q) = pqr - pr - qr + 1 < pqr$. Thus, finally we have obtained the identity

$$(\square) \qquad aqr - bpr + cpq = 1, \text{ where } 0 < a < p,\ 0 < b < q,\ 0 < |c| < r.$$

Now we consider two cases:

1. $c < 0$. Then the triple $\alpha = p - a$, $\beta = b$, $\gamma = -c$ is the solution that we seek for. In fact, α, β, γ clearly satisfy 0a, moreover $pqr - \alpha qr - \beta pr - \gamma pq = aqr - bpr + cpq = 1$. This is the optimal value one can hope for, so the Problem is solved in this case. Hence $\lambda^{\mathrm{nd}}(f_0) = 1$ and the deformation $(f_s^{p-a,b,-c})$ realizes the jump 1.

2. $c > 0$. Under this condition, we claim that there is no point (α, β, γ) satisfying both 0a and 0b and for which the minimum in the Problem is equal to 1. In fact, if there existed such a point, then from the relation $pqr - \alpha qr - \beta pr - \gamma pq = 1$ we would get $(p - \alpha)qr - \beta pr - \gamma pq = 1$, which together with $(\square)$ would imply that $(p - (\alpha + a))qr = (\beta - b)pr + (\gamma + c)pq$. But since $\mathrm{GCD}(p, r) = \mathrm{GCD}(p, q) = 1$ and $|p - (\alpha + a)| < p$, this is only possible when $\alpha = p - a$. Hence, we would get $(\beta - b)r + (\gamma + c)q = 0$. Similarly, since $\mathrm{GCD}(r, q) = 1$ and $|\beta - b| < q$, we would obtain $\beta = b$ and consequently $\gamma = -c < 0$, contradictory to 0a.

The above observation means that in case (2) we must further continue our search for α, β, γ solving the Problem. Accordingly, we repeat the above reasoning for the identity

$$aqr + bpr + cpq = 2,$$

and so on up to

$$aqr + bpr + cpq = r - 2.$$

If in one of the above steps we find integers a, b, c such that

$$aqr + bpr + cpq = i_0,$$

where $1 \leqslant i_0 \leqslant r - 2$, $0 < a < p$, $-q < b < 0$ and $-r < c < 0$, then we stop the procedure and the triple $\alpha = p - a$, $\beta = -b$, $\gamma = -c$ solves the Problem with minimum equal to i_0. Hence, $\lambda^{\mathrm{nd}}(f_0) = i_0$ and the deformation $(f_s^{(p-a,-b,-c)})$ realizes this jump.

If the above search fails, we conclude that $\lambda^{\mathrm{nd}}(f_0) = r - 1$ because the deformation $(f_s^{(b_1, q-a_1, 0)})$, where $a_1 p - b_1 q = 1$, $0 < a_1 < q$, $0 < b_1$, realizes this jump.

We may sum up the above considerations in the following theorem.

Theorem 4.1. *Let $f_0 \in \mathcal{O}_3$ be a convenient and non-degenerate singularity with only one two-dimensional face in its Newton diagram. Assume that the vertices*

$(p,0,0)$, $(0,q,0)$, $(0,0,r)$ of this face are such that $p \geqslant q \geqslant r \geqslant 2$ and the numbers p, q, r are pairwise coprime. Then

$$\lambda^{nd}(f_0) = \begin{cases} i_0 & \begin{array}{l} \textit{if there exist integers } a,b,c \textit{ such that} \\ aqr + bpr + cpq = i_0,\ 1 \leqslant i_0 \leqslant r-2, \\ \ 0 < a < p,\ 0 < -b < q,\ 0 < -c < r,\ i_0 \textit{ -- minimal}, \end{array} \\ \\ r-1 & \textit{otherwise.} \end{cases}$$

Moreover, i_0 can be found algorithmically using only Euclid's algorithm.

Corollary 4.2. *Under the assumptions of Theorem 4.1, if $r = 2$ then $\lambda^{nd}(f_0) = 1$.*

Example. For $f_0(x,y,z) := x^{11} + y^6 + z^5$ we have $p = 11$, $q = 6$, $r = 5$ and

$$\begin{array}{rl} 7 \cdot qr - 5 \cdot pr + 1 \cdot pq = 1 & \text{– does not satisfy the conditions in the theorem} \\ 3 \cdot qr - 4 \cdot pr + 2 \cdot pq = 2 & \text{– does not satisfy the conditions in the theorem} \\ 10 \cdot qr - 3 \cdot pr - 2 \cdot pq = 3 & \text{– do satisfy the conditions in the theorem.} \end{array}$$

Hence, $\lambda^{\mathrm{nd}}(f_0) = 3$. This jump is realized by the deformation

$$f_s^{(1,3,2)}(x,y,z) := x^{11} + y^6 + z^5 + sxy^3z^2.$$

The minimal jump realized by the points lying either in coordinate planes or on axes is equal to $r - 1 = 4$.

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References

[AGZV85] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts*, volume 82 of *Monographs in Mathematics*, Birkhäuser Boston Inc., Boston, MA, 1985. Translated from the Russian by Ian Porteous and Mark Reynolds.

[BK14] Sz. Brzostowski and T. Krasiński, *The jump of the Milnor number in the X_9 singularity class*, Cent. Eur. J. Math. **12**, no. 3 (2014), 429–435.

[BKW14] Sz. Brzostowski, T. Krasiński, and J. Walewska, *Milnor numbers in deformations of homogeneous singularities*, ArXiv e-prints, http://arxiv.org/abs/ 1404.7704v1, April 2014.

[Bod07] A. Bodin, *Jump of Milnor numbers*, Bull. Braz. Math. Soc. (N.S.) **38**, no. 3 (2007), 389–396.

[Fur04] M. Furuya, *bound of Newton number*, Tokyo J. Math. **27** (2004), 177–186.

[GLS07] G.-M. Greuel, Ch. Lossen, and E. Shustin, *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer, Berlin 2007.

[Gwo08] J. Gwoździewicz, *Note on the Newton number*, Univ. Iagel. Acta Math. **46** (2008), 31–33.

[GZ93] S. M. Gusein-Zade, *On singularities from which an A_1 can be split off*, Funct. Anal. Appl. **27**, no. 1 (1993), 57–59.

[Kou76] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32**, no. 1 (1976), 1–31.

[Oka79] M. Oka, *On the bifurcation of the multiplicity and topology of the Newton boundary*, J. Math. Soc. Japan **31**, no. 3 (1979), 435–450.

[Pło90] A. Płoski, *Newton polygons and the Łojasiewicz exponent of a holomorphic mapping of* $\mathbf{C}^2$, Ann. Polon. Math. **51** (1990), 275–281.

[Pło99] A. Płoski, *Milnor number of a plane curve and Newton polygons*, Univ. Iagel. Acta Math. **37** (1999), 75–80. Effective methods in algebraic and analytic geometry (Bielsko-Biała 1997).

[Wal10] J. Walewska, *The second jump of Milnor numbers*, Demonstratio Math. **43**, no. 2 (2010), 361–374.

[Wal13] J. Walewska, *Jumps of Milnor numbers in families of non-degenerate and non-convenient singularities*, in: *Analytic and Algebraic Geometry*, Faculty of Mathematics and Computer Science, University of Łódź, Łódź 2013, 141–153.

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NIEZDEGENEROWANY SKOK LICZB MILNORA OSOBLIWOŚCI POWIERZCHNI

S t r e s z c z e n i e

Skok liczby Milnora izolowanej osobliwości f_0 jest najważniejszą niezerową różnicą między liczbami Milnora rozmaitości f_0 i jedną z jej deformacji (f_z). Znajdujemy wzór na skok w pewnej klasie osobliwości powierzchni w przypadku deformacji i niezdegenerowanych.

Słowa kluczowe: liczba Milnora, deformacja osobliwości osobliwość niezdegenerowana, wielościan Newtona

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Roman Vorobel

CONSTRUCTION OF ADJUSTABLE PARAMETERIZED ALGEBRAIC MODEL FOR GRAY LEVEL IMAGE PROCESSING

Summary

This article describes a construction of adjustable, parameterized algebraic model for processing of gray level images. Algebraic structures are constructed that generalize known algebraic model. The proposed method determines analytical expressions for the realization of arithmetic operations that simultaneously model the human perception of images in the presence of constant intensity light source. To improve the efficiency of the new algebraic structure its flexibility is provided by using a parameterization. The analytical expressions for the construction of algebraic structures on two intervals are obtained. Flexibility of built algebraic structures is demonstrated.

Keywords and phrases: algebraic structure, real vector space, image processing

1. Introduction

Algebraic structures are one of constructing means of mathematical models for image processing. It is caused by the fact that images are mostly meant for human analysis or for the decision making by automated systems. As the human visual system is characterized by the best properties of the image perception, this property is used as the base of the image processing methods. For today there are many fields, where pixels are represented by gray levels – it is medicine (the roentgenography and the computed axial tomography), the non-destructive quality testing of materials and products, microscopic investigations. And here the human image perception is based on the reaction of his visual system to the light influence. A well-known psycho-physical Weber-Fechner law states that the perception of the human visual system is proportional to the logarithm of the stimulus intensity [1]. This logarithmic relation became the base for the construction of different algebraic structures used in image processing.

Oppenheim [2, 3] initiated using of logarithmic models in 1965. He applied the logarithmic scale of the data representation, set in the interval $(0, \infty)$, described using the multiplication as the operation of the addition, division as the operation of the subtraction and raising to the power as multiplication by real scalar. Isomorphism was defined by logarithmic function, and inverse function was the exponential one. Oppenheim realized this model as homomorphic filtering, but he didn't use operations with gray levels for images. From the other side Jourlin and Pinoli in 1985 in [4–6] showed that it was possible to represent the image as light passing through translucent environment. They called this approach as Logarithmic Image Processing (LIP) and it was based on algebraic model, in which abstract gray level was a value from interval $(-\infty, M)$, where $M > 0$. There were defined such operations for set $U = (-\infty, M)$ of gray levels and $\forall u_1, u_2 \in U$:

addition

$$u_1 \oplus u_2 = u_1 + u_2 - \frac{u_1 u_2}{M},$$

subtraction

$$u_1 \Theta u_2 = M \frac{u_1 - u_2}{M - u_2},$$

negative element

$$\Theta = -\frac{M \cdot u_1}{M - u_1},$$

neutral element

$$e = 0 : u_1 \oplus e = e \oplus u_1,$$

scalar multiplication by $\beta > 0$:

$$\beta \otimes u_1 = M \left[1 - \left(1 - \frac{u_1}{M} \right)^{\beta} \right],$$

isomorphism ϕ:

$$R \to U, \quad \phi(x) = \ln \left(\frac{M}{M - x} \right)^{M}$$

and the inverse function $\phi^{-1}(y)$:

$$U \to R, \quad \phi^{-1}(y) = M \left[1 - \exp \left(\frac{y}{M} \right) \right].$$

One should notice, that developing homomorphic systems and Oppenheim generalized addition, Shvayster and Peleg suggested log-ratio approaches in [7, 8], considering images as elements in a vector space, in which operation $\bar{\oplus}$ of addition X and Y was defined by expression

$$X \bar{\oplus} Y = \psi^{-1}[\psi(X) + \psi(Y)], \tag{1}$$

multiplication $\bar{\otimes}$ by real scalar $\gamma \in R$

$$\gamma \bar{\otimes} X = \psi^{-1}[\gamma \psi(X)]. \tag{2}$$

It was shown in [9], that equations (1) and (2) became the basis of constructing of algebraic structures, where arithmetical operations are modelling logarithmic properties of the human vision system. Described structures have limited options, because they aren't flexible. Therefore the aim of the given article is a construction of adjustable parameterised algebraic models for gray level image processing, which will consider conditions of human image analysis. Firstly we will describe some known algebraic models, which can be parameterised, or have interesting properties. Then we will present new algebraic models, which are generalizing the well-known one, and are flexible in the practical application.

2. Selected algebraic LIP models

In [10, 11] was suggested parameterization of LIP model for $g_1, g_2 \in [0, M)$, $M > 0$ with such operations of the addition $\tilde{\oplus}$, subtraction $\tilde{\Theta}$ and the multiplication $\tilde{*}$:

$$g_1 \tilde{\oplus} g_2 = g_1 + g_2 - \frac{g_1 g_2}{\gamma(M)},$$

$$g_1 \tilde{\Theta} g_2 = k(M) \frac{g_1 - g_2}{k(M) - g_2 + \varepsilon},$$

$$g_1 \tilde{*} g_2 = \varphi^{-1}[\varphi(g_1) \cdot \varphi(g_2)],$$

with isomorphism

$$\varphi(g) = -\lambda(M) \cdot \ln^{\beta}\left(1 - \frac{g}{\lambda(M)}\right),$$

and the inverse function

$$\varphi^{-1}(g) = \lambda(M) \cdot \left[1 - \exp\left(\frac{-g}{\lambda(M)}\right)^{\frac{1}{\beta}}\right],$$

where $\gamma(M)$, $k(M)$ and $\lambda(M)$are linear function of the type $\gamma(M) = AM + B$; A, B – constant parameters; ε is very small constant; $\beta > 0$. However this created parameterized algebraic model doesn't provide symmetrical gray level processing of images. When processing the input and inverted images the processed input image will differ from inverted processing of inverted image. This drawback is present in another LIP model, named logarithmic-like image processing model [10] that uses such operations for grey levels of image $q_1, q_2 \in G = [0, M)$:

addition:

$$q_1 \hat{\oplus} q_2 = 1 - \frac{(1 - q_1)(1 - q_2)}{1 - q_1 q_2}$$

subtraction:

$$q_1 \hat{\Theta} q_2 = \frac{q_1 - q_2}{1 + q_1 \cdot q_2 - 2q_2}, \qquad q_1 \geq q_2$$

multiplication by scalar $\alpha > 0$

$$\alpha \hat{\otimes} q_1 = \frac{\alpha \cdot q_1}{1 + (\alpha - 1) q_1}.$$

More effective LIP model based on the algebraic structure was proposed by Patrascu in [12–14]. Developing the Shvayster and Peleg approach, being based on Oppenheim works [2, 3] as well as Jourlin and Pinoli [4–6] Patrascu considered the vector space of gray levels as set $E \in (-1,\ 1)$. For this set in [13–15] was built algebraic structure of the vector space $(E, \langle + \rangle_1, \langle \times \rangle_1)$ with operation of addition $\langle + \rangle_1$ and multiplication by scalar $\langle \times \rangle_1$. These arithmetical operations are described by such expressions:

for addition

$$u \langle + \rangle_1 v = \frac{u + v}{1 + uv}, \quad \forall u, v \in E, \tag{3}$$

for multiplication by real scalar

$$\alpha \langle \times \rangle_1 u = \frac{(1+u)^\alpha - (1-u)^\alpha}{(1+u)^\alpha + (1-u)^\alpha}, \quad \forall \alpha \in R, \tag{4}$$

for subtraction

$$u \langle - \rangle_1 v = \frac{u - v}{1 - uv}, \quad \forall u, v \in E. \tag{5}$$

Such a vector structure is characterized by an isomorphism

$$\varphi_1 :\ E \to R, \varphi_1(x) = \ln \frac{1 + x}{1 - x} \tag{6}$$

with the inverse function

$$\varphi_1^{-1} :\ R \to E, \varphi_1^{-1}(y) = \ln \frac{\exp(y) + 1}{\exp(y) - 1}.$$

Note that function $\varphi_1(x)$ (6) is additive generator for a special case of parametric Hamacher triangular s-norm for $x \in [0,\ 1]$ [16].

In [9, 17] were generalized the mentioned above LIP models based on algebraic structures of the vector space. It is shown in [18, 19] that the base of construction of such models and obtaining of analytical expressions for the implementation of addition, multiplication on the real scalar and subtraction operations serve generator functions of strict triangular s-norm [16, 20, 21]. Therefore described algebraic structures of the vector space are opening the possibility of constructing LIP models. They are constructed using generator functions of logarithmic type. It is a base of the fact that received arithmetical operations are modelling the properties of the Weber-Fechner law of human perception of light. Because of Weber-Fechner law is psychophysical, it is reflecting only the character of the reaction of the human visual system and is not valid in the wide range of the light intensity influence. It allows to exploit different logarithmic functions as generators and to construct corresponding algebraic structures of vector space for different LIP models. Known for today algebraic LIP models are not taking into account the possibility of the presence of the additional light source when modelling the human perception of the image. Therefore farther we will build the new algebraic models, which are taking into account the presence of the additional source of the permanent light when modelling the human perception of the image.

3. New algebraic LIP models

Isomorphism represents the function of human perception of the light in algebraic model. However, as follows from (1) and (2) and as shown in [9, 18, 19], knowing mapping function can design logarithmic type algebraic structure. Therefore, to build new algebraic LIP models, will take as a basis the algebraic model Patrascu (3), (4) [13–15]. In this first construct an algebraic model that reflects a light source in the perception of image rights and then add this feature to manage its properties – that make it flexible. Researchers in image processing use data presentation in the interval $(-1, 1)$ and the interval $(0, 1)$. So we construct algebraic structures for both intervals.

3.1. Adjustable parameterized algebraic LIP model

To build an adjustable algebraic model we use construction technology of logarithmic type algebraic structures described in [9, 18]. We will consider the set $E \in (-1,\, 1)$ and will build for it algebraic structure $(E, \langle + \rangle_2, \langle \times \rangle_2)$ of the vector space with operations of addition $\langle + \rangle_2$ and multiplication by scalar $\langle \times \rangle_2$. Therefore, for the modelling of the additional source of lighting by the algebraic structure we will present isomorphism as two-component function

$$\varphi_2(x) = a + \varphi_1(x) = a + \ln \frac{1+x}{1-x}. \tag{7}$$

On this basis we obtain for set E of gray level pixels such expression for arithmetic operations:

addition

$$x \langle + \rangle_2 y = \frac{(b-1)(1+xy) + (b+1)(x+y)}{(b-1)(x+y) + (b+1)(1+xy)}, \quad \forall x, y \in E, \tag{8}$$

where $a = \ln(b)$, $b > 0$,
subtraction

$$x \langle - \rangle_2 y = \frac{(b-1)(1-xy) + (b+1)(x-y)}{(b-1)(x-y) + (b+1)(1-xy)}, \quad \forall x, y \in E, \tag{9}$$

multiplication by real scalar $\alpha \in R$

$$\alpha \langle \times \rangle_2 x = \frac{b^{\alpha-1}(1+x)^{\alpha} - (1-x)^{\alpha}}{b^{\alpha-1}(1+x)^{\alpha} + (1-x)^{\alpha}}. \tag{10}$$

Inverse function $\varphi_2^{-1}(x)$ is defined as

$$\varphi_2^{-1}(x) = \frac{\exp(x-a)-1}{\exp(x-a)+1}.$$

Comparing (3) to (8), (4) to (10) and (5) to (9) we see that when $b = 1$ these expressions coincide and Patrascu algebraic model is a partial case of the proposed model.

If we will consider interval $(0, 1)$ as the range of variation of gray levels of pixels $x, y \in (0,\ 1) = G$, then from formulas (8)–(10) we obtain the following expressions for the arithmetic operations for new tuned algebraic structures:

for addition

$$x\langle\hat{+}\rangle_2 y = 0,5 + 0,5\frac{(b-1)(1+x_1y_1)+(b+1)(x_1+y_1)}{(b-1)(x_1+y_1)+(b+1)(1+x_1y_1)}, \tag{11}$$

where $x_1 = 2x - 1$, $y_1 = 2y - 1$;

for subtraction

$$x\langle\hat{-}\rangle_2 y = 0,5 + 0,5\frac{(b-1)(1-x_1y_1)+(b+1)(x_1-y_1)}{(b-1)(x_1-y_1)+(b+1)(1-x_1y_1)},$$

for multiplication by real scalar $\alpha \in R$

$$\alpha\langle\hat{\times}\rangle_2 x = 0,5 + 0,5\frac{b^{\alpha-1}(1+x_1)^\alpha-(1-x_1)^\alpha}{b^{\alpha-1}(1+x_1)^\alpha+(1-x_1)^\alpha}. \tag{12}$$

The set E with operations $\langle+\rangle_2$ of addition (8) and the operation of multiplication $\langle\times\rangle_2$ by real scalar (10) form a real vector space, as the set G with operations $\langle\hat{+}\rangle_2$ of addition (11) and the operation $\langle\hat{\times}\rangle_2$ of multiplication by real scalar (12).

3.2. Flexible adjustable parameterized algebraic LIP model

To create adjustable algebraic model the flexible one, we use the ability to control the change character of the argument value of function $\varphi_2(x)$ (7) applying a power transformation and then receiving as x^β, where $\beta > 0$.

In this way we get a new function

$$\varphi_3(x) = b + \ln\frac{1 + sign(x)\cdot|x|^\beta}{1 - sign(x)\cdot|x|^\beta}.$$

Then, we use the similar technology of constructing logarithmic type algebraic structures described in [9, 18]. We will consider the set $E \in (-1,\ 1)$ and build it's algebraic structure $(E, \langle+\rangle_3, \langle\times\rangle_3)$ of the vector space with operation of addition $\langle+\rangle_3$ and multiplication by scalar $\langle\times\rangle_3$.

On this basis we obtain for set E of gray level pixels such expression for arithmetic operations:

addition

$$x\langle+\rangle_3 y = sign(s_1)\cdot\left(\frac{\exp|s_1|-1}{\exp|s_1|+1}\right)^{\frac{1}{\beta}},\ \ \forall x, y \in E, \tag{13}$$

where

$$s_1 = \ln(c \cdot x_2 \cdot y_2), b = \ln(c);$$

$$x_2 = \frac{1 + sign(x) \cdot |x|^\beta}{1 - sign(x) \cdot |x|^\beta}, y_2 = \frac{1 + sign(y) \cdot |y|^\beta}{1 - sign(y) \cdot |y|^\beta},$$

subtraction

(14) $$x \langle - \rangle_3 y = sign(s_2) \cdot \left(\frac{\exp |s_2| - 1}{\exp |s_2| + 1} \right)^{\frac{1}{\beta}}, \ \forall x, y \in E,$$

where

$$s_2 = \ln \left(\frac{c \cdot x_2}{y_2} \right);$$

multiplication by real scalar $k \in R$

(15) $$k \langle \times \rangle_3 x = sign(x_k) \cdot \left(\frac{\exp |x_k| - 1}{\exp |x_k| + 1} \right)^{\frac{1}{\beta}},$$

where

$$x_k = b \cdot (k-1) + k \ln(x_2) = (k-1) \ln(c) + k \ln(x_2) = \ln(c^{k-1} x_2^k).$$

Inverse function $\varphi_3^{-1}(x)$ is defined as

$$\varphi_3^{-1}(x) = sign(x - b) \cdot \left[\frac{\exp(x - b) - 1}{\exp(x - b) + 1} \right]^{\frac{1}{\beta}}.$$

When $\beta = 1$ new algebraic model (13)–(15) corresponds to adjustable algebraic model (8)–(10).

If we will consider interval $(0, 1)$ as the range of variation of gray levels of pixels $x, y \in (0,\, 1) = G$, then from formulas (13)–(15) we obtain the following expressions for the arithmetic operations for new tuned algebraic structures:

addition

(16) $$x \langle \hat{+} \rangle_3 y = 0,5 + 0,5 \cdot sign(s_3) \cdot \left(\frac{\exp |s_3| - 1}{\exp |s_3| + 1} \right)^{\frac{1}{\beta}},$$

where

$$s_3 = \ln(c \cdot x_3 \cdot y_3);$$

$$x_3 = \frac{1 + sign(x_1) \cdot |x_1|^\beta}{1 - sign(x_1) \cdot |x_1|^\beta}, y_3 = \frac{1 + sign(y_1) \cdot |y_1|^\beta}{1 - sign(y_1) \cdot |y_1|^\beta},$$

$$x_1 = 2x - 1, y_1 = 2y - 1,$$

subtraction

(17) $$x \langle \hat{-} \rangle_3 y = 0,5 + 0,5 \cdot sign(s_4) \cdot \left(\frac{\exp |s_4| - 1}{\exp |s_4| + 1} \right)^{\frac{1}{\beta}},$$

where

$$s_4 = \ln(\frac{c \cdot x_3}{y_3});$$

multiplication by real scalar $k \in R$

$$k\langle\hat{\times}\rangle_3 x = 0,5 + 0,5 \cdot sign(x_4) \cdot \left(\frac{\exp|x_4| - 1}{\exp|x_4| + 1}\right)^{\frac{1}{\beta}}, \tag{18}$$

where

$$x_4 = b(k-1) + k\ln(x_3) = \ln(c)(k-1) + k\ln(x_3) = \ln(c^{k-1} \cdot x_3^k).$$

The set E with operations $\langle+\rangle_3$ of addition (13) and the operation of multiplication $\langle\times\rangle_3$ by real scalar (15) form a real vector space, as the set G with operations $\langle\hat{+}\rangle_3$ of addition (16) and the operation $\langle\hat{\times}\rangle_3$ of multiplication by real scalar (18).

4. Results

New algebraic structures proposed in Section 3 make possible to simulate the presence of a light source by the human visual system perception of images through the use of constant component b in expressions that describe arithmetic operations (8)–(12) and (13)–(18). The use of control parameter β makes the new algebraic structure flexible. The Fig. 1 shows the first hyperplane, which is formed by the addition function (13) for the values $A = 3$ and $\beta = 0,7$, as in Fig. 2 – for the values $A = 0,7$ and $\beta = 1,5$.

These figures confirm adjustability of built algebraic structure and flexibility through its ability to change the values of control parameter β.

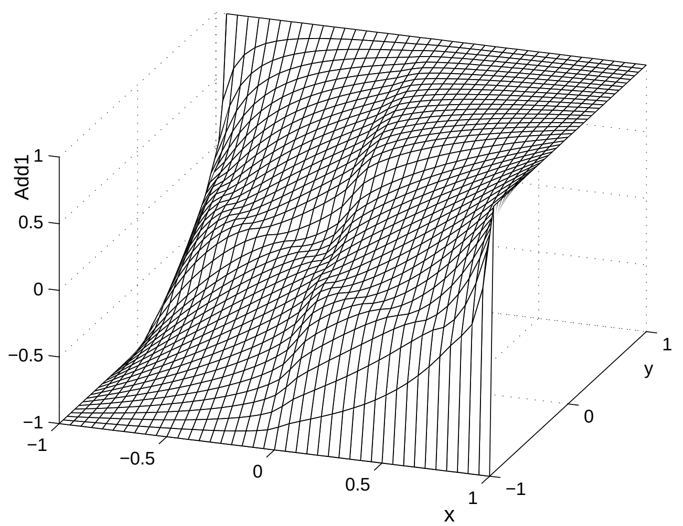

Fig. 1: Addition function (13) as the hyperplane Add1 (a) with values $A = 3$ and $\beta = 0,7$.

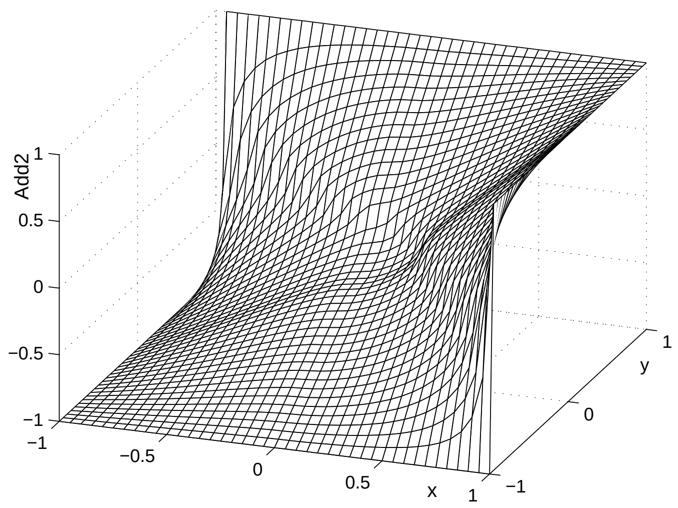

Fig. 2: Addition function (13) as the hyperplane Add2 (b) with values $A = 0,7$ and $\beta = 1,5$.

5. Conclusions

Application of built algebraic structures improves the efficiency of image processing by more accurate and precize modelling of human visual image analysis in the presence of light source of constant intensity. Parameterization of this structure offers opportunities of adaptive settings of such structures for better image processing.

References

[1] G. T. Fechner, *Elements of Psychophysics.* Vol. 1. Holt, Rinehart & Winston, New York 1960.

[2] A. V. Oppenheim, *Superposition in a class of non-linear system*, Tech. Rep. 432. Research Laboratory of Electronics M. I. T., Cambridge Ma. 1965.

[3] A. V. Oppenheim, *Generalized superposition*, Information and Control **11**, no. 5&6 (1967), 528–536.

[4] M. Jourlin and J.-C. Pinoli, *A model for logarithmic image processing*, Journal of Microscopy **149**, no. 1 (1988), 21–35.

[5] M. Jourlin and J.-C. Pinoli, *Logarithmic image processing*, Advances in Imaging and Electron Physics **115** (2001), 129–196.

[6] J.-C. Pinoli, *A general comparative study of the multiplicative homomorphic, log-ratio and logarithmic image processing approaches*, Signal Processing **58**, no. 1 (1997), 11–45.

[7] H. Shvayster and S. Peleg, *Pictures as elements in a vector spaces*, Proc. IEEE Conf. Comput. Vision Pattern Recogn, Washington 1983, 442–446.

[8] H. Shvayster and S. Peleg, *Inversion on picture operators*, Pattern Recognition Letters **5**, no. 1 (1987), 43–61.

[9] R. A. Vorobel, *Logarithmic image processing*, Naukova Dumka, Kyiv 2012.

[10] K. Panetta, E. Wharton, and S. Agaian, *Parameterization of Logarithmic Image Processing Models*, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics **41**, no. 2 (2011), 460–473.

[11] K. Panetta, Yicong Zhou, S. Agaian, Hongwei Jia, *Nonlinear unsharp masking for mammogram enhancement*, IEEE Trans. on Inform. Technol. in Biomedicine **15**, no. 6 (2011), 918–928.

[12] C. Vertan, A. Oprea, C. Florea, and L. Florea, *A pseudo-logarithmic image processing framework for edge detection*, Lecture Notice Computer Sciences **5259** (2008), 637–644.

[13] V. Patrascu and V. Buzuloiu. *A mathematical model for logarithmic image processing*, The 5th World Multi-Conference on Systemics, Cybernetics and Informatics, Orlando, USA, **13** (2001), 117–122.

[14] V. Patrascu and I. Voicu, *An algebraical model for gray level images*, Proc. of the 7th International Conference, Exhibition on Optimization of Electrical and Electronic Equipment, Brasov, Romania, (2000), 809–812.

[15] V. Patrascu, *Gray level image enhancement method using the logarithmic model*, Acta Tehnica Napocensis **44**, no. 2 (2003), 39–50.

[16] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, Kluwer Acad Publ., Dordrecht 2000.

[17] R. A. Vorobel, *Logarithmic image processing. Part 2: Generalized model*, Information Extraction and Processing **31**, no. 107 (2009), 36–46.

[18] R. A. Vorobel, *Construction of logarithmic type algebras*, Information Extraction and Processing, **32** no. 108 (2010), 131–143.

[19] R. Vorobel, *Logarithmic type image processing algebras*, 2010 International Kharkov Symposium on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves. Kharkov, Ukraine, (2010), Session C16. IEEE Catalog Number: CFP10780–CDR.

[20] C. Alsina, M. J. Frank, and B. Schweizer, *Associative functions: triangular norms and copulas*, World Scientific, Hackensack, London, Singapore, 2006.

[21] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation functions*, Cambridge University Press, Cambridge 2009.

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KONSTRUOWANIE NASTAWIALNEGO PARAMETRYZOWANEGO MODELU ALGEBRAICZNEGO DLA OPRACOWANIA OBRAZU O WIELU POZIOMACH SZAROŚCI

S t r e s z c z e n i e

W niniejszej pracy opisano konstruowanie nastawialnego parametryzowanego modelu algebraicznego przeznaczonego dla opracowania obrazów rastrowych. Opracowano struktury algebraiczne uogólniające znany model algebraiczny. Opisana została metoda otrzymania w postaci analitycznej wzorów dla realizacji operacji arytmetycznych, które jednocześnie modelują percepcję obrazu człowiekiem przy obecności źródła,a światła o stałym natężeniu. W celu poprawy wydajności nowej struktury algebraicznej zapewniono jej elastyczność przez użycie parametryzacji. Również zostały otrzymane wzory analityczne dla konstruowania struktur algebraicznych na dwóch przedziałach. Zademonstrowano elastyczność nowych struktur algebraicznych.

Słowa kluczowe: struktury algebraiczne, przestrzeń wektorowa liczb rzeczywistych, przetwarzanie obrazów

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Cristina Flaut and Vitalii Shpakivskyi

SOME REMARKS ABOUT FIBONACCI ELEMENTS IN AN ARBITRARY ALGEBRA

Summary

In this paper, we prove some relations between Fibonacci elements in an arbitrary algebra. Moreover, we define imaginary Fibonacci quaternions and imaginary Fibonacci octonions and we prove that always three arbitrary imaginary Fibonacci quaternions are linear independents and the mixed product of three arbitrary imaginary Fibonacci octonions is zero.

Keywords and phrases: Fibonacci quaternions, Fibonacci octonions, Fibonacci elements

1. Introduction

Fibonacci elements over some special algebras were intensively studied in the last time in various papers, as for example: [1]– [13]. All these papers studied properties of Fibonacci elements in complex numbers, or in quaternions and octonions, or in generalized Quaternion and Octonion algebras, or studied dual vectors or dual Fibonacci quaternions.

In this paper, we will prove that some of the obtained identities can be obtained over an arbitrary algebras. We introduce the notions of imaginary Fibonacci quaternions and imaginary Fibonacci octonions and we prove, using the structure of the quaternion algebras and octonion algebras, that three arbitrary imaginary Fibonacci quaternions are linear dependents and the mixed product of three arbitrary imaginary Fibonacci octonions is zero. For other details, properties and applications regarding quaternion algebras and octonion algebras, the reader is referred, for example, to [15], [14].

2. Fibonacci elements in an arbitrary algebra

Let A be a unitary algebra over K ($K=\mathbb{R},\mathbb{C}$) with a basis $\{e_0=1,e_1,e_2,...,e_n\}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the Fibonacci sequence

$$f_n=f_{n-1}+f_{n-2},\ n\geq 2,\ f_0=0,\ f_1=1.$$

In algebra A, we define the Fibonacci element as follows:

$$F_m=\sum_{k=0}^{n} f_{m+k}\, e_k.$$

Proposition 2.1. *With the above notations, the following relations hold:*

1) $F_{m+2}=F_{m+1}+F_m$;

2) $\sum\limits_{i=1}^{p} F_i=F_{p+2}-F_2$.

