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DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

SÉRIE: RECHERCHES SUR LES DÉFORMATIONS

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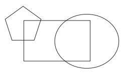

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TABLE DES MATIÈRES

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Ryszard J. Pawlak

ENTROPY OF NONAUTONOMOUS DISCRETE DYNAMICAL SYSTEMS CONSIDERED IN GTS AND GMS

Summary
In this paper we define and study the generalization of the notion of entropy which can be used for nonautonomous discrete dynamical system in various phase spaces. We focus special attention on considering a new approach in the case of discontinuous dynamical systems in generalized topological and metric spaces.

Keywords and phrases: nonautonomous discrete dynamical system, cover, topological entropy, turbulent function, generalized topological space (GTS), generalized metric space (GMS), almost compactness, separated set, span set, Lebesgue number

1. Introduction and preliminaries

In 1967, H. Furstenberg named all dynamical systems with zero entropy *deterministic* [17]. As the property of the dynamical system opposite to that of being deterministic one can accept the property of being chaotic. The concept of chaos is connected with many definitions, often non-equivalent. Some information on this subject one can find in the papers [27], [3]. In the article [18] the authors considering Devaney's definition of chaos stated that the basic criterion for chaotic dynamical system should be positive entropy of this system. At this point it is worth quoting a part of the paper [3]: "Regarding positive topological entropy as a chaotic feature finds a heuristic justification in ergodic theory, or more precisely in the part entropy plays in information theory. There measure theoretic entropy is interpreted as a measure of the indeterminacy of an invariant process: a process with discrete states is called deterministic if its past states almost surely determine its state at time 0, which is equivalent to its entropy being null. The link with topological entropy is the Variational Principle: the topological entropy of the system (X, T) is the supremum of the entropies of all f-invariant probability measures on X".

Key considerations on entropy for discrete dynamical systems are based on two classic definitions: the definition formulated by R. L. Adler, A. G. Konheim and M. H. McAndrew [1] using the properties of open cover of compact spaces and by Bowen [7] and Dinaburg [13] version for compact metric spaces. In the case of compact metric spaces these two definitions coincide. Since the entropy has become an important tool for the study of dynamical systems, many authors created a generalization of this notion. These generalizations were concentrated on both attempts of considering noncompact spaces (e.g. [8], [15]) and analyzing some families of discontinuous functions (e.g. [9], [38], [32]), as well as the transition of these considerations to the case of nonautonomous dynamical systems (e.g. [24], [23]). In the meanwhile, new challenges appeared in this area. At the end of the 20th century Á. Császár introduced the new notions: generalized topology and generalized topological space, in brief GT and GTS ([10], [11]). This new concept has quickly become the subject of intense research. Generalized topological spaces are studied by many mathematicians (e.g. [5], [35]). These studies are, for example, associated with different types of continuity in generalized topological spaces (e.g. [30], [6], [34]), connectedness (e.g. [12], [14]) or compactness of generalized topological spaces (e.g. [36], [21]), study of the consequences of the Baire property of spaces [25]. In addition to the theoretical results connected with generalized topological space we have J. Lee's observations in [26] concerning relationships between this theory and computer science.

A version of the generalization of the notion of entropy for these spaces can be found in [28]. Moreover, in 2013 there appeared the paper [25] which contains a definition of generalized metric spaces (briefly GMS) associated with generalized topological spaces. Both, in the case of GTS, as well as the GMS the considerations connected with discrete dynamical systems seems to be interesting (see [29]).

In this paper we will show that the notion of *entropy* can be transferred to more general spaces (which either may not be equipped with any topological structure or will be based on the structure of GTS or GMS). In addition, we will consider nonautonomous discrete dynamical systems without the assumption of continuity of the functions constituting these systems. It should be emphasized that in the case of classical spaces (compact topological or metric spaces) entropy in the new approach coincides with the usual notion of entropy. We will prove basic properties of entropy in the new approach. This way we will have generalization of greater scope for applications than those that existed before.

We will use mostly standard notations. In particular, by the letters $\mathbb{R}$ and $\mathbb{N}$ we will denote the sets of all real numbers and positive integers, respectively.

Basic symbols on nonautonomous discrete dynamical systems will be adopted in the same way as in [24]. Let X be a nonempty set. If $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of functions mapping X into itself then $f_i^0 = f_i^{-0} = \mathrm{id}_X$ (the identity map on X) for any $i \in \mathbb{N}$. Moreover, for any $i, n \in \mathbb{N}$ we put

$$f_i^n = f_{i+(n-1)} \circ \cdots \circ f_{i+1} \circ f_i \quad \text{and} \quad f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+(n-1)}^{-1}$$

(the last notations will be applied to sets i.e. the upper index -1 means preimage of a set under respective function).

Let (X, ϱ) be a metric space. We will use the symbol $\mathrm{cl}_\varrho(A)$ to denote the closure of the set $A \subset X$. The cardinality of the set A will be denoted by $\#(A)$.

Now we are going to present basic notions and properties connected with generalized topological and generalized metric spaces.

Let X be a nonempty set. We shall say that a family γ of subsets of X is a *generalized topology* in X (GT for short) iff

$$\emptyset \in \gamma \quad \text{and} \quad \bigcup_{t \in T} G_t \in \gamma \quad \text{whenever} \quad \{G_t : t \in T\} \subset \gamma.$$

The sets belonging to γ will be called γ-*open sets*. The pair (X, γ) is called a *generalized topological space* and it is denoted by GTS. Moreover, if $X \in \gamma$ we shall say that (X, γ) is a *strong generalized topological space* (briefly sGTS) and γ is a *strong generalized topology* (sGT for short). The complements of γ-open sets will be called γ-*closed* (or briefly: *closed*) sets.

The basic definitions in generalized topological spaces are usually formulated in the same way as in topological spaces. However, despite identical definitions the properties of some mathematical objects in the case of standard topology may be quite different than the properties of respective objects in generalized topology. One of the examples of such a situation is the fact that (in GTS) $\emptyset$ may not be a closed set.

Similarly to the case of topological spaces also in GTS one can consider special kind of spaces: generalized metric spaces.

Let $X \neq \emptyset$. The symbol Π stands for the family of metrics defined on subsets of X, i.e. $\varrho \in \Pi$ means that one can find a nonempty set $A_\varrho \subset X$ such that ϱ is a metric on A_ϱ. The set A_ϱ is named a *domain* of ϱ. We will use the symbol $\mathrm{dom}(\varrho)$ to denote the domain of a metric ϱ. The space (X, Π) is called a *generalized metric space* (GMS for short).

Because almost all results connected with GMS focused on finite subsets of the family Π (e.g. [25] in the context of kernel and [33] in the case of base in GMS), from now on we will assume that Π is a finite family of metrics.

Let (X, Π) be a GMS. We will say that a set $A \subset X$ is open if for each $x \in A$ there exist $\varrho \in \Pi$ and $\varepsilon > 0$ such that $x \in \mathrm{dom}(\varrho)$ and the set $B_\varrho(x, \varepsilon) = \{y \in \mathrm{dom}(\varrho) : \varrho(x, y) < \varepsilon\}$ is contained in A. We will denote by γ_Π the family of all open sets in (X, Π). It is easy to check that if (X, Π) is a GMS then (X, γ_Π) is a GTS.

Let (X, Π) be a GMS and $x, y \in X$. Let us adopt the following notation: $\mathcal{K}(x, y) = \{j : x, y \in \mathrm{dom}(\varrho_j)\}$. For abbreviation we write $\mathcal{K}(x)$ instead of $\mathcal{K}(x, x)$.

We call a point $x \in X$ a *node* if $\mathrm{dom}(\varrho_i) \cap \mathrm{dom}(\varrho_j) = \{x\}$, for $i, j \in \mathcal{K}(x)$, $i \neq j$. The set of all nodes will be denoted by $\mathcal{N}(X, \Pi)$. Moreover, in this paper we will consider the number $\varepsilon(X,\Pi)$ defined as follows: $\varepsilon(X,\Pi)=\min\{\varrho_j(x,y)\colon x,y\in\mathcal{N}(X,\Pi),\ x\neq y,\ j\in\mathcal{K}(x,y)\}$, if there exist $x,y\in\mathcal{N}(X,\Pi)$ such that $x\neq y$, $\mathcal{K}(x,y)\neq\emptyset$ and $\varepsilon(X,\Pi)=1$ in other cases.