Proof. 1)

$$\begin{aligned}F_{m+1}+F_m&=\sum_{k=0}^{n}f_{m+k+1}e_k+\sum_{k=0}^{n}f_{m+k}e_k=\sum_{k=0}^{n}(f_{m+k+1}+f_{m+k})e_k\\&=\sum_{k=0}^{n}f_{m+k+2}\,e_k=F_{m+2}.\end{aligned}$$

2)

$$\begin{aligned}\sum_{i=1}^{p}F_i=F_1+F_2+...+F_p&==\sum_{k=0}^{n}f_{k+1}\,e_k+\sum_{k=0}^{n}f_{k+2}\,e_k+...+\sum_{k=0}^{n}f_{k+p}\,e_k\\&=e_0\,(f_1+...+f_p)+e_1\,(f_2+...+f_{p+1})+e_2\,(f_3+...+f_{p+2})+...\\&\quad+e_n\,(f_{k+n}+...+f_{p+n})\\&=e_0\,(f_{p+2}-1)+e_1\,(f_{p+3}-1-f_1)+e_2\,(f_{p+4}-1-f_1-f_2)\\&\quad+e_3\,(f_{p+5}-1-f_1-f_2-f_3)+...\\&\quad+e_n\,(f_{p+n+2}-1-f_1-f_2-...-f_n)=F_{p+2}-F_2.\end{aligned}$$

We used the identity $\sum\limits_{i=1}^{p}f_i=f_{p+2}-1$ (for usual Fibonacci numbers) and $1+f_1+f_2+...+f_n=f_{n+2}$. □

Remark 2.2. The equalities 1, 2 from the above proposition generalize the corresponding formulae from [2, 7–9]

Proposition 2.3. *We have the following formula (Binet formula):*

$$F_m = \frac{\alpha^* \alpha^m - \beta^* \beta^m}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\alpha^* = \sum_{k=0}^{n} \alpha^k e_k$, $\beta^* = \sum_{k=0}^{n} \beta^k e_k$.

Proof. Using the formula for the real Fibonacci numbers, $f_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$, we obtain
$F_m = \sum_{k=0}^{n} f_{m+k} e_k = \frac{\alpha^m - \beta^m}{\alpha - \beta} e_0 + \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} e_1 + \frac{\alpha^{m+2} - \beta^{m+2}}{\alpha - \beta} e_2 + \dots$
$\dots + \frac{\alpha^{m+n} - \beta^{m+n}}{\alpha - \beta} e_n = \frac{a^m}{\alpha - \beta} \left(e_0 + \alpha e_1 + \alpha^2 e_2 + \dots + \alpha^n e_n \right) +$
$+ \frac{\beta^m}{\alpha - \beta} \left(e_0 + \beta e_1 + \beta^2 e_2 + \dots + \beta^n e_n \right) = \frac{\alpha^* \alpha^m - \beta^* \beta^m}{\alpha - \beta}$. □

Remark 2.4. The above result generalizes the Binet formulae from the papers [1, 2, 6–9].

Theorem 2.5. *The generating function for the Fibonacci number over an algebra is of the form*

$$G(t) = \frac{F_0 + (F_1 - F_0)\, t}{1 - t - t^2}.$$

Proof. We consider the generating function of the form

$$G(t) = \sum_{m=0}^{\infty} F_m t^m.$$

We consider the product
$G(t) \left(1 - t - t^2 \right) = \sum_{m=0}^{\infty} F_m t^m = \sum_{m=0}^{\infty} F_m t^m - \sum_{m=0}^{\infty} F_m t^{m+1} - \sum_{m=0}^{\infty} F_m t^{m+2} =$
$= F_0 + F_1 t + F_2 t^2 + F_3 t^3 + \dots - F_0 t - F_1 t^2 - F_2 t^3 - \dots -$
$- F_0 t^2 - F_1 t^3 - F_2 t^4 - \dots = F_0 + (F_1 - F_0)\, t.$ □

Remark 2.6. The above Theorem generalizes results from the papers [1, 2, 6–8].

Proposition 2.7.

$$F_{-m} = (-1)^{m+1} f_m F_1 + (-1)^m f_{m+1} F_0.$$

Proof. We use induction. For $m = 1$, we obtain $F_{-1} = f_1 F_1 - f_2 F_0$, which is true. Now, we assume that it is true for an arbitrary integer k

$$F_{-k} = (-1)^{k+1} f_k F_1 + (-1)^k f_{k+1} F_0$$

For $k + 1$, we obtain
$F_{-(k+1)} = (-1)^{k+2} f_{k+1} F_1 + (-1)^{k+1} f_{k+2} F_0 = (-1)^k f_k F_1 + (-1)^k f_{k-1} F_1 +$
$+ (-1)^{k-1} f_{k+1} F_0 + (-1)^{k-1} f_k F_0 = F_{-(n-1)} - F_{-n}.$
Therefore, this statement is true. □

Theorem 2.8. (**Cassini identity**). *With the above notations, we have the following formula*

$$F_{m-1}F_{m+1}-F_m^2=(-1)^m\left(F_{-1}F_1-F_0^2\right).$$

Proof. We consider
$F_{m-1}=f_{m-1}e_0+f_me_1+f_{m+1}e_2+f_{m+2}e_3+...+f_{m+n-1}e_n,$
$F_{m+1}=f_{m+1}e_0+f_{m+2}e_1+f_{m+3}e_2+f_{m+4}e_3+...+f_{m+n+1}e_n,$
$F_m=f_me_0+f_{m+1}e_1+f_{m+2}e_2+f_{m+2}e_3+...+f_{m+n}e_n.$

We compute

$$F_{m-1}F_{m+1}=\Big[f_{m-1}f_{m+1}e_0^2+f_{m-1}f_{m+2}e_0e_1+f_{m-1}f_{m+3}e_0e_2+$$
$$+f_{m-1}f_{m+4}e_0e_3+\ldots+f_{m-1}f_{m+n+1}e_0e_n\Big]+\Big[f_mf_{m+1}e_1e_0+f_mf_{m+2}e_1^2+$$
$$+f_mf_{m+3}e_1e_2+f_mf_{m+4}e_1e_3+\ldots+f_mf_{m+n+1}e_1e_n\Big]+\Big[f_{m+1}^2e_2e_0+$$
$$+f_{m+1}f_{m+2}e_2e_1+f_{m+1}f_{m+3}e_2^2+f_{m+1}f_{m+4}e_1e_3+\ldots+f_{m+1}f_{m+n+1}e_2e_n\Big]+$$
$$+\Big[f_{m+2}f_{m+1}e_3e_0+f_{m+2}^2e_3e_1+f_{m+2}f_{m+3}e_3e_2+f_{m+2}f_{m+4}e_3^2+\ldots$$
$$\ldots+f_{m+2}f_{m+n+1}e_3e_n\Big]+\ldots+\Big[f_{m+n-1}f_{m+1}e_ne_0+f_{m+n-1}f_{m+2}e_ne_1+$$
$$f_{m+n-1}f_{m+3}e_ne_2+f_{m+n-1}f_{m+4}e_ne_3+\ldots+f_{m+n-1}f_{m+n+1}e_n^2\Big].$$

Now, we compute

$$F_m^2=\Big[f_m^2e_0^2+f_mf_{m+1}e_0e_1+f_mf_{m+2}e_0e_2+f_mf_{m+3}e_0e_3+\ldots$$
$$\ldots+f_mf_{m+n}e_0e_n\Big]+\Big[f_{m+1}f_me_1e_0+f_{m+1}^2e_1^2+f_{m+1}f_{m+2}e_1e_2+$$
$$+f_{m+1}f_{m+3}e_1e_3+...+f_{m+1}f_{m+n}e_1e_n\Big]+\Big[f_{m+2}f_me_2e_0+f_{m+2}f_{m+1}e_2e_1+$$
$$+f_{m+2}^2e_2^2+f_{m+2}f_{m+3}e_2e_3+\ldots+f_{m+2}f_{m+n}e_2e_n\Big]+\Big[f_{m+2}f_me_2e_0+$$
$$+f_{m+2}f_{m+1}e_2e_1+f_{m+2}^2e_2^2+f_{m+2}f_{m+3}e_2e_3+...+f_{m+2}f_{m+n}e_2e_n\Big]+$$
$$+\Big[f_{m+3}f_me_3e_0+f_{m+3}f_{m+1}e_3e_1+f_{m+3}f_{m+2}e_3e_2+f_{m+3}^2e_3^2+\ldots$$
$$\ldots+f_{m+3}f_{m+n}e_3e_n\Big]+...+\Big[f_{m+n}f_me_ne_0+f_{m+n}f_{m+1}e_ne_1+f_{m+n}f_{m+2}e_ne_2+$$
$$+f_{m+n}f_{m+3}e_ne_3+...+f_{m+n}^2e_n^2\Big].$$

We compute the difference

$$F_{m-1}F_{m+1}-F_m^2=e_0\Big[e_0\left(f_{m-1}f_{m+1}-f_m^2\right)+e_1\left(f_{m-1}f_{m+2}-f_mf_{m+1}\right)+\ldots$$
$$\ldots+e_n\left(f_{m-1}f_{m+n+1}-f_mf_{m+n}\right)\Big]+e_1\Big[e_0\left(f_mf_{m+1}-f_{m+1}f_m\right)+$$
$$+e_1\left(f_mf_{m+2}-f_{m+1}^2\right)+\ldots+e_n\left(f_mf_{m+n+1}-f_{m+1}f_{m+n}\right)\Big]+$$
$$+e_2\Big[e_0\left(f_{m+1}^2-f_{m+2}f_m\right)+e_1\left(f_{m+1}f_{m+2}-f_{m+2}f_{m+1}\right)+\ldots$$
$$\ldots+e_n\left(f_{m+1}f_{m+n+1}-f_{m+2}f_{m+n}\right)\Big]+e_3\Big[e_0\left(f_{m+2}f_{m+1}-f_{m+3}f_m\right)+$$
$$+e_1\left(f_{m+2}^2-f_{m+3}f_{m+1}\right)+\ldots+e_n\left(f_{m+2}f_{m+n+1}-f_{m+3}f_{m+n}\right)\Big]+\ldots+$$

$+e_n\Big[e_0\left(f_{m+n-1}f_{m+1}-f_{m+n}f_m\right)+e_1\left(f_{m+n-1}f_{m+2}-f_{m+n}f_{m+1}\right)+\ldots$
$\ldots+e_n\left(f_{m+n-1}f_{m+n+1}-f_{m+n}^2\right)\Big].$

Using formula $f_i f_j - f_{i+k}f_{j-k} = (-1)^{j-k} f_{i+k-j}f_k$ (see [16, p. 87], formula 2) and the identities $f_1 = 1, f_{-m} = (-1)^{m+1} f_m$ (see [16, p. 84]), we obtain

$$F_{m-1}F_{m+1}-F_m^2 = e_0\left(-1\right)^{m+1}\Big[e_0f_1+e_1f_2+e_2f_3+\ldots+e_nf_{n+1}\Big]+$$
$$+e_1\left(-1\right)^{m+1}\Big[e_0f_0+e_1f_1+e_2f_2+\ldots+e_nf_n\Big]+$$
$$+e_2\left(-1\right)^{m}\Big[e_0f_{-1}+e_1f_0+e_2f_1+\ldots+e_nf_{n-1}\Big]+$$
$$+e_3\left(-1\right)^{m}\Big[e_0f_{-2}+e_1f_{-1}+e_2f_0+\ldots+e_nf_{n-2}\Big]+\ldots+$$
$$+\left(-1\right)^{m+n}e_n\Big[e_0f_{-n+1}+e_1f_{-n+2}+e_2f_{-n+3}+\ldots+e_nf_1\Big]=$$
$$=\left(-1\right)^m\Big(e_0F_1-e_1F_0+e_2F_{-1}-e_3F_{-2}+\ldots+\left(-1\right)^n e_nF_{-n+1}\Big).$$

Using Proposition 2.7, we have

$$F_{m-1}F_{m+1}-F_m^2 = \left(-1\right)^m\Big[e_0F_1-e_1F_0+e_2\left(F_1-F_0\right)-e_3\left(2F_0-F_1\right)+$$
$$+e_4\left(2F_1-3F_0\right)-e_5\left(-3F_1+5F_0\right)+\ldots$$
$$\ldots+e_n\left(-1\right)^n\Big(\left(-1\right)^n f_{n-1}F_1+\left(-1\right)^{n-1}f_nF_0\Big)\Big]=$$
$$=\left(-1\right)^m\Big[(e_0f_{-1}+e_1f_0+e_2f_1+...+e_nf_{n-1})F_1-$$
$$-\left(f_0e_0+f_1e_1+f_2e_2+...+f_ne_n\right)F_0\Big]=\left(-1\right)^m\Big[F_{-1}F_1-F_0^2\Big].$$

The theorem is now proved. □

Remark 2.9. i) Similarly, we can prove an analogue of Cassini's formula:

$$F_{m+1}F_{m-1}-F_m^2=\left(-1\right)^m\left[F_1F_{-1}-F_0^2\right].$$

ii) Theorem 2.8 generalizes Cassini's formula for all real algebras.

iii) If the algebra A is algebra of the real numbers $\mathbb{R}$, we have $F_m = f_m$. From the above theorem, it results that

$$f_{m+1}f_{m-1}-f_m^2=\left(-1\right)^m\left[f_1f_{-1}-f_0^2\right]=\left(-1\right)^m,$$

which it is the classical Cassini's identity.

3. Imaginary Fibonacci quaternions and imaginary Fibonacci octonions

Let $\mathbb{H}\left(\alpha,\beta\right)$ be the generalized real quaternion algebra, the algebra of the elements of the form

$$a = a_1\cdot 1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k},$$

where

$$a_i\in\mathbb{R}, \mathbf{i}^2=-\alpha, \mathbf{j}^2=-\beta, \quad \mathbf{k}=\mathbf{ij}=-\mathbf{ji}.$$

We denote by $\mathbf{t}(a)$ and $\mathbf{n}(a)$ the trace and the norm of a real quaternion a. The norm of a generalized quaternion has the following expression $\mathbf{n}(a) = a_1^2 + \alpha a_2^2 + \beta a_3^2 + \alpha\beta a_4^2$ and the trace is $\mathbf{t}(a) = 2a_1$. It is known that for $a \in \mathbb{H}(\alpha, \beta)$, we have $a^2 - \mathbf{t}(a)\, a + \mathbf{n}(a) = 0$. The quaternion algebra $\mathbb{H}(\alpha, \beta)$ is a *division algebra* if for all $a \in \mathbb{H}(\alpha, \beta)$, $a \neq 0$, we have $\mathbf{n}(a) \neq 0$, otherwise $\mathbb{H}(\alpha, \beta)$ is called a *split algebra.*

Let $\mathbb{O}(\alpha, \beta, \gamma)$ be a generalized octonion algebra over $\mathbb{R}$, with basis $\{1, e_1, ..., e_7\}$, the algebra of the elements of the form $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$ and the multiplication given in the following table:

Table 1.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-\alpha$	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	e_2	$-e_3$	$-\beta$	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	e_3	αe_2	$-\beta e_1$	$-\alpha\beta$	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-\gamma$	γe_1	γe_2	γe_3
e_5	e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	$-\alpha\gamma$	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	$-\beta\gamma$	$-\beta\gamma e_1$
e_7	e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	$-\alpha\beta\gamma$

The algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is non-commutative and non-associative.

If

$$a \in \mathbb{O}(\alpha, \beta, \gamma), \quad a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$$

then

$$\bar{a} = a_0 - a_1e_1 - a_2e_2 - a_3e_3 - a_4e_4 - a_5e_5 - a_6e_6 - a_7e_7$$

is called the *conjugate* of the element a. The scalars $\mathbf{t}(a) = a + \overline{a} \in \mathbb{R}$ and

$$\mathbf{n}(a) = a\overline{a} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 + \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 + \alpha\beta\gamma a_7^2 \in \mathbb{R},$$

are called the *trace*, respectively, the *norm* of the element $a \in A$. It follows that $a^2 - \mathbf{t}(a)\, a + \mathbf{n}(a) = 0, \forall a \in A$.The octonion algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is a *division algebra* if for all $a \in \mathbb{O}(\alpha, \beta, \gamma)$, $a \neq 0$ we have $\mathbf{n}(a) \neq 0$, otherwise $\mathbb{O}(\alpha, \beta, \gamma)$ is called a *split algebra.*

Let V be a real vector space of dimension n and $<,>$ be the inner product. The *cross product* on V is a continuous map

$$X : V^s \to V, s \in \{1, 2, ..., n\}$$

with the following properties:

1) $< X(x_1, ...x_s), x_i >= 0, i \in \{1, 2, ..., s\}$;

2) $< X(x_1, ...x_s), X(x_1, ...x_s) >= \det(< x_i, x_j >)$ (see [17]).

In [18], was proved that if $d = \dim_{\mathbb{R}} V$, therefore $d \in \{0, 1, 3, 7\}$ (see [18], Proposition 3).

The values $0, 1, 3$ and 7 for dimension are obtained from Hurwitz's theorem, since the real Hurwitz division algebras $\mathcal{H}$ exist only for dimensions $1, 2, 4$ and 8. In this situations, the cross product is obtained from the product of the normed division algebra, restricting it to imaginary subspace of the algebra $\mathcal{H}$, which can be of dimension $0, 1, 3$ or 7 (see [19]). It is known that the real Hurwitz division algebras are only: the real numbers, the complex numbers, the quaternions and the octonions (see [14]).

In $\mathbb{R}^3$ with the canonical basis $\{i_1, i_2, i_3\}$, the cross product of two linearly independent vectors $x = x_1 i_1 + x_2 i_2 + x_3 i_3$ and $y = y_1 i_1 + y_2 i_2 + y_3 i_3$ is a vector, denoted by $x \times y$, which can be expressed computing the following formal determinant

$$x \times y = \begin{vmatrix} i_1 & i_2 & i_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

The cross product can also be described using the quaternions and the basis $\{i_1, i_2, i_3\}$ as a standard basis for $\mathbb{R}^3$. If a vector $x \in \mathbb{R}^3$ has the form $x = x_1 i_1 + x_2 i_2 + x_3 i_3$ and it is represented as the quaternion $x = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$, therefore the cross product of two vectors has the form $x \times y = xy + < x, y >$, where $< x, y >= x_1 y_1 + x_2 y_2 + x_3 y_3$ is the inner product.

A cross product for 7-dimensional vectors can be obtained in the same way, by using the octonions instead of the quaternions. If

$$x = \sum_{i=0}^{7} x_i e_i \quad \text{and} \quad y = \sum_{i=0}^{7} y_i e_i$$

are two imaginary octonions, therefore

$$\begin{aligned} x \times y \;=\; & (x_2 y_4 - x_4 y_2 + x_3 y_7 - x_7 y_3 + x_5 y_6 - x_6 y_5)\, e_1 + \\ & +(x_3 y_5 - x_5 y_3 + x_4 y_1 - x_1 y_4 + x_6 y_7 - x_7 y_6)\, e_2 + \\ & +(x_4 y_6 - x_6 y_4 + x_5 y_2 - x_2 y_5 + x_7 y_1 - x_1 y_7)\, e_3 + \\ & +(x_5 y_7 - x_7 y_5 + x_6 y_3 - x_3 y_6 + x_1 y_2 - x_2 y_1)\, e_4 + \\ & +(x_6 y_1 - x_1 y_6 + x_7 y_4 - x_4 y_7 + x_2 y_3 - x_3 y_2)\, e_5 + \\ & +(x_7 y_2 - x_2 y_7 + x_1 y_5 - x_5 y_1 + x_3 y_4 - x_4 y_3)\, e_6 + \\ & +(x_1 y_3 - x_3 y_1 + x_2 y_6 - x_6 y_2 + x_4 y_5 - x_5 y_4)\, e_7, \end{aligned} \tag{1}$$

see [20] and [21].

Let $\mathbb{H}$ be the real division quaternion algebra (obtained for $\alpha = \beta = 1$) and $\mathbb{H}_0 = \{x \in \mathbb{H} \mid \mathbf{t}(x) = 0\}$. An element $F_n \in \mathbb{H}_0$ is called an *imaginary Fibonacci quaternion element* if it is on the form

$$F_n = f_{n+1}\mathbf{i} + f_{n+2}\mathbf{j} + f_{n+3}\mathbf{k},$$

where $(f_n)_{n \in \mathbb{N}}$ is the Fibonacci numbers sequence.

In the proof of the following results, we will use some relations between Fibonacci numbers, namely:

D'Ocagne's identity

$$f_m f_{n+1} - f_n f_{m+1} = (-1)^n f_{m-n} \tag{2}$$

see relation (33) from [22], and

Johnson's identity

$$f_a f_b - f_c f_d = (-1)^r \left(f_{a-r} f_{b-r} - f_{c-r} f_{d-r}\right), \tag{3}$$

for arbitrary integers a, b, c, d, and r with $a + b = c + d$, see relation (36) from [22].

Let F_k, F_m, F_n be three imaginary Fibonacci quaternions. We have the following results.

Proposition 3.1. *With the above notations, for three arbitrary Fibonacci imaginary quaternions, we have*

$$< F_k \times F_m, F_n >= 0.$$

Therefore, the vectors F_k, F_m, F_n are linear dependents.

The above result is similar with the result for dual Fibonacci vectors obtained in [6], Theorem 11.

Let $\mathbb{O}$ be the real division octonion algebra (obtained for $\alpha = \beta = \gamma = 1$) and

$$\mathbb{O}_0 = \{x \in \mathbb{H} \mid \mathbf{t}(x) = 0\}.$$

An element $F_n \in \mathbb{O}_0$ is called an *imaginary Fibonacci octonion element* if it is of the form

$$F_n = f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3 + f_{n+4}e_4 + f_{n+5}e_5 + f_{n+6}e_6 + f_{n+7}e_7,$$

where $(f_n)_{n\in\mathbb{N}}$
is the Fibonacci numbers sequence. Let F_k, F_m, F_n be three imaginary Fibonacci octonions.

Proposition 3.2. *With the above notations, for three arbitrary Fibonacci imaginary octonions, we have*

$$< F_k \times F_m, F_n >= 0.$$

Proof. Using formulae (1), (2) and (3), we will compute $F_k \times F_m$.

The coefficient of e_1 is

$f_{m+2}f_{k+4} - f_{k+2}f_{m+4} + f_{m+3}f_{k+7} - f_{k+3}f_{m+7} + f_{m+5}f_{k+6} - f_{k+5}f_{m+6} =$
$= f_m f_{k+2} - f_k f_{m+2} - f_m f_{k+4} + f_k f_{m+4} - f_m f_{k+1} + f_k f_{m+1} =$
$= f_m (f_{k+2} - f_{k+4} - f_{k+1}) + f_k (-f_{m+2} + f_{m+4} + f_{m+1}) =$
$= f_m (f_k - f_{k+4}) + f_k (f_{m+4} - f_m) =$
$= -f_m (3f_{k+1} + f_k) + f_k (3f_{m+1} + f_m) =$
$= -3 (f_m f_{k+1} - f_k f_{m+1}) = -3 (-1)^k f_{m-k}.$

The coefficient of e_2 is
$f_{m+3}f_{k+5} - f_{k+3}f_{m+5} + f_{m+4}f_{k+1} - f_{k+4}f_{m+1} + f_{m+6}f_{k+7} - f_{k+6}f_{m+7} =$
$= -f_m f_{k+2} + f_k f_{m+2} - f_{m+3}f_k + f_{k+3}f_m + f_m f_{k+1} - f_k f_{m+1} =$
$= f_m\left(-f_{k+2} + f_{k+3} + f_{k+1}\right) + f_k\left(f_{m+2} - f_{m+3} - f_{m+1}\right) =$
$= 2\left(f_m f_{k+1} - f_k f_{m+1}\right) = 2\left(-1\right)^k f_{m-k}.$

The coefficient of e_3 is
$f_{m+4}f_{k+6} - f_{m+3}f_{k+5} + f_{m+5}f_{k+2} - f_{m+2}f_{k+5} + f_{m+7}f_{k+1} - f_{k+7}f_{m+1} =$
$= f_m f_{k+2} - f_{m+2}f_k + f_{m+3}f_k - f_m f_{k+3} - f_{m+6}f_k + f_m f_{k+6} =$
$= f_m\left(f_{k+2} - f_{k+3} + f_{k+6}\right) + f_k\left(-f_{m+2} + f_{m+3} - f_{m+6}\right) =$
$= 7\left(f_m f_{k+1} - f_k f_{m+1}\right) = 7\left(-1\right)^k f_{m-k}.$

The coefficient of e_4 is
$f_{m+5}f_{k+7} - f_{k+5}f_{m+7} + f_{m+6}f_{k+3} - f_{k+6}f_{m+3} + f_{m+1}f_{k+2} - f_{m+2}f_{k+1} =$
$= -f_m f_{k+2} + f_k f_{m+2} - f_{m+3}f_k + f_{k+3}f_m - f_m f_{k+1} + f_k f_{m+1} =$
$= f_m\left(-f_{k+2} + f_{k+3} - f_{k+1}\right) = 0.$

The coefficient of e_5 is
$f_{m+6}f_{k+1} - f_{k+6}f_{m+1} + f_{m+7}f_{k+4} - f_{k+7}f_{m+4} + f_{m+2}f_{k+3} - f_{k+2}f_{m+3} =$
$= -f_{m+5}f_k + f_{k+5}f_m + f_{m+3}f_k - f_{k+3}f_m + f_m f_{k+1} - f_k f_{m+1} =$
$= f_m\left(f_{k+5} - f_{k+3} + f_{k+1}\right) + f_k\left(-f_{m+5} + f_{m+3} - f_{m+1}\right) =$
$= 4\left(f_m f_{k+1} - f_k f_{m+1}\right) = 4\left(-1\right)^k f_{m-k}.$

The coefficient of e_6 is
$f_{m+7}f_{k+2} - f_{k+7}f_{m+2} + f_{m+1}f_{k+5} - f_{k+1}f_{m+5} + f_{m+3}f_{k+4} - f_{k+3}f_{m+4} =$
$= f_{m+5}f_k - f_{k+5}f_m - f_m f_{k+4} + f_k f_{m+4} - f_m f_{k+1} + f_k f_{m+1} =$
$= f_m\left(-f_{k+5} - f_{k+4} - f_{k-1}\right) + f_k\left(f_{k+5} + f_{k+4} + f_{k-1}\right) =$
$= -9\left(f_m f_{k+1} - f_k f_{m+1}\right) = -9\left(-1\right)^k f_{m-k}.$

The coefficient of e_7 is
$f_{m+1}f_{k+3} - f_{k+1}f_{m+3} + f_{m+2}f_{k+6} - f_{k+2}f_{m+6} + f_{m+4}f_{k+5} - f_{k+4}f_{m+5} =$
$= f_m\left(-f_{k+2} + f_{k+4} + f_{k+1}\right) + f_k\left(f_{m+2} - f_{m+4} - f_{m+1}\right) =$
$= 3\left(f_m f_{k+1} - f_k f_{m+1}\right) = 3\left(-1\right)^k f_{m-k}.$

We obtain that
$F_k \times F_m = \left(-1\right)^k f_{m-k}\left(-3e_1 + 2e_2 + 7e_3 + 4e_5 - 9e_6 + 3e_7\right).$

Therefore
$< F_k \times F_m, F_n >= \left(-1\right)^k f_{m-k}\big(- 3f_{n+1} + 2f_{n+2} + 7f_{n+3} + 4f_{n+5} -$
$-9f_{n+6} + 3f_{n+7}\big) = -2f_{n+2} + 2f_{n+1} + 2f_n = 0.$
The proposition is proved. □

Conclusions

In this paper, we proved that some of the identities obtained for Fibonacci quaternions and Fibonacci octonions can be obtained in an arbitrary algebras. In the same manner, similar identities and their applications, as for example D'Ocagne's identity or Johnson's identity, can be obtained. We introduced the notions of imaginary

Fibonacci quaternions and imaginary Fibonacci octonions and we proved that three arbitrary imaginary Fibonacci quaternions are linear dependents and the mixed product of three arbitrary imaginary Fibonacci octonions is zero.

References

[1] I. Akkus and O. Keçilioğlu, *Split Fibonacci and Lucas Octonions*, accepted in Adv. Appl. Clifford Algebras, DOI 10.1007/s00006-014-0515-8.

[2] O. Keçilioğlu and I. Akkus, *The Fibonacci Octonions,* Adv. Appl. Clifford Algebras **25**, no. 1 (2015), 151–158.

[3] C. Flaut and D. Savin, *Quaternion Algebras and Generalized Fibonacci-Lucas Quaternions*, accepted in Adv. Appl. Clifford Algebras.

[4] C. Flaut and V. Shpakivskyi, *Real matrix representations for the complex quaternions*, Adv. Appl. Clifford Algebras **23**, no. 3(2013), 657–671.

[5] C. Flaut and V. Shpakivskyi, *On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions*, Adv. Appl. Clifford Algebras **23**, no. 3 (2013), 673–688.

[6] I. A. Guren and S. K. Nurkan, *A new approach to Fibonacci, Lucas numbers and dual vectors*, accepted in Adv. Appl. Clifford Algebras, DOI 10.1007/s00006-014-0516-7.

[7] S. K. Nurkan and I. A. Guren, *Dual Fibonacci quaternions*, accepted in Adv. Appl. Clifford Algebras, DOI 10.1007/s00006-014-0488-7.

[8] S. Halici, *On Fibonacci Quaternions*, Adv. in Appl. Clifford Algebras **22**, no. 2 (2012), 321–327.

[9] S. Halici, *On dual Fibonaci quaternions*, Selcuk J. Appl. Math, accepted.

[10] A. F. Horadam, *A Generalized Fibonacci Sequence*, Amer. Math. Monthly **68** (1961), 455–459.

[11] A. F. Horadam, *Complex Fibonacci Numbers and Fibonacci Quaternions*, Amer. Math. Monthly **70** (1963), 289–291.

[12] J. L. Ramirez, *Some Combinatorial Properties of the k-Fibonacci and the k-Lucas Quaternions*, An. St. Univ. Ovidius Constanta, Mat. Ser. **2** (2015).

[13] M. N. S. Swamy, *On generalized Fibonacci Quaternions*, The Fibonacci Quaterly **11**, no. 5 (1973), 547–549.

[14] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New-York 1966.

[15] R. D. Schafer, *On the algebras formed by the Cayley-Dickson process*, Amer. J. Math. **76** (1954), 435–446.

[16] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience publication, U.S.A. 2001.

[17] R. Brown and A. Gray, *Vector cross products*, Commentarii Mathematici Helvetici **42**, no. 1 (1967), 222–236.

[18] M. Rost, *On the dimension of a composition algebra*, Doc. Math. J. **1** (1996), 209–214.

[19] N. Jacobson, *Basic algebra I*, Freeman 1974 2nd ed., 1974, p. 417–427.

[20] P. McLoughlin, *When does a cross product on $\mathbb{R}^n$ exist?*, http://arxiv.org/pdf/1212.3515.pdf.

[21] Z. K. Silagadze, *Multi-dimensional vector product*, arxiv.

[22] http://mathworld.wolfram.com/FibonacciNumber.html

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UWAGI O ELEMENTACH FIBONACCIEGO DOWOLNEJ ALGEBRY

S t r e s z c z e n i e

W pracy dowodzimy pewnych relacji między elementami Fibonacciego w dowolnej algebrze. Ponadto definiujemy urojone kwaterniony Fibonacciego i urojone oktoniony Fibonacciego oraz dowodzimy, że zawsze trzy dowolne urojone kwaterniony Fibonacciego są liniowo niezależne, a mieszane iloczyny trzech dowolnych urojonych oktonionów Fibonacciego są równe zeru.

Słowa kluczowe: kwaterniony Fibonacciego, oktoniony Fibonacciego, elementy Fibonacciego

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Anna Urbaniak-Kucharczyk, Iwona Łużniak, and Andrzej Korejwo

SPIN WAVE RESONANCE PROFILES IN MAGNETIC TRIPLE LAYERS

Summary

Spin wave resonance spectra for the system of three ferromagnetic layers divided by nonmagnetic spacers have been calculated. The Green function method has been used to calculate basics characteristics of spin wave resonance spectra. The effects of damping due spin-spin interaction leading to non-zero line-width of ferromagnetic resonance peaks have been additionally taken into account. The influence of interaction parameters appearing in used model on the spin wave patterns and the shape of resonance lines has been shown.

Keywords and phrases: spin wave resonance, Green function method, magnetic layered systems

1. Introduction

Interest in properties of magnetic ultrathin metallic films exchange coupled by nonmagnetic spacer has been growing considerably in last two decades due to increasing ability to produce samples of controlled quality and their technical importance (see e.g. [1, 2]). Beside of the problem of interlayer exchange coupling, which has been investigated by means of various theoretical methods also, basic magnetic properties of multilayer systems have been examined both by experimentalists and theoretician [3–10].

In particular the problem of elementary magnetic excitations in multilayers has been considered in many papers, where magnon dispersion relation or spin wave spectra have been obtained [11–17]. Recently, theoretical and experimental approaches dedicated to layered systems showed the role of the anisotropic factors is very important for proper description of their properties [18–20]. However, little attention has been up to now paid to the problem of magnon damping effects and its influence on the shape of spin wave resonance lines. In presented paper a Green's function method allowing to calculate spin wave spectra including profiles of FMR lines [21] is applied to a triple layer system.

2. Method and calculations

A system consisting of three homogeneous ferromagnetic layers separated by non-magnetic spacers is considered. Each ferromagnetic sublayer is made of N_l ($l = 1, 2, 3$) monolayers. To avoid the problem connected with detailed magnetic structure and rearrangement [3] we assume that an externally applied static magnetic field of the strength in the range corresponding to the ferromagnetic resonance condition is oriented perpendicularly to the film surface and all the spins can be considered statically as parallel to the external field.

We focus our attention on the exchange modes that can be separated from the magnetostatic ones by the proper choice of radiofrequencies. The effective field H_{eff} acting on a spin is taken as a sum of the external uniform field, the demagnetising field and the uniaxial bulk anisotropy field. The system is described by Heisenberg Hamiltonian consisting of the exchange, single ion anisotropy, Zeeman and dipolar coupling terms. We denote by J_l the exchange integrals for sublayers, while J_{12} and J_{23} stand for the parameters of exchange interaction between spins belonging to interface layers in different magnetic sublayers.

Below we will focus our on low temperature properties of layered composite and use in calculations the Green function method in Random Phase Approximation (RPA) [10, 21]. Magnetisation of the monolayer layer ν in the layered system consisting of $N_1 + N_2 + N_3$ monolayers is given by:

$$\langle S_\nu^z \rangle = S - \varphi_\nu,$$

(1)

$$\varphi_\nu = \frac{1}{n} \sum_{\vec{h}} \sum_{i=1}^{N_1+N_2+N_3} \frac{b_\nu^2(k_i)}{e^{\left(\frac{E(k_i,\vec{h})}{k_B T}\right)} - 1},$$

where $b_\nu(k_i)$ stand for amplitudes of spin waves with wave vectors k_i and energy $E(k_i, \mathbf{h})$ propagating in the system. The following set of equations for coefficients $b_\nu(k_i)$ can be obtained:

$$\begin{aligned}
&\left[\frac{D_1}{J} - \alpha(k_i0\right] b_1(k_i) + b_2(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{\nu-1}(k_i) - \alpha(k_i) b_\nu(k_i) + b_{\nu+1}(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{N_1-1}(k_i) - \left[\frac{A_{12}}{J} - \alpha(k_i)\right] b_{N_1}(k_i) + \frac{J_{12}}{J} b_{N_1+1}(k_i) = 0,\\
&\frac{J_{12}}{J} b_{N_1-1}(k_i) - \left[\frac{A_I}{J} - \alpha(k_i)\right] b_{N_1+1}(k_i) + b_{N_1+2}(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{N_1+\nu-1}(k_i) - \alpha(k_i) b_{N_1+\nu}(k_i) + b_{N_1+\nu+1}(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{N_2-1}(k_i) - \left[\frac{A_{23}}{J} - \alpha(k_i)\right] + \frac{J_{23}}{J} b_{N_2+1}(k_i) = 0,\\
&\frac{J_{23}}{J} b_{N_2} - \left[\frac{A_{23}}{J - \alpha(k_i)}\right] b_{N_2+1}(k_i) + b_{N_2+2}(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{N_2+\nu-1}(k_i) - \alpha(k_i) b_{N_2+\nu}(k_i) + b_{N_2+\nu+1}(k_i) = 0,\\
&\dots\dots\dots\dots\\
&b_{N_3-1}(k_i) - \left[\frac{D_3}{J} - \alpha(k_i)\right] b_{N_3}(k_i) = 0.
\end{aligned} \tag{2}$$

The anisotropy in the layer ν is assumed to be in the following form:

$$A_{\nu'} = A + A_{12}\delta_{N,N+1} + A_{23}\delta_{N_2,N_2+1} + D_1\delta_{1,2} + D_3\delta_{N_3-1,N_3}. \tag{3}$$

A_{12} and A_{23} are the anisotropy at the interface between fist and second and second and third magnetic layer, respectively. D_1 and D_3 denote surface anisotropy at the surface belonging to external layers. The term $\alpha(k_i) = 2\cos(k_i)$ is proportional to the energy of elementary excitation [10]. The set of allowed values of k_i can be found by solving the characteristic equation obtained employing the transfer matrix method [22]. For the sake of simplicity we assume $N_1 = N_2 = N_3 = N$. Then the characteristics equation reads:

$$\begin{aligned}
&[X + (1 - D_1)]\,[X + (1 - D_3)]\\
\times\ &\{x_1 - (1 - D_1)x_2 + (1 - D_3)\,[(1 - D_1)x_3 - x_2]\}\\
-\ &[X + (1 - D_3)]\,\{x_1 - (1 - D_3)x_2 + [X + 2(1 - D_1)]\\
&\times [(1 - D_3)x_3 - x_2]\}\\
-\ &J_{23}\,[X + (1 - D_1)]\,\{x_1 - (1 - D_1)x_2 + [X + 2(1 - D_3)]\\
&\times [(1 - D_1)x_3 - x_2]\}\\
+\ &J_{12}J_{23}\,\{x_1 - 2\,[X + (1 - D_1) + (1 - D_3)x_2]\\
&+ [X + 2(1 - D_1)]\ [X + 2(1 - D_3)]\,x_3\} = 0,
\end{aligned} \tag{4}$$

where

$$X = \frac{\sin(N+1)k - \sin Nk}{\sin(N-1)k - \sin Nk}, \tag{5}$$

$$x_1 = \sin(N+1)k, \tag{6}$$

$$x_2 = \sin Nk, \tag{7}$$

$$x_3 = \sin(N-1)k. \tag{8}$$

The characteristic equation for the wave vectors k_i is convenient for calculation of the positions and relative intensities of spin wave resonance modes. Described up to now method doesn't, however, allow to obtain the line shape of the resonance picture.

To take into account the effects of damping which may be due to spin-spin interaction, existence of magnetic surface single-ion anisotropy and interaction of the magnetic system with lattice vibrations, the two-dimensional Fourier transform $g_{\nu j \nu' j'}(E)$ of the Green function $G_{\nu j \nu' j'}(t-t')$ should be written [21, 23–24] as:

$$g_{\nu j \nu' j'}(E) = \frac{A_\nu\left(\langle S^z \rangle\right)}{E - \tilde{E}_{\nu j \nu' j'}}, \tag{9}$$

with

$$\tilde{E}_{\nu j \nu' j'} = E_{\nu j \nu' j'} + i\Gamma_{\nu j \nu' j'}. \tag{10}$$

Then the transformation coefficients to the momentum space take the form [21]:

$$Q_{\nu\nu'}(E) = \frac{1}{2\pi} \sum_{i=1}^{3N} \frac{b_\nu(k_i) b_{\nu'}(k_i)}{E - \tilde{E}(k_i, \vec{h})}, \tag{11}$$

where

$$\tilde{E}(k_i \vec{h}) = E(k_i, \vec{h}) + i\Gamma(k_i, \vec{h}). \tag{12}$$

The imaginary part of energy term can be calculated on the base of relaxation equation [25]:

$$\Gamma(k_i, \vec{h}) = \frac{1}{2\tau} \sum_{i=1}^{3N} b_\nu^2(k_i) A_\nu \left[\langle S_\nu^z \rangle - \langle S_\nu^z \rangle_{\mathrm{eq}}\right], \tag{13}$$

where the parameter τ stands for relaxation time. It has been, for example, estimated by Wesselinova [26, 27] for damping due to magnon-magnon interaction.

Equation (13) allows one to write the spectral density as [23]:

$$I(E, k_i) = \frac{\langle S^z \rangle}{\pi} \frac{1}{e^{\frac{\tilde{E}(k_i, \vec{h})}{kT}} - 1} \frac{\Gamma(k_i, \vec{h})}{\left(E - E(k_i, \vec{h})\right)^2 + \Gamma^2(k_i \vec{h})}. \tag{14}$$

The spectral density can be used to calculate the spin wave resonance spectra in the way described in [21]. For the magnetic field polarized in x direction the formula for absorbed power is in the form:

$$W(E) \propto \sum_l \frac{E^2 E_{\mathrm{lrez}} \Gamma_l}{\left(E^2 - E_{\mathrm{lrez}}^2 - \Gamma^2\right)^2 + (2E\Gamma_l)^2}, \tag{15}$$

where l denotes the number of resonance line. Equation (15) gives a continuous distribution of resonance intensity, therefore it reflects better the real situations observed in FMR experiments than calculations neglecting damping effects.

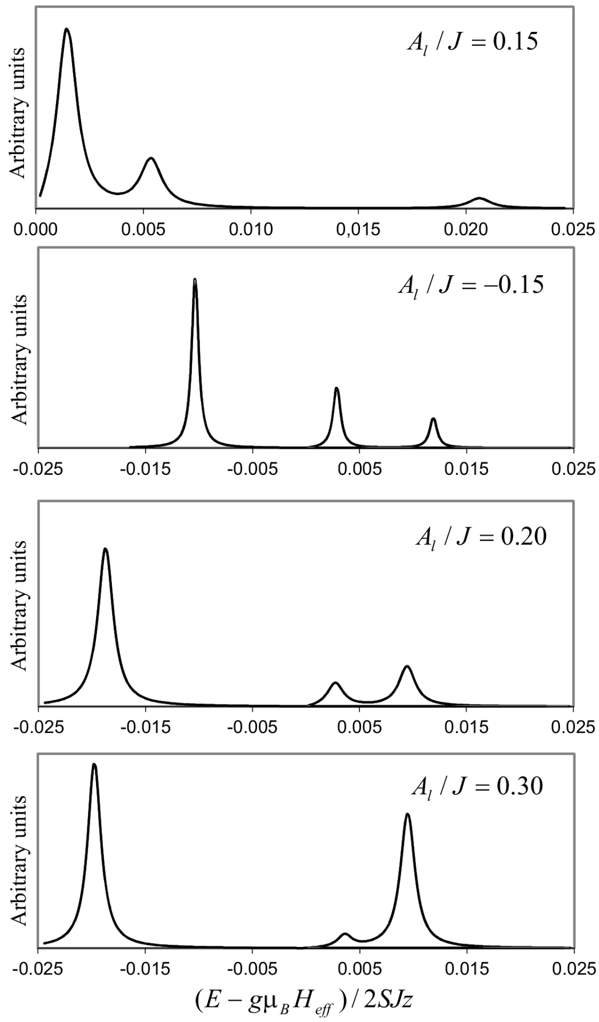

Fig. 1: Intensity distribution for the magnetic field polarized in x direction for triple layer consisting of 60 magnetic monolayers ($N1 = N2 = N3 = 20$) for $J_{12}/J = J_{23}/J = 0.1$ and $D_1/J = D_3/J = 1.0$.

The numerical calculations based on the presented above formalism have been carried out for the exchange triple layer system. Positions of resonance peaks have been calculated employing method proposed in [28, 29] and next the spin wave spectrum has been obtained including the damping term derived on the basis of results of Wesselinowa [26, 27]. The results obtained are presented in Figs 1–4 as a function of interaction parameters.

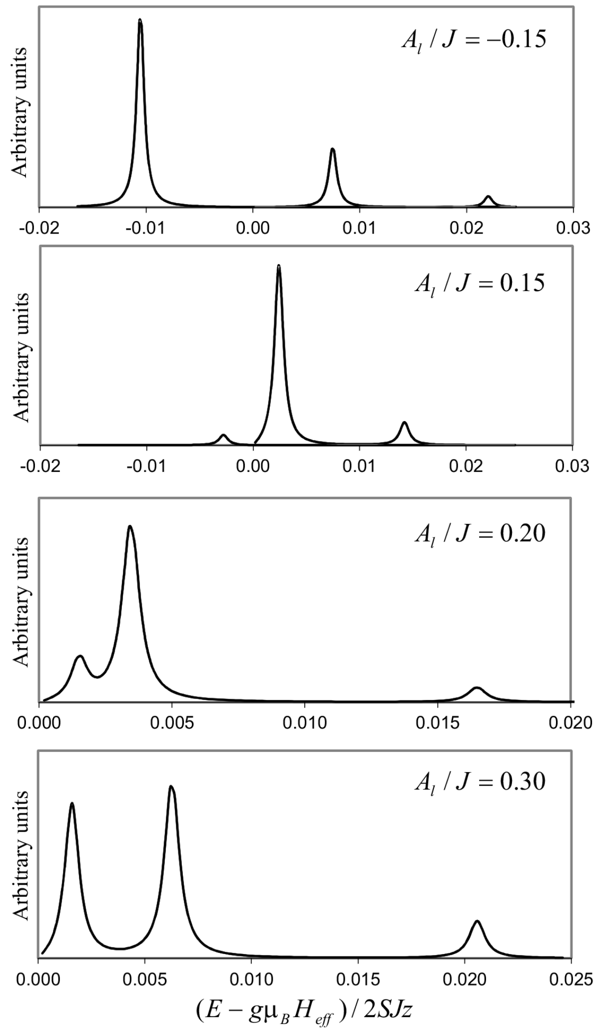

Fig. 2: Intensity distribution for the magnetic field polarized in x direction for triple layer consisting of 60 magnetic monolayers ($N1 = N2 = N3 = 20$) for $J_{12}/J = J_{23}/J = -0.1$ and $D_1/J = D_3/J = 1.0$.

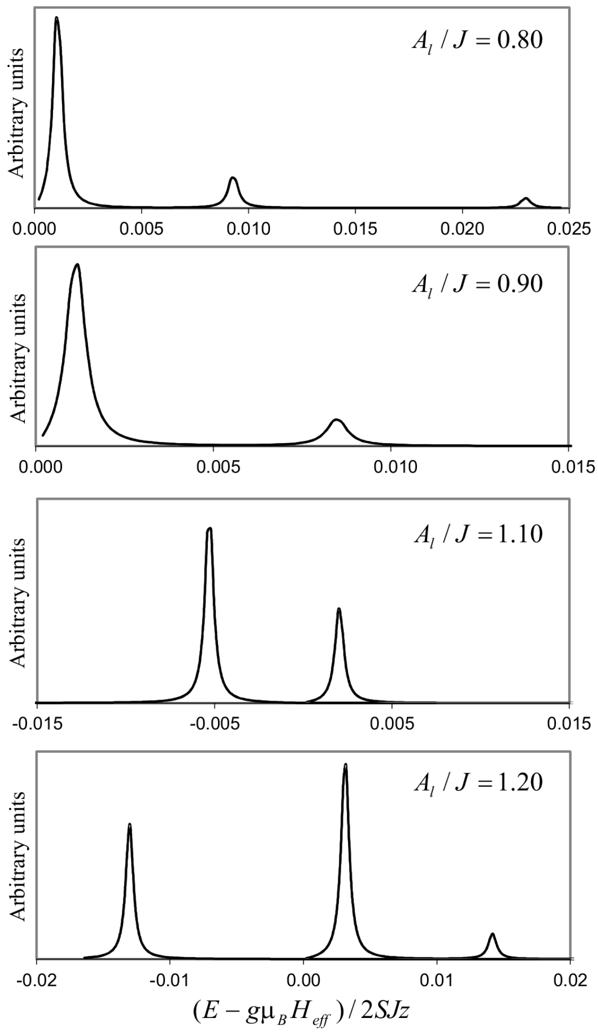

Fig. 3: Intensity distribution for the magnetic field polarized in x direction for triple layer consisting of 60 magnetic monolayers ($N1 = N2 = N3 = 20$) for $J_{12}/J = J_{23}/J = 0.25$ and $D_1/J = D_3/J = 0.0$.

3. Final remarks

The results presented in this paper show that introducing damping effect even on the basis of phenomenological relaxation equation gives possibility of calculation of more realistic resonance spectra with non-zero line-width. It would be interesting

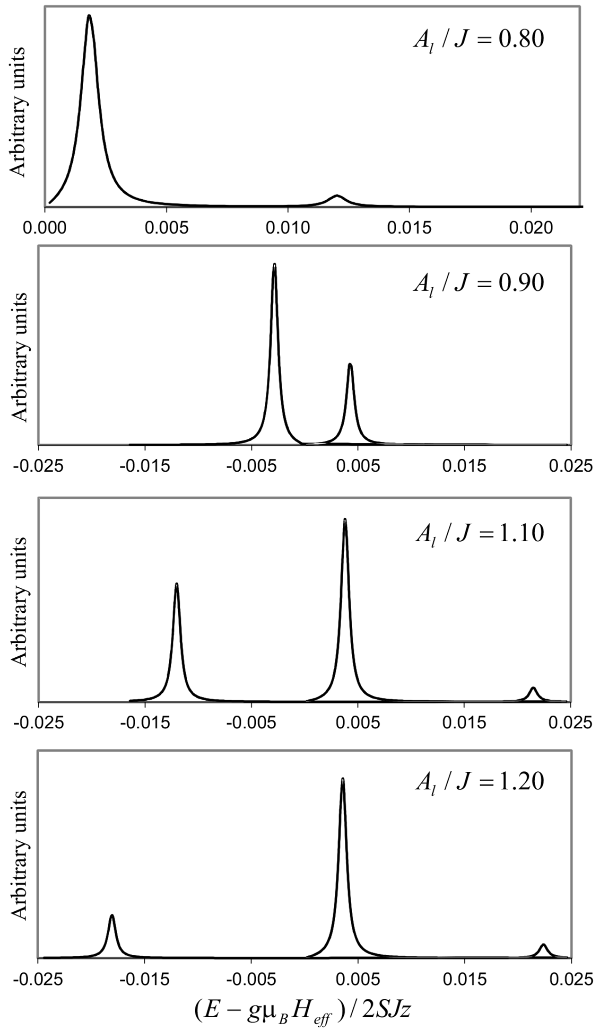

Fig. 4: Intensity distribution for the magnetic field polarized in x direction for triple layer consisting of 60 magnetic monolayers ($N1 = N2 = N3 = 20$) for $J_{12}/J = J_{23}/J = -0.25$ and $D_1/J = D_3/J = 0.0$.

to compare spectra obtained introducing different sources of dumping. An attempt to calculate spin wave characteristics for materials of anisotropy distribution across layers has been done in [30]. The results obtained which are only of qualitative character show that introducing of non-uniform anisotropy leads to modification of resonance spectra which is similar to the effect caused by the existence of roughness at the surface and interfaces of the sample.

References

[1] P. Bruno, Phys. Rev. B **52** (1995), 411.

[2] S. Krompiewski, F. Süss, and U. Krey, Europhys. Lett. **26** (1994), 303.

[3] D. Mercier, Y. C. S. Levy, M. L. Watson, J. S. S. Whiting, and A. Chambers, Phys. Rev. B **43** (1991), 3311.

[4] H. Puszkarski, Phys. Rev. B **46** (1992), 8926.

[5] F. L. Castillo Alvarado, J. H. Rutkowski, A. Urbaniak-Kucharczyk, and L. Wojtczak, Thin Solid Films **324** (1998), 225.

[6] F. L. Castillo Alvarado, S. Machowski, and A. Urbaniak-Kucharczyk, Vacuum **63** (2001), 323.

[7] G. Bayreuther, F. Bensch, and V. Kottler, J. Appl. Phys. **79** (1996), 4509.

[8] A. Urbaniak-Kucharczyk, phys. stat. sol. (b) **203** (1997), 195.

[9] P. Poulopoulos, V. Bovensiepen, M. Farle, and K. Baberschke, Phys. Rev. B **57** (1998), R15036.

[10] S. Machowski and A. Urbaniak-Kucharczyk, Surf. Sci. **507–510**C (2002), 551.

[11] M. Vohl, J. Barnas, and P. Grünberg, Phys. Rev. B **39** (1989), 12003.

[12] B. Hillebrands, Phys. Rev. B **41** (1990), 530.

[13] H. Puszkarski, J. Magnetism and Magnetic Mater. **93** (1991), 290 (1991).

[14] H. Puszkarski, phys. stat. solidi (b) **171** (1992), 205.

[15] Q. Hong and Q. Yong, J. Phys.: Condensed Matter. **4** (1992), 7979.

[16] A. Urbaniak-Kucharczyk, phys. stat. solidi (b) **157** (1996), 455.

[17] S. T. B. Goennenwein, T. Graf, T. Wassner, M. S. Brandt, M. Stutzmann, J. B. Philipp, R. Gross, M. Krieger, K. Zürm, P. Ziemann, A. Koeder, S. Frank, W. Stoch, and A. Waag, Appl. Phys. Lett. **82** (2003), 730.

[18] X. Liu, Y. Y. Zhou, and J. K. Furdyna, Phys. Rev. B **75** (2007), 195220.

[19] A. Urbaniak-Kucharczyk, Acta Physica Polonica, A **126** (2014), 208.

[20] A. Urbaniak-Kucharczyk, Acta Phys. Superficierum **13** (2012), 162.

[21] H. Puszkarski, M. Krawczyk, J. C. S. Levy, and D. Mercier, Acta Physica Polonica A **100** (2001), 195.

[22] S. W. Tyablikov, *Methods of quantum theory of magnetism*, Nauka, Moscow 1965.

[23] L. Wojtczak, *Cienkie warstwy magnetyczne*, WUL Łódź 2009.

[24] T. Oguchi, in: *Summer School on Critical Phenomena and Phase Transition in Magnetics, Cetniewo 1973*, PWN Warsaw 1974.

[25] J. M. Wesselinowa, J. Phys. Condens. Matter **17** (2005), 6687.

[26] J. M. Wesselinowa, J. Phys. Condens. Matter **18** (2006), 8169.

[27] A. Urbaniak-Kucharczyk, phys. stat. sol. (b) **188** (1995), 795.

[28] I. Staniucha and A. Urbaniak-Kucharczyk, phys. stat. sol. (c) **3** (2006), 65.

[29] A. Urbaniak-Kucharczyk, Acta Physica Polonica, A **127** (2015), 543.

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PROFILE REZONANSU FAL SPINOWYCH W POTRÓJNYCH WARSTWACH MAGNETYCZNYCH

S t r e s z c z e n i e

W pracy wyliczone są widma rezonansu fal spinowych dla układu trzech warstw ferromagnetycznych przedzielonych niemagnetycznymi przekładkami. Metoda funkcji Greena jest zastosowana dla wyznaczenia podstawowych charakterystyk rezonansu fal spinowych. Efekty tłumienia związane z oddziaływaniem spin-spin, które prowadzą do niezerowej szerokości linii rezonansowych, zostały dodatkowo wzięte pod uwagę. Pokazano jaki wpływ na widma rezonansowe i kształt linii mają parametry oddziaływań występujace w stosowanym modelu.

Słowa kluczowe: rezonans fal spinowych, metoda funkcji Greena, magnetyczne układy warstwowe

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Contribution to the jubilee volume, dedicated
to Professors J. Ławrynowicz and L. Wojtczak

Natalia Zorii

CONSTRAINED GAUSS VARIATIONAL PROBLEM FOR CONDENSERS WITH TOUCHING PLATES

Summary

We study a constrained minimum energy problem with an external field relative to the α-Riesz kernel $|x-y|^{\alpha-n}$ of an arbitrary order $\alpha \in (0, n)$ for a generalized condenser $\mathbf{A} = (A_1, A_2)$ with touching oppositely-charged plates in $\mathbb{R}^n$, $n \geqslant 2$. Conditions sufficient for the solvability of the problem are obtained. Our arguments are mainly based on the definition of an appropriate metric structure on a set of vector measures associated with $\mathbf{A}$ and the establishment of a completeness theorem for the corresponding metric space.

Keywords and phrases: Minimum Riesz energy problems, external field, constraint, condenser with touching plates, strong completeness theorem for vector measures

1. Introduction

This paper is devoted to the well-known Gauss variational problem of minimizing the α-Riesz energy, $\alpha \in (0, n)$, in the presence of an external field, treated for a generalized condenser $\mathbf{A}$ with touching oppositely-charged plates $A_1, A_2 \subset \mathbb{R}^n$, $n \geqslant 2$. In the case where the Euclidean distance $\operatorname{dist}(A_1, A_2)$ between A_1 and A_2 is nonzero (which might happen if A_1 and A_2 touch each other *only* at the Alexandroff point $\omega_{\mathbb{R}^n}$), a fairly complete investigation of this problem has been provided in [17, 18] (see also the bibliography therein; see Section 3.3 below for a short review).

However, the results obtained in [17,18] and the approach developed are no longer valid if $\operatorname{dist}(A_1, A_2) = 0$ (e.g, if A_1 and A_2 touch each other at a *finite* point $x \in \mathbb{R}^n$).

Then the infimum of the Gauss functional can not, in general, be attained among the admissible measures. Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x-y|^{-1}$ on $\mathbb{R}^3$, a short-circuit between A_1 and A_2 might occur. Therefore, it is meaningful to ask what kind of additional requirements on the charges (measures) under consideration would prevent this phenomenon.

A natural idea, to be exploited below, is to impose an upper constraint on vector measures associated with $\mathbf{A}$ so that the infimum of the Gauss functional over the corresponding (narrower) class of constrained admissible vector measures would be already an actual minimum. See Section 3.4 for a precise formulation of the constrained problem; as for the history of the question, cf. Remarks 3.10–3.12.

A statement on the solvability of the constrained Gauss variational problem is given by Theorem 4.1, the main result of the study. Its proof is based on the definition of an appropriate metric structure on a set of vector measures associated with $\mathbf{A}$ and the establishment of a completeness theorem for the corresponding metric space (see Theorem 5.1). The results obtained are illustrated by Example 4.2.

2. Preliminaries

Let X be a locally compact Hausdorff space, to be specified below, and $\mathfrak{M}(\mathrm{X})$ the linear space of all real-valued scalar Radon measures μ on X, equipped with the *vague* topology, i.e. the topology of pointwise convergence on the class $\mathrm{C}_0(\mathrm{X})$ of all real-valued continuous functions on X with compact support. We denote by μ^+ and μ^- the positive and the negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathfrak{M}(\mathrm{X})$, respectively, and by S^μ_{X} its support. These and other notions of the theory of measures and integration in a locally compact space, to be used throughout the paper, can be found in [3, 8] (see also [9] for a short review).

A *kernel* $\kappa(x, y)$ on X is a symmetric, lower semicontinuous function $\kappa : \mathrm{X}\times\mathrm{X} \to [0,\infty]$. Given $\mu, \mu_1 \in \mathfrak{M}(\mathrm{X})$, let $E_\kappa(\mu, \mu_1)$ and $U^\mu_\kappa(\cdot)$ denote the *mutual energy* and the *potential* relative to the kernel κ, respectively, i.e.

$$E_\kappa(\mu, \mu_1) := \int \kappa(x, y)\, d(\mu \otimes \mu_1)(x, y),$$

$$U^\mu_\kappa(x) := \int \kappa(x, y)\, d\mu(y), \quad x \in \mathrm{X}.$$

(When introducing notation, we assume the corresponding object on the right to be well defined — as a finite number or $\pm\infty$.)

For $\mu = \mu_1$, the mutual energy $E_\kappa(\mu, \mu_1)$ defines the *energy* $E_\kappa(\mu) := E_\kappa(\mu, \mu)$. Let $\mathcal{E}_\kappa(\mathrm{X})$ consist of all $\mu \in \mathfrak{M}(\mathrm{X})$ whose energy $E_\kappa(\mu)$ is finite.

Having denoted by $\mathfrak{M}^+(\mathrm{X})$ the convex cone of all nonnegative $\mu \in \mathfrak{M}(\mathrm{X})$, we write

$$\mathcal{E}^+_\kappa(\mathrm{X}) := \mathfrak{M}^+(\mathrm{X}) \cap \mathcal{E}_\kappa(\mathrm{X}).$$

Given a set $B \subset \mathrm{X}$, $B \neq \mathrm{X}$, let $\mathfrak{M}^+(B; \mathrm{X})$ consist of all $\mu \in \mathfrak{M}^+(\mathrm{X})$ concentrated in B, and let $\mathcal{E}^+_\kappa(B; \mathrm{X}) := \mathcal{E}_\kappa(\mathrm{X}) \cap \mathfrak{M}^+(B; \mathrm{X})$.

Observe that, if B is closed, then $\mu \in \mathfrak{M}^+(\mathrm{X})$ belongs to $\mathfrak{M}^+(B;\mathrm{X})$ if and only if the set $\mathrm{X} \setminus B$ is μ-negligible (or, equivalently, if $S^\mu_{\mathrm{X}} \subset B$). Furthermore, then $\mathfrak{M}^+(B;\mathrm{X})$ and $\mathcal{E}^+_\kappa(B;\mathrm{X})$ are closed in the induced vague topology (see, e.g., [9]).

Let $C_\kappa(B)$ be the *interior capacity* of B relative to the kernel κ, given by

$$C_\kappa(B) := \Bigl[\inf_{\mu\in\mathcal{E}^+_\kappa(B;\mathrm{X}):\ \mu(B)=1} E_\kappa(\mu)\Bigr]^{-1};$$

see, e.g., [9,13]. Then $0 \leqslant C_\kappa(B) \leqslant \infty$. (Here, as usual, the infimum over the empty set is taken to be $+\infty$. We also put $1\big/(+\infty) = 0$ and $1\big/0 = +\infty$.)

A kernel κ is called *strictly positive definite* if the energy $E_\kappa(\mu)$, $\mu \in \mathfrak{M}(\mathrm{X})$, is nonnegative whenever defined and $E_\kappa(\mu) = 0$ implies $\mu = 0$. Then $\mathcal{E}_\kappa(\mathrm{X})$ forms a pre-Hilbert space with the scalar product $E_\kappa(\mu,\mu_1)$ and the norm $\|\mu\|_\kappa := \sqrt{E_\kappa(\mu)}$ (see [9]). The topology on $\mathcal{E}_\kappa(\mathrm{X})$ defined by $\|\cdot\|_\kappa$ is said to be *strong*.

Following Fuglede [9], we call a strictly positive definite kernel κ *perfect* if any strong Cauchy sequence in $\mathcal{E}^+_\kappa(\mathrm{X})$ converges strongly and, in addition, the strong topology on $\mathcal{E}^+_\kappa(\mathrm{X})$ is finer than the induced vague topology on $\mathcal{E}^+_\kappa(\mathrm{X})$. Note that then $\mathcal{E}^+_\kappa(\mathrm{X})$ is a strongly complete metric space.

3. Unconstrained and constrained Gauss variational problems

Throughout the paper, let $n \geqslant 2$, $n \in \mathbb{N}$, and $\alpha \in (0,n)$ be fixed. In $\mathrm{X} = \mathbb{R}^n$, consider the α-*Riesz kernel* $\kappa_\alpha(x,y) := |x-y|^{\alpha-n}$ of order α, where $|x-y|$ denotes the Euclidean distance between x and y in $\mathbb{R}^n$. The α-Riesz kernel is known to be strictly positive definite and, moreover, perfect (see [5,6]); hence, the metric space $\mathcal{E}^+_{\kappa_\alpha}(\mathbb{R}^n)$ is complete in the induced strong topology. However, by Cartan [4] (see also [12, Theorem 1.19]), the whole pre-Hilbert space $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ for $\alpha \in (1,n)$ is strongly incomplete (compare with Theorem 5.1 and Remark 5.2 below).

From now on we shall write simply α instead of κ_α if it serves as an index. E.g., $C_\alpha(\cdot) = C_{\kappa_\alpha}(\cdot)$ denotes the α-Riesz interior capacity of a set. An expression $\mathcal{U}(x)$, involving a variable point $x \in \mathbb{R}^n$, is said to subsist *nearly everywhere* (n.e.) in a set $B \subset \mathbb{R}^n$ if $C_\alpha(N) = 0$, where N consists of all $x \in B$ for which $\mathcal{U}(x)$ fails to hold.

3.1. Generalized condensers. Vector measures and their α-Riesz energies

Given $B \subset \mathbb{R}^n$, write $B^c := \mathbb{R}^n \setminus B$. Recall that a (*standard*) *condenser* in $\mathbb{R}^n$ is usually meant as an ordered pair of nonempty, closed (though not necessarily compact), nonintersecting sets in $\mathbb{R}^n$. We extend this notion as follows.

Definition 3.1. An ordered pair $\mathbf{A} := (A_1, A_2)$ of nonempty sets in $\mathbb{R}^n$ is called a generalized condenser if the following two conditions are fulfilled for every $i = 1,2$:

(a) $A_i \subset D_i$, where $D_i := \bigl(\mathrm{C}\ell_{\mathbb{R}^n} A_j\bigr)^c$, $j \neq i$;

(b) A_i is closed in the relative topology of the (open) set D_i.

Observe that the notion of a generalized condenser $\mathbf{A} = (A_1, A_2)$ is reduced to that of a standard one if and only if the sets A_i, $i = 1, 2$, are closed in $\mathbb{R}^n$.

In the example below, $n = 3$ and $\overline{B}(x, 1)$ is the closed three-dimensional ball of radius 1 centered at $x \in \mathbb{R}^3$.

Example 3.2. *Consider $\overline{B}(\xi_1, 1)$ and $\overline{B}(\xi_2, 1)$ with $\xi_1 = (0,0,0)$ and $\xi_2 = (2,0,0)$; these balls intersect each other at $\xi_0 = (1,0,0)$. Then the sets $A_i := \overline{B}(\xi_i, 1) \setminus \{\xi_0\}$, $i = 1, 2$, satisfy both assumptions* (a) *and* (b) *from Definition* 3.1 *and, hence, form a generalized condenser $\mathbf{A}$ in $\mathbb{R}^3$, which certainly is not a standard one.*

In all that follows, fix a generalized condenser $\mathbf{A} = (A_1, A_2)$ such that $A_i \neq D_i$ for all $i = 1, 2$. To avoid triviality, suppose

$$\prod_{i=1,2} C_\alpha(A_i) > 0.$$

Let $\mathfrak{M}^+(\mathbf{A})$ stand for the Cartesian product $\prod_{i=1,2} \mathfrak{M}^+(A_i; D_i)$, where D_i is thought of as a locally compact space. Then $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ is a nonnegative *vector measure* $(\nu^i)_{i=1,2}$ with the components $\nu^i \in \mathfrak{M}^+(A_i; D_i)$; it is said to be *associated* with the condenser $\mathbf{A}$.

Definition 3.3. The $\mathbf{A}$-vague topology on $\mathfrak{M}^+(\mathbf{A})$ is the topology of the product space $\prod_{i=1,2} \mathfrak{M}^+(A_i; D_i)$, where each of the factors $\mathfrak{M}^+(A_i; D_i)$, $i = 1, 2$, is endowed with the vague topology induced from $\mathfrak{M}(D_i)$.

As A_i is closed in D_i, $\mathfrak{M}^+(\mathbf{A})$ is $\mathbf{A}$-vaguely closed. Besides, since every $\mathfrak{M}(D_i)$ is Hausdorff, so is $\mathfrak{M}^+(\mathbf{A})$ (see [11, Chapter 3, Theorem 5]). Hence, an $\mathbf{A}$-vague limit of any $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}^+(\mathbf{A})$ belongs to $\mathfrak{M}^+(\mathbf{A})$ and is unique (provided it exists).

If $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ and a vector-valued function $\boldsymbol{u} = (u_i)_{i=1,2}$ with the ν^i-measurable components $u_i : A_i \to [-\infty, \infty]$ are given, then we write

$$\langle \boldsymbol{u}, \boldsymbol{\nu} \rangle := \sum_{i=1,2} \int u_i \, d\nu^i.$$

We call A_1 and A_2 the *positive* and the *negative plates* of $\mathbf{A}$, respectively. In accordance with the electrostatic interpretation of a condenser, assume that the interaction between the charges lying on the conductors A_i, $i = 1, 2$, is characterized by the matrix $(s_i s_j)_{i,j=1,2}$, where

$$s_i := \operatorname{sign} A_i = \begin{cases} +1 & \text{if } \quad i = 1, \\ -1 & \text{if } \quad i = 2. \end{cases}$$

Then the *α-Riesz mutual energy* of $\boldsymbol{\nu}, \boldsymbol{\nu}_1 \in \mathfrak{M}^+(\mathbf{A})$ is given formally by

$$E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1) := \sum_{i,j=1,2} s_i s_j \int |x - y|^{\alpha - n} \, d(\nu^i \otimes \nu_1^j)(x, y). \tag{3.1}$$

For $\boldsymbol{\nu} = \boldsymbol{\nu}_1$, $E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$ defines the *α-Riesz energy* $E_\alpha(\boldsymbol{\nu}) := E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu})$ of $\boldsymbol{\nu}$. We denote by $\mathcal{E}^+_\alpha(\mathbf{A})$ the set of all $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ whose energy $E_\alpha(\boldsymbol{\nu})$ is finite.

3.2. Metric structure on classes of vector measures

Let $\breve{\mathfrak{M}}^+(\mathbf{A})$ consist of all $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ such that each of its components ν^i, $i = 1, 2$, can be extended to a Radon measure on $\mathbb{R}^n$ (denote it again by ν^i) by setting

$$\nu^i(\varphi) := \langle \chi_{D_i}\varphi, \nu^i\rangle \quad \text{for all} \quad \varphi \in \mathrm{C}_0(\mathbb{R}^n),$$

where χ_{D_i} is the characteristic function of D_i. A sufficient condition for $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ to belong to $\breve{\mathfrak{M}}^+(\mathbf{A})$ is that $\nu^i(A_i) < \infty$ for all $i = 1, 2$. Also note that

$$\breve{\mathfrak{M}}^+(\mathbf{A}) = \mathfrak{M}^+(\mathbf{A}) \iff \mathbf{A} \text{ is standard;} \tag{3.2}$$

otherwise, $\breve{\mathfrak{M}}^+(\mathbf{A})$ forms a proper subset of $\mathfrak{M}^+(\mathbf{A})$ that is not $\mathbf{A}$-vaguely closed.