2. Adler-Konheim-McAndrew definition in the case of an arbitrary set and in GTS

As a base of our considerations we take a nonempty set X called *space* or *phase set* and a sequence of functions (*actions*) $(f_{1,\infty})=(f_i)_{i=1}^{\infty}$ mapping X into itself. Let $\Lambda(X)$ denote the family of all covers of X having finite subcover. If $\lambda\in\Lambda(X)$ then we will denote by $\mathcal{N}(\lambda)$ the minimal possible cardinality of subcover of λ.

For a finite sequence of covers $\lambda_1,\ldots,\lambda_n\in\Lambda(X)$ we denote

$$\bigvee_{i=1}^{n}\lambda_i=\lambda_1\vee\lambda_2\vee\ldots\vee\lambda_n=\{\bigcap_{i=1}^{n}A_i : A_i\in\lambda_i, i=1,2,\ldots,n\}.$$

In order to define the entropy we should distinguish some properties of covers belonging to $\Lambda(X)$. Proofs of some of them are either very simple or similar to the proofs of the respective properties discussed in the papers [2], [4], [9], [24]. For that reason, we will omit these proofs, but for the convenience of the reader, we will present all the properties that are necessary for the coherence of our considerations. To formulate them, we will need the following notations.

Let $\lambda\in\Lambda(X)$. For any $i,n,k=1,2,\ldots$ we will write

$$f_i^{-k}(\lambda)=\{f_i^{-k}(A)\colon A\in\lambda\}\ \text{ and }\ \lambda_i^n=\bigvee_{k=0}^{n-1}f_i^{-k}(\lambda).$$

We will say that a cover λ_1 is finer that a cover λ_2 and write $\lambda_1\geq\lambda_2$, when each element of λ_1 is contained in some element of λ_2.

Property 2.1.

[P1] *If* $\lambda_1,\ldots,\lambda_n\in\Lambda(X)$, *then* $\bigvee_{i=1}^{n}\lambda_i\in\Lambda(X)$.

[P2] *If* $\lambda\in\Lambda(X)$, *then* $f_i^{-n}(\lambda)\in\Lambda(X)$, *for* $i,n=0,1,2,\ldots$.

[P3] *If* $\lambda_1,\lambda_2\in\Lambda(X)$, *then* $\mathcal{N}(\lambda_1\vee\lambda_2)\leq\mathcal{N}(\lambda_1)\cdot\mathcal{N}(\lambda_2)$.

[P4] *If* $\lambda\in\Lambda(X)$, *then* $\mathcal{N}(f_i^{-n}(\lambda))\leq\mathcal{N}(\lambda)$.

[P5] *If* $\lambda_1,\lambda_2\in\Lambda(X)$, *then* $f^{-1}(\lambda_1\vee\lambda_2)=f^{-1}(\lambda_1)\vee f^{-1}(\lambda_2)$.

[P6] *If* $\{x_n\}$ *is a subadditive sequence (i.e* $x_{k+n}\leq x_k+x_n$*), then the limit* $\lim_{n\to\infty}\frac{x_n}{n}$ *exists and is equal to* $\inf_n\frac{x_n}{n}$ [2].

[P7] *If* $\lambda_1,\lambda_2\in\Lambda(X)$ *and* $\lambda_1\geq\lambda_2$, *then* $N(\lambda_1)\geq N(\lambda_2)$.

[P8] *If* $\lambda_1,\lambda_2\in\Lambda(X)$ *and* $\lambda_1\geq\lambda_2$, *then* $(\lambda_1)_i^n\geq(\lambda_2)_i^n$.

Taking into account the need to define entropy connected with the fixed cover we will assume the following definition:

$$h(f_{1,\infty}, \lambda) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\lambda_1^n).$$

Obviously, our definition agrees with the classical one for open cover of topological spaces (see [1] and [2] for autonomous discrete dynamical system; [9] for discontinuous function; [24] for nonautonomous discrete dynamical system). The properties [P8] and [P7] imply the following:

[P9] If $\lambda_1, \lambda_2 \in \Lambda(X)$ and $\lambda_1 \geq \lambda_2$ then $h(f_{1,\infty}, \lambda_1) \geq h(f_{1,\infty}, \lambda_2)$.

Now we will introduce the notion of entropy of nonautonomous discrete dynamical system $(f_{1,\infty})$ with respect to a family of covers.

> Let us assume that $\Lambda \subset \Lambda(X)$. Then the entropy of $(f_{1,\infty})$ with respect to Λ is the number
> $$h_\Lambda(f_{1,\infty}) = \sup\{h(f_{1,\infty}, \lambda)\colon \lambda \in \Lambda\}.$$

Obviously, the value of entropy strongly depends on the family of covers, but in this paper the considerations connected with this topic will be omitted.

2.1. Two basic properties

In this section we will give two basic properties of the entropy for nonautonomous discrete dynamical system considered in abstract set. The first one (with theoretical character) is connected with a sequence of covers which is cofinal with a family of covers being base of considerations concerning entropy. For the convenience of the reader we recall the definition.

> Let Λ be a family of covers of the set X. A sequence of covers $(\lambda_k)_{k=1}^\infty$ is cofinal with Λ provided for each $\lambda \in \Lambda$ there exists $k_0 \in \mathbb{N}$ such that $\lambda_{k_0} \geq \lambda$.

Theorem 2.2. *Let $\Lambda \subset \Lambda(X)$ and let $\lambda_k \in \Lambda$ $(k = 1, 2, \dots)$ be a sequence of covers of X cofinal with Λ such that $\lambda_1 \leq \lambda_2 \leq \dots$. Then $h_\Lambda(f_{1,\infty}) = \lim_{k\to\infty} h(f_{1,\infty}, \lambda_k)$.*

Proof. On account of the property [P9] we have

$$h(f_{1,\infty}, \lambda_k) \leq h(f_{1,\infty}, \lambda_{k+1}), \text{ for any } k = 1, 2, \dots$$

which means that the limit of the sequence $\{h(f_{1,\infty}, \lambda_k)\}$ exists (perhaps equal to ∞).

The inequality

$$\lim_{k\to\infty} h(f_{1,\infty}, \lambda_k) \leq h_\Lambda(f_{1,\infty})$$

is obvious. So, taking into account the existence of the limit $\lim_{k\to\infty} h(f_{1,\infty}, \lambda_k)$, it will be sufficient to show that

$$\text{for any } \eta > 0 \text{ there exists a positive integer } k_\eta \\ \text{such that } h(f_{1,\infty}, \lambda_{k_\eta}) > h_\Lambda(f_{1,\infty}) - \eta.$$

By the definition of the entropy one can infer that there exists $\lambda_0 \in \Lambda$ such that

$$h(f_{1,\infty}, \lambda_0) > h_\Lambda(f_{1,\infty}) - \eta.$$

The assumpion that (λ_k) is cofinal with Λ permits to conclude the existence of $k_\eta \in \mathbb{N}$ such that $\lambda_{k_\eta} \geq \lambda_0$. We now apply the property [P9] again, to obtain

$$h(f_{1,\infty}, \lambda_{k_\eta}) \geq h(f_{1,\infty}, \lambda_0) > h_\Lambda(f_{1,\infty}) - \eta.$$

□

For many considerations related to the theory of entropy it is important to be able to reduce these considerations to determining values of the entropy on subsets of the phase space. It is therefore necessary to modify the preceding definitions to make them useful in the case of phase subsets.

So, let $\emptyset \neq Z \subset X$. If $\lambda \in \Lambda(X)$, then we will denote by $\lambda \mid Z$ the family of intersections $\{A \cap Z \colon A \in \lambda\}$. Obviously $\mathcal{N}(\lambda) \geq \mathcal{N}(\lambda \mid Z)$. Then, one can formulate the definition of the entropy of $(f_{1,\infty})$ on Z with respect to the cover λ in the following way:

$$h(f_{1,\infty}, \lambda \mid Z) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathcal{N}(\lambda_1^n \mid Z)).$$

Let $\Lambda \subset \Lambda(X)$. The entropy of discrete nonautonomous dynamical system $(f_{1,\infty})$ on $Z \subset X$ is the number

$$h_\Lambda(f_{1,\infty} \mid Z) = \sup\{h(f_{1,\infty}, \lambda \mid Z) \colon \lambda \in \Lambda\}.$$

Taking into account the inequality $\mathcal{N}(\lambda) \geq \mathcal{N}(\lambda \mid Z)$, we conclude that

$$h_\Lambda(f_{1,\infty}) \geq h_\Lambda(f_{1,\infty} \mid Z).$$

Before formulating the next theorem, we will signal some equality in the form of lemma (see [2], Lemma 5.1.9 p. 195).