For any $\boldsymbol{\nu} \in \breve{\mathfrak{M}}^+(\mathbf{A})$, write

$$R\boldsymbol{\nu} := \sum_{i=1,2} s_i\nu^i; \tag{3.3}$$

then $R\boldsymbol{\nu}$ is a *signed* scalar Radon measure on $\mathbb{R}^n$. Since $A_1 \cap A_2 = \varnothing$, R is a one-to-one mapping between $\breve{\mathfrak{M}}^+(\mathbf{A})$ and its R-image,

$$R\bigl(\breve{\mathfrak{M}}^+(\mathbf{A})\bigr) = \bigl\{\nu \in \mathfrak{M}(\mathbb{R}^n):\ \nu^+ \in \mathfrak{M}^+(A_1; D_1),\ \nu^- \in \mathfrak{M}^+(A_2; D_2)\bigr\}.$$

Lemma 3.4. *For any $\boldsymbol{\nu}, \boldsymbol{\nu}_1 \in \breve{\mathfrak{M}}^+(\mathbf{A})$, $E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$ is well defined if and only if so is $E_\alpha(R\boldsymbol{\nu}, R\boldsymbol{\nu}_1)$, and then they coincide:*

$$E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1) = E_\alpha(R\boldsymbol{\nu}, R\boldsymbol{\nu}_1). \tag{3.4}$$

Proof. Indeed, this can be obtained directly from (3.1) and (3.3). □

In view of the strict positive definiteness of the α-Riesz kernel, Lemma 3.4 yields that $E_\alpha(\boldsymbol{\nu})$, $\boldsymbol{\nu} \in \breve{\mathfrak{M}}^+(\mathbf{A})$, is $\geqslant 0$ whenever defined, and it is zero only for $\boldsymbol{\nu} = \mathbf{0}$. Write $\breve{\mathcal{E}}^+_\alpha(\mathbf{A}) := \mathcal{E}^+_\alpha(\mathbf{A}) \cap \breve{\mathfrak{M}}^+(\mathbf{A})$. Having defined

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})} := \Bigl[\sum_{i,j=1,2} s_is_jE_\alpha(\nu^i - \nu^i_1, \nu^j - \nu^j_1)\Bigr]^{1/2} \quad \text{for all} \quad \boldsymbol{\nu}, \boldsymbol{\nu}_1 \in \breve{\mathcal{E}}^+_\alpha(\mathbf{A}),$$

we also see from (3.4) by means of a straightforward calculation that, in fact,

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})} = \|R\boldsymbol{\nu} - R\boldsymbol{\nu}_1\|_\alpha, \tag{3.5}$$

so that $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$ forms a metric space with the metric $\|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})}$. Since, in consequence of (3.5), $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$ and its R-image are isometric, similar to the terminology in $\mathcal{E}_\alpha(\mathbb{R}^n)$ we shall call the topology of the metric space $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$ *strong*.

3.3. Unconstrained f-weighted minimum α-Riesz energy problem

Given a locally compact space X, let $\Phi(\mathrm{X})$ consist of all lower semicontinuous functions $\psi : \mathrm{X} \to (-\infty, \infty]$ such that $\psi \geqslant 0$ unless X is compact. Then for any $\psi \in \Phi(\mathrm{X})$, the map

$$\mu \mapsto \langle \psi, \mu\rangle, \quad \mu \in \mathfrak{M}^+(\mathrm{X}),$$

is vaguely lower semicontinuous (see, e.g., [9, Section 1.1]).

Fix a vector-valued function $\mathbf{f} = (f_i)_{i=1,2}$, where each $f_i : A_i \to [-\infty, \infty]$ is universally measurable and it is treated as an *external field* acting on the charges from $\mathfrak{M}^+(A_i; D_i)$. Then the $\mathbf{f}$-*weighted* α-*Riesz energy* of $\boldsymbol{\nu} \in \mathcal{E}^+_\alpha(\mathbf{A})$ is defined by

$$G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}) := E_\alpha(\boldsymbol{\nu}) + 2\langle \mathbf{f}, \boldsymbol{\nu}\rangle; \tag{3.6}$$

$G_{\alpha,\mathbf{f}}(\cdot)$ is also known as the *Gauss functional* (see, e.g., [13]). Let $\mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A})$ consist of all $\boldsymbol{\nu} \in \mathcal{E}^+_\alpha(\mathbf{A})$ with finite $G_{\alpha,\mathbf{f}}(\boldsymbol{\nu})$.

In this paper, we tacitly assume that one of the following Cases I or II holds:

I. *For every $i = 1, 2$, $f_i \in \Phi(A_i)$, where A_i is thought of as a locally compact space*;

II. *For every $i = 1, 2$, $f_i = s_i U^\zeta_\alpha\big|_{A_i}$, where a (signed) scalar measure $\zeta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ is given.*

For any $\boldsymbol{\nu} \in \breve{\mathcal{E}}^+_\alpha(\mathbf{A})$, $G_{\alpha,\mathbf{f}}(\boldsymbol{\nu})$ is then well defined in both Cases I and II. Furthermore, if Case II takes place, then, by (3.6) and (3.4),

$$\begin{aligned} G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}) &= \|R\boldsymbol{\nu}\|^2_\alpha + 2\sum_{i=1,2} s_i E_\alpha(\zeta, \nu^i) \\ &= \|R\boldsymbol{\nu}\|^2_\alpha + 2E_\alpha(\zeta, R\boldsymbol{\nu}) = \|R\boldsymbol{\nu} + \zeta\|^2_\alpha - \|\zeta\|^2_\alpha \end{aligned} \tag{3.7}$$

and, consequently,

$$-\infty < -\|\zeta\|^2_\alpha \leqslant G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}) < \infty \quad \text{for all} \ \ \boldsymbol{\nu} \in \breve{\mathcal{E}}^+_\alpha(\mathbf{A}). \tag{3.8}$$

Also fix a numerical vector $\mathbf{a} = (a_i)_{i=1,2}$ with $a_i > 0$ and a vector-valued function $\mathbf{g} = (g_i)_{i=1,2}$, where all the $g_i : D_i \to (0, \infty)$ are continuous and such that

$$g_{i,\inf} := \inf_{x \in A_i} g_i(x) > 0. \tag{3.9}$$

Write

$$\begin{aligned} \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \bigl\{\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A}) : \ \langle g_i, \nu^i\rangle = a_i \quad \text{for all} \ \ i = 1,\, 2\bigr\}, \\ \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A}), \\ G_{\alpha,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \inf_{\boldsymbol{\nu} \in \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})} G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}). \end{aligned}$$

Observe that, because of (3.9),

$$\nu^i(A_i) \leqslant a_i g^{-1}_{i,\inf} < \infty \quad \text{for all} \ \ \boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$$

and, therefore,

$$\mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \breve{\mathfrak{M}}^+(\mathbf{A}), \quad \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \breve{\mathcal{E}}^+_\alpha(\mathbf{A}). \tag{3.10}$$

Combined these with Lemma 3.4 and the fact that a lower semicontinuous function is bounded from below on a compact set, in Case I we obtain

$$G_{\alpha,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) > -\infty.$$

The same holds true in Case II as well, which is obvious from (3.8) and (3.10).

If the class $\mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is nonempty or, equivalently, if

$$G_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty, \tag{3.11}$$

then the following (unconstrained) $\mathbf{f}$-weighted minimum α-Riesz energy problem, also known as the *Gauss variational problem* (see [10, 13]), makes sense.

Problem 3.5. *Does there exist $\boldsymbol{\lambda}_{\mathbf{A}} \in \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ with $G_{\alpha,\mathbf{f}}(\boldsymbol{\lambda}_{\mathbf{A}}) = G_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$?*

Remark 3.6. *Analysis similar to that for a standard condenser (cf. Lemma* 6.2 *in* [17]*) shows that assumption* (3.11) *is equivalent to the following one:*

$$f_i(x) < \infty \quad \text{n.e. in} \;\; A_i, \quad i = 1, 2.$$

In turn, this yields that (3.11) *holds automatically whenever Case* II *takes place, for the α-Riesz potential of $\zeta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ is finite n.e. in $\mathbb{R}^n$.*

Remark 3.7. *In the case where every A_i is compact in D_i (i.e., $\mathbf{A}$ is a compact standard condenser) and Case* I *takes place, the solvability of Problem* 3.5 *can easily be established by exploiting the $\mathbf{A}$-vague topology only, since then $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ is $\mathbf{A}$-vaguely compact, while $G_{\alpha,\mathbf{f}}(\cdot)$ is $\mathbf{A}$-vaguely lower semicontinuous on $\mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A})$ (see* [13, *Theorem* 2.30]*). However, these arguments break down if any of the two requirements is not satisfied, and then Problem* 3.5 *becomes rather nontrivial. E.g., $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ is no longer $\mathbf{A}$-vaguely compact if some of the A_i is noncompact in D_i.*

Remark 3.8. *Assume that $\mathbf{A}$ is still a standard condenser, though now, in contrast to Remark* 3.7*, its plates might be noncompact in $\mathbb{R}^n$. Under the assumption*

$$\operatorname{dist}(A_1, A_2) := \inf_{x\in A_1,\ y\in A_2} |x - y| > 0, \tag{3.12}$$

in [17, 18] *we worked out an approach based on both the $\mathbf{A}$-vague and the strong topologies on $\mathcal{E}^+_\alpha(\mathbf{A})$ and a certain strong completeness result, which made it possible to provide a fairly complete analysis of Problem* 3.5*. In more detail, it has been shown that, if $g_i|_{A_i}$, $i = 1, 2$, are bounded from above, then, in both Cases* I *and* II*,*

$$C_\alpha(A_1 \cup A_2) < \infty \tag{3.13}$$

is sufficient for Problem 3.5 *to be (uniquely) solvable for every $\mathbf{a}$ (see* [17]*, Theorem* 8.1*). However, if* (3.13) *does not hold, then, in general, there exists a vector $\mathbf{a}'$ such that the Gauss variational problem admits no solution* [17]*. Therefore, it was interesting to give a description of the set of all vectors $\mathbf{a}$ for which the problem would be, nevertheless, solvable. Such a characterization has been established in* [18].

In the rest of the paper, except for Remark 3.10, we do not assume (3.12) necessarily to hold. Then the results obtained in [17, 18] and the approach developed are no longer valid. In particular, assumption (3.13) does not guarantee anymore that $G_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is attained among $\boldsymbol{\nu} \in \mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Using the electrostatic interpretation, a short-circuit between the touching oppositely-charged plates of the condenser might occur. Therefore, it is meaningful to ask what kind of additional requirements on the measures under consideration would prevent this phenomenon, and a solution to the corresponding **f**-weighted minimum α-Riesz energy problem would, nevertheless, exist.

The idea discussed below is to find out such an upper constraint on the measures from $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ which would not allow the "blow-up" effect between A_1 and A_2.

3.4. Constrained f-weighted minimum α-Riesz energy problem

Let $\mathfrak{C}(\mathbf{A})$ consist of all $\boldsymbol{\sigma}=(\sigma^i)_{i=1,2}\in\mathfrak{M}^+(\mathbf{A})$ such that

$$S_{D_i}^{\sigma^i}=A_i \quad\text{and}\quad \langle g_i,\sigma^i\rangle>a_i \quad\text{for all}\quad i=1,2; \tag{3.14}$$

these $\boldsymbol{\sigma}$ will serve as *constraints* for $\boldsymbol{\nu}\in\mathfrak{M}^+(\mathbf{A})$. Given $\boldsymbol{\sigma}\in\mathfrak{C}(\mathbf{A})$, write

$$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}):=\bigl\{\boldsymbol{\nu}\in\mathfrak{M}^+(\mathbf{A}):\ \nu^i\leqslant\sigma^i\quad\text{for all}\quad i=1,\,2\bigr\},$$

where $\nu^i\leqslant\sigma^i$ means that $\sigma^i-\nu^i$ is a nonnegative scalar measure, and

$$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g}):=\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})\cap\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g}),$$
$$\mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}):=\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})\cap\mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A}).$$

Since $\mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\subset\mathcal{E}^+_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, we get

$$-\infty<G_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\leqslant G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}):=\inf_{\boldsymbol{\nu}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})}G_{\alpha,\mathbf{f}}(\boldsymbol{\nu})\leqslant\infty.$$

If the class $\mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is nonempty or, equivalently, if

$$G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})<\infty, \tag{3.15}$$

then the following constrained **f**-weighted minimum α-Riesz energy problem, also known as the *constrained Gauss variational problem*, makes sense.

Problem 3.9. *Given $\boldsymbol{\sigma}\in\mathfrak{C}(\mathbf{A})$, does there exist $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ with*

$$G_{\alpha,\mathbf{f}}(\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}})=G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})?$$

Remark 3.10. *Assume for a moment that* (3.12) *holds. It has been shown by* [16, *Theorem* 6.2] *that if, in addition, $g_i|_{A_i}$, $i=1,2$, are bounded from above and conditions* (3.13) *and* (3.15) *are satisfied, then, in both Cases* I *and* II, *Problem* 3.9 *is* (*uniquely*) *solvable. But this does not remain true if requirement* (3.12) *is dropped.*

Remark 3.11. *If $0<\alpha\leqslant 2<n$, $a_1=a_2$, $\mathbf{g}=\mathbf{1}$, A_2 is not α-thin at $\omega_{\mathbb{R}^n}$, $f_2=0$ and $\sigma^2=\infty$ (i.e., no external field and no constraint act on the measures concentrated in A_2), then sufficient and/or necessary conditions for the solvability of Problem* 3.9 *have been established in* [7]. *Crucial to the arguments exploited in* [7] *is that, in this special case, Problem* 3.9 *can be reduced to the problem of minimizing the f_1-weighted $g^\alpha_{D_1}$-Green energy over the class $\mathcal{E}^+_{g^\alpha_{D_1}}(A_1;D_1)$. However, under the assumptions of the present study, such an observation is no longer valid.*

Remark 3.12. *If $a_1 = a_2$, $\mathbf{g} = \mathbf{1}$, $\mathbf{f} = \mathbf{0}$ and A_i, $i = 1, 2$, are bounded, then the constrained minimum logarithmic energy problem for a condenser with touching plates in $\mathbb{C}$ has been investigated by Beckermann and Gryson* (*see* [1, *Theorem* 2.2]). *Our paper is related to the α-Riesz kernels, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geqslant 2$, and the results obtained and the approaches developed are rather different from those in* [1].

4. Sufficient conditions for the solvability of Problem 3.9

Denote by $\overline{B}$ the closure of $B \subset \mathbb{R}^n$ in $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\omega_{\mathbb{R}^n}\}$, the one-point compactification of $\mathbb{R}^n$.

Theorem 4.1. *Let $\mathbf{A}$, $\mathbf{f}$, $\mathbf{g}$ and $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ possess the following four properties:*

(a′) *$\overline{A_1} \cap \overline{A_2}$ consists of at most one point, i.e., $\overline{A_1} \cap \overline{A_2} = \varnothing \vee \{x_0\}$ where $x_0 \in \overline{\mathbb{R}^n}$;*

(b′) *$f_i(x) < \infty$ n.e. in A_i, $i = 1, 2$;*

(c′) *$E_\alpha\big(\sigma^i\big|_{K_i}\big) < \infty$ for every compact $K_i \subset A_i$, $i = 1, 2$;*

(d′) *$\langle g_i, \sigma^i \rangle < \infty$, $i = 1, 2$.*

Then, in both Cases I *and* II, *Problem* 3.9 *is uniquely solvable for every vector* $\mathbf{a}$.

The proof of Theorem 4.1 is given in Section 6; it is based on Theorem 5.1, which provides a strong completeness result for metric subspaces of $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$.

Example 4.2. *Let $\mathbf{A} = (A_1, A_2)$ be as in Example* 3.2. *Having fixed $\alpha \in (0, 3)$, assume that $\mathbf{g} = \mathbf{1}$ and either Case* II *holds or $f_i(x) < \infty$ n.e. in A_i, $i = 1, 2$. For any $\mathbf{a} = (a_i)_{i=1,2}$ define $\sigma^i := c_i m_3|_{A_i}$, where $c_i \in (a_i, \infty)$ is chosen arbitrarily and m_3 denotes the* 3*-dimensional Lebesgue measure on $\mathbb{R}^3$. Then, by Theorem* 4.1, *Problem* 3.9 *admits a solution; hence, no short-circuit between A_1 and A_2 occurs, though these conductors touch each other at the point ξ_0* (*see Example* 3.2).

5. Strong completeness theorem for metric subspaces of $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$

Let $\mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ consist of all $\boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A})$ such that $\langle g_i, \nu^i \rangle \leqslant a_i$ for all $i = 1, 2$. In view of (3.9),

$$\nu^i(A_i) \leqslant a_i g_{i,\inf}^{-1} < \infty \quad \text{for all} \quad \boldsymbol{\nu} \in \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}). \tag{5.1}$$

Hence, $\mathcal{E}^+_\alpha(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) := \mathcal{E}^+_\alpha(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ can be thought of as a metric subspace of $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$; its topology will likewise be called *strong*.

Theorem 5.1. *Suppose that a generalized condenser $\mathbf{A}$ satisfies condition* (a′) *of Theorem* 4.1. *Then the metric space $\mathcal{E}^+_\alpha(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is strongly complete and the strong topology on this space is finer than the induced $\mathbf{A}$-vague topology.*

Remark 5.2. *In view of the fact that the metric space $\mathcal{E}^+_\alpha(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is isometric to its R-image, Theorem* 5.1 *has singled out a strongly complete topological subspace of the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$, whose elements are signed Radon measures. This is of independent interest since, according to a well-known counterexample by Cartan, the whole pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$ is, in general, strongly incomplete.*

5.1. Auxiliary results

Based on the definition of the **A**-vague topology (see Definition 3.3), we call a set $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ **A**-*vaguely bounded* if, for every $i = 1, 2$ and every $\varphi \in \mathrm{C}_0(D_i)$,

$$\sup_{\boldsymbol{\nu} \in \mathfrak{F}} |\nu^i(\varphi)| < \infty.$$

Lemma 5.3. *If $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ is* **A**-*vaguely bounded, then it is* **A**-*vaguely relatively compact.*

Proof. Since by [3, Chapter III, Section 2, Proposition 9] any vaguely bounded part of $\mathfrak{M}^+(D_i)$ is vaguely relatively compact, the lemma follows from Tychonoff's theorem on the product of compact spaces (see, e.g., [11, Chapter 5, Theorem 13]). □

Lemma 5.4. $\mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ *is* **A**-*vaguely bounded and* **A**-*vaguely closed; hence, it is* **A**-*vaguely compact.*

Proof. Indeed, it is obvious from (5.1) that $\mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$ is **A**-vaguely bounded. Fix an arbitrary $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$; then, by Lemma 5.3, it has an **A**-vague cluster point $\boldsymbol{\nu}_0$. In fact, $\boldsymbol{\nu}_0 \in \mathfrak{M}^+(\mathbf{A})$, for $\mathfrak{M}^+(\mathbf{A})$ is **A**-vaguely closed. Choose a subsequence $\{\boldsymbol{\nu}_{k_m}\}_{m\in\mathbb{N}}$ of $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ that converges **A**-vaguely to $\boldsymbol{\nu}_0$. As g_i is positive and continuous, we get

$$\langle g_i, \nu_0^i \rangle \leqslant \liminf_{m\to\infty} \langle g_i, \nu_{k_m}^i \rangle \leqslant a_i \quad \text{for all} \quad i = 1, 2,$$

and the lemma follows. □

Lemma 5.5. *Assume that* **A** *is a standard condenser; i.e., $\overline{A_1} \cap \overline{A_2} = \varnothing \vee \{\omega_{\mathbb{R}^n}\}$. Then the metric space $\mathcal{E}^+_\alpha(\mathbf{A})$ $\bigl(= \breve{\mathcal{E}}^+_\alpha(\mathbf{A})\bigr)$ is strongly complete. In more detail, any strong Cauchy sequence $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^+_\alpha(\mathbf{A})$ converges both strongly and* **A**-*vaguely to some $\boldsymbol{\nu}_0 \in \mathcal{E}^+_\alpha(\mathbf{A})$, and this limit is unique.*

Proof. It is clear from (3.2) that, for a standard **A**,

$$\mathcal{E}^+_\alpha(\mathbf{A}) = \breve{\mathcal{E}}^+_\alpha(\mathbf{A}).$$

Since $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$ and $R\bigl(\breve{\mathcal{E}}^+_\alpha(\mathbf{A})\bigr)$, the latter being treated as a metric subspace of the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$, are isometric to each other by (3.5), the lemma follows from [15] (see Theorem 1 and Corollary 1 therein). □

5.2. Proof of Theorem 5.1

Fix a strong Cauchy sequence $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^+_\alpha(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$. According to Lemma 5.4, it has an **A**-vague cluster point $\boldsymbol{\nu}_0 \in \mathfrak{M}^+(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g})$. Let $\{\boldsymbol{\nu}_{k_m}\}_{m\in\mathbb{N}}$ be a (strong Cauchy) subsequence of $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ that converges **A**-vaguely to $\boldsymbol{\nu}_0$, i.e.

$$\nu^i_{k_m} \to \nu^i_0 \quad \text{vaguely in } \ \mathfrak{M}(D_i), \ i = 1, 2. \tag{5.2}$$

We proceed by showing that $E_\alpha(\boldsymbol{\nu}_0)$ is finite, so that

$$\boldsymbol{\nu}_0 \in \mathcal{E}^+_\alpha(\mathbf{A}, \leqslant \mathbf{a}, \mathbf{g}) \quad \bigl(\subset \breve{\mathcal{E}}^+_\alpha(\mathbf{A})\bigr), \tag{5.3}$$

and, moreover, $\boldsymbol{\nu}_{k_m} \to \boldsymbol{\nu}_0$ strongly as $m \to \infty$, i.e.

$$\lim_{m\to\infty} \|\boldsymbol{\nu}_{k_m} - \boldsymbol{\nu}_0\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})} = 0. \tag{5.4}$$

To establish these assertions, it is enough to analyze the case

$$\overline{A_1} \cap \overline{A_2} = \{x_0\} \quad \text{where} \ \ x_0 \in \mathbb{R}^n, \tag{5.5}$$

since otherwise they are obtained directly from Lemma 5.5.

Consider the inversion I with respect to the $(n-1)$-dimensional unit sphere centered at x_0; namely, each point $x \neq x_0$ is mapped to the point x^* on the ray through x which issues from x_0, determined uniquely by

$$|x - x_0| \cdot |x^* - x_0| = 1.$$

This is a one-to-one, bicontinuous mapping of $\mathbb{R}^n \setminus \{x_0\}$ onto itself; furthermore,

$$|x^* - y^*| = \frac{|x - y|}{|x_0 - x||x_0 - y|}. \tag{5.6}$$

Extend it to a one-to-one, bicontinuous map of $\overline{\mathbb{R}^n}$ onto itself by setting $I(x_0) = \omega_{\mathbb{R}^n}$.

To each signed scalar measure $\nu \in \mathfrak{M}(\mathbb{R}^n)$ with $\nu\bigl(\{x_0\}\bigr) = 0$ there corresponds the Kelvin transform $\nu^* \in \mathfrak{M}(\mathbb{R}^n)$ by means of the formula

$$d\nu^*(x^*) = |x - x_0|^{\alpha-n}\, d\nu(x), \quad x^* \in \mathbb{R}^n$$

(see [14] or [12, Chapter IV, Section 5, n° 19]). Then, in view of (5.6),

$$U^{\nu^*}_\alpha(x^*) = |x - x_0|^{n-\alpha} U^\nu_\alpha(x), \quad x^* \in \mathbb{R}^n,$$

and therefore

$$E_\alpha(\nu^*) = E_\alpha(\nu). \tag{5.7}$$

It is clear that the Kelvin transformation is additive and it is an involution, i.e.

$$\bigl(\nu_1 + \nu_2\bigr)^* = \nu^*_1 + \nu^*_2, \tag{5.8}$$

$$(\nu^*)^* = \nu. \tag{5.9}$$

Write $A^*_i := I\bigl(\overline{A_i}\bigr) \cap \mathbb{R}^n$, $i = 1, 2$; then $\mathbf{A}^* = (A^*_1, A^*_2)$ forms a standard condenser in $\mathbb{R}^n$, which is obvious from (5.5) and the above-mentioned properties of I.

Applying the Kelvin transformation to each of the components of any given $\boldsymbol{\nu} = (\nu^i)_{i=1,2} \in \breve{\mathfrak{M}}^+(\mathbf{A})$, we get $\boldsymbol{\nu}^* := \bigl((\nu^i)^*\bigr)_{i=1,2} \in \mathfrak{M}^+(\mathbf{A}^*)$; and the other way around. Based on Lemma 3.4 and relations (3.5) and (5.7)–(5.9), we also see that the α-Riesz energy of $\boldsymbol{\nu} \in \breve{\mathfrak{M}}^+(\mathbf{A})$ is well defined if and only if so is that of $\boldsymbol{\nu}^*$, and then they coincide; and, furthermore,

$$\|\boldsymbol{\nu}_1^* - \boldsymbol{\nu}_2^*\|_{\mathcal{E}_\alpha^+(\mathbf{A}^*)} = \|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\|_{\breve{\mathcal{E}}_\alpha^+(\mathbf{A})} \quad \text{for all } \ \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \breve{\mathcal{E}}_\alpha^+(\mathbf{A}). \tag{5.10}$$

Summarizing what has thus been observed, we conclude that the Kelvin transformation is a one-to-one, isometric mapping of $\breve{\mathcal{E}}_\alpha^+(\mathbf{A})$ onto $\mathcal{E}_\alpha^+(\mathbf{A}^*)$.

Let $\boldsymbol{\nu}_{k_m}$, $m \in \mathbb{N}$, and $\boldsymbol{\nu}_0$ be as above. In view of (5.1) and (5.2), for each $i = 1, 2$ one can apply [12, Lemma 4.3] to $\nu_{k_m}^i$, $k \in \mathbb{N}$, and ν_0^i, and consequently

$$\boldsymbol{\nu}_{k_m}^* \to \boldsymbol{\nu}_0^* \quad \mathbf{A}\text{-vaguely as } \ m \to \infty. \tag{5.11}$$

But $\{\boldsymbol{\nu}_{k_m}^*\}_{m\in\mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}_\alpha^+(\mathbf{A}^*)$, which is clear from (5.10). This together with (5.11) implies, by Lemma 5.5, that $\boldsymbol{\nu}_0^* \in \mathcal{E}_\alpha^+(\mathbf{A}^*)$ and

$$\lim_{m\to\infty} \|\boldsymbol{\nu}_{k_m}^* - \boldsymbol{\nu}_0^*\|_{\mathcal{E}_\alpha^+(\mathbf{A}^*)} = 0.$$

Repeated application of (5.10) then leads to relations (5.3) and (5.4) as claimed.

In turn, (5.4) yields $\boldsymbol{\nu}_k \to \boldsymbol{\nu}_0$ strongly as $k \to \infty$, for $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ is strongly fundamental. It has thus been established that $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ converges strongly to any of its $\mathbf{A}$-vague cluster points. As $\|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\|_{\breve{\mathcal{E}}_\alpha^+(\mathbf{A})}$ is a metric, $\boldsymbol{\nu}_0$ has to be the unique $\mathbf{A}$-vague cluster point of $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$. Since the $\mathbf{A}$-vague topology is Hausdorff, $\boldsymbol{\nu}_0$ is actually also the $\mathbf{A}$-vague limit of $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ (cf. [2, Chapter I, Section 9, n° 1]). This completes the proof. □

6. Proof of Theorem 4.1

We start by observing that $\mathcal{E}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty and, hence, (3.15) holds. Indeed, it is seen from assumptions (3.14) and (b′) in consequence of [9, Lemma 1.2.2] that, for every $i = 1, 2$, there is a compact set $K_i \subset A_i$ such that $\langle g_i, \sigma^i|_{K_i}\rangle > a_i$ and $f_i(x) \leqslant M < \infty$ for all $x \in K_i$. Define $\theta^i := \sigma^i|_{K_i} / \langle g_i, \sigma^i|_{K_i}\rangle$. Due to assumption (c′) and Lemma 3.4, we then obtain $\boldsymbol{\theta} := (\theta^i)_{i=1,2} \in \mathcal{E}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ as claimed.

Therefore, the class $\mathbb{M}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ of all $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with

$$\lim_{k\to\infty} G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_k) = G_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \tag{6.1}$$

is nonempty. Fix arbitrary $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{\mu}_m\}_{m\in\mathbb{N}}$ in $\mathbb{M}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Taking (3.10) into account, we proceed by proving that

$$\lim_{k,m\to\infty} \|\boldsymbol{\nu}_k - \boldsymbol{\mu}_m\|_{\breve{\mathcal{E}}_\alpha^+(\mathbf{A})} = 0. \tag{6.2}$$

Based on the convexity of $\mathcal{E}_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, from (3.4) and (3.6) we get

$$4G_{\alpha,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leqslant 4G_{\alpha,\mathbf{f}}\Big(\frac{\boldsymbol{\nu}_k + \boldsymbol{\mu}_m}{2}\Big) = \|R\boldsymbol{\nu}_k + R\boldsymbol{\mu}_m\|_\alpha^2 + 4\langle \mathbf{f}, \boldsymbol{\nu}_k + \boldsymbol{\mu}_m\rangle.$$

On the other hand, applying the parallelogram identity in the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$ to $R\boldsymbol{\nu}_k$ and $R\boldsymbol{\mu}_m$ and then adding and subtracting $4\langle \mathbf{f}, \boldsymbol{\nu}_k + \boldsymbol{\mu}_m\rangle$, we have

$$\|R\boldsymbol{\nu}_k - R\boldsymbol{\mu}_m\|_\alpha^2 = -\|R\boldsymbol{\nu}_k + R\boldsymbol{\mu}_m\|_\alpha^2 - 4\langle \mathbf{f}, \boldsymbol{\nu}_k + \boldsymbol{\mu}_m\rangle + 2G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_k) + 2G_{\alpha,\mathbf{f}}(\boldsymbol{\mu}_m).$$

When combined with the preceding relation, this gives

$$0 \leqslant \|R\boldsymbol{\nu}_k - R\boldsymbol{\mu}_m\|_\alpha^2 \leqslant -4G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) + 2G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_k) + 2G_{\alpha,\mathbf{f}}(\boldsymbol{\mu}_m).$$

On account of (3.5), (6.1) and the fact that $G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is finite, we derive (6.2) from the very relation by letting $k, m \to \infty$.

Assuming now $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{\mu}_m\}_{m\in\mathbb{N}}$ in (6.2) to be equal, we see that any fixed sequence $\{\boldsymbol{\nu}_k\}_{k\in\mathbb{N}} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is strongly fundamental in the metric space $\mathcal{E}^+_\alpha(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$. Thus, by Theorem 5.1, there exists the unique $\boldsymbol{\nu}_0 \in \mathcal{E}^+_\alpha(\mathbf{A},\leqslant\mathbf{a},\mathbf{g})$ such that

$$\boldsymbol{\nu}_k \to \boldsymbol{\nu}_0 \quad \mathbf{A}\text{-vaguely (as } k \to \infty), \tag{6.3}$$

$$\lim_{k\to\infty} \|\boldsymbol{\nu}_k - \boldsymbol{\nu}_0\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})} = 0. \tag{6.4}$$

We assert that this $\boldsymbol{\nu}_0$ gives a solution to Problem 3.9, i.e.

$$\boldsymbol{\nu}_0 \in \mathcal{E}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) \quad \text{and} \quad G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_0) = G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{6.5}$$

Observe that

$$G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_0) \leqslant \liminf_{k\to\infty} G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_k).$$

Indeed, if Case I holds, then this inequality can be obtained directly from (6.3) and (6.4), while otherwise it follows from (6.4) with the help of (3.7). Combining it with (6.1) and (3.15), we get $G_{\alpha,\mathbf{f}}(\boldsymbol{\nu}_0) \leqslant G^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty$.

As $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$ is $\mathbf{A}$-vaguely closed, we therefore conclude that relation (6.5) will have been established once for each $i = 1, 2$ we show

$$\langle g_i, \nu_0^i\rangle = a_i. \tag{6.6}$$

Consider an exhaustion of A_i by an increasing sequence of compact sets $K_\ell \subset A_i$, $\ell \in \mathbb{N}$. In view of the positivity and continuity of g_i on A_i, from (6.3) and [9, Lemma 1.2.2] we get

$$\begin{aligned} a_i \geqslant \langle g_i, \nu_0^i\rangle &= \lim_{\ell\to\infty} \bigl\langle g_i\chi_{K_\ell}, \nu_0^i\bigr\rangle \geqslant \lim_{\ell\to\infty} \limsup_{k\to\infty} \bigl\langle g_i\chi_{K_\ell}, \nu_k^i\bigr\rangle \\ &= a_i - \lim_{\ell\to\infty} \liminf_{k\to\infty} \bigl\langle g_i\chi_{A_i\setminus K_\ell}, \nu_k^i\bigr\rangle. \end{aligned}$$

Hence, to prove (6.6), it is enough to verify the relation

$$\lim_{\ell\to\infty} \liminf_{k\to\infty} \bigl\langle g_i\chi_{A_i\setminus K_\ell}, \nu_k^i\bigr\rangle = 0. \tag{6.7}$$

Since, by (d′),

$$\infty > \langle g_i, \sigma^i\rangle = \lim_{\ell\to\infty} \bigl\langle g_i\chi_{K_\ell}, \sigma^i\bigr\rangle,$$

we have

$$\lim_{\ell\to\infty} \bigl\langle g_i\chi_{A_i\setminus K_\ell}, \sigma^i\bigr\rangle = 0.$$

When combined with

$$\bigl\langle g_i\chi_{A_i\setminus K_\ell}, \nu_k^i\bigr\rangle \leqslant \bigl\langle g_i\chi_{A_i\setminus K_\ell}, \sigma^i\bigr\rangle \quad \text{for all} \quad \ell, k \in \mathbb{N},$$

this implies (6.7), hence (6.6), and consequently (6.5).

It is left to establish the statement on the uniqueness. Let, on the contrary, $\widehat{\boldsymbol{\nu}}_0$ be an other solution of Problem 3.9. Then trivial sequences $\{\boldsymbol{\nu}_0\}$ and $\{\widehat{\boldsymbol{\nu}}_0\}$ are both elements of $\mathbb{M}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and therefore, by (6.2), $\|\boldsymbol{\nu}_0-\widehat{\boldsymbol{\nu}}_0\|_{\breve{\mathcal{E}}^+_\alpha(\mathbf{A})}=0$. As $\breve{\mathcal{E}}^+_\alpha(\mathbf{A})$ is a metric space, this results in $\boldsymbol{\nu}_0=\widehat{\boldsymbol{\nu}}_0$, and the proof is complete. $\square$

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References

[1] B. Beckermann, A. Gryson, *Extremal rational functions on symmetric discrete sets and superlinear convergence of the ADI method*, Constr. Approx. **32** (2010), 393–428.

[2] N. Bourbaki, *Elements of Mathematics. General Topology. Chap.* 1–4. Springer, Berlin, 1989.

[3] N. Bourbaki, *Elements of Mathematics. Integration. Chap.* 1–6, Springer, Berlin, 2004.

[4] H. Cartan, *Théorie du potentiel Newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. Fr. **73** (1945), 74–106.

[5] J. Deny, *Les potentiels d'énergie finite*, Acta Math. **82** (1950), 107–183.

[6] J. Deny, *Sur la définition de l'énergie en théorie du potentiel*, Ann. Inst. Fourier Grenoble **2** (1950), 83–99.

[7] P.D. Dragnev, D.P. Hardin, E.B. Saff, N. Zorii, *Minimum Riesz energy problems for a condenser with "touching plates"*, ArXive:1504.03805 (2015), 32 p.

[8] R. Edwards, *Functional analysis. Theory and applications*, Holt. Rinehart and Winston, New York, 1965.

[9] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215.

[10] C.F. Gauss, *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs– und Abstoßungs–Kräfte* (1839), Werke **5** (1867), 197–244.

[11] J.L. Kelley, *General Topology*, Princeton, New York (1957).

[12] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer, Berlin (1972).

[13] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 **25** (1961), 135–352.

[14] M. Riesz, *Intégrales de Riemann–Liouville et potentiels*, Acta Szeged, **9** (1938), 1–42.

[15] N. Zorii, *A noncompact variational problem in Riesz potential theory*. I; II, Ukr. Math. J. **47** (1995), 1541–1553; **48** (1996), 671–682.

[16] N. Zorii, *Constrained energy problems with external fields for vector measures*, Math. Nachr. **285** (2012), 1144–1165.

[17] N. Zorii, *Equilibrium problems for infinite dimensional vector potentials with external fields*, Potential Anal. **38** (2013), 397–432.

[18] N. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem for infinite dimensional vector measures*, Potential Anal. **41** (2014), 81–115.

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WARIACYJNY PROBLEM GAUSSA Z WARUNKAMI POBOCZNYMI DLA KONDENSATORÓW ZE STYKAJĄCYMI SIĘ OKŁADKAMI

S t r e s z c z e n i e

Badany problem minimum energii z warunkami pobocznymi przy zewnętrznym polu związanym z jądrem α-Riesza $|x-y|^{\alpha-n}$ dowolnego rzędu $\alpha \in (0,n)$ dla uogólnionego kondensatora $\mathbf{A} = (A_1, A_2)$ ze stykającymi się przeciwnie naładowanymi okładkami w $\mathbb{R}^n$, $n \geqslant 2$. Uzyskujemy warunki wystarczające dla rozwiązalności tak postawionego problemu. Nasze rozumowanie opiera się głównie na definicji stosowanej struktury metrycznej na zbiorze miar wektorowych stowarzyszonych z kondensatorem $\mathbf{A}$ i na uzyskaniu twierdzenia o zupełności dla odpowiedniej przestrzeni metrycznej.

Słowa kluczowe: zagadnienia minimalizacji energii typu Riesza, pole zewnętrzne, warunek poboczny, kondensator ze stykającymi się okładkami, twierdzenie o silnej zupełności dla miar wektorowych

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EXACTLY MAXIMALLY CONVERGENT SEQUENCES OF MULTIPOINT PADÉ APPROXIMANTS

Summary

Given a regular compact set E in $\mathbb{C}$, a unit measure μ supported by E, a triangular point set $\beta := \{\{\beta_{n,k}\}_{k=1}^{n}\}_{n=1}^{\infty}, \beta \subset E$ and a function f, holomorphic on E, let $\pi_{n,m}^{\beta,f}$ be the associated multipoint β-Padé approximant of order (n,m). Under the condition that the points β are uniformly distributed relatively to the measure μ, we provide results about the existence of exactly maximally convergent sequences $\pi_{n,m}^{\beta,f}$ as $n \to \infty, m-$ fixed relatively to μ and the domain of the $m-$ meromorphy of the function f.

Keywords and phrases: multipoint Padé approximants, maximal convergence, domain of m-meromorphy

1. Introduction

We first introduce some needed notations.

Let Π_n, $n \in \mathbb{N}$ be the class of the polynomials of degree $\leq n$ and $\mathcal{R}_{n,m} := \{r = p/q, p \in \Pi_n, q \in \Pi_m, q \not\equiv 0\}$.

Given a compact set E, we say that E is *regular*, if the unbounded component of the complement $E^c := \overline{\mathbb{C}} \setminus E$ is solvable with respect to Dirichlet problem. We will assume throughout the paper that E possesses a connected complement E^c. In what follows, we will be working with the max-norm $||...||_E$ on E; that is $||...||_E := \max_{z\in E} |...|(z)$.

Let $\mathcal{B}(E)$ be the class of the unit measures supported on E; that is supp$(...) \subseteq E$. We say that the infinite sequence of Borel measures $\{\mu_n\} \in \mathcal{B}(E)$ converges in the weak topology to a measure μ and write $\mu_n \longrightarrow \mu$, if

$$\int g(t)d\mu_n \to \int g(t)d\mu$$

for every function g continuous on E. We associate with a measure $\mu \in \mathcal{B}(E)$ the logarithmic potential $U^\mu(z)$; that is,

$$U^\mu(z) := \int \log \frac{1}{|z-t|} d\mu.$$

Recall that U^μ ([1]) is a function superharmonic in $\mathbb{C}$, subharmonic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$, harmonic in $\mathbb{C} \setminus \operatorname{supp}(\mu)$ and

$$U^\mu(z) = \ln \frac{1}{|z|} + o(1), z \to \infty.$$

We now associate with a polynomial $p \in \Pi_n$ the normalized counting measure μ_p of p, that is

$$\mu_p(F) := \frac{\text{number of zeros of } p \text{ on } F}{\deg p},$$

where F is a point set in $\mathbb{C}$.

Given a domain $B \subset \mathbb{C}$, a function g and a number $m \in \mathbb{N}$, we say that g is m-meromorphic in B ($g \in \mathcal{M}_m(B)$) if g has no more than m poles in B (poles are counted with their multiplicities). We say that a function f is holomorphic on the compactum E and write $f \in \mathcal{A}(E)$, if it is holomorphic in some open neighborhood of E.

Let β be an infinite triangular table of points, $\beta := \{\{\beta_{n,k}\}_{k=1}^n\}_{n=1,2,\ldots}, \beta_{n,k} \in E$, with no limit points outside E (we write $\beta \in E$). Set

$$\omega_n(z) := \prod_{k=1}^n (z - \beta_{n,k}).$$

Let $f \in \mathcal{A}(E)$ and (n, m) be a fixed pair of nonnegative integers. The rational function $\pi_{n,m}^{\beta,f} := p/q$, where the polynomials $p \in \Pi_n$ and $q \in \Pi_m$ are such that

$$\frac{fq - p}{\omega_{n+m+1}} \in \mathcal{A}(E)$$

is called *a β-multipoint Padé approximant of f of order* (n, m). As is well known, the function $\pi_{n,m}^{\beta,f}$ always exists and is unique ([2], [3]). In the particular case when $\beta \equiv 0$, the multipoint Padé approximant $\pi_{n,m}^{\beta,f}$ coincides with the classical Padé approximant $\pi_{n,m}^f$ of order (n, m) ([4]).

Set

$$\pi_{n,m}^{\beta,f} := \frac{P_{n,m}^{\beta,f}}{Q_{n,m}^{\beta,f}}, \tag{1}$$

where the polynomials $P_{n,m}^{\beta,f}$ and $Q_{n,m}^{\beta,f}$ do not have common divisors. The zeros of $Q_{n,m}^{\beta,f}$ are called *free zeros* of $\pi_{n,m}^{\beta,f}$; $\deg Q_{n,m} \le m$.

We say that the points $\beta_{n,k}$ are *uniformly distributed relatively to the measure* μ, if

$$\mu_{\omega_n} \longrightarrow \mu,\ n \to \infty.$$

We recall the notion of m_1-Hausdorff measure (cf. [5]). For $\Omega \subset \mathbb{C}$, we set

$$m_1(\Omega) := \inf \left\{ \sum_{\nu} |V_\nu| \right\}$$

where the infimum is taken over all coverings $\{\sum V_\nu\}$ of Ω by disks and $|V_\nu|$ is the radius of the disk V_ν.

Let D be a domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in D, is said to converge to a function φ m_1-*almost uniformly inside* D if for any compact subset $K \subset D$ and every $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \backslash K_\varepsilon) < \varepsilon$ and the sequence $\{\varphi_n\}$ converges uniformly to φ on K_ε.

For $\mu \in \mathcal{B}(E)$, define

$$\rho_{\min} := \inf_{z \in E} e^{-U^\mu(z)}$$

and

$$\varrho_{\max} := \max_{z \in E} e^{-U^\mu(z)};$$

(U^μ is superharmonic on E; hence it attains its minimum (on E)). As is known ([6], [1]),

$$e^{-U^\mu(z)} \geq \rho_{\min},\ z \in E^c.$$

Set, for $r > \rho_{\min}$,

$$E_\mu(r) := \{z \in \mathbb{C}, e^{-U^\mu(z)} < r\}.$$

Because of the upper semicontinuity of the function $e^{-U^\mu(z)}$, the set $E_\mu(r)$ is open; clearly $E_\mu(r_1) \subset E_\mu(r_2)$ if $r_1 \leq r_2$ and $E_\mu(r) \supset E$ if $r > \varrho_{\max}$.

Let $f \in \mathcal{A}(E)$ and $m \in \mathbb{N}$ be fixed. Let $R_{m,\mu}(f) = R_{m,\mu}$ and $D_{m,\mu}(f) = D_{m,\mu} := E_\mu(R_{m,\mu})$ denote, respectively, *the radius and domain of m-meromorphy with respect to μ;* that is

$$R_{m,\mu} := \sup\{r, f \in \mathcal{M}_m(E_\mu(r))\}.$$

Furthermore, we introduce the notion of a *μ-maximal convergence to f with respect to the m-meromorphy of a sequence of rational functions* $\{r_{n,\nu}\}$ (a μ-maximal convergence): that is, for any $\varepsilon > 0$ and each compact set $K \subset D_m$, there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and

$$\limsup_{n+\nu \to \infty} ||f - r_{n,\nu}||_{K_\varepsilon}^{1/n} \leq \frac{||e^{-U^\mu}||_K}{R_{m,\mu}(f)}.$$

Hernandez and Calle Ysern proved the following:

Theorem A [7]. *Let E, μ, β and $\omega_n, n = 1, 2, ...,$ be defined as above. Suppose that $\mu_{\omega_n} \longrightarrow \mu$ as $n \to \infty$ and $f \in \mathcal{A}(E)$. Then, for each fixed $m \in \mathbb{N}$, the sequence $\pi_{n,m}^{\beta,f}$ converges to f μ-maximally with respect to the m-meromorphy.*

Theorem A generalizes E. B. Saff's theorem of Montessus de Ballore's type about multipoint Padé approximants (see [2]).

From Theorem A, it follows that for every compact set $K \subset D_{m,\mu}$ which does not contain poles of f and concentration points of free poles, as $n \to \infty$, the estimation

$$\limsup_{n+\nu\to\infty} ||f - r_{n,\nu}||_{K_\varepsilon}^{1/n} \leq \frac{||e^{-U^\mu}||_K}{R_{m,\mu}(f)}$$

holds.

We now utilize the *normalization of the polynomials* $Q_{n,m}(z)$ *with respect to a given open set* $D_{m,\mu}$; that is,

$$Q_{n,m}(z) = \prod(z - \alpha'_{n,k}) \prod(1 - z/\alpha''_{n,k}), \tag{2}$$

where $\alpha'_{n,k}$, $\alpha''_{n,k}$ are the zeros lying inside, resp. outside $D_{m,\mu}$. Under this normalization, for every compact set K and n large enough there holds

$$\|Q^{\beta,f}_{n,m}\|_K \leq C_1,$$

where $C_1 = C_1(K)$ is a positive constant, depending on K. In the sequel, we denote by C_i positive constant, independent on n and different at different occurrences.

Let Q be the monic polynomial, the zeros of which coincide with the poles of f in $D_{m,\mu}$; $\deg Q \leq m$. It was proved in [7] (Proof of Lemma 2.3) that for every compact subset K of $D_{m,\mu}$

$$\limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|_K^{1/n} \leq \frac{||e^{-U^\mu}||_K}{R_{m,\mu}}. \tag{3}$$

Hence, $-U^\mu(z) - \ln R_{m,\mu}$ is a harmonic majorant of the family

$$\{|(fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m})(z)|^{1/n}\}_{n=1}^{\infty} \quad \text{in} \quad D_{m,\mu}.$$

In the present paper, we pose the question about sufficient conditions of the function above to be an exact harmonic majorant, with other words,

$$\limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|_K^{1/n} = \frac{||e^{-U^\mu}||_K}{R_{m,\mu}}$$

on every compactum in $D_{m,\mu}$. Clearly, if $-U^\mu - \ln R_{m,\mu}$ is an exact harmonic majorant, then there is a infinite sequence Λ such that

$$\lim_{n\to\infty, n\in\Lambda} \|QfQ^{\beta,f}_{n,m} - P^{\beta,f}_{n,m}Q\|_K^{1/n} = \|e^{-U^\mu}\|_K / R_{m,\mu} \tag{4}$$

(see [9], [10]) for a discussion of exact harmonic majorant)). We will refer to the sequences Λ as to *an exactly maximally convergent sequence relatively the measure* μ *with respect to the* m*-meromorphy of* f.

In [8], the validity of the following result was established:

Theorem B. *Under the same conditions on* E*, assume that* $\mu \in B(E)$ *and that* $\beta \subset E$ *is a triangular set of points uniformly distributed relatively to the measure* μ*. Let* $m \in \mathbb{N}$ *be fixed,* $f \in \mathcal{A}(E)$ *and* $\varrho_{max} < R_{m,\mu} < \infty$*. Suppose that* $D_{m,\mu}$

is connected. Then the function $-U^\mu - \ln R_{m,\mu}$ *is an exact harmonic majorant in* $D_{m,\mu} - E_{\varrho max}$ *of the family* $\{|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}|^{1/n}\}$

Before announcing the next result in the named area, we introduce the notion on a triangle point set of *Newtonian type.*

Given a triangle point set ω with no concentration points outside E, we say that it is of *Newtonian type*, if ω_n/ω_{n+1} for every $n \in \mathbb{N}$.

The next result was established in [7].

Theorem C. *Preserving the conditions on* E *and* ω *from Theorem B, assume that* ω *is of Newtonian type and* $m \in \mathbb{N}$ *is fixed. Then on each compactum* $K \subset D_{m,\mu}$ *which does not contain poles of* f *and concentration points of free poles of* $\pi^\beta_{n,m}$ *as* $n \to \infty$ *and such that* $\rho_\mu(K)$ *is not attained at a point belonging to* E *there holds*

$$\limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_K = \frac{||e^{-U^\mu}||_K}{R_{m,\mu}}.$$

2. Main results and Proofs

Before presenting the new result, we introduce the term of a multivalued singularity.

Given a function g and an point $z_0 \in \mathbb{C}$ we say that z_0 is a multivalued singularity of g if g can not be continued as a holomorhic function (analytic and single valued) in any neighborhood of z_0.

The main result of the present paper is

Theorem 1. *Under the above conditions on* E, *assume that* $\mu \in \mathcal{B}(E)$ *and that* $\beta \subset E$ *is a triangular set of points uniformly distributed relatively the measure* μ. *Let* $m \in \mathbb{N}$ *be fixed,* $f \in \mathcal{A}(E)$ *and* $\varrho_{max} < R_{m,\mu} < \infty$. *Assume that* $D_{m,\mu}$ *is a domain and* f *has at least one multivalued singularity on* $\partial D_{m,\mu}$. *Then the function* $-U^\mu - \ln R_{m,\mu}$ *is an exact harmonic majorant in* $D_{m,\mu}$ *of the family* $\{|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}|^{1/n}\}$.

As a consequence of Theorem 1, we derive

Theorem 2. *Under the conditions of Theorem 1, there is a sequence* $\Lambda \subset \mathbb{N}$ *such that on each compact set* $K \subset D_{m,\mu} \setminus E_{\varrho max}$ *and non containing poles of* f *and free poles of* $\{\pi^\beta_{n,m}\}$ *there holds*

$$\limsup_{n\in\Lambda} ||f - \pi^\beta_{n,m}||^{1/n}_K = \frac{||e^{-U^\mu}||_K}{R_{m,\mu}}.$$

In what follows, we lay out the main ideas of the proof of Theorem 1. As noticed above, Theorem 2 follows directly from Theorem 1.

From (3), it follows that

$$\lim_{r\to R_{m,\mu}} \limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_{E_\mu(r)} \le 1.$$

Let us suppose that we have a strong inequality, i.e.,

$$\lim_{r\to R_{m,\mu}} \limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_{E_\mu(r)} < 1.$$

Using now Theorem 1 in [12], and repeating the proof on Theorem 4 in [13], we conclude that the function f should be singlevalued in an appropriate neighborhood of the set $\partial D_{m,\mu}$. This contradicts the assertion that f has at least one multivalued singularity on $\partial D_{m,\mu}$ - contradiction to the conditions of Theorem 2. Therefore,

$$\lim_{r\to R_{m,\mu}} \limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_{E_\mu(r)} = 1. \tag{5}$$

We now prove, that for every $r < R_{m,\mu}$

$$\limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_{E_\mu(r)} = 1.$$

Indeed, suppose that for some $r, r \in (\rho_{max}, R_{m,\mu}$

$$\limsup_{n\to\infty} \|fQQ^{\beta,f}_{n,m} - QP^{\beta,f}_{n,m}\|^{1/n}_{E_\mu(r)} \le e^{-\tau}\frac{r}{R_{m,\mu}}.$$

The functions

$$\chi_n(z) := \frac{1}{n+m+1}\ln|QQ_{n+1}P^{\beta,f}_{n,m} - QQ_nP^{\beta,f}_{n+1,m}| - U^\mu(z)$$

are subharmonic in $E_\mu(r)^c$, and thus obey the maximum principle. Then it is easy to see that $\limsup_n \chi_n(z) < 1$ for $z \in \partial D_{m,\mu}$. This opposes (5).

On this, the proof of Theorem 1 is completed.

References

[1] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Springer, Grundlehren der Mathematischen Wissenschaften, New York-Berlin 1997, 316.

[2] E. B. Saff, *An extension of Montessus de Ballore theorem on the convergence of interpolation rational functions*, J. Approx. Theory **6** (1972), 63–67.

[3] R. K. Kovacheva, *Generalized Padé approximants of Kakehashi's type and meromorphic continuation of functions*, Deformation of Mathematical Structures, Kluwer Academic Publishers, 1989, 151–159.

[4] O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner, Leipzig 1929.

[5] A. A. Gonchar, *On the convergence of generalized Padé approximants of meromorphic functions*, Mat. Sbornik **98**, no. 140 (1975), 564–577; English translation in Math. USSR Sbornik **27**, no. 4 (1975), 503–514.

[6] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo 1959.

[7] M. Bello Hernándes, De la Calli Ysern, *Meromorphic continuation of functions and arbitrary distribution of interpolation points*, J. of Mathematical Analysis and Applications **403** (2013), 107–119.

[8] R. K. Kovacheva, *On the distribution of exactly maximally convergent sequences of multipoint Padé approximants*, Applied Mathematics **6** (2015), 5.

[9] J. L. Walsh, *Overconvergence, degree of convergence, and zeros of sequences of analytic functions*, Duke Math. J. **13** (1946), 195–234.

[10] J. L. Walsh, *The analogue for maximally convergent polynomials of Jentzsch's theorem*, Duke Math. J. **26** (1959), 605–616.

[11] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc. Colloq. Pub. **20**, New York 1969.

[12] A. A. Gonchar, *The rate of rational approximation and the property of single valuedness of an analytic function in a neighborhood of an isolated singular point*, Matem. Sb. **94**, no. 136 (1974), 265–282; English translation in Math. UdSSR Sb. **23**, no. 2 (1974), 254–270.

[13] H. P. Blatt and R. K. Kovacheva, *Growth behavior and zero distribution of rational approximants* Constructive approximation **34** (2011), 393–420.

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DOKŁADNIE MAKSYMALNIE ZBIEŻNE CIĄGI WIELOPUNKTOWYCH APPROKSYMANT PADÉ

S t r e s z c z e n i e

Przy danym regularnym zwartym zbiorze E na płaszczyźnie $\mathbb{C}$, mierze jednostkowej μ o nośniku E, trjkątnym zbiorze punktów $\beta := \{\{\beta_{n,k}\}_{k=1}^{n}\}_{n=1}^{\infty}, \beta \subset E$ i funkcji f holomorficznej na zbiorze E, niech $\pi_{n,m}^{\beta,f}$ będzie stowarzyszoną β-aproksymantą Padé rzędu (n, m). Przy warunku, że punkty β są jednostajnie rozmieszczone relatywnie do miary μ, uzyskujemy wyniki o istnieniu dokładnie maksymalnie zbieżnych ciągów $\pi_{n,m}^{\beta,f}$ przy $n \to \infty$, zaś m-liczbą naturalną ustaloną relatywnie do μ i obszaru m-meromorficzności funkcji f.

Słowa kluczowe: wielopunktowe aproksymanty Padé, zbieżność maksymalna, obszar m-meromorficzności

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Contribution to the jubilee volume, dedicated to Professors J. Ławrynowicz and L. Wojtczak

Alfonso Hernández Montes and Lino Feliciano Reséndis Ocampo

MOISIL-THÉODORESCU QUATERNIONIC $F(p,q,s)$ FUNCTION SPACES

Summary

In this paper we define differentiability in the sense of Moisil-Théodorescu associated to a particular embeding of $\mathbb{R}^3$ in the quaternionic space $\mathbb{H}$. Using the Moisil-Théodorescu derivative we introduce and study the analogous to the function spaces $F(p,q,s)$ and $F_0(p,q,s)$ introduced in the paper [30] R. Zhao. We obtain similar results that in the monogenic case, see [9, 11] and [21].

Keywords and phrases: Moisil-Théodorescu, $\mathcal{Q}_p^i$, i-Bloch and $F_i(p,q,s)$ spaces.

1. Introduction

In [30], R. Zhao defined and studied the $F(p,q,s)$ spaces that consist of analytic functions $f:\mathbb{D}\to\mathbb{C}$ such that

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q g^s(z,a)\,dx\,dy<\infty$$

where $0<p<\infty$, $-2<q<\infty$, $0<s<\infty$ and g is the Green's function of the unit disk $\mathbb{D}$, given by

$$g(z,a)=\ln\left|\frac{1-\overline{a}z}{a-z}\right|.$$

These spaces are the generalization of the $\mathcal{Q}_s=F(2,0,s)$ spaces introduced by R. Aulaskari and Lappan in [1] for $1\leq s<\infty$ and for $0<s<1$ by R. Aulaskari,

J. Xiao and R. Zhao in [2]. The family $F(p,q,s)$ is quite general, includes, among others, $BMOA$, Dirichlet and α-Bloch spaces. In this articles was proved that the weight function $g(z,a)$ can be replace by the weight

$$1-\left|\frac{a-z}{1-\overline{a}z}\right|^2 .$$

Analogous $\tilde{F}(p,q,s)$ Bergman spaces were studied in [17] and [5].

There are several approaches to study these spaces in higher dimensions, see [6–12,15] and [21] in the quaternionic case, [16] in hyperkählerian case and [22,24,25] in the holomorphic case.

Let $\mathbb{H}$ be the skew field of real quaternions, that is, each element $a \in \mathbb{H}$ can be written in the form

$$a := a_0 + a_1 i + a_2 j + a_3 k, \qquad a_l \in \mathbb{R},\ l = 0,\ 1,\ 2,\ 3$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$, with the multiplication rules

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j\ .$$

The product is extended by linearity. The quaternionic conjugation defined by $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$ permits to define the norm $|a|$ of $a \in \mathbb{H}$ by

$$|a|^2 = a\overline{a} = \overline{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2\ .$$

Therefore, if $a \in \mathbb{H} \setminus \{0\}$, the quaternion

$$a^{-1} := \frac{1}{|a|^2}\overline{a}$$

is the multiplicative inverse of a. Also, the norm satisfies $|ab| = |a||b|$ for each $a,\ b \in \mathbb{H}$.

The standard Moisil-Théodorescu operator (MT-operator) and its conjugate are given by

$$D_{MT} := i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3},$$

$$\overline{D}_{MT} := -i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

Let $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{H}$ be a function of (C^1, Ω). We say that f is Moisil-Théodorescu hyperholomorphic (MT hyperholomorphic) if $D_{MT}f = 0$ and MT_Ω denote the kernel of D_{MT}.

This operator does not have a good derivative for MT hyperholomorphic funtions as we will show below.

One way is extending the domain of the function and to use the Fueter operator, that has a good derivative, see [28] and [20]. Other one is using differential forms to define a good derivative.

Let $\widetilde{\Omega} := \mathbb{R} \times \Omega \subset \mathbb{H}$ and $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{H}$. Define $\widetilde{f} : \widetilde{\Omega} \longrightarrow \mathbb{H}$ by

$\widetilde{f}(x_0,x_1,x_2,x_3) := f(x_1,x_2,x_3)$ for all $x_0 \in \mathbb{R}$. If $f \in MT_\Omega$, then

$$D_F[\widetilde{f}] = \frac{\partial \widetilde{f}}{\partial x_0}(x_0,x_1,x_2,x_3) + D_{MT}[\widetilde{f}](x_0,x_1,x_2,x_3) = 0$$

and

$$\overline{D}_F[\widetilde{f}] = \frac{\partial \widetilde{f}}{\partial x_0}(x_0,x_1,x_2,x_3) + \overline{D}_{MT}[\widetilde{f}](x_0,x_1,x_2,x_3) = 0.$$

where D_F is the Fueter operator and $\overline{D}_F$ its conjugate. Then $f(x) = \tilde{h}(z) + \tilde{l}(z)j$ where $\tilde{h}$ and $\tilde{l}$ are two holomorphic complex functions of the complex variable $z = x_1 + x_2 i$.

By other way, following [20], if $f \in MT_\Omega$ and there exist a 1-differential form σ_x^1 and a 2-differential form σ_x^2 such that

$$\begin{aligned} d(\sigma_x^1 f(x)) &= \frac{1}{2}\sigma_x^2 \overline{D}_{MT}[f](x) + \frac{1}{2}\overline{\sigma}_x^2 D_{MT}[f](x) \\ &= -\frac{1}{2}\sigma_x^2 D_{MT}[f](x) + \frac{1}{2}\overline{\sigma}_x^2 D_{MT}[f](x) \\ &= \frac{1}{2}(-\sigma_x^2 + \overline{\sigma}_x^2) D_{MT}[f](x) \\ &= 0. \end{aligned}$$

For this reason all MT hyperholomorphic functions have derivative zero.

Now, we define an analogous of the Moisil-Théodorescu operator

$$D_{MT}^i := \frac{\partial}{\partial x_1} - k\frac{\partial}{\partial x_2} + j\frac{\partial}{\partial x_3}$$

Observe that $D_{MT}^i := iD_{MT}$, then $f \in MT_\Omega$ if and only if f belongs to the kernel of D_{MT}^i.

The conjugate of D_{MT}^i is given by

$$\overline{D}_{MT}^i := \frac{\partial}{\partial x_1} + k\frac{\partial}{\partial x_2} - j\frac{\partial}{\partial x_3}.$$

In general if $f \in MT_\Omega$ then $\overline{D}_{MT}^i \neq 0$ as $f(x_1,x_2,x_3) = x_1 - kx_2$ shows.

To justify that $\overline{D}_{MT}^i$ is a good derivative we will use differential forms. In this way we need to embed $\mathbb{R}^3$ in $\mathbb{H}$. Motivated by the definition of the D_{MT}^i operator we choose the following isometric embedding: $\mathbb{R}^3 \ni x = (x_1,x_2,x_3) \mapsto x_1 - kx_2 + jx_3 = x \in \mathbb{H}$ that will denote by the i-embedding of $\mathbb{R}^3$ in $\mathbb{H}$. We define $\mathbb{R}_i^3 := \{x \in \mathbb{H} : x = x_1 - kx_2 + jx_3\}$ with basis $\{1, -k, j\}$.

Thus if we use differential forms to write the normal vector on a 2-surface in $\mathbb{R}_3^i$ we get

$$\sigma_i^2 := dx_2 \wedge dx_3 + k dx_1 \wedge dx_3 + j dx_1 \wedge dx_2.$$

Let $f:\mathbb{R}_i^3 \longrightarrow \mathbb{H}$ be a C^1 function and $\omega = \sigma_i^2 f$. The total differential of ω is:

$$\begin{aligned} dw &= d(\sigma_i^2 f) = \sigma_i^2 \wedge df \\ &= (dx_2 \wedge dx_3 + k dx_1 \wedge dx_3 + j dx_1 \wedge dx_2) \wedge \left[\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3\right] \\ &= D^i_{MT}[f] dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Then $f \in MT_\Omega$ if and only if $d(\sigma_i^2 f) = 0$. By other way, similarly to σ_i^2, we define

$$\tau_i := -k dx_3 - j dx_2$$

and $\omega_1 = \tau_i f$. Thus

$$\begin{aligned} d\omega_1 &= d(\tau_i f) = \tau_i \wedge df \\ &= k\left(\frac{\partial f}{\partial x_1}dx_1 \wedge dx_3 + \frac{\partial f}{\partial x_2}dx_2 \wedge dx_3\right) + j\left(\frac{\partial f}{\partial x_1}dx_1 \wedge dx_2 - \frac{\partial f}{\partial x_3}dx_2 \wedge dx_3\right). \end{aligned}$$

By other way:

$$\begin{aligned} &\sigma_i^2 \overline{D}^i_{MT}[f] - \overline{\sigma}_i^2 D^i_{MT}[f] \\ &= (dx_2 \wedge dx_3 + k dx_1 \wedge dx_3 + j dx_1 \wedge dx_2)\left(\frac{\partial f}{\partial x_1} + k\frac{\partial f}{\partial x_2} - j\frac{\partial f}{\partial x_3}\right) \\ &- (dx_2 \wedge dx_3 - k dx_1 \wedge dx_3 - j dx_1 \wedge dx_2)\left(\frac{\partial f}{\partial x_1} - k\frac{\partial f}{\partial x_2} + j\frac{\partial f}{\partial x_3}\right) \\ &= 2\left[k\left(\frac{\partial f}{\partial x_1}dx_1 \wedge dx_3 + \frac{\partial f}{\partial x_2}dx_2 \wedge dx_3\right) + j\left(\frac{\partial f}{\partial x_1}dx_1 \wedge dx_2 - \frac{\partial f}{\partial x_3}dx_2 \wedge dx_3\right)\right]. \end{aligned}$$

If we used the previous results we obtain

$$\frac{1}{2}\left[\sigma_i^2 \overline{D^i_{MT}}[f] - \overline{\sigma}_i^2 D^i_{MT}[f]\right] = d(\tau_i f).$$

Therefore if $f \in MT_\Omega$ and following [20], we define the i-hyper derivative of f as

$$f'^i := \overline{D}^i_{MT}[f].$$

Proposition 1.1. *Let $f:\mathbb{R}_i^3 \to \mathbb{H}$ be a MT-hyperholomorphic function. Then*

$$f'^i = 2\frac{\partial f}{\partial x_1}.$$

Proof. By definition

$$i\overline{D^i_{MT}}[f] = i\left[\frac{\partial f}{\partial x_1} + k\frac{\partial f}{\partial x_2} - j\frac{\partial f}{\partial x_3}\right] = i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3},$$

thus

$$\begin{aligned} -2i\frac{\partial f}{\partial x_1} + i\overline{D^i_{MT}}[f] &= -2i\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = -i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} \\ &= \overline{D_{MT}}[f] = -D_{MT}[f] = 0, \end{aligned}$$

since $f \in MT_\Omega$. This concludes the proof. $\square$

The operator D^i_{MT} and its conjugate factorize the Laplace operator in $\mathbb{R}^3$, that is.

$$D^i_{MT} \circ \overline{D^i_{MT}} = \Delta_{\mathbb{R}^3}.$$

As a consequence each $f \in MT_\Omega$ is a harmonic function. Let $a \in \mathbb{R}^3_i$ with $|a| < 1$, we define the Möbius transform

$$\varphi^i_a : \mathbb{R}^3_i - \left\{ \frac{a}{|a|^2} \right\} \to \mathbb{H}$$

by

$$\varphi^i_a(x) := (a - x)(1 - \overline{a}x)^{-1}.$$

For $R > 0$, we define

$$B_i(R) := \{x \in \mathbb{R}^3_i : |x| < R\}, \quad B_i := B_i(1), \quad S_i := \partial B_i$$

and $A_i(R) := B_i \setminus B_i(R)$.

Proposition 1.2. *Let $a \in B_i$, then φ^i_a maps conformally the unit ball B_i onto itself.*

Proof. Its well known that φ^i_a is a conformal mapping, but we will give other proof, see ([4], Theorem 3.2.7). For $x \in \mathbb{R}^3_i$, $a \in B_i$ $a \neq 0$ let

$$T_0(x) = \frac{\overline{a}}{|a|} x \frac{\overline{a}}{|a|}, \quad T_1(x) = \left(\frac{|a|^2}{1 - |a|^2} \right) x, \quad T_2(x) = x + \frac{\overline{a}}{|a|^2 - 1},$$

$$T_3(x) = x^{-1},\ x \neq 0, \quad T_4(x) = \frac{a}{|a|^2} + x,$$

then

$$\big(T_4 \circ T_3 \circ T_2 \circ T_1 \circ T_0\big)(x) = (a - x)(1 - \overline{a}x)^{-1} = \varphi^i_a(x).$$

It is easy to see that each T_i preserves cross ratios. For example, by definition of cross ratios, for T_3 and $x, y, z, w \in \mathbb{R}^3_i$ we have:

$$\begin{aligned} [T_3(x), T_3(y), T_3(z), T_3(w)] &= \frac{|T_3(x) - T_3(z)||T_3(y) - T_3(w)|}{|T_3(x) - T_3(y)||T_3(z) - T_3(w)|} \\ &= \frac{\left| \frac{\overline{x}}{|x|^2} - \frac{\overline{z}}{|z|^2} \right| \left| \frac{\overline{y}}{|y|^2} - \frac{\overline{w}}{|w|^2} \right|}{\left| \frac{\overline{x}}{|x|^2} - \frac{\overline{y}}{|y|^2} \right| \left| \frac{\overline{z}}{|z|^2} - \frac{\overline{w}}{|w|^2} \right|}. \end{aligned}$$

As

$$\left| \frac{\overline{x}}{|x|^2} - \frac{\overline{z}}{|z|^2} \right| = \left| \frac{|z|^2\overline{x} - |x|^2\overline{z}}{|x|^2|z|^2} \right| = \left| \frac{\overline{x}|z|^2 - |x|^2\overline{z}}{|x|^2|z|^2} \right| = \frac{1}{|x|^2|z|^2} |\overline{x}z\overline{z} - \overline{x}x\overline{z}|$$

$$= \frac{1}{|x|^2|z|^2} |\overline{x}(z - x)\overline{z}| = \frac{|x - z|}{|x||z|}$$

then

$$[T_3(x), T_3(y), T_3(z), T_3(w)] = [x, y, z, w].$$

We now proof that $\varphi_a^i(x) \in \mathbb{R}_i^3$. Let $a, x \in \mathbb{R}_i^3$ with $a = a_1 - ka_2 + ja_3$ and $x = x_1 - kx_2 + jx_3$ with $x \neq \frac{a}{|a|^2}$, then

$$\begin{aligned}\varphi_a^i(x) =&(a-x)\overline{(1-\overline{a}x)} = (a_1 - x_1)\big(1 - (a_1x_1 + a_2x_2 + a_3x_3)\big)\\ &- < (0, a_3 - x_3, x_2 - a_2), (a_3x_2 - a_2x_3, a_1x_3 - a_3x_1, a_2x_1 - a_1x_2) >\\ &+ (a_1 - x_1)\big(i(a_3x_2 - a_2x_3) + j(a_1x_3 - a_3x_1) + k(a_2x_1 - a_1x_2)\big)\\ &+ \big(1 - (a_1x_1 + a_2x_2 + a_3x_3)\big)\big((x_2 - a_2)k + (a_3 - x_3)j\big)\\ &+ \big((x_2 - a_2)k + (a_3 - x_3)j\big) \times \big(i(a_3x_2 - a_2x_3) + j(a_1x_3 - a_3x_1)\\ &+ k(a_2x_1 - a_1x_2)\big)\end{aligned}$$

and its i-component is

$$\begin{aligned}&(a_1 - x_1)(a_3x_2 - x_3a_2) + (a_3 - x_3)(x_1a_2 - a_1x_2) - (a_1x_3 - x_1a_3)(x_2 - a_2)\\ &= a_1a_3x_2 - a_1x_3a_2 + x_1x_3a_2 + a_3x_1a_2 - a_3a_1x_2 - x_3x_1a_2 + x_3a_1x_2\\ &- a_1x_3x_2 + a_1x_3a_2 + x_1a_3x_2 - x_1a_3a_2 = 0.\end{aligned}$$

Since $\overline{x}a + \overline{a}x = a\overline{x} + x\overline{a}$, then

$$|a|^2 - (a\overline{x} + x\overline{a}) + 1 = 1 - (\overline{x}a + \overline{a}x) + |a|^2$$

or equivalently

$$(a - x)(\overline{a} - \overline{x}) = (1 - \overline{a}x)(1 - \overline{x}a).$$

That is

$$|\varphi_a^i(x)| = 1, \quad \text{if} \quad |x| = 1.$$

Since $\varphi_a^i(0) = a \in B_i$ then $\varphi_a^i(x) \in B_i$ for all $x \in B_i$ and we finished the proof. $\square$

The set of MT-hyperholomorphic functions defined on the unit ball B_i is denoted by $\mathfrak{M} := MT_{B_i}$.