Lemma 2.3. *If $a_{n,i}$, $i = 1, 2, \ldots, m$, $n = 0, 1, 2, \ldots$ are non-negative numbers, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{m} a_{n,i} = \max\{\limsup_{n\to\infty} \frac{1}{n} \log a_{n,i} \colon i = 1, 2, \ldots m\}.$$

□

Theorem 2.4. *Let $X = \bigcup_{i=1}^m Z_i$ be a phase set, where $Z_i \neq \emptyset$, for $i = 1, \ldots, m$ and let $(f_{1,\infty})$ be a nonautonomous discrete dynamical system on X. Then*

$$h_\Lambda(f_{1,\infty}) = \max\{h_\Lambda(f_{1,\infty} \mid Z_i) \colon i = 1, \ldots, m\}.$$

Proof. The definition of entropy on a set makes it clear that

$$\max\{h_\Lambda(f_{1,\infty} \mid Z_i) \colon i = 1, \ldots, m\} \leq h_\Lambda(f_{1,\infty}).$$

Hence, it is sufficient to prove

$$h_\Lambda(f_{1,\infty}) \le \max\{h_\Lambda(f_{1,\infty} \mid Z_i)\colon i = 1, \dots, m\}.$$

So, let $\lambda \in \Lambda$. We will show that

$$\mathcal{N}(\lambda_1^n) \le \mathcal{N}(\lambda_1^n \mid Z_1) + \ldots + \mathcal{N}(\lambda_1^n \mid Z_m). \tag{2.1}$$

Indeed. Let us consider $\lambda \mid Z_i$ and let λ_i be subcovers of $\lambda_1^n \mid Z_i$ such that $\#(\lambda_i) = \mathcal{N}(\lambda_1^n \mid Z_i)$, for $i = 1, \dots, m$. To deduce (2.1), we begin by proving that

(2.2) $\lambda^\star = \lambda_1 \cup \ldots \cup \lambda_m$ is a finite cover of X such that $\lambda^\star \ge \lambda_1^n$ for $n = 1, 2, ...$

Obviously, $\lambda^\star$ is a finite cover of X.

Therefore, to obtain (2.2) it is sufficient to show that

$$\lambda^\star \ge \lambda_1^n.$$

Let A be na arbitrary set belonging to $\lambda^\star$. Then there exists $i_A \in \{1, \dots, m\}$ such that $A \in \lambda_{i_A} \subset \lambda_1^n \mid Z_{i_A}$. It is immediate that $A = A_0 \cap Z_{i_A}$, where $A_0 \in \lambda_1^n$. This completes our argument for (2.2).

Taking into account the property [P7] we observe that

$$\mathcal{N}(\lambda_1^n) \le \#(\lambda_1) + \ldots + \#(\lambda_m) = \mathcal{N}(\lambda_1^n \mid Z_1) + \ldots + \mathcal{N}(\lambda_1^n \mid Z_m),$$

which completes the proof of (2.1).

Now, using Lemma 2.3 one can calculate

$$\begin{aligned} h(f_{1,\infty}, \lambda) &\le \limsup_{n\to\infty} \frac{1}{n} \log(\mathcal{N}(\lambda_1^n \mid Z_1) + \ldots + \mathcal{N}(\lambda_1^n \mid Z_m)) \le \\ &\le \max\{\limsup_{n\to\infty} \frac{1}{n} \log(\mathcal{N}(\lambda_1^n \mid Z_i))\colon i = 1,\ \dots, m\} = \\ &= \max\{h_\Lambda(f_{1,\infty} \mid Z_i)\colon i = 1, \dots, m\}. \end{aligned} \tag{2.3}$$

This completes the proof of Theorem 2.4. □

2.2. Turbulent function

Making the definitions of an entropy the only basis of examinations would lead to very complicated considerations. For that reason, some additional notions are often introduced and statements connected with them are used. Now we introduce the concept of turbulent function in sGTS, which is some generalization either of the notion of horseshoe which is used mainly in issues concerning one dimensional dynamics [2] or turbulent functions which are frequently examined in topological (metric) spaces [9], [4].

Before going to the main definition and Theorem 2.6, we shall present useful notations.

Here and subsequently, (X, γ) stands for sGTS. We call the family $\mathcal{F}$ of subsets of X *closed*, if $\bigcup \mathcal{F}$ is a closed set in (X, γ). We will denote by Λ_γ the family of all γ-open covers of X with finite subcovers.

Let $\mathcal{P} = \{A_1, \ldots, A_k\}$ be a nonsingleton family of sets. We will say that a family $\mathfrak{G} \subset \gamma$ weakly separates $\mathcal{P}$ if there exist $V_1, \ldots, V_k \in \mathfrak{G}$ such that $A_i \subset V_i$ ($i = 1, \ldots, k$) and $A_i \cap V_j = \emptyset$ for $i \neq j$ such that $i, j \in \{1, \ldots, k\}$. If we have a compact sGTS (X, γ) which is a normal space, then γ does not need to weakly separate even points which would be close in this space.

The next corollary shows, among other things, that if generalized topology weakly separates some closed family $\mathcal{P}$ then there is a finite γ-open cover (being subcover of γ) which weakly separates $\mathcal{P}$.

Lemma 2.5. *Let γ weakly separate closed family $\mathcal{P} = \{A_1, \ldots, A_k\}$. Then there exists γ-open cover $\lambda = \{V_1, \ldots, V_k\}$ weakly separating $\mathcal{P}$.*

Proof. Let $U_1, \ldots, U_k \in \gamma$ denote sets such that $A_i \subset U_i$ and $A_i \cap U_j = \emptyset$ for $i \neq j$, $i, j \in \{1, \ldots, k\}$. Put $V_i = U_i$ if $i = 1, \ldots, k-1$ and $V_k = U_k \cup (X \setminus (A_1 \cup \ldots \cup A_k))$. It is evident that $\lambda = \{V_1, \ldots, V_k\}$ is a γ-open cover of X, which weakly separates $\mathcal{P}$. □

We will say that nonautonomous discrete dynamical system $(f_{1,\infty})$ is *k-turbulent* if there exists a closed family $\mathcal{P} = \{A_1, \ldots, A_k\}$ consisting of nonempty sets such that γ weakly separates $\mathcal{P}$ and, moreover,

$$\bigcup_{j=1}^{k} A_j \subset \bigcap_{j=1}^{k} f_i(A_j), \text{ for } i = 1, 2, \ldots .$$

Theorem 2.6. *If $(f_{1,\infty}) = \{f_i\}_{i=1}^{\infty}$ is a k-turbulent nonautonomous discrete dynamical system, then $h_{\Lambda_\gamma}(f_{1,\infty}) \geq \log k$.*

Proof. Let $\mathcal{P} = \{A_1, \ldots, A_k\}$ be a closed family of nonempty sets such that γ weakly separates $\mathcal{P}$ and

$$\bigcup_{j=1}^{k} A_j \subset \bigcap_{j=1}^{k} f_i(A_j), \quad \text{for} \quad i = 1, 2, \ldots .$$

According to Lemma 2.5 there exists γ-open cover $\lambda = \{V_1, V_2, \ldots, V_k\}$ such that λ weakly separates $\mathcal{P}$. In the remainder of this proof we assume that $A_i \subset V_i$ and $A_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. It is evident that $\lambda \in \Lambda_\gamma$.