Our previous definitions are used to generalize the $\mathcal{Q}_s$ type spaces (see [10,21]). More precisely, we have the following definitions. Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $f \in \mathfrak{M}$. Define $J_{p,q,s}^i f : B_i \to [0, \infty)$ by

$$J_{p,q,s}^i f(a) = \int_{B_i} \left|\overline{D}_{MT}^i f(x)\right|^p \left(1 - |x|^2\right)^q \left(1 - \left|\varphi_a^i(x)\right|^2\right)^s dx \ .$$

The sets $F_\varphi^i(p, q, s)$ and $F_{\varphi,0}^i(p, q, s)$ are defined as

$$F_\varphi^i(p, q, s) = \{ \ f \in \mathfrak{M} \ : \ \sup_{a \in B} J_{p,q,s}^i f(a) < \infty \} \ ,$$

and

$$F_{\varphi,0}^i(p, q, s) = \{ \ f \in \mathfrak{M} \ : \ \lim_{|a| \to 1^-} J_{p,q,s}^i f(a) = 0 \}.$$

The corresponding Besov spaces $B^{i,p}$ and $B^{i,q,p}$ are

$$F_\varphi^i(p, \frac{3p}{2} - 3, 3) \quad \text{and} \quad F_\varphi^i(p, \frac{3p}{2} - q, q)$$

respectively. In this definitions we are using the weight function $1-\left|\varphi_a^i(x)\right|^2$ and we will proof in Theorem 4.2 that the weight function can by replace by a modified Green's function in quaternionic sense.

The set $F_\varphi^i(p,q,s)$ is a $\mathbb{H}$-right module (also is left). This can be easily seen by the inequality $(x+y)^p \leq 2^p(x^p+y^p)$. From the definition of the sets they are right $\mathbb{H}$ modules.

For $0<p<\infty$, $-1<q<\infty$, define the $D^i_{p,q}$ weighted Dirichlet space, as the set of $f \in \mathfrak{M}$ satisfying

$$\int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^q \, dx < \infty \ .$$

From the definition of $F_\varphi^i(p,q,s)$ space the following result became immediate with $a=0$.

Lemma 1.1. *Let $0<p<\infty$, $-1<q<\infty$ and $0<s<\infty$. Then $F_\varphi^i(p,q,s) \subset D^i_{p,q+s}$.*

The i-Bloch spaces will be motivated and defined more later.

2. Preliminaries

Given $a \in B_i$, the Möbius transform $\varphi_a^i : B_i \to B_i$ satisfies

$$\frac{1-\left|\varphi_a^i(x)\right|^2}{1-|x|^2} = \frac{1-|a|^2}{|1-\overline{a}x|^2} = |J\varphi_a^i(x)|^{\frac{1}{3}} \quad \text{for all } x \in B_i, \tag{2.1}$$

where $J\varphi_a^i$ denotes the Jacobian of the function φ_a^i. For $0<R<1$ the pseudohyperbolic ball $D_i(a,R)$ is defined by

$$D_i(a,R) = \left\{x \in B_i : \left|\varphi_a^i(x)\right| < R\right\} \ .$$

This is an euclidean ball, with center and radius given respectively by

$$c = \frac{1-R^2}{1-R^2|a|^2} a \ , \qquad r = \frac{1-|a|^2}{1-R^2|a|^2} R \ . \tag{2.2}$$

The next result is a consequence of Cauchy-Schwartz inequality.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain, $f:\Omega \to \mathbb{R}^n$ be an integrable function on Ω. Then*

$$\left|\int_\Omega f\right| \leq \int_\Omega |f| \ .$$

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain and $f:\Omega \to \mathbb{R}^n$ with $f=(f_1,\ldots,f_n)$ and $1 \leq p < \infty$. If each coordinate function $f_i : \Omega \to \mathbb{R}$ is subharmonic, then $|f|^p$ is subharmonic on Ω.*

The following result was proved in [12].

Lemma 2.1. *Let $0 < p \leq 2$, $0 < r < 1$, $|a| < 1$ and $S \subset \mathbb{R}$ be the unit sphere. Then there exists $C > 0$ such that*

$$\int_S \frac{d\sigma(\zeta)}{|1 - \overline{a}r\zeta|^{2p}} \leq \frac{C}{(1 - |a|\, r)^p} \leq \frac{C}{(1 - |a|)^p} \ . \tag{2.3}$$

We reformulate the following result proved in [21].

Proposition 2.1. *Let $0 < R < 1$ and $h : B_i \to \mathbb{R}$ be a continuous function. If $-2 < q < \infty$, $0 < s < \infty$ with $-1 < q + s$ then*

$$\sup_{a \in B_i} \int_{B_i(R)} h(x)(1 - |\varphi_a^i(x)|^2)^s \, dx < \infty, \quad \lim_{|a| \to 1^-} \int_{B_i(R)} h(x)(1 - |\varphi_a^i(x)|^2)^s \, dx = 0$$

and

$$\sup_{a \in B_i} \int_{B_i} (1 - |x|^2)^q (1 - |\varphi_a^i(x)|^2)^s \, dx < \infty, \quad \lim_{|a| \to 1^-} \int_{B_i} (1 - |x|^2)^q (1 - |\varphi_a^i(x)|^2)^s \, dx = 0.$$

Like in [29], we have (see [23]) :

Lemma 2.2. *Let $1 \leq p < \infty$, $a \in B_i$ and $f : B_i \to \mathbb{H}$ be a MT-hyperholomorphic function. Let $\psi_{f,a}^i : B_i \to \mathbb{H}$ given by*

$$\psi_{f,a}^i(x) = \frac{1 - \overline{x}a}{|1 - \overline{a}x|^3} \overline{D}_{MT}^i f(\varphi_a^i(x)) \ . \tag{2.4}$$

Then $\psi_{f,a}^i$ is a MT-hyperholomorphic function and $|\psi_{f,a}^i|^p$ is a subharmonic function.

The following result was proved in [26].

Lemma 2.3. *Let $q(r)$ and $p(r)$ be two integrable and nonnegative functions on $[0, 1)$. If exists τ' with $0 < \tau' < 1$ and a positive constant C such that $q(r) \leq Cp(r)$ for $r \in [\tau', 1)$. Then for all τ with $\tau' < \tau \leq 1$ and all nondecreasing and nonnegative function $h(r)$ on $[0, 1)$, there exists a constant $K = K(\tau) \geq C$ independent of τ' and h, such that*

$$\int_0^\tau h(r)q(r)\, dr \leq K \int_0^\tau h(r)p(r)\, dr.$$

The hyperholomorphic constants of the i-Moisil-Théodorescu operator are characterized by the following result.

Lemma 2.4. *Let $f : B_i \to \mathbb{H}$ such that $\overline{D}_{MT}^i f(x) = 0 = D_{MT}^i f(x)$ for all $x \in B_i$. Then $f(x) = \tilde{h}(z) + \tilde{l}(z)j$ where $\tilde{h}$ and $\tilde{l}$ are two holomorphic complex functions of the complex variable $z = x_2 + ix_3$.*

Proof. If

$$D^i_{MT}f(x) = 0 = \overline{D}^i_{MT}f(x)$$

for all $x \in B_i$, then

$$0 = D^i_{MT}f(x) + \overline{D}^i_{MT}f(x) = 2\frac{\partial f}{\partial x_1},$$

for this reason f does not depend of x_1. Those

$$0 = D^i_{MT}f(x) = -k\frac{\partial f}{\partial x_2} + j\frac{\partial f}{\partial x_3}$$

or equivalently

$$k\frac{\partial f}{\partial x_2} = j\frac{\partial f}{\partial x_3}.$$

If $f = f_1 + if_2 + jf_3 + kf_4$ then we get the following equations

$$\frac{\partial f_3}{\partial x_2} = -\frac{\partial f_4}{\partial x_3} \quad ; \quad \frac{\partial f_4}{\partial x_2} = \frac{\partial f_3}{\partial x_3}$$

$$\frac{\partial f_1}{\partial x_2} = -\frac{\partial f_2}{\partial x_3} \quad ; \quad \frac{\partial f_2}{\partial x_2} = \frac{\partial f_1}{\partial x_3}.$$

Define $\tilde{h}(z) = f_4(z) + if_3(z)$ and $\tilde{l}(z) = f_2(z) + if_1(z)$, where $z = x_2 + ix_3$. These functions are holomorphic by the previous relations. □

We say that $f,\ g \in \mathfrak{M}$ are equivalents ($\sim$) if $f(x) - g(x) = \tilde{h}(z) + j\tilde{l}(z)$ where $\tilde{h}$ and $\tilde{l}$ are two holomorphic complex functions of the complex variable $z = x_2 + ix_3$. If we consider $\mathfrak{M}$ with this equivalence relation then for $1 \le p < \infty$, by Minkowski's inequality

$$\begin{aligned}\| f \| &= \sup_{a \in B_i} \left(I_{p,q,s}f(a)\right)^{\frac{1}{p}} \\ &= \sup_{a \in B_i} \left(\int_{B_i} \left|\overline{D}^i_{MT}f(x)\right|^p \left(1 - |x|^2\right)^q (1 - |\varphi^i_a(x)|^2)^s\, dx\right)^{\frac{1}{p}}\end{aligned}$$

defines a norm in $F^i_\varphi(p,q,s)$.

3. Properties of MT-spaces

In this section we present several basic properties and some examples of different quaternionic spaces.

Proposition 3.1. *Let $1 \le p < \infty$ and $-2 < q < \infty$. If $0 < s < \infty$ and $q + s \le -1$ then $F^i_\varphi(p,q,s)$ consists only of constant functions.*

Proof. Let $f \in F^i_\varphi(p,q,s)$ be a non constant function. Then there exist $x_0 \in B_i$ and $0 < R < 1$ such that $|\overline{D}^i_{MT}f(x)| > 0$ for all $x \in B_i(x_0,R) \subset B_i$. Thus by subharmonicity of $|\overline{D}^i_{MT}f|^p$ we have

$$\begin{aligned}
\infty &> \int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^q (1-|x|^2)^s\, dx \\
&\geq \int_{A_i(|x_0|)} |\overline{D}^i_{MT} f(x)|^p (1-r^2)^{(q+s)}\, dx \\
&\geq \int_{|x_0|}^1 (1-r^2)^{(q+s)} r^2 \int_{S_i} |\overline{D}^i_{MT} f(r\zeta)|^p\, d\sigma(\zeta)\, dr \\
&\geq \int_{S_i} |\overline{D}^i_{MT} f(|x_0|\zeta)|^p\, d\sigma(\zeta) \int_{|x_0|}^1 (1-r^2)^{(q+s)} r^2\, dr = \infty
\end{aligned}$$

as $q+s \leq -1$, we get a contradiction; therefore f is constant. $\square$

From now on, we will suppose $-1 < q+s < \infty$.

Example 3.1. *Let $-2 < q < \infty$, $0 < s < \infty$ with $-1 < q+s < \infty$. The spaces $F^i_\varphi(p,q,s)$ and $F^i_{\varphi,0}(p,q,s)$ are not empty. More precisely, let $f : B_i \to \mathbb{H}$, $f \in \mathfrak{M}$. If there exists $M > 0$ such that $|\overline{D}^i_{MT} f(x)| < M$ for all $x \in B_i$, then*

$$\int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^q (1-|\varphi^i_a(x)|^2)^s\, dx \leq M \int_{B_i} (1-|x|^2)^q (1-|\varphi^i_a(x)|^2)^s\, dx$$

and apply Proposition 2.1. Thus f belongs to the quoted spaces. The previous condition is satisfied, for example, if $f \in \mathfrak{M} \cap C^1(\overline{B_i})$.

Theorem 3.1. *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. Then the function $a \mapsto J^i_{p,q,s} f(a)$ is continuous.*

Proof. Let $a \in B^i$ be fixed and $\varepsilon > 0$. Let

$$r = \frac{1-|a|}{2}$$

and define the function

$$h(x,b) = \frac{(1-|b|^2)^s}{|1-\bar{b}x|^{2s}} \quad \text{for all} \quad (x,b) \in \overline{B_i} \times \overline{B_i(a,r)}\ .$$

Since $h(x,b)$ is uniformly continuous, there exists $\delta > 0$ such that if $|b-b'| < \delta$ then

$$|J^i_{p,q,s} f(b) - J^i_{p,q,s} f(b')| \leq \frac{\varepsilon}{J^i_{p,q,s} f(0)}.$$

Thus if $|b-a| < \delta$ then

$$|J^i_{p,q,s} f(b) - J^i_{p,q,s} f(a)| \leq \int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^{q+s} |h(x,b)-h(x,a)|\, dx < \varepsilon.$$

$\square$

Theorem 3.2. *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. Then $F^i_{\varphi,0}(p,q,s) \subset F^i_\varphi(p,q,s)$.*

Proof. Let $f \in F^i_{\varphi,0}(p,q,s)$. Thus, by Theorem 3.1 we can extend continuously the definition of $J^i_{p,q,s}f$ to $\overline{B_i}$ by setting $J^i_{p,q,s}f(a) = 0$ when $|a| = 1$. Then

$$\sup_{a\in B} J^i_{p,q,s}f(a) = \max_{a\in\overline{B^i}} J^i_{p,q,s}f(a) < \infty \ .$$

Moreover, there is $b \in B_i$, such that $\max_{a\in\overline{B_i}} J^i_{p,q,s}f(a) = J^i_{p,q,s}f(b)$. □

4. Characterizations of $F^i_\varphi(p,q,s)$ spaces

Let $a \in B_i$ fix, and for $x \neq a$ define

$$g\,(x,a) = \frac{1}{|\varphi^i_a\,(x)|} - 1.$$

Thus $g(x,a)$ is a translation of a multiple scalar of the fundamental solution of the Laplacian in $\mathbb{R}^3_i$ applied to the Möbius transform φ^i_a, i.e. $g(x,a)$ is the modified Green's function in quaternion sense. We prove in this section that the spaces $F^i_\varphi(p,q,s)$ can be characterized using the $g(x,a)$ as a weight function. Likewise we give a characterization of these spaces using Carleson measures on boxes.

The next results are more general than the analogous results in (see [11]) Lemma 5.1, Theorem 5.1, Proposition 5.1 and Theorem 2.2 from [9], like in [21].

The following result is other characterization of the Dirichlet type spaces.

Theorem 4.1. *Let $1 \leq p < \infty$, $-2 < q < \infty$ and $0 < s < 3$. Let $f : B_i \to \mathbb{H}$ a MT-hyperholomorphic function. Then f belongs to the Dirichlet space $\mathcal{D}^i_{p,q+s}$ if and only if*

$$\int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^q g^s(x,0)\, dx < \infty \ .$$

Proof. We will prove that

$$\int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^{q+s}\, dx \approx \int_{B_i} |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^q g^s(x,0)\, dx \ .$$

Applying spherical coordinates we get

$$\begin{aligned} &\int_0^1 \int_{S_i} |\overline{D}^i_{MT} f(r\zeta)|^p (1-r^2)^{q+s} r^2\, d\sigma(\zeta) dr \\ &\quad\approx \int_0^1 \int_{S_i} |\overline{D}^i_{MT} f(r\zeta)|^p (1-r^2)^q (1-r)^s r^{2-s}\, d\sigma(\zeta)\, dr \ . \end{aligned}$$

Since $1 \leq 1+r \leq 2$ and $|\overline{D}^i_{MT} f|^p$ is subharmonic the result follows if we apply Proposition 2.3 with $\tau = 1$ and

$$h(r) = \int_{S_i} |\overline{D}^i_{MT} f(r\zeta)|^p\, d\sigma(\zeta) \ .$$

□

The previous result motivates the following characterization of the spaces $F^i_\varphi(p,q,s)$.

Theorem 4.2. *Let* $1 \le p < \infty$, $-2 < q < \infty$ *and* $0 < s < 3$. *Then* $f \in F^i_\varphi(p,q,s)$ *if and only if*

$$\sup_{a\in B_i} \int_{B_i} \left|\overline{D}^i_{MT} f\,(x)\right|^p \left(1-|x|^2\right)^q g^s\,(x,a)\;dx < \infty$$

and $f \in F^i_{\varphi,0}(p,q,s)$ *if and only if*

$$\lim_{|a|\to 1^-} \int_{B_i} \left|\overline{D}^i_{MT} f\,(x)\right|^p \left(1-|x|^2\right)^q g^s\,(x,a)\;dx = 0\ .$$

Proof. Since $1 - |\varphi^i_a(x)| \le g(x,a)$ we prove only

$$F^i_\varphi(p,q,s) \subset F^i_g(p,q,s) \qquad \text{and} \qquad F^i_{\varphi,0}(p,q,s) \subset F^i_{g,0}(p,q,s)\ .$$

By the change of variable $x = \varphi^i_a(w)$ and (2.4), we have

$$\begin{aligned} I_{p,q,s}f(a) &= \int_{B_i} \left|\overline{D}^i_{MT} f\,(x)\right|^p \left(1-|x|^2\right)^q g^s\,(x,a)\;dx \\ &= \int_{B_i} |\psi^i_{f,a}(w)|^p (1-|\varphi^i_a(w)|^2)^q \frac{(1-|w|)^s}{|w|^s} \frac{(1-|a|^2)^3}{|1-\overline{a}w|^{6-2p}}\,dw \end{aligned}$$

while

$$\begin{aligned} &\int_{B_i} \left|\overline{D}^i_{MT} f\,(x)\right|^p \left(1-|x|^2\right)^q \left(1-\left|\varphi^i_a(x)\right|^2\right)^s dx \\ &= \int_{B_i} |\psi^i_{f,a}(w)|^p (1-|\varphi^i_a(w)|^2)^q (1-|w|^2)^s \frac{(1-|a|^2)^3}{|1-\overline{a}w|^{6-2p}}\,dw. \end{aligned}$$

Since $\frac{1}{2} \le |w| < 1$ implies $1 < \dfrac{1}{|w|^s} \le 2^s$ then

$$\begin{aligned} &\int_{B_i\setminus B_i(\frac{1}{2})} |\psi^i_{f,a}(w)|^p (1-|\varphi_a(w)|^2)^q \frac{(1-|w|)^s}{|w|^s} \frac{(1-|a|^2)^3}{|1-\overline{a}w|^{6-2p}}\,dw \\ &\approx \int_{B_i\setminus B_i(\frac{1}{2})} |\psi^i_{f,a}(w)|^p (1-|\varphi_a(w)|^2)^q (1-|w|)^s \frac{(1-|a|^2)^3}{|1-\overline{a}w|^{6-2p}}\,dw\ . \end{aligned}$$

Since $0 \le |w| \le \dfrac{1}{2}$ implies

$$\frac{1}{2^{2|q-p+3|}} \le \frac{1}{|1-\overline{a}w|^{2(q-p+3)}} \le 2^{2|q-p+3|},$$

thus, by (2.1) and changing to spherical coordinates, it is enough to prove

$$\begin{aligned} &\int_0^{\frac{1}{2}} \int_{S_i} |\psi^i_{f,a}(r\zeta)|^p (1-r^2)^q (1-r)^s r^{2-s}\,d\sigma(\zeta)\,dr \\ &\approx \int_0^{\frac{1}{2}} \int_{S_i} |\psi^i_{f,a}(r\zeta)|^p (1-r^2)^{q+s} r^2\,d\sigma(\zeta)\,dr. \end{aligned}$$

We have $1 \le 1+r \le 2$ and by (2.4), $|\psi^i_{f,a}|^p$ is subharmonic. Now the result follows applying Proposition 2.3 with $\tau = \dfrac{1}{2}$ and

$$h(r) = \int_{S_i} |\psi^i_{f,a}(r\zeta)|^p \, d\sigma(\zeta) \ .$$

□

Proposition 4.1. *The next inclusions are true*

i) If $0 < p' < p < \infty$ *then*

$$F^i_\varphi(p,q,s) \quad \subset \quad F^i_{\varphi,0}(p',q,s) \qquad \textit{for} \quad -2 < q < \infty,\ 0 < s < \infty \ .$$

ii) If $-2 < q' < q < \infty$ *then*

$$F^i_\varphi(p,q',s) \quad \subset \quad F^i_\varphi(p,q,s), \qquad \textit{for} \quad 0 < p < \infty,\ 0 < s < \infty \ .$$

iii) If $-2 < q < \infty$ *and* $0 < p < \infty$ *then*

$$F^i_\varphi(p,q,s') \quad \subset \quad F^i_\varphi(p,q,s), \qquad \textit{for} \quad 0 < s' < s < \infty \ .$$

Proof. We prove only *i*). Let $f \in F^i_\varphi(p,q,s)$, $d\mu(x) = (1-|x|^2)^q(1-|\varphi^i_a(x)|^2)^s\,dx$ and $0 < p' < p < \infty$. By Hölder's inequality we have

$$\int_{B_i} |\overline{D}^i_{MT} f(x)|^{p'} \, d\mu(x) \quad \le \quad \left(\int_{B_i} |\overline{D}^i_{MT} f(x)|^p \, d\mu(x) \right)^{\frac{p'}{p}} (\mu(B))^{\frac{p-p'}{p}} \ .$$

By Proposition 4.1 we get the result. □

Corollary 4.1. *Let* $0 < p < \infty$, $-2 < q < \infty$ *and* $0 < s < \infty$. *Then:*

i) $F^i_\varphi(p,q,s) \subset \bigcap_{0<p'<p} F^i_{\varphi,0}(p',q,s)$;

ii) $F^i_\varphi(p,q',s) \subset \bigcap_{q'<q} F^i_\varphi(p,q,s)$;

iii) $F^i_\varphi(p,q,s') \subset \bigcap_{s'<s} F^i_\varphi(p,q,s)$.

Now, we give a characterization of $F^i_\varphi(p,q,s)$ in terms of Carleson measures.

We assume definitions and results of [14]. Let $a \in B_i$. Define the Carleson tube by

$$E(a) = \left\{ \ x \in B_i \ : \ \left| x - \frac{a}{|a|} \right| < 1 - |a| \ \right\} \ .$$

For $0 < s < \infty$, a positive Borel measure μ on B_i is a bounded s-Carleson measure if

$$\sup_{a \in B_i} \frac{\mu(E(a))}{(1-|a|)^s} < \infty \ ,$$

and μ is a s-compact Carleson measure if

$$\lim_{|a|\to 1^-} \frac{\mu(E(a))}{(1-|a|)^s} = 0 \ .$$

Theorem 4.3. (Theorems 3.1 and 3.2, [14]). *Let $0 < s < \infty$ and $0 < \tau < \infty$. A positive Borel measure μ on B_i is a bounded s-Carleson measure if and only if*

$$\sup_{a\in B_i} \int_{B_i} \left(\frac{(1-|a|^2)^\tau}{(1+|a|^2|x|^2 - 2(x,a))^{\frac{1+\tau}{2}}} \right)^s d\mu(x) < \infty$$

and μ is a compact s-Carleson measure if and only if

$$\lim_{|a|\to 1^-} \int_{B_i} \left(\frac{(1-|a|^2)^\tau}{(1+|a|^2|x|^2 - 2(x,a))^{\frac{1+\tau}{2}}} \right)^s d\mu(x) = 0 \ .$$

From this result we obtain the next characterization for $F^i_\varphi(p,q,s)$ spaces in terms of Carleson measures.

Theorem 4.4. *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. Then*

i) $f \in F^i_\varphi(p,q,s)$ if and only if $d\mu(x) = |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^{q+s}\, dx$ is a bounded p-Carleson measure.

ii) $f \in F^i_{\varphi,0}(p,q,s)$ if and only if $d\mu(x) = |\overline{D}^i_{MT} f(x)|^p (1-|x|^2)^{q+s}\, dx$ is a compact s-Carleson measure.

Proof. Consider Theorem 4.3, with $\tau = 1$, the identity (2.1) and the fact that

$$|1-\overline{a}x|^2 = (1-\overline{a}x)(1-\overline{x}a) = 1 + |a|^2|x|^2 - 2\,\mathrm{Re}\,(x\overline{a}) \ .$$

So the result follows from the definition of the spaces. □

For additional information on this topic see [3].

5. The i-Bloch and i-Dirichlet spaces

In this section we define the i-Bloch space and characterize these spaces by some special family of $F^i_\varphi(p,q,s)$ spaces. Similar results can be proved using the function $g^s(z,a)$ as weight, instead of the weight $(1-|\varphi^i_a(z)|^2)^s$. The start point is the following result.

Proposition 5.1. *Let $1 \le p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $0 < R < 1$ be fixed. If $f \in F^i_\varphi(p,q,s)$, then there exists $\widetilde{C} = \widetilde{C}(R)$ such that*

$$\left(1-|a|^2\right)^{q+3} \left|\overline{D}^i_{MT} f\,(a)\right|^p \le \widetilde{C} J_{p,q,s} f(a) \qquad \textit{for all } a \in B_i.$$

Proof. Let $0<R<1$ be fixed and $a\in B_i$. By the change of variable $x=\varphi_a^i(w)$ and Lemma 2.2 we have for $0<s<\infty$

$$\begin{aligned}
&J_{p,q,s}^i f(a)\\
&\geq \int_{D_i(a,R)} \left|\overline{D}_{MT}^i f(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi_a^i(x)|^2)^s\,dx\\
&\geq (1-R^2)^s \int_{B_i(R)} \left|\overline{D}_{MT}^i f\left(\varphi_a^i(w)\right)\right|^p \left(1-\left|\varphi_a^i(w)\right|^2\right)^q \left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^3 dw\\
&= (1-R^2)^s(1-|a|^2)^{q+3} \int_{B_i(R)} \left|\frac{1-\overline{w}a}{|1-\bar{a}w|^3}\overline{D}_{MT}^i f\left(\varphi_a^i(w)\right)\right|^p \frac{\left(1-|w|^2\right)^q}{|1-\bar{a}w|^{2q+6-2p}}dw\\
&\geq \frac{(1-R^2)^s\left(1-|a|^2\right)^{q+3}}{2^{2(q-p+3)}} \int_{B_i(R)} \left|\frac{1-\overline{w}a}{|1-\bar{a}w|^3}\overline{D}_{MT}^i f\left(\varphi_a^i(w)\right)\right|^p \left(1-|w|^2\right)^q dw\\
&= \frac{(1-R^2)^s\left(1-|a|^2\right)^{q+3}}{2^{2(q-p+3)}} \int_0^R \int_{S_i} \left|\psi_{f,a}^i(\rho\zeta)\right|^p \left(1-\rho^2\right)^q \rho^2 d\sigma(\zeta)d\rho\\
&\geq \frac{(1-R^2)^s\left(1-|a|^2\right)^{q+3}}{2^{2(q-p+3)}} \left|\overline{D}_{MT}^i f(a)\right|^p \int_0^R \left(1-\rho^2\right)^q \rho^2 d\rho\\
&= C(R)\left(1-|a|^2\right)^{q+3}\left|\overline{D}_{MT}^i f(a)\right|^p .
\end{aligned}$$

Now the proposition follows from this estimation, where we have used the subharmonicity of $|\psi_{f,a}^i|^p$. □

The previous result motivates the definition of i-Bloch spaces.

Let $\alpha>0$. Define the α,i-Bloch space $\mathcal{B}_\alpha^i$ as the set of $\mathfrak{M}$ functions $f: B_i\to\mathbb{H}$ such that

$$\sup_{a\in B_i}(1-|x|^2)^\alpha|\overline{D}_{MT}^i f(x)|<\infty$$

and the little α-Bloch space $\mathcal{B}_{\alpha,0}^i$ as the set of $\mathfrak{M}$ functions $f: B_i\to\mathbb{H}$ such that

$$\lim_{|a|\to 1^-}(1-|x|^2)^\alpha|\overline{D}_{MT}^i f(x)|=0.$$

We observe that if $0<\alpha<\alpha'$, then $\mathcal{B}_\alpha^i\subset\mathcal{B}_{\alpha'}^i$. By Proposition 5.1 we obtain.

Corollary 5.1. *Let $1\leq p<\infty$, $-2<q<\infty$ and $0<s<\infty$. Then*

$$F_\varphi^i(p,q,s)\subset\mathcal{B}_{\frac{q+3}{p}}^i \qquad \textit{and} \qquad F_{\varphi,0}^i(p,q,s)\subset\mathcal{B}_{\frac{q+3}{p},0}^i\ .$$

We have the following partial reciprocal result of Corollary 5.1.

Proposition 5.2. *Let* $0 < p < \infty$*,* $-2 < q < \infty$ *and* $2 < s < \infty$*. If* $f \in \mathcal{B}^i_{\frac{q+3}{p}}$ *(respectively* $f \in \mathcal{B}^i_{\frac{q+3}{p},0}$*), then* $f \in F^i_{\varphi}(p,q,s)$*, (respectively* $f \in F^i_{\varphi,0}(p,q,s)$ *).*

Proof. Let $f \in \mathcal{B}^i_{\frac{q+3}{p}}$ be a non constant function. Then, there exists $0 < M < \infty$ such that

$$\left(1-|x|^2\right)^{\frac{q+3}{p}} \left|\overline{D}^i_{MT} f\,(x)\right| \le M \tag{5.5}$$

for all $x \in B_i$. Using the change of variable $x = \varphi^i_a(w)$ we get

$$\begin{aligned} J_{p,q,s}f(a) &\le \int_{B_i} \frac{M^p}{(1-|x|^2)^{q+3}} (1-|x|^2)^q (1-|\varphi^i_a(x)|^2)^s \, dx \\ &= M^p \int_{B_i} \frac{1}{(1-|\varphi^i_a(w)|^2)^3} (1-|w|^2)^s \frac{(1-|a|^2)^3}{|1-\overline{a}w|^6} \, dw \\ &= M^p \int_{B_i} (1-|w|^2)^{s-3} \, dw \end{aligned}$$

but the last integral is finite since $2 < s < \infty$ and so $f \in F^i_{\varphi}(p,q,s)$.

We suppose now that $f \in \mathcal{B}^i_{\frac{q+3}{p},0}$. Then there exists $0 < R < 1$ such that for all $R < |x| < 1$

$$\left(1-|x|^2\right)^{\frac{q+3}{p}} \left|\overline{D}^i_{MT} f\,(x)\right| \le \frac{\varepsilon^{\frac{1}{p}}}{\left(\int_{B_i} (1-|w|^2)^{s-3} \, dw\right)^{\frac{1}{p}}} .$$

By Proposition 2.1 is enough to estimate

$$\begin{aligned} &\int_{A_i(R)} \left|\overline{D}^i_{MT} f\,(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s \, dx \\ &\le \frac{\varepsilon}{\int_{B_i} (1-|w|^2)^{s-3} \, dw} \int_{A_i(R)} \frac{1}{(1-|x|^2)^{q+3}} (1-|x|^2)^q (1-|\varphi^i_a(x)|^2)^s \, dx \\ &< \frac{\varepsilon}{\int_{B_i} (1-|w|^2)^{s-3} \, dw} \int_{B_i} (1-|w|^2)^{s-3} \, dw < \varepsilon \end{aligned}$$

so this concludes the proof. □

Combining Corollary 5.1 and Proposition 5.2 we have the following theorem:

Theorem 5.1. *Let* $1 \le p < \infty$*,* $-2 < q < \infty$*. The following conditions are equivalent:*

i) $f \in \mathcal{B}^i_{\frac{q+3}{p}}$ *(respectively* $f \in \mathcal{B}^i_{\frac{q+3}{p},0}$*).*

ii) $f \in F^i_{\varphi}(p,q,s)$*, (respectively* $f \in F^i_{\varphi,0}(p,q,s)$*) for all* $s > 2$*.*

iii) $f \in F^i_{\varphi}(p,q,s)$*, (respectively* $f \in F^i_{\varphi,0}(p,q,s)$*) for some* $s > 2$*.*

Proposition 5.3. *Let $0<p<\infty$, $-1<q<\infty$, $0<s<2$ and*
$$\frac{q+1}{p}<\alpha<\frac{q+s+1}{p}.$$
If $f\in\mathcal{B}^i_\alpha$, (respectively $f\in\mathcal{B}^i_{\alpha,0}$) then $f\in F^i_\varphi(p,q,s)$ (respectively $f\in F^i_{\varphi,0}(p,q,s)$). In particular
$$\bigcup_{0<\alpha<\frac{q+s+1}{p}}\mathcal{B}^i_\alpha\subset F^i_\varphi(p,q,s)\subset\mathcal{B}^i_{\frac{q+3}{p}}$$

Proof. Let $f\in\mathcal{B}^i_\alpha$ be a non constant function. Then, there exists $0<M<\infty$ such that $\left(1-|x|^2\right)^\alpha\left|\overline{D}^i_{MT}f\,(x)\right|\le M$ for all $x\in B_i$, and so by the change of variable $x=\varphi^i_a(w)$ and by Lemma 2.1 we have
$$\begin{aligned}
J_{p,q,s}f(a) &\le \int_{B_i}\frac{M^p}{(1-|x|^2)^{\alpha p}}(1-|x|^2)^q(1-|\varphi^i_a(x)|^2)^s\,dx\\
&= M^p\int_{B_i}(1-|\varphi^i_a(w)|^2)^{q-\alpha p}(1-|w|^2)^s\frac{(1-|a|^2)^3}{|1-\overline{a}w|^6}\,dw\\
&= M^p(1-|a|^2)^{q+3-\alpha p}\int_{B_i}\frac{(1-|w|^2)^{q+s-\alpha p}}{|1-\overline{a}w|^{6+2q-2\alpha p}}\,dw\\
&= M^p(1-|a|^2)^{q-\alpha p+3}\int_0^1(1-r^2)^{q+s-\alpha p}r^2\int_{S^i}\frac{d\sigma(\zeta)}{|1-\overline{a}r\zeta|^{2(q-\alpha p+3)}}\,dr\\
&\approx M^p2^{q-\alpha p+3}\lambda\int_0^1(1-r)^{q-p\alpha+s}r^2dr,
\end{aligned}$$
and the last integral is finite. For the little spaces imitate the proof of Proposition 5.2. $\square$

Now we prove some results about Dirichlet spaces $D^i_{p,q}$.

Theorem 5.2. *Let $1\le p<\infty$, $-1<q<\infty$. Then $D^i_{p,q}\subset\mathcal{B}^i_{\frac{q+3}{p},0}$.*

Proof. Let $0<R<1$ be fixed. Imitating the proof of Proposition 5.1 we obtain
$$\int_{D_i(a,R)}|\overline{D}^i_{MT}f(x)|^p(1-|x|^2)^q\,dx\ge C(R)(1-|a|^2)^{q+3}|\overline{D}^i_{MT}f(a)|^p. \tag{5.6}$$
Since
$$\int_{B_i}|\overline{D}^i_{MT}f(x)|^p(1-|x|^2)^q\,dx<\infty$$
then given $\varepsilon>0$, there exists $0<\tilde{R}<1$ such that
$$\int_{A_i(\tilde{R})}|\overline{D}^i_{MT}f(x)|^p(1-|x|^2)^q\,dx<\varepsilon\ .$$
By (2.2) there exists $\tilde{R}<R'<1$ such that $D_i(a,R)\subset A_i(\tilde{R})$ for all $a\in B_i$ with $R'<|a|<1$. From (5.6) we get our result. $\square$

Theorem 5.3. *Let* $1 \leq p < \infty$, $-1 < q < \infty$. *Then*

$$D^i_{p,q} \subset \bigcap_{0<s<\infty} F^i_{\varphi,0}(p,q,s) \ .$$

Proof. Let $f \in D^i_{p,q}$ and $\varepsilon > 0$. By Theorem 5.2, there exists $0 < R'' < 1$ such that

$$(1-|x|^2)^{q+3}|\overline{D}^i_{MT}f(x)|^p < \varepsilon \quad \text{for all} \quad 0 < R'' < |x| < 1. \tag{5.7}$$

Let R, $\tilde{R}$, R' be as in the previous theorem, where R is fixed and we can choose $R'' < \tilde{R} < R' < 1$. Then we have

$$\begin{aligned} J^i_{p,q,s}f(a) &= \int_{B_i(\tilde{R})} \left|\overline{D}^i_{MT}f\,(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s\,dx \\ &\quad + \int_{A_i(\tilde{R})} \left|\overline{D}^i_{MT}f\,(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s\,dx. \end{aligned}$$

By Proposition 2.1 the first integral goes to 0 when $|a| \to 1^-$. We consider $R' < |a| < 1$ and the second integral is divided in two integrals. Thus

$$\begin{aligned} &\int_{A_i(\tilde{R})\setminus D_i(a,R)} \left|\overline{D}^i_{MT}f\,(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s\,dx \\ &\leq\ (1-R^2)^s \int_{A^i(\tilde{R})\setminus D^i(a,R)} \left|\overline{D}^i_{MT}f\,(x)\right|^p \left(1-|x|^2\right)^q dx < \varepsilon\ . \end{aligned}$$

Finally by (5.7), the change of variable $\varphi^i_a(w) = x$ and (2.1) we have

$$\begin{aligned} &\int_{D_i(a,R)} \left|\overline{D}^i_{MT}f\,(x)\right|^p \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s\,dx \\ &\leq \int_{D_i(a,R)} \frac{\varepsilon}{(1-|x|^2)^{q+3}} \left(1-|x|^2\right)^q (1-|\varphi^i_a(x)|^2)^s\,dx \\ &\leq \varepsilon \int_{B_i(R)} \frac{1}{(1-|\varphi^i_a(w)|^2)^3}(1-|w|^2)^s \frac{(1-|a|^2)^3}{|1-\overline{a}w|^6}\,dw \\ &= \varepsilon \int_{B_i(R)} (1-|w|)^{s-3}\,dw \end{aligned}$$

so we finish the proof. □

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References

[1] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic funtion to be normal,* Complex Analysis and its Applications,(Hong Kong 1993), Pitman Res. Notes Math. Ser. 305, Longman Scientific and Technical. Harlow 1994, 136–146.

[2] R. Aulaskari, J. Xiao,and R. Zhao, *On subspaces and subsets of BMOA and UBC.* Analysis **15** (1995), 101–102.

[3] S. Bernstein and P. Cerejeiras, *Carleson Measures and Monogenic functions*, (English) Stud. Math. **180**, no. 1, (2007), 11–25

[4] A. F. Beardon, *The geometry of discrete groups,* Graduate Texts in Mathematics no. 91, Springer Verlag, pp. 337.

[5] L. Carmona, L. Javier; R. Ocampo, L. Feliciano; T. Sánchez, L. Manuel, *Weighted Bermang Spaces,* Hypercomplex Analysis: New Perspectives and Applications, Trends in Mathematics (2014), 89–110.

[6] J. Cnops and R. Delanghe, *Möbius invariant spaces in the unit ball,* Appl. Anal. **73** (2000), 45–64.

[7] J. Cnops, R. Delanghe, K. Güerlebeck, and M. V. Shapiro, *$\mathcal{Q}_p$-spaces in Clifford Analysis*, Advances in Applied Clifford Algebras **11** (2001), 201–218

[8] A. El-Sayed Ahmed and K. Gürlebeck, *Integral norms for Hyperholomorphic Bloch functions in te unit ball of* $\mathbb{R}^3$, in: H. Begehr et al (Eds.) Progress in Analysis, vol. 1, Singapore: World Scientific public, 253–266

[9] A. El-Sayed Ahmed, K. Gürlebeck, L. F. Reséndis, S. Tovar, M. Luis M, *Characterizations for $B^{p,q}$ spaces in Clifford Analysis*, Complex Variables and Elliptic Equations **51**, no. 2 (2006), 119–136.

[10] A. El-Sayed , O. Saleh, *Weighted classes of quaternion-valued functions*, Banach Journal of Mathematical Analysis **6**, no. 2 (2012), 180–191.

[11] K. Gürlebeck, U. Kähler, M. V. Shapiro, and L. M. Tovar, *On $\mathcal{Q}_p-$Spaces of Quaternion-Valued Functions,* As part of *Complex Variables*, **39**, OPA (Overseas Publishers Associations) N. Y. 1999, 115–135.

[12] K. Gürlebeck and H. R. Malonek, *On Strict Inclusions of Weighted Dirichlet Spaces of Monogenic Functions*, Bull. Austral Math. Soc. **64** (2001), 33–50.

[13] K. Gürlebeck and W. Sprössig, *Quaternionic analysis and elliptic boundary value problems,* Int. Ser. Num. Math. **89**, Birkhäuser, Basel 1990.

[14] M. Kotilainen, V. Latvala, and J. Rättyä, *Carleson measures and Jacobian of conformal self-mappings of the unit ball*, mathstat.helsinki.fi/analysis/seminar/esitelmat/VL171005.pdf, 19th October 2005.

[15] J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *$\mathcal{Q}_p-$hyperholomorphic function and harmonic majorants in the quaternionic case*, Bull. Soc. Sci. Lettres Łódź **53**, Sér. Rech. Déform. **41** (2003), 37–47.

[16] J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *$\mathcal{Q}_p-$hyperholomorphic function and harmonic majorants in the hyperkählerian case,* Revue Roumanie de Mathématiques Pures and Appliquées **51** (2006).

[17] J. Ławrynowicz, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Like-Hiperbolic Bloch-Bergman Classes*, Contemporary mathematics, **509** (2010), 103–117.

[18] H. Malonek, *Power series representation for monogenic functions in* $\mathbb{R}^{m+1}$ *based on a permutational product*, Compl. Variab. Theory Applic. **15** (1990), 181–191.

[19] M. Mateljevic and M. Pavlovic, *L^p -bahaviour of power series with coefficients and Hardy spaces*, Proc. Amer. Math. Soc. **87** (1983), 309–316.

[20] Differentiation of the Martinelli-Bochner integrals and the notion of the hyperderivability.Internal report 107, August 1992. Departamento de Matematicas, CINVETAV del IPN, México-City, México. p. 29. Mathematische Nachrichten **172** (1995), 211–238.

[21] P. A. M. Miss, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Quaternionic $F(p,q,s)$function spaces* , Complex Variables and Operator Theory, 2015.

[22] P. A. M. Miss, L. F. Reséndis Ocampo, L. M. Tovar Sánchez, *Holomorphic $F(p,q,s)$ function spaces* , Functions Spaces, 2015.

[23] A. Montes, *Sobre un nuevo espacio de funciones Moisil-Théodorescu hiperholomorfas*, Ms. Thesis, 2011.

[24] C. Ouyang, W. Yang, and R. Zhao, *Characterizatins of Bergman spaces and bloch space in the unit ball of $\mathbb{C}^n$*, Transactions of the American Mathematical Society, **347**, no. 11 (1995), 4301–4313.

[25] C. Ouyang, W. Yang, and R. Zhao, *Möbius Invariant $\mathcal{Q}_p$ Spaces associated with the Green's Function on the Unit Ball of $\mathbb{C}^n$*, Pacific Journal of Mathematics **182**, no. 1 (1998), 69–99.

[26] H. J. Pérez, L. F. Reséndis Ocampo, and L. M. Tovar Sanchez, *Some hyperbolic classes of analytic functions in the unit disk*, preprint 2015.

[27] J. Ryan, *Conformally covariant operators in Clifford Analysis*, ZAA **14** (1995), 677–704.

[28] *On the quaternionic-monogenic functions. Methods of solution of the directand inverse geoelectrical problems*, Moscow 1987

[29] A. Sudbery, *Quaternionic analysis*, Math. Proc. Camb. Phil. Soc. **85** (1979), 199–225.

[30] R. Zhao, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. **105** (1996).

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KWATERNIONOWE PRZESTRZENIE FUNKCYJNE $F(p,q,s)$ MOISILA-THÉODORESCU

S t r e s z c z e n i e

W pracy definiujemy różniczkowalność w sensie Moisila-Théodororescu przyporządkowaną szczególnemu włożeniu przestrzeni $\mathbb{R}^3$ w przestrzeń kwaternionową $\mathbb{H}$. Przy użyciu pochodnej Moisila-Théodororescu wprowadzamy i badamy odpowiednik przestrzeni funkcyjnej $F(p,q,s)$ oraz $F_0(p,q,s)$ wprowadzonych w pracy R. Zhao (1996). Uzyskujemy wyniki podobne do otrzymanych w przypadku monogenicznym; zob. [9, 11, 21].

Słowa kluczowe: Moisila-Théodororescu, $\mathcal{Q}_p^i$, przestrzenie i-Blocha i $F_i(p,q,s)$

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Massimo Vaccaro

ORBITS IN THE REAL GRASSMANNIAN OF 2-PLANES UNDER THE ACTION OF THE GROUPS $Sp(n)$ AND $Sp(n) \cdot Sp(1)$

Summary

The natural action of the unitary group $U(n)$ on $\mathbb{C}^n$ induces its action on the Grassmann manifold $G_k^{\mathbb{R}}(\mathbb{C}^n)$ consisting of real k-dimensional subspaces in $\mathbb{C}^n$. In [9] it has been shown that the Kähler angle, used by Chern and Wolfson in the theory of minimal surfaces, determines the orbit of a 2-plane of a complex vector space in the real Grassmannian under the action of the unitary group. Generalizing such notion in [15], the multiple Kähler angle $\theta(U)$ of a real subspace U of a complex vector space is defined and it is shown that it is a complete invariant with respect to the natural action of the unitary group, that is, for two real subspaces V and W of same dimension in $\mathbb{C}^n$, there exists g in $U(n)$ which satisfies $W = g \cdot V$ if and only if $\theta(V) = \theta(W)$. In this article we determine a complete invariant for a real subspace of dimension 2 with respect to the action of $Sp(n)$ and $Sp(n) \cdot Sp(1)$ – the groups of automorphisms of a real vector space endowed respectively with an Hermitian hypercomplex and an Hermitian quaternionic structure. A model of such spaces is the n-dimensional quaternionic numerical space $\mathbb{H}^n$, a vector space of dimension $4n$ over $\mathbb{R}$. We introduce the *imaginary measure* and the *characteristic deviation* of a 2-plane and prove that they characterize completely the orbit in a 2-plane in $G_2^{\mathbb{R}}(\mathbb{H}^n)$ under the action of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$, respectively.

Keywords and phrases: hermitian quaternionic structure, principal angles, Kähler angles, $Sp(n)$, $Sp(n) \cdot Sp(1)$

1. The Hermitian quaternionic structure

Let V be a real vector space of dimension $4n$.

Definition 1.1. 1. A triple $\mathcal{H} = \{J_1, J_2, J_3\}$ of anticommuting complex structures on V with $J_1 J_2 = J_3$ is called a *hypercomplex structure* on V.

2. The 3-dimensional subalgebra

$$\mathcal{Q} = \mathrm{span}_{\mathbb{R}}(\mathcal{H}) = \mathbb{R}J_1 + \mathbb{R}J_2 + \mathbb{R}J_3 \approx \mathfrak{sp}_1$$

of the Lie algebra $\mathrm{End}(V)$ is called a *quaternionic structure* on V.

Note that two hypercomplex structures $\mathcal{H} = \{J_1, J_2, J_3\}$ and $\mathcal{H}' = \{J'_1, J'_2, J'_3\}$ generate the same quaternionic structure $\mathcal{Q}$ iff they are related by a rotation, i.e.

$$J'_\alpha = \sum_\beta A_\alpha^\beta J_\beta, \qquad (\alpha = 1, 2, 3)$$

with $(A_\alpha^\beta) \in SO(3)$. A hypercomplex structure generating $\mathcal{Q}$ is called an *admissible* (hypercomplex) *basis* of $\mathcal{Q}$. We denote by $S(\mathcal{Q})$ the 2-sphere of complex structures $J \in \mathcal{Q}$, i.e.

$$S(\mathcal{Q}) = \{aJ_1 + bJ_2 + cJ_3,\ a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1\}.$$

Definition 1.2. An Euclidean scalar product $<,>$ in V is called *Hermitian* with respect to a hypercomplex structure $\mathcal{H} = (J_\alpha)$ (resp. the quaternionic structure $\mathcal{Q} = \mathrm{span}_{\mathbb{R}}(\mathcal{H})$) if and only if, for any $X, Y \in V$,

$$< J_\alpha X, J_\alpha Y >=< X, Y >, \qquad (\alpha = 1, 2, 3)$$

(respectively

$$< JX, JY >=< X, Y >, \qquad (\forall J \in S(\mathcal{Q}))).$$

Definition 1.3. A hypercomplex structure $\mathcal{H}$ (resp. quaternionic structure $\mathcal{Q}$) together with an Hermitian scalar product $<,>$ is called an *Hermitian hypercomplex* (resp. *Hermitian quaternionic*) *structure* on V and the triple $(V^{4n}, \mathcal{H}, <,>)$ (resp. $(V^{4n}, \mathcal{Q}, <,>)$) is an *Hermitian hypercomplex* (resp. *quaternionic*) *vector space.*

For a survey of some results on Hermitian hypercomplex and Hermitian quaternionic structures one can refer among others to [2] and [12].

The prototype of an Hermitian hypercomplex vector space is the n-dimensional quaternionic numerical space $\mathbb{H}^n$ which is a real vector space of dimension $4n$, a $\mathbb{H}$-module with respect to right (resp. left) multiplication by quaternions and is endowed with the canonical positive definite Hermitian product

$$\text{(1)} \qquad \begin{aligned} &\mathbf{h} \cdot \mathbf{h}' = \sum_{\alpha=1}^n \overline{h_\alpha} h'_\alpha \quad (\text{resp. } \mathbf{h} \cdot \mathbf{h}' = \sum_{\alpha=1}^n h_\alpha \overline{h'_\alpha}), \\ &\qquad\qquad \mathbf{h} = (h_1, \ldots, h_n),\ \mathbf{h}' = (h'_1, \ldots, h'_n) \in \mathbb{H}^n. \end{aligned}$$

The real part of the Hermitian product defines an Euclidean scalar product $<,>= \mathrm{Re}(\cdot)$ on the real vector space $\mathbb{H}^n \simeq \mathbb{R}^{4n}$.

If we consider the basis $(1, i, j, k)$ of $\mathbb{H}$ satisfying the multiplication table obtainable from the conditions

$$i^2 = j^2 = k^2 = -1; \; ij = -ji = k, \tag{2}$$

one has that the right multiplications by $-i, -j, -k$ (resp. left multiplication by i, j, k) induce real endomorphisms $(I = R_{-i}, J = R_{-j}, K = R_{-k})$ of the $\mathbb{H}$-module $\mathbb{H}^n$ satisfying $I^2 = J^2 = K^2 = -\mathrm{Id}$, $IJ = K = -JI$ and skew-symmetric with respect to the metric $<,>$ i.e. a Hermitian hypercomplex structure on $\mathbb{H}^n$. The basis $(1, i, j, k)$ of $\mathbb{H}$ is not the only one satisfying the relations (2).

Proposition 1.4. [3] *A new basis $(1, i', j', k')$ of $\mathbb{H}$ gives rise to the same multiplication table iff $(i', j', k') = (i, j, k)C$ with $C \in SO(3)$.*

Proof. It is possible to prove the above proposition by a direct calculation. Alternatively we recall that all automorphisms of the algebra $\mathbb{H}$ are internal i.e. given by an application such as

$$\alpha_p : q \mapsto pqp^{-1}, \tag{3}$$

where p is a suitable quaternion that we can always assume unitary (hence $p^{-1} = \bar{p}$).

Such an application has a simple geometrical interpretation. By representing any $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$ on the Euclidean space E^4, the automorphism α_p acts clearly as the identity on the axis where are represented quaternions which reduce to real number and give rise to a rotation of the E^3 orthogonal to such axis. In fact, for $X, Y \in \mathbb{H}$,

$$\begin{aligned} < \alpha_q X, \alpha_q Y > \; &= \mathrm{Re}(\overline{qX\bar{q}}qY\bar{q}) = \mathrm{Re}(q\overline{qX}qY\bar{q}) = \mathrm{Re}(q\bar{X}\bar{q}qY\bar{q}) = \\ &= \mathrm{Re}(q\bar{X}Y\bar{q}) = \mathrm{Re}(\bar{X}Y\bar{q}q) = \mathrm{Re}(\bar{X}Y) = < X, Y > . \end{aligned}$$

Then $\alpha_q \in O(3)$. Moreover, being $Sp(1) \equiv S^3$ connected, it is immediate to verify that $Sp(1)$ is in the connected components of the identity of $O(3)$. The application $q \to \alpha_q$ is an epimorphism of $Sp(1)$ in $SO(3)$. In fact

$$\alpha'_q \alpha_q(X) = q'qX\bar{q}\bar{q}' = q'qX\overline{q'q} = \alpha_{q'q}(X).$$

To see that it is surjective we observe that for $q = \cos\theta + i\sin\theta$ (resp. $q = \cos\theta + j\sin\theta$, $q = \cos\theta + k\sin\theta$) we obtain all rotation around the axis i (resp. j, k) of an angle 2θ. The Kernel of the homomorphism $Sp(1) \to SO(3)$ is $\mathbb{Z}_2 = \{1, -1\}$. □

Making the identification $E^3 \cong (\mathrm{Im}\,\mathbb{H}, \mathrm{Re}(\,\cdot\,))$ we have then proved the following well known

Proposition 1.5.

$$SO(3) = \{\alpha_q : x' = qx\bar{q}, \; x \in \mathrm{Im}\,\mathbb{H}, \; q \in Sp(1)\}; \quad SO(3) \cong Sp(1)/\mathbb{Z}_2$$

We shall denote by $\mathcal{B}$ the set of bases of $\mathbb{H}$ satisfying the relations (2) and call it *canonical system of bases* (see [3]).

If $q = a + bi + cj + dk \in Sp(1)$, the orthogonal matrix C_q associated to α_q such that $(i', j', k') = (i, j, k)C_q$ is given by

$$(4)\quad SO(3) \ni C_q = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

In [3] it has been proved that

Proposition 1.6. [3] *Both the Hermitian product and the scalar product of* $\mathbb{H}^n$ *have intrinsic meaning with respect to the canonical system of bases* $\mathcal{B}$.

Let $(1, i, j, k) \in \mathcal{B}$ be a chosen basis in $\mathbb{H}$ and denote by $I = R_{-i},\ J = R_{-j},\ K = R_{-k}$ the real endomorphisms of the $\mathbb{H}$-module $\mathbb{H}^n$. Let $\mathcal{Q} = \mathrm{span}_{\mathbb{R}}(I, J, K)$.

Proposition 1.7. *For the scalar product and the Hermitian product of a pair of vectors* $L, M \in \mathbb{H}^n$ *the following relation holds:*

$$(5)\qquad L \cdot M = < L, M > + < L, IM > i + < L, JM > j + < L, KM > k$$

Proof. We prove that

$$< L, IM >, < L, JM >, < L, KM >$$

are respectively the coefficients of i, j, k in the Hermitian product $L \cdot M$. In fact $< L, IM >= \mathrm{Re}(L \cdot -Mi) = -\mathrm{Re}(L \cdot M)i$ which is exactly the coefficient of i of the quaternion $L \cdot M$ and analogously for $< L, JM >$ and $< L, KM >$. □

If $(1, i', j', k') \in \mathcal{B}$ and

$$I' = R_{-i'}, J' = R_{-j'}, K' = R_{-k'}$$

is an admissible basis of $\mathcal{Q}$, from Proposition 1.6 one has

$$\begin{aligned} L \cdot M &= \mathrm{Re}(L \cdot M) + \mathrm{Re}(L \cdot I'M)i' + \mathrm{Re}(L \cdot J')j' + \mathrm{Re}(L \cdot K'M)k' \\ &= < L, M > + < L, I'M > i' + < L, J'M > j' + < L, K'M > k' \\ &= < L, M > + < L, IM > i + < L, JM > j + < L, KM > k \end{aligned}$$

The coefficients of the ($\mathbb{H}$-valued) Hermitian product defined in (5) depend clearly on the chosen basis of $\mathbb{H}$. Since a pair of bases $B_1, B_2 \in \mathcal{B}$ of $\mathbb{H}$ is related by an orthogonal transformation which fixes the real axis and is the real part of $(L \cdot M)$ independent from the admissible hypercomplex basis, we can state the

Proposition 1.8. *The quantity*

$$\begin{gathered} < L, R_{-i}M >^2 + < L, R_{-j}M >^2 + < L, R_{-k}M >^2 \\ =< L, IM >^2 + < L, JM >^2 + < L, KM >^2 \end{gathered}$$

does not depend on the admissible basis (I, J, K) *of* $\mathcal{Q}$.

Later on we see a geometrical consequence of this proposition.

2. The groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$

Let V be a $4n$-dimensional real vector space. We regard $V \cong \mathbb{R}^{4n}$ as a right module over the skew-field $\mathbb{H}$ of quaternions by identifying (in the canonical way) $\mathbb{R}^{4n}$ with $\mathbb{H}^n$ and by letting $\mathbb{H}$ act by right multiplication. For any basis $(1, i, j, k) \in \mathcal{B}$ of $\mathbb{H}$ the scalar right multiplications

$$I = R_{-i}, J = R_{-j}, I = R_{-k}$$

define an Hermitian hypercomplex structure on V (clearly depending on the identification $V \cong \mathbb{R}^{4n}$) and consequently an Hermitian quaternionic structure $\mathcal{Q} = \mathrm{span}_{\mathbb{R}}(I, J, K)$.

We first recall the following definition appearing in [3] and [5].

Definition 2.1. The subspaces $U^{4h} \subset \mathbb{H}^n$ of real dimension $4h$, being the real image of the subspaces of $\mathbb{H}^n$ of quaternionic dimension h, are called *characteristic* (*quaternionic*) *subspaces.* A subspace U^p of $V^{4n} \simeq \mathbb{H}^n$ is *pseudo-characteristic* if it is contained in a $\tilde{U}^{4m}$ characteristic being m the smallest integer such that $4m \geq p$. Any subspace U^p is contained in some characteristic subspaces and the real dimension $4t$ of the smallest among them ranges between p and $4p$. A subspace U^p is *almost characteristic* if $4t < 4p$.

Observe that for instance a subspace of real dimension 1 is always pseudo-characteristic and never almostcharacteristic. In the following we shall call characteristic line, (resp. plane, 3-plane, ...) a characteristic subspace of dimension 4 (resp. $8, 12, \ldots$). Moreover, for a subspace $U \subset V$ we denote by $U^{\mathbb{H}}$ the smallest characteristic subspace containing U, i.e. the subspace spanned over $\mathbb{H}$ by a basis of U.

The group $Sp(1)$ is the group with multiplication of unitary quaternions. It is a Lie group whose Lie algebra $\mathfrak{sp}_1 = \mathrm{Im}\,\mathbb{H} \simeq \mathcal{Q}$. For any quaternion $q \in Sp(1)$, let us consider the unitary homothety in the $\mathbb{H}$-module V:

$$q : X \mapsto Xq, \quad X \in V.$$

For instance the automorphisms $I = R_{-i}, J = R_{-j}, K = R_{-k}$ belong yo these transformations .

Proposition 2.2. [4] *The unitary homotheties are rotations of V^{4n} that leave invariant any characteristic line. Moreover for any $X \in V$ the angle $\widehat{X, Xq}$ does not depend on X and it is*

$$\cos \widehat{X, Xq} = \mathrm{Re}(q)$$

Restricting to the action of $Sp(1)$ determines then an inclusion

$$\lambda : Sp(1) \hookrightarrow SO(4n).$$

We define $Sp(n)$ to be the subgroup of $SO(4n)$ commuting with $\lambda(Sp(1))$ i.e. $Sp(n)$ is the centralizer of $\lambda Sp(1)$ in $SO(4n)$. From (5), by

$$AIA^{-1} = I, AJA^{-1} = J, AKA^{-1} = K, \ \forall A \in Sp(n),$$

it follows that $Sp(n)$, besides preserving any admissible basis of $\mathcal{Q}$, preserves the Hermitian product (1), i.e. it is the (quaternionic) unitary group of $\mathbb{H}^n$. It acts transitively on orthonormal (with respect to the Hermitian product (1)) bases $\{X_1, \ldots, X_n\}$ of the right quaternionic vector space $\mathbb{H}^n$. Moreover, with respect to the structure of a $4n$-dimensional real vector space, for any admissible basis (I, J, K) of $\mathcal{Q}$ it acts transitively on orthonormal (with respect to the Euclidean scalar product $\mathrm{Re}(\cdot)$) bases such as

$$\{X_1, IX_1, JX_1, KX_1, \ldots, X_n, IX_n, JX_n, KX_n\}.$$

Let now consider in $\mathbb{H}^n$ the transformations $T_{(A,q)} : X \mapsto AXq$ with $A \in Sp(n),\ q \in Sp(1),\ X \in \mathbb{H}^n$. We denote by $Sp(n) \cdot Sp(1)$ the group of these transformations. We can write

$$X \mapsto AXq = A(q\bar{q})Xq = (Aq)(\bar{q}Xq).$$

Observe that $(Aq) \in Sp(n)$ since q is unitary. In fact

$$B \in Sp(n) \Leftrightarrow B\bar{B}^t = \mathrm{Id}$$

and

$$Aq(\overline{Aq})^t = Aq(\bar{q}\bar{A})^t = Aq\bar{q}(\bar{A})^t = A\bar{A}^t = \mathrm{Id}.$$

In order to study the transformation $X \mapsto \bar{q}Xq$, let $\{X_1, \ldots, X_n\}$ be a basis of $\mathbb{H}^n$ over $\mathbb{H}$ and (I, J, K) an admissible basis of $\mathcal{Q}$.

The vectors $X_i, IX_i, JX_i, KX_i,\ i = 1, \ldots, n$ form a basis over $\mathbb{R}$ of the characteristic lines $X_i^{\mathbb{H}} \simeq \mathbb{H} = \mathrm{span}_{\mathbb{R}}(1, i, j, k)$. From Proposition (1.5) the action of α_q on $\mathbb{H}$ preserves the real axis and rotate the basis (i, j, k). Then

$$\bar{q}\ X_i^{\mathbb{H}}\ q : (X_i, IX_i, JX_i, KX_i) \mapsto (X_i, I'X_i, J'X_i, K'X_i), \quad i = 1, \ldots, n$$

where

$$I' = R_{-i'}, J' = R_{-j'}, K' = R_{-k'} \quad \text{with} \quad (i', j', k') = q(i, j, k)\bar{q}.$$

Therefore $Sp(n) \cdot Sp(1)$ is the normalizer of $\lambda Sp(1)$ in $SO(4n)$ which is isomorphic to the product $Sp(n) \times_{\mathbb{Z}_2} Sp(1)$ where $\mathbb{Z}_2 = \{1, -1\}$. Note that $Sp(1) \cdot Sp(1)$ is precisely $SO(4)$, whereas, for $n \geq 2$, $Sp(n) \cdot Sp(1)$ is a maximal Lie subgroup of $SO(4n)$. Observe that $Sp(n) \cdot Sp(1)$ is not a subgroup of $U(2n)$.

For a deeper understanding of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$ one can refer among others to [12] and [7].

In light of what we said we have that the groups of automorphisms of an Hermitian hypercomplex and an Hermitian quaternionic vector space are isomorphic to $Sp(n)$ and $Sp(n) \cdot Sp(1)$ ([2]), respectively. They may be characterized by the property of preserving some class of <u>admissible bases</u> of V. For an Hermitian hypercomplex structure $\mathcal{H} = (I, J, K)$ of V the admissible bases are orthonormal basis of the form

$$\{X_1, IX_1, JX_1KX_1, \ldots, X_n, IX_n, JX_nKX_n\}.$$

For an Hermitian quaternionic structure $\mathcal{Q}$ on V the class of admissible bases is the union of those corresponding to the Hermitian hypercomplex structures generating $\mathcal{Q}$.

3. Angles between subspaces of a Euclidean vector space

Following [13] and [10], we define the (Euclidean) angle between two subspaces of dimension p and q of a Euclidean vector space by using an exterior algebra. Let E^n be an n-dimensional vector space endowed with a Euclidean scalar product. For a pair of vectors $a, b \in E^n$ we denote by $a \cdot b$ their inner product.

For $1 \leq p \leq n$, $\Lambda^p E^n$ denotes the vector space consisting of p-vectors, i.e. linear combinations over $\mathbb{R}$ of wedges of p-vectors. A p-vector is called decomposable if it can be decomposed as a single wedge of p-vectors of E^n.

We extend the scalar product $(\,\cdot\,)$ in E^n to a scalar product $<,>$ in the vector space $\Lambda^p E^n$ by defining

$$< \alpha, \beta >= det(a_i \cdot b_j)$$

for a pair of decomposable vectors

$$\alpha = a_1 \wedge \ldots, \wedge a_p; \qquad \beta = b_1 \wedge \ldots, \wedge b_p, \quad a_i, b_i \in E^n$$

and then extending for linearity to any pair of vectors in $\Lambda^p E^n$.

It is definite positive and non degenerate; then the pair $(\Lambda^p E^n, <>)$ is a Euclidean vector space. In particular, for the angle between α and β,

$$\cos \widehat{\alpha\beta} = \frac{< \alpha, \beta >}{\sqrt{< \alpha, \alpha >}\sqrt{< \beta, \beta >}} = \frac{\det(a_i \cdot b_i)}{\text{mis } \alpha \text{ mis } \beta}. \tag{6}$$

with

$$\text{mis } \alpha = |\alpha| = \sqrt{< \alpha, \alpha >}.$$

Any decomposable p-vector $\alpha = a_1 \wedge \ldots \wedge a_p$ corresponds to a subspace $A^p \in E^n$ and precisely to that spanned by $a_1, \ldots, a_p$. Conversely, for any basis of A^p the wedge of these vectors is a multiple of α (i.e. it is equal to $k\alpha$, with $k \in \mathbb{R}, k \neq 0$).

Given A^p and B^q with $\alpha = a_1 \wedge \ldots, \wedge a_p \in \Lambda^p E^n$ associated to A and $\beta = b_1 \wedge \ldots, \wedge b_q \in \Lambda^q E^n$ associated to B, we consider the orthogonal projections of $a_1, \ldots, a_p$ on B and $B^\perp$. Then $a_i = a_i^H + a_i^V$, and $\alpha = \alpha_H + \alpha_V + \alpha_M$ (M means mixed part).

Lemma 3.1. *If we choose another basis in A (then $\alpha' = k\alpha$), we have*

$$\alpha'_H = k\alpha_H, \quad \alpha'_V = k\alpha_V, \quad \alpha'_M = k\alpha_M.$$

Definition 3.2. The *angle between A^p and B^q*, $p \leq q$ is the usual angle (between two lines, a line and a plane, two planes) i.e. the angle between one subspace and its orthogonal projection onto the other, i.e.

$$\theta = \arccos \frac{|\alpha_H|}{|\alpha|}.$$

Then $\theta \in [0, \pi/2]$ and, from the previous lemma, it is independent of the chosen basis in A. In particular, if $p = q$ then we can write

$$\theta = \arccos \frac{|\det(a_i \cdot b_j)|}{|\alpha| \cdot |\beta|} \tag{7}$$

i.e. *the cosine of the angle between a pair of p-planes $A, B \subset E^n$ equals the absolute value of the cosine of the angle between any pair of p-vectors $\alpha, \beta \in \Lambda^p E^n$ corresponding to A and B.*

We recall the definition of the principal angles between a pair of subspaces of a real vector space V (see [6], [11] among others).

Definition 3.3. Let $A, B \subseteq V$ be subspaces, $\dim k = \dim(A) \leq \dim(B) = l \geq 1$. The *principal angles* $\theta_i \in [0.\pi/2]$ are recursively defined for $i = 1, \ldots, k$ by

$$\cos\theta_i = \frac{< a_i, b_i >}{||a_i||\,||b_i||} = \max\{\frac{< a, b >}{||a||\,||b||} : a \perp a_m,\, b \perp b_m,\, m = 1, 2, \ldots, i-1\},$$

where $a_j \in A$, $b_j \in B$.

In words, the procedure is to find the unit vector $a_1 \in A$ and the unit vector $b_1 \in B$ which minimize the angle between them and call this angle θ_1. Now take the orthogonal complement in A to a_1 and the orthogonal complement in B to b_1 and iterate.

The principal angles $\theta_1, \ldots, \theta_k$ between the pair of subspaces A, B are some of the critical values of the angular function

$$\phi_{A,B} = A \times B \to \mathbb{R}$$

associating with each pair of non-zero vectors $a \in A$, $b \in B$ the angle between them. (Other critical values of this function are for instance $\pi - \theta_i$).

We recall the theorem of Afriat ([8], [1]) which states:

Theorem 3.4. [8], [1] *In any pair of subspaces A^k and B^l there exist orthonormal bases $\{u_i\}_{i=1}^k$ and $\{v_j\}_{j=1}^l$ such that*

$$< u_i, v_i >\geq 0 \quad \textit{and} \quad < v_i \times v_j >= 0 \quad \textit{if} \quad i \neq j.$$

Then, from Definition 3.3, it follows that the values $< u_i, v_i >$, $i = 1, \ldots, k$, are clearly the cosines of the principal angles between the subspaces A and B.

The principal angles between a pair of subspaces A, B of V are also defined as the singular value of the orthogonal projector $P^A : B \to A$. (or, equivalently, the singular value of the orthogonal projector $P^B : A \to B$).

We recall the following well known theorem of linear algebra.

Theorem 3.5. (SVD: Singular Value Decomposition) *Let M be an $m \times n$-matrix whose entries come from the field K, which is either the field of real numbers or the field of complex numbers. Then there exists a factorization of the form*

$$M = U\Sigma V^*$$

where U is an $m \times m$-unitary matrix over K, the matrix Σ is $m \times n$-diagonal matrix with nonnegative real numbers on the diagonal, and V^* denotes the conjugate transpose of V, an $n \times n$-unitary matrix over K. [Such a factorization is called a *singular-value decomposition* of M (SVD)].

A common convention is to order the diagonal entries Σ_{ii} in non-increasing fashion. In this case, the diagonal matrix Σ is uniquely determined by M (though the matrices U and V are not).

Definition 3.6. A non-negative real number σ is a *singular value* for M iff there exist unit-length vectors u in K^m and v in K^n such that

$$Mv = \sigma u, \qquad M^*u = \sigma v$$

The vectors u and v are called respectively *left-singular and right-singular* vectors associated to the singular value σ.

In any singular value decomposition

$$M = U\Sigma V^*$$

the diagonal entries of Σ are necessarily equal to the singular values of M. The columns of U and V are, respectively, left- and right-singular vectors for the corresponding singular values. Consequently, the SVD theorem states that an $m \times n$ matrix M has at least one and at most $p = min(m, n)$ distinct singular values, and that it is always possible to find a unitary basis for K^m and a unitary basis for K^n consisting respectively of left-singular and right-singular vectors vectors of M.