Fix a positive integer n and let us consider λ_1^n. We shall show that

$$\text{for each } A \in \lambda_1^n \text{ there exist } \alpha_A \in A \text{ and a sequence } \{i_j\}_{j=0}^{n-1} \subset \{1, 2 \ldots, k\}$$
$$\text{such that } f_1^j(\alpha_A) \in A_{i_j} \text{ for } j \in \{0, \ldots, n-1\}. \tag{2.4}$$

Indeed. Let $A \in \lambda_1^n$. The set A can be written as $A = V_{i_0} \cap f_1^{-1}(V_{i_1}) \cap \ldots \cap f_1^{-(n-1)}(V_{i_{n-1}})$, where $V_{i_j} \in \lambda$, and so $A_{i_j} \subset V_{i_j}$ for $j \in \{0, \ldots, n-1\}$. Fix $\alpha_{i_{n-1}} \in A_{i_n-1}$. We conclude from the inclusion $A_{i_{n-1}} \subset f_{n-1}(A_{i_{n-2}})$ that there exists $\alpha_{i_{n-2}} \in A_{i_{n-2}}$ such that $f_{n-1}(\alpha_{i_{n-2}}) = \alpha_{i_{n-1}}$. Similarly, there exists $\alpha_{i_{n-3}} \in A_{i_{n-3}}$ such that

$f_{n-2}(\alpha_{i_{n-3}}) = \alpha_{i_{n-2}}$. Continuing this process we find an element $\alpha_A = \alpha_{i_0} \in A_{i_0}$ such that $f_1^{n-1}(\alpha_A) = \alpha_{i_{n-1}}$ and, moreover, $f_1^j(\alpha_A) \in A_{i_j}$ for $j = 0, 1, \ldots, n-1$. Obviously, $\alpha_A \in A$. This finishes the proof of (2.2).

Now, we shall show that

$$\alpha_A \notin P \in \lambda_1^n, \text{ for } P \neq A. \tag{2.5}$$

For this purpose, let us consider $P \in \lambda_1^n$, such that $P \neq A$. Obviously,

$$P = V'_{i_0} \cap f_1^{-1}(V'_{i_1}) \cap \ldots \cap f_1^{-(n-1)}(V'_{i_{n-1}}),$$

where $V'_{i_j} \in \lambda$, for $j = 0, 1, \ldots, n-1$. Since $P \neq A$, it follows that there exists $j_0 \in \{0, 1, \ldots, n-1\}$ such that $V_{i_{j_0}} \neq V'_{i_{j_0}}$. Consequently $A_{i_{j_0}} \subset V_{i_{j_0}}$ and $A_{i_{j_0}} \cap V'_{i_{j_0}} = \emptyset$. By (2.2) we have $f_1^{j_0}(\alpha_A) \in A_{i_{j_0}}$ and so $f_1^{j_0}(\alpha_A) \notin V'_{i_{j_0}}$. The relationship (2.5) is proved.

From (2.2) and (2.5) one can deduce that there is no subcover of λ_1^n different from λ_1^n.

Let $\mathbb{N}_n^k$ denote the set of all sequences of n-terms of the set $\{1, 2, \ldots, k\}$. Note that $\#(\mathbb{N}_n^k) = k^n$.

Let us define a function $\Psi\colon \mathbb{N}_n^k \to \lambda_1^n$ in the following way:

$$\Psi((y_0, y_1, \ldots, y_{n-1})) = \bigcap_{j=0}^{n-1} f_1^{-j}(V_{y_j}).$$

It is easy to check that Ψ is a bijection, and so

$$\mathcal{N}(\lambda_1^n) = \#(\mathbb{N}_n^k) = k^n.$$

Thus, at the end

$$h_{\Lambda_\gamma}(f_{1,\infty}) \geq h(f_{1,\infty}, \lambda) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\lambda_1^n) = \log k.$$

$\square$

The above theorem may be applied even in the case when we consider discontinuous functions mapping the unit interval into itself, while the earlier known statements did not allow to obtain any information about an entropy of the function ξ presented below.

$$\xi(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]; \\ \frac{3}{4} & \text{if } x \in [\frac{3}{8}, \frac{1}{2}]; \\ \frac{1}{4} & \text{if } x \in (\frac{1}{2}, \frac{5}{8}]; \\ 4x - \frac{3}{4} & \text{if } x \in [\frac{1}{4}, \frac{3}{8}]; \\ 4x - \frac{9}{4} & \text{if } x \in [\frac{5}{8}, \frac{3}{4}]. \end{cases}$$

Let $\mathcal{P} = \{\left[\frac{1}{4}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{3}{4}\right]\}$. We see at once that γ weakly separates $\mathcal{P}$ and, moreover,

$$\left[\frac{1}{4}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{3}{4}\right] \subset f\left(\left[\frac{1}{4}, \frac{3}{8}\right]\right) \cap f\left(\left[\frac{5}{8}, \frac{3}{4}\right]\right),$$

which means that autonomous discrete dynamical system (f) is 2-turbulent, and so $h_{\Lambda_\gamma}(f) > \log 2 > 0$.

Of course, this is only the simplest example. One can consider more complex examples in the more abstract spaces.

3. Entropy in GMS

In this chapter we will consider GMS (X, Π) such that

$$\Pi = \{\varrho_1, \ldots, \varrho_t\} \quad \text{and} \quad X = \bigcup_{i=1}^{t} \operatorname{dom}(\varrho_j).$$

It follows that X is an open set and consequently (X, γ_Π) is a strongly generalized topological space (sGTS). It simplifies the argument, and causes no loss of generality, to assume $\varrho_j(x, y) \le 1$ for $j = 1, 2, \ldots, t$ for $x, y \in \operatorname{dom}(\varrho_j)$.

In order to bypass some technicalities, we will assume that all considered GMSs have the following property: for each $x \notin \mathcal{N}(X, \Pi)$ there exists a dominant metric, i.e. there exists $j_0 \in \mathcal{K}(x)$ such that for each $r > 0$,

$$\bigcup_{j \in \mathcal{K}(x)} B_{\varrho_j}(x, r) \subset B_{\varrho_{j_0}}(x, r).$$

Now, we shall introduce a new definition:

> We call a GMS (X, Π) *almost compact* if $(\operatorname{dom}(\varrho_j), \varrho_j)$ is a compact space for $j = 1, 2, \ldots, t$.

It is obvious that almost compact GTS may be or may not be compact in the sense of Borel Lebesgue property even if we additionally assume that it is a Hausdorff space.

As we know, in the case of usual metric space there are two equivalent concepts of the notion of *entropy*. The first one was introduced by R. L. Adler, A. G. Konheim and M.H. McAndrew in the paper [1] (this concept was the subject of consideration in the previous chapter) and the second one is the Bowen-Dinaburg version of the definition of entropy of a function defined on a metric space [7, 13] (see also [24] for nonautonomous discrete dynamical systems or [9] for discontinuous function). Now let us concentrate on the Bowen-Dinaburg version of the definition of an entropy of a function in GMS. Obviously, current definition is more complex than its primal version.

We start with the notions of separated and span sets.

Let $(f_{1,\infty}) = \{f_1, f_2, \ldots\} = \{f_i\}_{i=1}^{\infty}$. Fix $\varepsilon \in (0, \varepsilon(X, \Pi))$ and $n \in \mathbb{N}$.

A set $S \supset \mathcal{N}(X, \Pi)$ is $(f_{1,\infty}, n, \varepsilon)$-separated if for each $x, y \in S$ such that $x \ne y$ there is $0 \le i < n$ such that either $\mathcal{K}(f_1^i(x), f_1^i(y)) = \emptyset$ or $\varrho_j(f_1^i(x), f_1^i(y)) > \varepsilon$, for every $j \in \mathcal{K}(f_1^i(x), f_1^i(y))$.

A set $M \supset \mathcal{N}(X, \Pi)$ is $(f_{1,\infty}, n, \varepsilon)$-span if for each $x \in X$ there is $y \in M$ such that for each $0 \le i < n$ there exists $j \in \mathcal{K}(f_1^i(x), f_1^i(y))$ such that $\varrho_j(f_1^i(x), f_1^i(y)) \le \varepsilon$.

Obviously $\varrho_j(f_1^i(x), f_1^i(y)) \leq \varepsilon$ means that $j \in \mathcal{K}(f_1^i(x), f_1^i(y))$ and, moreover, one can note that j is dependent on i.

We start with two propositions, which permit to define suitable numbers being a base of Bowen-Dinaburg version of an entropy.

Proposition 3.1. *For any $n \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon(X, \Pi))$ there exists a positive integer $\beta \geq 1$ such that if S is an $(f_{1,\infty}, n, \varepsilon)$-separated set, then $\#(S) \leq \beta$.* □

The above proposition allows one to define the following number (n is a nonnegative integer and $\varepsilon > 0$):

$s_{n,\varepsilon}(f_{1,\infty})$ is defined as the maximal cardinality of an $(f_{1,\infty}, n, \varepsilon)$-separated set.

Proposition 3.2. *If S is an $(f_{1,\infty}, n, \varepsilon)$-separated set such that $\#(S) = s_{n,\varepsilon}(f_{1,\infty})$, then S is an $(f_{1,\infty}, n, \varepsilon)$-span set.* □

Now, we are able to define second number connected with the notion of a *span* set (n is nonegative integer and $\varepsilon > 0$).

$r_{n,\varepsilon}(f_{1,\infty})$ is defined as the minimal cardinality of an $(f_{1,\infty}, n, \varepsilon)$-span set.

In this chapter we aim to prove theorem giving some kind of equivalence between definition of entropy presented in previous chapter and Bowen-Dinaburg concept of this notion in GMS. Before stating the result to be proved, we give some lemmas.

According to Proposition 3.2 it follows

Lemma 3.3. *For each $n \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon(X, \Pi))$ we have $r_{n,\varepsilon}(f_{1,\infty}) \leq s_{n,\varepsilon}(f_{1,\infty})$.* □

Now we shall prove

Lemma 3.4. *If $n \in \mathbb{N}$ and $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon(X, \Pi)$, then*
(a) $r_{n,\varepsilon_2}(f_{1,\infty}) \leq r_{n,\varepsilon_1}(f_{1,\infty})$;
(b) $s_{n,\varepsilon_2}(f_{1,\infty}) \leq s_{n,\varepsilon_1}(f_{1,\infty})$.

Proof. We begin by proving the first inequality. Let M be an $(f_{1,\infty}, n, \varepsilon_1)$-span set such that $\#(M) = r_{n,\varepsilon_1}(f_{1,\infty})$. Then M is also $(f_{1,\infty}, n, \varepsilon_2)$-span set and consequently $r_{n,\varepsilon_2}(f_{1,\infty}) \leq r_{n,\varepsilon_1}(f_{1,\infty})$.

We now turn to the second inequality. Let S be an $(f_{1,\infty}, n, \varepsilon_2)$-separated set such that $\#(S) = s_{n,\varepsilon_2}(f_{1,\infty})$. It is easy to see that S is also $(f_{1,\infty}, n, \varepsilon_1)$-separated set, which gives $s_{n,\varepsilon_2}(f_{1,\infty}) \leq s_{n,\varepsilon_1}(f_{1,\infty})$. □

Throughout this chapter, $\Lambda(X, \Pi)$ (as distinguished from all γ-open covers) denotes the family of all γ-open covers λ such that for any $j \in \{1, 2, \ldots, t\}$ there exists a family of ϱ_j-open covers $\lambda(j) \subset \{A \in \lambda \colon A \subset \operatorname{dom}(\varrho_j)\}$ of the space $(\operatorname{dom}(\varrho_j), \varrho_j)$ additionally fulfilling the following property:

if $x \in \mathcal{N}(X, \Pi)$ and $x \in A_j \in \lambda(j)$, for $j \in J \subset \mathcal{K}(x)$, then $\bigcup_{j \in J} A_j \in \lambda$.

First we note that

Lemma 3.5. *If $\lambda \in \Lambda(X, \Pi)$, then λ has a finite subcover.* □

Lemma 3.6. *If $\lambda \in \Lambda(X, \Pi)$, then there exists a positive number $\mathrm{L}(\lambda)$ such that for each $x \in X$ and an arbitrary $J \subset \mathcal{K}(x)$, we have that $\bigcup_{j \in J} B_{\varrho_j}(x, \mathrm{L}(\lambda))$ is contained in some set $U \in \lambda$.* □

By analogy to metric spaces the number $\mathrm{L}(\lambda)$ one can call the *Lebesgue number of the cover λ.*

Proof. Fix $\lambda(j) \subset \{A \in \lambda \colon A \subset \operatorname{dom}(\varrho_j)\}$, for $j \in \{1, 2, \ldots, t\}$ having property described in definition of $\Lambda(X, \Pi)$. With each $\lambda(j)$ there is associated a Lebesgue number $\mathrm{L}(\lambda(j))$ (in usual metric space $(\operatorname{dom}(\varrho_j), \varrho_j)$). Put

$$\mathrm{L}(\lambda) = \min\{\mathrm{L}(\lambda(j)) \colon j = 1, \ldots, t\}.$$

We shall prove that the assertion of our lemma is fulfilled for this number.

Fix $x \in X$ and $J \subset \mathcal{K}(x)$.

It is easy to observe that there exists $U_j \in \lambda(j)$ such that $B_{\varrho_j}(x, \mathrm{L}(\lambda(j))) \subset U_j$, for $j \in \mathcal{K}(x)$. There are two cases to consider.

1^0. $x \in \mathcal{N}(X, \Pi)$. Then we have

$$\bigcup_{j \in J} B_{\varrho_j}(x, \mathrm{L}(\lambda)) \subset \bigcup_{j \in J} B_{\varrho_j}(x, \mathrm{L}(\lambda(j))) \subset \bigcup_{j \in J} U_j \in \lambda.$$

2^0. $x \notin \mathcal{N}(X, \Pi)$. If $\#(\mathcal{K}(x)) = 1$, let $j_0 \in \{1, \ldots, t\}$ be an element such that $x \in \operatorname{dom}(\varrho_{j_0})$ and second possibility: if $\#(\mathcal{K}(x)) > 1$ let $j_0 \in \{1, \ldots, t\}$ be such that $\bigcup_{j \in \mathcal{K}(x)} B_{\varrho_j}(x, \mathrm{L}(\lambda)) \subset B_{\varrho_{j_0}}(x, \mathrm{L}(\lambda))$. Thus

$$\bigcup_{j \in J} B_{\varrho_j}(x, \mathrm{L}(\lambda)) \subset B_{\varrho_{j_0}}(x, \mathrm{L}(\lambda)) \subset U_{j_0} \in \lambda(j_0) \subset \lambda.$$

□

At the end of this paper we present theorem which connects, in the case of GMS, the definition of entropy presented in the first part (some version of Adler-Konheim-McAndrew definition) and the definition adopted in this chapter (some version of Bowen-Dinagurg definition).

Theorem 3.7. *Let (X, Π) be an almost compact GMS and $f_i \colon X \to X$ $(i = 1, 2, \ldots)$. In the case of limits in assertion of the theorem we consider $\varepsilon \in (0, \varepsilon(X, \Pi))$. Then*

$$h_{\Lambda(X,\Pi)}(f_{1,\infty}) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_{n,\varepsilon}(f_{1,\infty}) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{n,\varepsilon}(f_{1,\infty}).$$

If we additionally assume that (X,Π) contains no nodes ($\mathcal{N}(X,\Pi)=\emptyset$), then

$$h_{\Lambda(X,\Pi)}(f_{1,\infty}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\varepsilon}(f_{1,\infty}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\varepsilon}(f_{1,\infty}).$$

It should be note that, according to Lemma 3.4, the limits by $\varepsilon\to 0$ always exist (may be equal to $+\infty$).

Proof. We start with the proof of the first inequality

$$h_{\Lambda(X,\Pi)}(f_{1,\infty}) \le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\varepsilon}(f_{1,\infty}). \tag{3.1}$$

Let $\lambda\in\Lambda(X,\Pi)$. Fix $n\in\mathbb{N}$ and $\varepsilon\in(0,\min(\frac{\mathrm{L}(\lambda)}{2},\varepsilon(X,\Pi)))$. Let M be an $(f_{1,\infty},n,\varepsilon)$-span set such that

$$\#(M) = r_{n,\varepsilon}(f_{1,\infty}). \tag{3.2}$$

Since $\mathrm{L}(\lambda)$ is a Lebesgue number of λ, then according to Lemma 3.6 for any $p\in M$ and each $0\le i<n$ there exists such $V(p,i)\in\lambda$ that

$$\bigcup_{j\in\mathcal{K}(f_1^i(p))}\mathrm{cl}_{\varrho_j}(B_{\varrho_j}(f_1^i(p),\varepsilon)) \subset \bigcup_{j\in\mathcal{K}(f_1^i(p))} B_{\varrho_j}(f_1^i(p),\mathrm{L}(\lambda)) \subset V(p,i). \tag{3.3}$$

Let us define a function $\xi : M\to\lambda_1^n$ in the following way:

$$\xi(p) = \bigcap_{i=0}^{n-1} f_1^{-i}(V(p,i)) \text{ for } p\in M.$$

For the proof of the inequality (3.1), let us consider $\alpha_n=\{\xi(p)\colon p\in M\}$.