A singular value for which we can find two left (or right) singular vectors that are linearly independent is called *degenerate*.

Non-degenerate singular values always have unique left and right singular vectors, up to multiplication by a unit phase factor $e^{i\theta}$ (for the real case up to sign).

Consequently, if all singular values of M are non-degenerate and non-zero, then its singular value decomposition is unique, up to multiplication of a column of U by a unit phase factor and simultaneous multiplication of the corresponding column of V by the same unit phase factor.

Degenerate singular values, by definition, have non-unique singular vectors. Furthermore, if u_1 and u_2 are two left-singular vectors which both correspond to the singular value σ, then any normalized linear combination of the two vectors is also a left singular vector corresponding to the singular value σ. The similar statement is true for right singular vectors. Consequently, if M has degenerate singular values, then its singular value decomposition is not unique (while the diagonal matrix is always unique).

According to the SVD theorem, there exist orthonormal bases $(a_1, \dots, a_k)$ in A and $(b_1, \dots, b_k)$ in B with respect to which the matrix representing P^B (which is the Gram matrix $G(A \times B)$ assumes diagonal form (with non negative entries) plus a null block i.e

$$G(A \times B) = [\,\Gamma \quad 0\,], \quad (\Gamma = \mathrm{diag}).$$

Completing the basis of B with an orthonormal basis $(b_{k+1}, \dots, b_l)$ of $\mathrm{span}_{\mathbb{R}}(b_1, \dots, b_k)^{\perp}$ in case $l > k$, we have that the Gram matrix G with respect to the orthonormal bases $\{a_i,\ i = 1, \dots, k\}$ and $\{b_i,\ i = 1, \dots, l\}$ assumes the form above.

By the unicity of the diagonal form (setting the diagonal entries in non increasing order), the diagonal entries are then exactly the principal angles between of the pair A, B i.e. $\Gamma = diag(\cos\theta_i),\ i = 1, \dots, k$ whereas the vectors $\{a_i,\ i = 1, \dots, k\}$ and $\{b_i,\ i = 1, \dots, l\}$ are left and right singular vectors of the SVD.

Definition 3.7. Two subspaces A and B of same dimension are said to be *isoclinic* and the angle ϕ $(0 \leq \phi \leq \frac{\pi}{2})$ is said to be angle of isoclinicity between them if either of the following conditions hold:

1) the angle between any non-zero vectors of one of the subspaces and the other subspace is equal to ϕ;

2) $GG^t = \cos\phi\,\mathrm{Id}$ for the Gram matrix G with respect to any orthonormal basis of A and B;

3) all principal angles between A and B equals ϕ.

A practical way to determine the principal angles follows from the theory of eigenvalue decomposition of endomorphisms. Given in fact an SVD of M, as described above, the following two relations hold:

$$M^*M = V\Sigma^*U^*U\Sigma V^* = V(\Sigma^*\Sigma)V^*$$

$$MM^* = U\Sigma V^*V\Sigma^*U^* = U(\Sigma\Sigma^*)U^*$$

The right-hand sides of these relations describe the eigenvalue decompositions of the symmetric (or Hermitian if $K = \mathbb{C}$) matrices of the left hand sides. Consequently, *the squares of the non-zero singular values of M are equal to the non-zero eigenvalues of either M^*M or MM^* . Furthermore, the columns of U (left singular vectors) are eigenvectors of MM^* and the columns of V (right singular vectors) are eigenvectors of M^*M.*

In our case M is the Gram matrix G and the singular values are the cosines of the principal angles.

We underline the following relation between the angle and the principal angles between a pair of subspaces of a real vector space V.

Proposition 3.8. [13] *Let A^p and B^q be a pair of subspaces of V^n with $1 \leq p \leq q \leq n$. Let θ be the angle between the subspaces A^p and B^q of E^n and $\theta_1, \ldots, \theta_p$ the principal angles between them realized by the pairs of unitary vectors (a_i, b_i), $i = 1, \ldots, p$ (i.e. $\theta_i =< a_i, b_i >$). Then*

$$\cos\theta = \cos\theta_1 \cos\theta_2 \ldots \cos\theta_p.$$

Proof. Let $\alpha = a_1 \wedge \ldots, \wedge a_p$ the p-vector corresponding to A^p. Then $|\alpha| = 1$. Moreover $a_i^H = b_i cos\theta_i, \quad i = 1, \ldots, p$. Then

$$\alpha^H = \cos\theta_1 \cos\theta_2 \ldots \cos\theta_p \, b_1 \wedge b_2 \wedge \ldots \wedge b_q.$$

and

$$\cos\theta = |\alpha_H| = \cos\theta_1 \cos\theta_2 \ldots \cos\theta_p.$$

$\square$

This makes perfect sense because the principal angle θ_i is just the length of the projection of a_i onto B. If we consider a unit cube with edges given by the a_i then its projection onto B will have edges scaled by the appropriate $\cos\theta_i$. Thus, projecting the cube scales its volume by the product of the $\cos\theta_i$. In particular if $p = q$ the angle between A and B is given by the determinant of the Gram Matrix $G(A \times B)$ (i.e. the matrix of the projector $P^A : B \to A$).

From Afriat Theorem we derive the following

Corollary 3.9. *Let A^l and B^p a pair of subspaces in V^n, $l \leq p$, $l + p \leq n$, and θ_i, $i = 1, \ldots, p$ the principal angles between them. There exists an orthonormal basis $\{x_1, \ldots, x_n\}$ of V^n such that*

$$\{a_1, \ldots, a_l\} = \{x_1, \ldots x_l\}$$

is an orthonormal basis of A^l, and

$$\{b_1, \ldots, b_p\} = \{\cos\theta_1 x_1 + \sin\theta_1 x_{l+1}, \ldots, \cos\theta_l x_l + \sin\theta_l x_{2l}, x_{l+1}, \ldots, x_p\}$$

is an orthonormal basis for B^p.

This is a nice choice of the basis for A and B since the angle between a_i and b_i is exactly the principal angle θ_i for all $i = 1, \ldots, p$.

Proof. From Theorem (3.4), there exist orthonormal bases $\{a_1, a_2, \ldots, a_l\}$ and $\{b_1, b_2, \ldots, b_p\}$ such that $< a_i, b_i >= \cos\theta_i$, $i = 1, \ldots, p$, and $< a_i, b_j >= 0$ for $i = 1, \ldots, l$, $j = 1, \ldots, p$, $i \neq j$.

Let then $(x_1, x_2, \ldots, x_l) = (a_1, a_2, \ldots, a_l)$. Complete it to an orthonormal basis $\{\tilde{x}_{l+1}, \ldots, \tilde{x}_n\}$ Then

$$b_i = \cos\alpha_i x_i + h_{i1}\tilde{x}_{l+1} + h_{i2}\tilde{x}_{l+2} + \ldots + h_{i,n-l}\tilde{x}_n,\ i = 1,\ldots,l$$
$$b_i = h_{i1}\tilde{x}_{l+1} + h_{i2}\tilde{x}_{l+2} + \ldots + h_{i,n-l}\tilde{x}_n,\ i = l+1,\ldots,p$$

where

$$h_{i1}^2 + h_{i2}^2 + \ldots + h_{i,n-l}^2 = \sin^2\alpha_i,\ i = 1,\ldots,l$$
$$h_{i1}^2 + h_{i2}^2 + \ldots + h_{i,n-l}^2 = 1,\ i = l+1,\ldots,p$$

Moreover

$$h_{i1}h_{j1} + h_{i2}h_{j2} + \ldots + h_{i,n-l}h_{j,n-l} = 0,\ i,j = 1,\ldots,p, \quad i \neq j.$$

Consider the vectors

$$x_{l+i} = \frac{1}{\sin\alpha_i}(h_{i1}\tilde{x}_{k+1} + h_{i2}\tilde{x}_{k+2} + \ldots + h_{i,n-l}\tilde{x}_n), \quad i = 1\ldots l$$
$$x_{l+i} = h_{i1}\tilde{x}_{k+1} + h_{i2}\tilde{x}_{k+2} + \ldots + h_{i,n-l}\tilde{x}_n, \quad i = l+1\ldots p$$

Then $\{x_1,\ldots,x_{l+p},\tilde{x}_{l+p+1},\ldots,x_n\}$ is an orthonormal basis of $\mathbb{R}^n$ and

$$b_i = \cos\alpha_i x_i + \sin\alpha_i x_i, \quad i = 1,\ldots,l.$$
$$b_i = x_i, \quad i = l+1,\ldots,p.$$

□

Finally we recall the notion of Kähler angle which is defined in a real vector space V endowed with a complex structure I.

Definition 3.10. Let (V^{2n}, I) be a real vector space endowed with a complex structure I. For any pairs of vectors $X, Y \in V$ their *Kähler angle* is given by

$$\theta = \arccos\frac{< X, IY >}{|X|\,|Y|\sin\widehat{XY}} = \arccos\frac{< X, IY >}{\text{mis }(X \wedge Y)}. \tag{8}$$

Then $0 \leq \theta \leq \pi$. If one wants to disregard the orientation of the 2-plane $A = \text{span}_{\mathbb{R}}(X,Y)$ we can consider the absolute value of the right hand side of equation (8) restricting the Kähler angle to the interval $[0.\pi/2]$.

It is straightforward to check that the Kähler angle is an intrinsic property of the (oriented) 2-plane A. For this reason we will also speak of the Kähler angle of a 2-plane. The Kähler angle measures the deviation of a 2-plane from holomorphicity. Observe that the Kähler angle of the 2-plane A is one of the two identical principal angles between the pairs of 2-plane A and IA which are always isoclinic as one can immediately verify. Therefore for the angle between the pair of 2-planes A and IA one has:

$$\cos(\widehat{A, IA}) = \frac{< X, IY >^2}{\text{mis}^2\ (X \wedge Y)}. \tag{9}$$

From Proposition (1.8) it follows the

Corollary 3.11. *Let* $U \subset (V^{4n}, \mathcal{Q}, <,>)$ *be a* 2 *plane. The sum of the cosines of the angles between the pairs* (U, IU), (U, JU), (U, KU) *is constant for any admissible basis* (I, J, K) *of* $\mathcal{Q}$.

Proof. Let $U = \mathrm{span}_{\mathbb{R}}(X, Y)$ and (I, J, K) an admissible basis of $\mathcal{Q}$. By (9) one has

$$\cos(\widehat{U, IU}) + \cos(\widehat{U, JU}) + \cos(\widehat{U, KU})$$

$$= \frac{(< X, IY >^2 + < X, JY >^2 + < X, KY >^2)}{\mathrm{mis}^2\ (X \wedge Y)}.$$

The conclusion follows from Proposition (1.8). □

The following angular definitions shall apply in the context of an Hermitian quaternionic vector space $(V^{4n}, \mathcal{Q}, <,>)$.

For a 2-dimensional subspace $U \subset V$, in order to generalize the notion of Kähler angle, we will need to specify the complex structure we are considering. Therefore we will speak of *J-Kähler angle* of U with $J \in S(\mathcal{Q})$.

Definition 3.12. For a pair of vectors L, M of V and $I \in S(\mathcal{Q})$, we define their *I-complex characteristic angle* ϕ as the angle between the pseudo-characteristic 2-dimensional subspaces $\mathrm{span}_{\mathbb{R}}(L, IL)$ and $\mathrm{span}_{\mathbb{R}}(M, IM)$. Moreover we call the *quaternionic characteristic angle* φ between L and M the angle between the characteristic lines $L^{\mathbb{H}}$ and $M^{\mathbb{H}}$.

Proposition 3.13. *The* ***I-complex characteristic angle*** ϕ *between a pair of vectors* L, M *of* V *is given by*

$$\cos\phi = \frac{(< L, M >^2 + < L, IM >^2)}{< L, L >< M, M >}, \tag{10}$$

while the quaternionic characteristic angle φ *between the same pair of vectors is given by*

$$\begin{aligned} \cos\varphi &= \frac{[\mathcal{N}(L \cdot M)]^2}{\mathrm{mis}^4\ L\ \mathrm{mis}^4\ M} \\ &= \frac{(< L, M >^2 + < L, IM >^2 + < L, JM >^2 + < L, KM >^2)^2}{< L, L >^2 < M, M >^2} \end{aligned} \tag{11}$$

where $\mathcal{N}(q) = q\bar{q}$.

Proof. The proof follows immediately from (7). From Proposition (1.8) we derive the expected independence of the quaternionic characteristic angle from the admissible basis (I, J, K). □

As the Euclidean metric give rise to the consideration of Euclidean angles, the Hermitian metric defined in V allows us to introduce an Hermitian angle between a pair of vectors. In [4] the *Hermitian angle* between a pair of vectors of a quaternionic vector space V^{4n} is defined as

$$\cos\psi = \frac{|(L\cdot M)|}{|L||M|} \tag{12}$$
$$= \frac{\sqrt{(<L,M>^2+<L,IM>^2+<L,JM>^2+<L,KM>^2)}}{\sqrt{<L,L>}\sqrt{<M,M>}}$$

It does not depend on the admissible basis of $\mathcal{Q}$. Observe that the Hermitian angle ψ between a pair of vectors L, M is just the angle between such pair (computed by using the Hermitian product), whereas the characteristic angle φ is the angle between the 4-dimensional characteristic lines they span over $\mathbb{H}$. It is $\cos\varphi = \cos^4\psi$.

4. The imaginary measure and the characteristic deviation of a 2-plane

Let $(V^{4n}, \mathcal{Q}, <,>)$ be an Hermitian quaternionic vector space and $U \subset V$ a 2-plane. Consider the purely imaginary quaternion

$$\mathcal{IM}(U) = \frac{\mathrm{Im}(L\cdot M)}{\mathrm{mis}(L\wedge M)}, \quad L, M \in U.$$

Proposition 4.1. *$\mathcal{IM}(U)$ in an intrinsic property of a 2-plane $U \subset (V^{4n}, \mathcal{Q}, <,>)$ i.e. it does not depend neither on the chosen generators L, M nor on the admissible basis $\mathcal{H}$ of $\mathcal{Q}$. Moreover $Sp(n)$ preserves $\mathcal{IM}(U)$.*

Proof. If $L' = rL + sM, \quad M' = r'L + s'M, \quad r, s, r', s' \in \mathbb{R}$ then

$$\mathcal{IM}(U) = \frac{\mathrm{Im}(L'\cdot M')}{\mathrm{mis}(L'\wedge M')} = \frac{<L', IM'>i + <L', JM'>j + <L', KM'>k}{\sqrt{<(L'\wedge M'), (L'\wedge M')>}}$$
$$= \frac{(rs'-sr')\mathrm{Im}(L\cdot M)}{(rs'-sr')\sqrt{<(L\wedge M), (L\wedge M)>}}.$$

The second statements follows from Proposition (1.6). The invariance of $\mathcal{IM}(U)$ under the action of $Sp(n)$ on V is obvious being $Sp(n)$ the quaternionic unitary group. □

Definition 4.2. We call

$$\mathcal{IM}(U) = \frac{\mathrm{Im}(L\cdot M)}{\mathrm{mis}(L\wedge M)}$$

the *imaginary measure* of the 2-plane U in the Hermitian quaternionic vector space $(V^{4n}, \mathcal{Q}, <,>)$.

In particular, if the pair L, M is an orthonormal Euclidean basis of U, then $\mathcal{IM}(U) = L \cdot M$.

On the contrary, the group $Sn(n) \cdot Sp(1)$ does not preserve $\mathcal{IM}(U)$. If $q \in Sp(1)$, one has $ALq \cdot AMq = \bar{q}(AL \cdot AM)q = \bar{q}(L \cdot M)q$. Then the action of $Sp(n) \cdot Sp(1)$ performs a rotation on the imaginary quaternion $\mathcal{IM}(U)$ in $\operatorname{Im}\mathbb{H}$ (Proposition 1.5).

Extending to an Hermitian quaternionic vector space case some notions and results of a vector space endowed with a complex structure (see [10]), in [3] it has been introduced

$$\begin{aligned} \Delta(U) &= \mathcal{N}(\mathcal{IM}(U)) = \frac{\mathcal{N}[\operatorname{Im}(L \cdot M)]}{\operatorname{mis}^2(L \wedge M)} \\ &= \frac{< L, IM >^2 + < L, JM >^2 + < L, KM >^2}{\operatorname{mis}^2(L \wedge M)}. \end{aligned} \tag{13}$$

In particular, in case the basis L, M is orthonormal, $\Delta(U) = \mathcal{N}(L \cdot M)$.

From Proposition (1.8), one has the

Proposition 4.3. *The quantity $\Delta(U)$ is an intrinsic property of a 2-plane preserved by the action of the group $Sp(n) \cdot Sp(1)$ on V.*

Claim 4.4. [3] *The real number $\Delta(U) \in [0, 1]$ and equals* 1 *iff* $\dim U^{\mathbb{H}} = 1$.

Proof. Choosing a pair of orthonormal vectors L, M in U, it is $\Delta(U) = |(L \cdot M)|^2$ which equals the square of the cosine of the characteristic angle between the characteristic lines $L^{\mathbb{H}}, M^{\mathbb{H}}$. Then $\Delta(U) = 1$ iff U is pseudo-characteristic. □

Definition 4.5. The angle $\delta(U) \in [0, \pi/2]$ such that $\cos^2 \delta(U) = \Delta(U)$ is called the *characteristic deviation* of the real 2-plane $U \subset V$.

Lemma 4.6. [3]

$$\Delta(U) = cos^2\delta(U) = \cos(\widehat{U, IU}) + \cos(\widehat{U, JU}) + \cos(\widehat{U, KU}).$$

where $\cos(\widehat{U, IU})$ *(resp.* $\cos(\widehat{U, JU})$, $\cos(\widehat{U, KU})$*) denotes the cosine of the angle between the pairs of* 2*-planes* (U, IU) *(resp.* (U, JU), (U, KU)*).*

Proof. Let compute the angle between the pair of isoclinic 2-plane $U = \operatorname{span}_{\mathbb{R}}(L, M)$ and IU. Applying (7) one has

$$\cos(\widehat{U, IU}) = \frac{\begin{vmatrix} < L, IL > & < L, IM > \\ < M, IL > & < M, IM > \end{vmatrix}}{\operatorname{mis}(L \wedge M) \operatorname{mis}(IL \wedge IM)}$$

i.e.

$$\cos(\widehat{U, IU}) = \frac{< L, IM >^2}{\operatorname{mis}^2(L \wedge M)}$$

□

Observe that, from Proposition (3.8), $\cos(\widehat{U,IU})$ (resp. $\cos(\widehat{U,JU})$, $\cos(\widehat{U,KU})$) is the product of the cosines of the pair of identical principal angles between the pair (U,IU) (resp. (U,JU),(U,KU)) which in this case equal the I-Kähler angle (resp. J-Kähler, K-Kähler) given in (8).

The definition of characteristic deviation of a 2-plane given in [3] has been extended in [3] and [4] to 3 and 4-dimensional subspaces. In all this cases, the characteristic deviation is an angle $\delta \in [0,\pi/2]$ which measures the deviation for a subspace from being pseudo-characteristic. The definition of characteristic deviation given for subspaces of dimension 2,3 and 4 has been generalized in [5] to any $U^t \subseteq V^{4n}$ using the ratio between the Euclidean and Hermitian measure of a simple multivector that we can always associate to the given subspace. To this aim it has been used the theory of determinants on a non commutative field developed by Dieudonné. In this case it is an angle $\delta \in [0,\pi/2]$ which measures the deviation for the given subspace $U^p \subset V^{4n}$ from being almost characteristic. For our purposes we will introduce the following invariant to associate to a subspace U^p of any dimension p. To this aim we give the following lemma whose proof can be found in [3].

Lemma 4.7. *Let $\{X_1,\ldots,X_m\}$ be a set of linear independent vectors in V. For any orthogonal transformation*

$$T: X_r \mapsto X'_r = \sum_{s=1}^{m} c_{rs} X_s$$

with (c_{rs}) orthogonal matrix of order m, one has

$$\sum_{r<s} \mathcal{N}[\mathrm{Im}(X_r \cdot X_s)] = \sum_{r<s} \mathcal{N}[\mathrm{Im}(X'_r \cdot X'_s)]$$

The following definition generalizes the definition of the characteristic deviation given for a 2-plane.

Definition 4.8. Let $(X_1,\ldots,X_m)$ be an orthonormal basis of the subspace U^m. Denote by $U_{rs} =< X_r, X_s >_{\mathbb{R}}$. We call the quantity

$$\Delta(U) = \begin{pmatrix} m \\ 2 \end{pmatrix}^{-1} \sum_{r<s} \Delta(U_{rs}) \tag{14}$$

the *characteristic deviation of the subspace U^m.*

From Lemma (4.7) and Proposition (4.3) it follows that the characteristic deviation of a subspace $U \subset V$ depends neither on the admissible basis $\mathcal{H}$ of $\mathcal{Q}$ nor on the chosen orthonormal basis of U which determines the 2-planes of U appearing in the (14) then

Proposition 4.9. *The characteristic deviation $\Delta(U)$ of a subspace U of the Hermitian quaternionic vector space V is an intrinsic property of U.*

5. Orbit of a 2-plane of $\mathbb{H}^n$ under the action on $Sp(n)$ and $Sp(n) \cdot Sp(1)$

We fix one for all the basis $(1, i, j, k) \in \mathcal{B}$ of $\mathbb{H}$. Let consider the $4n$-dimensional Hermitian quaternionic vector space $(\mathbb{H}^n, \mathcal{Q} = \operatorname{span}_{\mathbb{R}}(I, J, K), <, >= \operatorname{Re}(\cdot))$ where $(I = R_{-i}, J = R_{-j}, K = R_{-k})$ is the Hermitian hypercomplex structure given by right multiplications by the imaginary units $(-i, -j, -k)$ on the $\mathbb{H}$-module $\mathbb{H}^n$. Let $(\mathbf{e_1}, \dots \mathbf{e_n})$ be the (Hermitian) orthonormal canonical basis of $\mathbb{H}^n$ over $\mathbb{H}$. We denote by $G_2^{\mathbb{R}}(\mathbb{H}^n)$ the real Grassmannian of the 2-planes of $\mathbb{H}^n$ and by

$$G_2^{\mathcal{IM}} = \{U \in G_2^{\mathbb{R}}(\mathbb{H}^n) \mid \mathcal{IM}(U) = \mathcal{IM}\}$$

the set of subspaces $U \subset \mathbb{H}^n$ of real dimension 2 sharing the same imaginary measure equal to $\mathcal{IM} \in \operatorname{Im}\mathbb{H}$. Recall that $\delta(U) \in [0, \pi/2]$ is the angle whose square cosine is the characteristic deviation of U i.e. $cos^2\delta(U) = \Delta(U)$. In particular, if (L, M) is an orthonormal basis of U, one has $\Delta(U) = \mathcal{N}(\mathcal{IM}(U)) = \mathcal{N}(L \cdot M)$.

Theorem 5.1. *The imaginary measure $\mathcal{IM}(U)$ determines completely the orbit of a 2-plane $U \subset \mathbb{H}^n$ in the Grassmannian $G_2^{\mathbb{R}}(\mathbb{H}^n)$ under the action of $Sp(n)$. By denoting*

$$W = \operatorname{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}\mathcal{IM}(U) + \sin\delta(U)\mathbf{e_2}),$$

we have $G_2^{\mathcal{IM}} = Sp(n) \cdot W$.

Proof. Let $\mathbb{H}^n \supset U = (\mathbf{u_1}, \mathbf{u_2})_{\mathbb{R}} \in G_2^{\mathcal{IM}}$ with $(\mathbf{u_1}, \mathbf{u_2})$ an orthonormal basis. Observe that $\mathcal{IM}(W) = \mathcal{IM}(U) = \mathcal{IM}$. From Proposition (4.1), the group $Sp(n)$ preserves $\mathcal{IM}(U)$ then $G_2^{\mathcal{IM}} \supseteq Sp(n) \cdot W$. We show that $G_2^{\mathcal{IM}} \subseteq Sp(n) \cdot W$.

The group $Sp(n)$ acts transitively on unitary vectors so there exist $A \in Sp(n)$ such that

$$\begin{aligned} A : \mathbf{u_1} &\mapsto \mathbf{e_1}, \\ A : \mathbf{u_2} &\mapsto A\mathbf{u_2} = \mathbf{Y} = (\mathbf{e_1}q_1 + \mathbf{e_2}q_2 + \dots + \mathbf{e_n}q_n) \\ &\phantom{\mapsto A\mathbf{u_2}} = (\mathbf{e_1}q_1) + (\mathbf{e_2}q_2 + \dots + \mathbf{e_n}q_n) \end{aligned}$$

with $\mathbf{Y}$ unitary. Let $\mathbf{Y_1} = \mathbf{e_1}q_1$ and $\mathbf{Y_2} = \mathbf{e_2}q_2 + \dots + \mathbf{e_n}q_n$.

By the action of $1 \oplus B$ with $B \in Sp(n-1)$ (since $Sp(n-1)$ acts transitively on vectors of $\mathbb{H}^{n-1}$ preserving norms):

$$B : (A\mathbf{u_2}) = \mathbf{Y} \mapsto (\mathbf{e_1}q_1 + \mathbf{e_2}|\mathbf{Y_2}|), \qquad |\mathbf{Y_2}| = \sqrt{(\mathbf{Y_2} \cdot \mathbf{Y_2})} = \sqrt{(\bar{q}_2 q_2 + \dots, \bar{q}_n q_n))}$$

i.e. by $(B \circ A) \in Sp(n)$ we have carrier U to $\operatorname{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}q_1 + \mathbf{e_2}|\mathbf{Y_2}|)$

Since $Sp(n) \subset SO(4n)$ preserves Hermitian measures and in particular $\mathcal{IM}(U)$ as well as all Euclidean measures in particular Euclidean norms and angles between vectors and between subspaces (in particular Kähler angles), it follows that $q_1 = \mathcal{IM}(U)$ and $|\mathbf{Y_2}| = \sqrt{1 - \Delta(U)} = \sin\delta(U)$. In fact:

$$0 = \operatorname{Re}(\mathbf{e_1} \cdot (\mathbf{e_1}q_1 + \mathbf{e_2}|\mathbf{Y_2}|)) = \operatorname{Re}(q_1)$$

i.e. q_1 is purely imaginary; moreover

$$\mathcal{IM}(U) = \mathcal{IM}(E) = \mathrm{Im}(\mathbf{e_1} \cdot (\mathbf{e_1} q_1 + \mathbf{e_2}|\mathbf{Y_2}|)) = \mathrm{Im}(q_1) = q_1,$$

i.e. $\mathbf{Y} = \mathcal{IM}(U)\mathbf{e_1} + \mathbf{e_2}|Y_2|$. Finally imposing

$$1 = |\mathbf{Y}| = (\mathbf{e_1} q_1 + \mathbf{e_2}|Y_2|)\cdot(\mathbf{e_1} q_1 + \mathbf{e_2}|Y_2|) = \bar{q_1}(\mathbf{e_1}\cdot\mathbf{e_1})q_1 + |Y_2|^2(\mathbf{e_2}\cdot\mathbf{e_2}) = \Delta(U) + |Y_2|^2$$

we have $|Y_2| = \sqrt{1-\Delta(U)}$. Then by $B \circ A \in Sp(n)$ we have carrier U to

$$W = \mathrm{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}\mathcal{IM}(U) + \sqrt{1-\Delta(U)}\mathbf{e_2}) = \mathrm{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}\mathcal{IM}(U) + \sin\delta(U)\mathbf{e_2})$$

□

We now study the orbit of the same 2-plane $U \subset \mathbb{H}^n$ under the action of the group $Sp(n) \cdot Sp(1)$. We recall that the characteristic deviation $\delta(U)$ is invariant under a change of the hypercomplex basis in $\mathbb{H}^n$ given by $(i', j', k') = p(i, j, k)p^{-1}$ with $p \in Sp(1)$.

Let denote by $G_2^\delta = \{U \subset G_2^{\mathbb{R}}(\mathbb{H}^n), \mid \delta(U) = \delta\}$ the set of 2-plane in $\mathbb{H}^n$ with characteristic deviation equal to δ.

Theorem 5.2. *The characteristic deviation δ determines completely the orbit of the 2-plane $U \subset \mathbb{H}^n$ in the Grassmannian $G_2^{\mathbb{R}}(\mathbb{H}^n)$ under the action of $Sp(n) \cdot Sp(1)$.*

Proof. From Proposition (4.3), the group $Sp(n)\cdot Sp(1)$ preserves δ then $G_2^\delta \supseteq [Sp(1)\cdot Sp(n)]\cdot W$. We prove the opposite inclusion. To this aim, let consider a pair of 2-planes U_1, U_2 such that $\delta(U_1) = \delta(U_2) = \delta$ whereas $\mathcal{IM}(U_1) \neq \mathcal{IM}(U_2)$. We prove that they belong to the same orbit. It is $\mathcal{IM}(U_1) = q(\mathcal{IM}(U_2))\bar{q}$ by some quaternion q that we can always assume to be unitary. We have seen that there exist some $A, A' \in Sp(n)$ such that

$$A \cdot U_1 = W_1 = \mathrm{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}\mathcal{IM}(U_1) + \sin\delta(U_1)\mathbf{e_2}),$$

$$A' \cdot U_2 = W_2 = \mathrm{span}_{\mathbb{R}}(\mathbf{e_1}, \mathbf{e_1}\mathcal{IM}(U_2) + \sin\delta(U_2)\mathbf{e_2})).$$

Left multiplication by q belongs to the group $Sp(n)$ then, by $q \circ A' \in Sp(n)$,

$$(q \circ A') \cdot U_2 = \mathrm{span}_{\mathbb{R}}(q\mathbf{e_1}, q\mathbf{e_1}\mathcal{IM}(U_2) + \sin\delta(U_2)q\mathbf{e_2}).$$

and through the left multiplication by $\bar{q}$ the conclusion follows. □

Conclusions

This article contains some of the results we have obtained so far in a research now in progress aimed to determine the orbits in $G_k^{\mathbb{R}}(\mathbb{H}^n)$, $0 < k < 4n$ under the action of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$. In particular here we consider $G_2^{\mathbb{R}}(\mathbb{H}^n)$ i.e. the real Grassmannian of 2-planes of the $4n$-dimensional real vector space $\mathbb{H}^n$ chosen as model of all Hermitian quaternionic vector spaces. The full set of invariants for the orbit of a 2-plane $U \subset \mathbb{H}^n$ is given by $\mathcal{IM}(U)$ in case we consider the action of $Sp(n)$ and by the characteristic deviation $\Delta(U)$ when the acting group is $Sp(n) \cdot Sp(1)$.

In the next step we will consider the action of the same groups on the subset of $G_k^{\mathbb{R}}(\mathbb{H}^n)$ consisting of $2k$-dimensional A-complex subspaces $\forall A \in S(\mathcal{Q})$ and $0 < 2k < 4n$ that is the subset of all complex subspaces by some compatible complex structure.

Finally, still considering the action of the groups $Sp(n)$ and $Sp(n) \cdot Sp(1)$, we intend to determine the full set of invariants for the orbit of a generic subspace of $\mathbb{H}^n$. For this reason we decided to include for completeness in this article, besides the results related to the 2-planes, also some definitions, like the characteristic deviation of a k-dimensional subspace of $\mathbb{H}^n$, the I-complex characteristic angle, etc .. , basic notions and some results that will be used in the following articles.

References

[1] S. Afriat, *Orthogonal and oblique projectors and the characteristics of pair of vector spaces*, Proc. Cambridge Philos. Soc. **53** (1957), 800–816.

[2] D. V. Alekseevsky and S. Marchiafava, *A report on quaternionic-like structures on a manifold*, Proc. Internat. Workshop on Differential Geometry and its Applications (Bucharest 1993). Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **55**, no. 3–4 (1993), 9–34.

[3] M. Bruni, *Su alcune proprietà di geometria euclidea ed Hermitiana in uno spazio vettoriale quaternionale*, Ann. Mat. Pura Appl. **72** (1965), 59–77.

[4] M. Bruni, *Misure angolari in uno spazio vettoriale quaternionale*, Rend. Mat. Appl. **25** (1966), 394–401.

[5] M. Bruni, *Misure euclidee, hermitiane, simplettiche e potenze esterne di uno spazio vettoriale quaternionale*, Ann. Mat. Pura Appl. **78** (1971), 71–97.

[6] A. Galantai and Cs. J. Hegedus, *Jordan's principal angles in omplex vetor spaes*, Numer. Linear Algebra Appl. **13** (2006), 589–598.

[7] A. Gray, *A note on manifold whose holonomy group is a subgroup of $Sp(n) \cdot Sp(1)$*, Mitch. Math. J. **16** (1969), 125–128.

[8] S. Gutmann and L. Shepp, *Orthogonal bases for two subspaces with all mutual angles acute*, Indiana Univ. Mathematics J. **27**, no. 1 (1978), 79–90.

[9] H. J. Kang and H. Tasaki, *Integral geometry of real surfaces in complex projective spaces*, Tsukuba J. Math. **25**, no. 1 (2001), 155–164.

[10] G. B. Rizza, *Deviazione caratteristica e proprietà locali delle $2q$-faccette di una V_{2n} a struttura complessa*, Rend. Acc. Naz. **40** (1959), 191–205.

[11] G. B. Rizza, *On the geometry of a pair of oriented planes*, Riv. Mat. Univ. Parma (6) **4** (2001), 217–228.

[12] S. M. Salamon, *Quaternion-Kähler geometry*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom. VI, Int. Press, Boston, MA 1999, pp. 83–121.

[13] K. Sharnhorst, *Angles in complex vector spaces*, Acta Applicandae Matematicae **69** (2001), 95–103.

[14] Sheng Jiang, *Angles between Eulidean subspaes*, Geometriae Dedicata **63** (1996), 113–121.

[15] H. Tasaki, *Generalization of Kähler angle and integral geometry*, in: Complex Projective Spaces, Steps in Differential Geometry, Proc. of Colloquium on Differential Geometry, Debrecen 2000, Inst. Math. Inform. Debrecen 2001, pp. 349–361.

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ORBITY W RZECZYWISTYM GRASSMANNIANIE 2-PŁASZCZYZN POD DZIAŁANIEM GRUP $Sp(n)$ ORAZ $Sp(n) \cdot Sp(1)$

S t r e s z c z e n i e

Naturalne działanie grupy unitarnej $U(n)$ w przestrzeni $\mathbb{C}^n$ indukuje jej działanie na rozmaitości Grassmanna $G_k^{\mathbb{R}}(\mathbb{C}^n)$ złożonej z k-wymiarowych podprzestrzeni rzeczywistych przestrzeni $\mathbb{C}^n$. W pracy wyznaczamy kompletny niezmiennik dla podprzestrzeni rzeczywistej wymiaru 2 ze względu na działanie grup $Sp(n)$ oraz $Sp(n) \cdot Sp(1)$ – grup automorfizmów rzeczywistej przestrzeni wektorowej wyposażonej odpowiednio w hermitowską strukturę hiperzespoloną i hermitowską strukturę kwaternionową. Wprowadzamy miarę urojoną i charakterystyczną dewiację 2-płaszczyzny i dowodzimy, że charakteryzują one kompletnie orbity w 2-płaszczyźnie w $G_2^{\mathbb{R}}(\mathbb{H}^n)$ odpowiednio przy działaniu grup $Sp(n)$ oraz $Sp(n) \cdot Sp(1)$

Słowa kluczowe: hermitowska struktura kwaternionowa, kąty pryncypalne, kąty Kählera, grupy $Sp(n)$, grupy $Sp(n) \cdot Sp(1)$

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