We shall show that

$$\alpha_n \text{ is a cover of } \lambda_1^n. \tag{3.4}$$

From the fact that $V(p,i)\in\lambda$ ($0\le i<n$ and $p\in M$) it follows that $\alpha_n\subset\lambda_1^n$ and so for the proof of (3.4) it is sufficient to show that α_n is a cover of X.

For this purpose, fix $x\in X$ and let p_x be an element of M such that for each $0\le i<n$ there exists a positive integer $j(i)\in[1,t]$ such that $j(i)\in\mathcal{K}(f_1^i(x),f_1^i(p_x))$ and

$$\varrho_{j(i)}(f_1^i(x),f_1^i(p_x))\le\varepsilon<1.$$

Taking into account (3.3) for $0\le i<n$ we have

$$f_1^i(x)\in\mathrm{cl}_{\varrho_{j(i)}}(B(f_1^i(p_x),\varepsilon)) \subset \bigcup_{j\in\mathcal{K}(f_1^i(p_x))}\mathrm{cl}_j(B_{\varrho_j}(f_1^i(p_x),\varepsilon)) \subset V(p_x,i).$$

Consequently,

$$x = f_1^0(x) =\in V(p_x,0)\cap f_1^{-1}(V(p_x,1))\cap\ldots\cap f_1^{-n+1}(V(p_x,n-1)).$$

This finishes the proof of (3.4).

Since $\xi(M)=\alpha_n$ we have $\#(\alpha_n)\le\#(M)$. On account of (3.4), [P7] and (3.2) it may be concluded that:

$$\mathcal{N}(\lambda_1^n)\le\mathcal{N}(\alpha_n)\le\#(M)=r_{n,\varepsilon}(f_{1,\infty}).$$

Therefore

$$h(f_{1,\infty},\lambda)=\limsup_{n\to\infty}\frac{1}{n}\log(\mathcal{N}(\lambda_1^n))\le\limsup_{n\to\infty}\frac{1}{n}\log(r_{n,\varepsilon}(f_{1,\infty})).$$

Hence

$$h_{\Lambda(X,\Pi)}(f_{1,\infty})=\sup\{h(f_{1,\infty},\lambda)\colon\lambda\in\Lambda(X,\Pi)\}\le\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\varepsilon}(f_{1,\infty}).$$

This completes our argument for the first inequality.

The innequality

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\varepsilon}(f_{1,\infty})\le\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\varepsilon}(f_{1,\infty})$$

follows immediately from Lemma 3.4.

The proof of this theorem will be finished if we show inequality

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\varepsilon}(f_{1,\infty})\le h_{\Lambda(X,\Pi)}(f_{1,\infty}),$$

by the assumption that $\mathcal{N}(X,\Pi)=\emptyset$.

Let $\varepsilon<\varepsilon(X,\Pi)$ and $n\in\mathbb{N}$. Let S be the $(f_{1,\infty},n,\varepsilon)$-separated set. Put

$$\lambda=\{B_{\varrho_j}(x,\frac{\varepsilon}{2})\colon x\in X\wedge j\in\mathcal{K}(x)\}.$$

Note that if λ' is an arbitrary subcover of λ, then for each $x,y\in S$ such that $x\ne y$ we have

$$\text{if } x\in A_x\in\bigvee_{i=0}^{n-1}f_1^{-i}(\lambda'),\ y\in A_y\in\bigvee_{i=0}^{n-1}f_1^{-i}(\lambda'),\ \text{then } A_x\ne A_y. \tag{3.5}$$

Indeed. Let us suppose (by our assumptions connected with x and y) that there exists

$$A\in\bigvee_{i=0}^{n-1}f_1^{-i}(\lambda')\subset\bigvee_{i=0}^{n-1}f_1^{-i}(\lambda)$$

such that $x,y\in A$. For each $0\le i<n$ there exists $U(i)\in\lambda'\subset\lambda$ fulfiing

$$f_1^i(x),f_1^i(y)\in U(i).$$

Fix $0\le i<n$. Then

$$U(i)=B_{\varrho_{j_0}}(z,\frac{\varepsilon}{2}),$$

for some $j_0\in\{1,\dots,t\}$. Then

$$\rho_{j_0}(f_1^i(x),f_1^i(y))\le\rho_{j_0}(f_1^i(x),z)+\rho_{j_0}(z,f_1^i(y))<\varepsilon.$$

This contradicts the fact that S is the $(f_{1,\infty},n,\varepsilon)$-separated set.

This finishes the proof of (3.5).

The relationship (3.5) leads us to the conclusion that $s_{n,\varepsilon}(f_{1,\infty}) \le \mathcal{N}(\lambda_1^n)$ and thus

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\varepsilon}(f_{1,\infty}) \le h(f_{1,\infty},\lambda) \le h_{\Lambda(X,\Pi)}(f_{1,\infty}).$$

□

Conclusion
– from thermodynamics through information theory to topology

In the introduction to the article a short history of the notion of topological entropy is presented. Such limitation may be justified by the article's broad range of subjects.

The notion of entropy was introduced with respect to the issues connected with thermodynamics. Let us cite [22]: *The word entropy, amalgamated from the Greek words energy and tropos (meaning "turning point"), was introduced in an 1864 paper of Rudolph Clausius, who defined the change in entropy of a body as heat transfer divided by temperature, and he postulated that overall entropy does not decrease (the second law of thermodynamics). Clausius was motivated by an earlier work of Sadi Carnot on an ideal engine in which the entropy (how Clausius understood it) would be constant.* One can find more information on the topic in [31]. Generally, we can say that the notion of entropy was introduced as a result of observations concerning energy loss. In the seventies of the XIX century a physicist J. W. Gibbs, on the basis of Clausius's results, submitted mathematical approach to the issue of energy loss. His considerations were later creatively developed by J. C. Maxwell and M. Planck.

In 1948, working in Bell Telephone Laboratories, C. Shannon was tending to fix a value of a "lost information". To that purpose, he devised a very general notion of information entropy ([37]) – a fundamental basis of information theory. More precisely, the name "entropy" was suggested to Shannon during discussion by J. von Neumann. Shannon's proposition was connected with a finite or countable measurable partition of a probability space. Topological entropy, mentioned in the introduction, was *imitated Shannon's development, replacing partitions by covers and, for lack of a notion of size, replacing weighted average of logarithmic size by the maximum of such expressions for a given number of elements* [22]. Mutual correspondence between these entropies is established by *variational principle* ([19], [16]).

Currently, entropy is widely used in many fields of knowledge. Similarly to the notion of chaos, it has different meanings. It should be emphasized that a lot of Polish physicists have made a significant contribution to building the entropy theory. In the relevant literature, one example of Polish contribution to the field is that of Roman Stanisław Ingarden [20]. Iwo Białynicki-Birula, J. Mycielski and many other Polish scientists have also made substantial achievements to this area of study.

The paper has a theoretical character. However, taking into account the origin of the notion of entropy, the substantial contribution of Polish physicists into development of this notion and the nature of the journal that the paper will be published in, one hopes that the results presented here will have their practical continuation.

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ENTROPIA NIEAUTONOMICZNYCH DYSKRETNYCH UKŁADÓW DYNAMICZNYCH ROZWAżANYCH W UOGÓLNIONYCH PRZESTRZENIACH TOPOLOGICZNYCH I METRYCZNYCH

S t r e s z c z e n i e

W pracy tej wprowadzamy i analizujemy uogólnienia pojęcia entropii w przypadku nieautonomicznych, dyskretnych układów dynamicznych w różnych przestrzeniach fazowych. Szczególną uwagę koncentrujemy na możliwości wykorzystania naszego ujęcia definicji entropii w przypadku układów rozważanych w uogólnionych przestrzeniach topologicznych i metrycznych.

Słowa kluczowe: nieautonomiczny dyskretny układ dynamiczny, pokrycie, entropia topologiczna, funkcja zaburzeń, uogólniona przestrzeń topologiczna (GTS), prawie zawartość, zbiór odosobniony, zbiór oddzielany, liczna Lebesquina

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Anna Chojnowska-Michalik and Emilia Fraszka-Sobczyk

ON THE UNIFORM CONVERGENCE OF COX-ROSS-RUBINSTEIN FORMULAS TO THE BLACK-SCHOLES FORMULA

Summary

We prove that the convergence of calibrated Cox-Ross-Rubinstein option price formulas to the Black-Scholes formula is uniform with respect to initial stock price $s_0 \in (0, \infty)$.

Keywords and phrases: Cox-Ross-Rubinstein model (CRR model), Black-Scholes formula, option pricing, uniform convergence

1. Introduction

We consider a European call option that is a contract which gives the holder the right to buy the underlying asset (for example stocks) for some strike price K that is determined at the moment 0. The holder can buy stocks only at the determined future date T, which is called the expiry date. Since the holder has the right but he does not have the obligation to buy the asset he will only exercise it if it is profitable for him. So he will exercise the option if the market price is greater than strike price K. Of course, the option holder pays the premium for having this right to buy stock. This premium is called the option price.

Formally, we have the following definition.

Definition. A European call option is a pair (T, C_T) where $T > 0$ and $C_T(\cdot) : \mathfrak{R}_+ \to \mathfrak{R}$ is the function

$$C_T(s) = (s-K)^+ = \begin{cases} s-K, & \text{if} \quad s > K \\ 0 & \text{if} \quad s \le K, \end{cases} \quad \text{for some } K \in \mathfrak{R}_+.$$

Option pricing theory has a rather long history. In 1973 Fischer Black and Myron Scholes considered a continuous-time market and they presented their famous formula for the option pricing.

In the Black-Scholes model a stock price $S(t)$ at time t is defined as

$$S(t) = s_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad t \geq 0,$$

where W_t is a Wiener process, $s_0 > 0$, $\sigma > 0$, and μ are constants, s_0 is the stock price observed at time 0 and σ is volatility. Moreover, the instantaneous interest rate $r > 0$ of a bank account is also assumed to be constant, i.e. $\exp(rt)$ is the value of the one unit of money in a bank account at time $t \geq 0$ (see e.g. [5] for details).

Theorem 1. (Black-Scholes option pricing, 1973 [1]). *The time* 0 *fair price* $C_0^{BS}(s_0)$ *of a European call option with strike price* K *and expiry date* τ *in the Black-Scholes model is given by*

$$C_0^{BS}(s_0) = s_0 \phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right) - \frac{K}{e^{r\tau}}\phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right),$$

where $\phi(\cdot)$ *is the cumulative normal distribution function.*

Next, in 1979 Cox, Ross and Rubinstein presented a discrete-time option pricing formula (CRR model). They assumed that in each step the upper and lower stock prices' changes are the same. So they got the following possible changes of the stock price:

$$S_0 = s_0, \quad S_t = u \cdot S_{t-1} \quad \text{or} \quad S_t = d \cdot S_{t-1}, \quad 1 \leq t \leq T, \quad t \in N,$$

where

T is a fixed natural number of short periods (the expiry date),

t is the number of the present step,

s_0 is a positive constant (the stock price at moment 0),

u is the upper stock price's change during one short period,

d is the lower stock price's change during one short period

(u and d are the only possible changes of stock prices during one short period).

Moreover, in the CRR model:

$\hat{r}^t$ is the value at time $t = 0, 1, \ldots, T$ of the one unit of money in a bank account and it is assumed that:

$$0 < d < \hat{r} < u \quad \text{(see e.g. [5] for details).}$$

Cox, Ross and Rubinstein proved the following theorem describing option pricing:

Theorem 2. (CRR option pricing, 1979 [2]). *In the CRR model the fair price* $C_0(s_0)$ *at moment* 0 *of an European call option with expiry date* T *and strike price* $K = s_0 u^{k_0} d^{T-k_0}$ *for a certain* $k_0 = 0, 1, \ldots, T$*, is given by:*

$$C_0(s_0) = s_0 \bar{D} - \frac{K}{\hat{r}^T} D^*,$$

where

$$\bar{D} = \sum_{k=k_0}^{T} \binom{T}{k} \cdot \bar{p}^k \cdot \bar{q}^{T-k}, \quad D^* = \sum_{k=k_0}^{T} \binom{T}{k} \cdot p^{*K} \cdot q^{*T-k},$$

$$k_0 = \frac{\ln \frac{K}{S_0} - T \cdot \ln d}{\ln \left(\frac{u}{d}\right)}, \quad p^* = \frac{\hat{r} - d}{u - d}, \quad q^* = \frac{u - \hat{r}}{u - d}, \quad \bar{p} = p^* \cdot \frac{u}{\hat{r}}, \quad \bar{q} = q^* \cdot \frac{d}{\hat{r}}.$$

This discrete time formula is simpler and useful in numerical calculation.

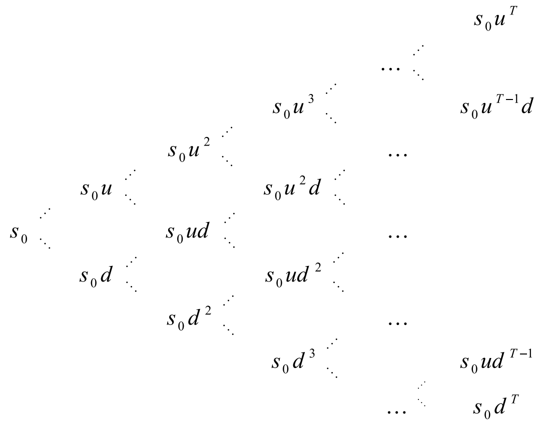

Fig. 1: Event tree in CRR model.

Moreover, Cox, Ross and Rubinstein proved that their formula converges to the Black-Scholes formula when the number of moments of the portfolio's change is large.

2. Calibration of CRR model, notation and formulation of main result

From now on, we take for simplicity the expiry date $\tau = 1$ and we consider n moments of the stock price's change: $j/n,\ j = 0, 1, \ldots, n$.

For the interest rate r and the volatility σ such as in the Black-Scholes model we define:

$$u_n = e^{\frac{\sigma}{\sqrt{n}}}, \quad d_n = e^{-\frac{\sigma}{\sqrt{n}}}, \quad \hat{r}_n = e^{\frac{r}{n}}. \tag{1}$$

Then for sufficiently large n we have $0 < d_n < \hat{r}_n < u_n$, and we can consider the n-th CRR model with $T = n$ and the parameters $u_n,\ d_n$, and $\hat{r}_n$ defined above. It follows from Theorem 2 that the price $C_{0,n}(s_0)$ at the time 0 of a European call option with strike price K, the expiry date $\tau = 1$ and the initial stock price $s_0 > 0$, described above, is given in the n-th CRR model by the formula:

(2) $$C_{0,n}(s_0) = s_0 \bar{D}_n - \frac{K}{e^r} D_n^*,$$

where

$$\bar{D}_n = \sum_{k=k_{0,n}}^{n} \binom{n}{k} \cdot \bar{p}_n^k \cdot \bar{q}_n^{n-k}, \quad D_n^* = \sum_{k=k_{0,n}}^{n} \binom{n}{k} \cdot p_n^{*k} \cdot q_n^{*n-k},$$

(3) $$k_{0,n} = \frac{\sqrt{n} \ln \frac{K}{s_0} + \sigma \cdot n}{2\sigma},$$

(we assume tacitly that the numbers k in the above sums are nonnegative integers),

(4) $$p_n^* = \frac{\hat{r}_n - d_n}{u_n - d_n}, \quad q_n^* = \frac{u_n - \hat{r}_n}{u_n - d_n}, \quad \bar{p}_n = p_n^* \cdot \frac{u_n}{\hat{r}_n}, \quad \bar{q}_n = q_n^* \cdot \frac{d_n}{\hat{r}_n}.$$

Cox, Ross and Rubinstein proved the following theorem:

Theorem 3. (CRR, 1979). *Under the above assumptions let* $C_{0,n}(s_0)$ *be given by formula* (2). *Then for any fixed* $s_0 > 0$ *we have*

$$\lim_{n\to\infty} C_{0,n}(s_0) = C_0^{B-S}(s_0) = s_0 \phi(d_+) - \frac{K}{e^r} \phi(d_-),$$

where

$$d_+ = \frac{\ln \frac{s_0}{K} + \left(r + \frac{\sigma^2}{2}\right)}{\sigma}, \qquad d_- = \frac{\ln \frac{s_0}{K} + \left(r - \frac{\sigma^2}{2}\right)}{\sigma}.$$

Our result is the following:

Main Theorem. *The convergence in Theorem* 3 is uniform *with respect to the initial stock price* s_0 *on the half-line* $(0, \infty)$ $(s_0 \in (0, \infty))$.

Finally, let us note that this uniform convergence is important when we consider certain generalized CRR models with time dependent parameters and their limit formulas for option pricing.

3. Auxiliary facts

In the proof of the uniform convergence of CRR formulas to the Black-Scholes formula we use the following well-known Bernstein inequality for Bernoulli scheme:

Theorem 4. (The Bernstein inequality). *If* S_n *is the number of successes in the Bernoulli scheme with* n *samples and the probability* p *of the success in each sample, then we have the following inequalities:*

(5) $$P\left(\frac{S_n}{n} \le p - \varepsilon\right) \le e^{-n\frac{\varepsilon^2}{4}}, \quad P\left(\frac{S_n}{n} \ge p + \varepsilon\right) \le e^{-n\frac{\varepsilon^2}{4}}, \quad \varepsilon > 0.$$

(*see e.g.* [7]).

We also use a certain version of Dini's theorem and the lemma below:

Lemma 1. *For large n we have the following asymptotic behavior of the probabilities p_n^* and $\bar{p}_n$ defined by formula (4) above:*

$$p_n^* = \frac{1}{2} + \left(\frac{r}{2\sigma} - \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{2},$$

$$\bar{p}_n = \frac{1}{2} + \left(\frac{r}{2\sigma} + \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{2},$$

(*see* [4] *and* [7]).

Theorem 5. (The Dini Theorem). *Let $-\infty < a < b < \infty$ and $(F_n)_{n=1}^{\infty}$ be a sequence of non-decreasing and bounded functions $F_n : [a,b] \to \mathfrak{R}$. Assume that*

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \textit{for every} \quad x \in [a,b],$$

where $F : [a,b] \to \mathfrak{R}$ is a continuous function. Then

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \textit{uniformly on} \quad [a,b].$$

Remark 1. If $f(\cdot)$ is the density of the standard normal distribution and $\gamma < 0$, then we have the estimate:

$$\int_{-\infty}^{\gamma} f(x)dx = (u = x - \gamma) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\gamma+u)^2} du$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma^2} \int_{-\infty}^{0} \cdot e^{-\gamma u} e^{-\frac{1}{2}u^2} du \le \frac{1}{2} e^{-\frac{1}{2}\gamma^2} du.$$

4. Proof of the Main Theorem

We consider the following two cases.

I. $s_0 > \tilde{s}$,

where $\tilde{s} > 0$ is a sufficiently large number that will be chosen later.

From (2) we have the estimate

$$\begin{aligned}
&\left|C_{0,n}(s_0) - \left(s_0 \cdot \phi(d_+) - Kr^{-r} \cdot \phi(d_-)\right)\right| \\
&= \left| s_0 \cdot \sum_{k=k_{0,n}}^{n} \binom{n}{k} \cdot \bar{p}_n^k \cdot \bar{q}_n^{n-k} - Ke^{-r} \cdot \sum_{k=k_{0,n}}^{n} \binom{n}{k} \cdot p_n^{*k} \cdot q_n^{*n-k} \right. \\
&\quad \left. - s_0 \cdot \phi(d_+) + Ke^{-r} \cdot \phi(d_-) \right| \\
(6)\ &= \left| s_0 \cdot P\left(\bar{S}_n \ge k_{0,n}\right) - Ke^{-r} \cdot P\left(S_n^* \ge k_{0,n}\right) - s_0 \cdot \phi(d_+) + Ke^{-r} \cdot \phi(d_-) \right| \\
&= \left| s_0 - s_0 \cdot P\left(\bar{S}_n < k_{0,n}\right) - Ke^{-r} + Ke^{-r} \cdot P\left(S_n^* < k_{0,n}\right) - s_0 \right. \\
&\quad \left. + s_0 \cdot \phi(-d_+) + Ke^{-r} - K^{-r} \cdot \phi(-d_-) \right| \\
&\le s_0 \cdot P\left(\bar{S}_n \le k_{0,n}\right) + s_0 \cdot \phi(-d_+) + Ke^{-r} \cdot P\left(S_n^* \le k_{0,n}\right) + Ke^{-r} \cdot \phi(-d_-),
\end{aligned}$$

where $\bar{S}_n$ is the number of successes in Bernoulli scheme with n samples and the probability $\bar{p}_n$ of the success in each sample, S_n^* is the number of successes in Bernoulli scheme with n samples and the probability p_n^* of the success in each sample.

From (3) we obviously have:

$$\frac{k_{0,n}}{n} = \frac{\sqrt{n}\ln\frac{K}{s_0} + \sigma\cdot n}{2\sigma\cdot n} = \frac{\ln\frac{K}{s_0}}{2\sigma\cdot\sqrt{n}} + \frac{1}{2} < \frac{1}{2} \quad \text{for} \quad s_0 > K. \tag{7}$$

Since

$$\frac{o(1/\sqrt{n})}{1/\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0,$$

there exists a number m (that depends on a particular sequence $o(1/\sqrt{n})$) such that

$$\underset{m\le n\in N}{\forall} -\frac{1}{4}\sigma < \frac{o(1/\sqrt{n})}{1/\sqrt{n}} < \frac{1}{4}\sigma. \tag{7a}$$

We take such a number m that (7a) is satisfied for sequences $o(1/\sqrt{n})$ in the formulas on $\bar{p}_n$ and p_n^* in Lemma 1.

Then from Lemma 1 and (7) we obtain the following inequality:

$$\begin{aligned} p_n^* - \frac{k_{0,n}}{n} &= \frac{1}{2} + \frac{r}{2\sigma\sqrt{n}} - \frac{1}{4}\sigma\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{2} - \frac{\ln K - \ln s_0}{2\sigma\sqrt{n}} \ge \ (\text{for } n \ge m) \\ &\ge \frac{\ln s_0 + r - \ln K}{2\sigma\sqrt{n}} - \frac{\sigma}{2\sqrt{n}} = \frac{\ln s_0 + r - \ln K - \sigma^2}{2\sigma\sqrt{n}}. \end{aligned}$$

Observe that

$$\underset{s''\ge K}{\exists}\ \underset{s''<s_0}{\forall} \ln s_0 + r - \ln K - \sigma^2 > 0.$$

Therefore

$$\begin{gathered} p_n^* - \frac{k_{0,n}}{n} \ge \frac{\ln s_0 + r - \ln K - \sigma^2}{2\sigma\sqrt{n}} > 0, \\ \text{for every} \quad s_0 > s'' \quad \text{and every} \quad n \ge m. \end{gathered} \tag{8}$$

Similarly, for $n \ge m$ and $s_0 > s''$ we have

$$\begin{aligned} \bar{p}_n - \frac{k_{0,n}}{n} &= \frac{1}{2} + \left(\frac{r}{2\sigma} + \frac{1}{4}\sigma\right)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) - \frac{\sqrt{n}\ln\frac{K}{s_0} + \sigma\cdot n}{2\sigma\cdot n} \\ &= \frac{r + 0,5\cdot\sigma^2}{2\sigma\cdot\sqrt{n}} - \frac{\ln\frac{K}{s_0}}{2\sigma\cdot\sqrt{n}} + \frac{o\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}}\cdot\frac{1}{\sqrt{n}} \\ &\ge \frac{\ln s_0 + r - \ln K}{2\sigma\sqrt{n}} + \frac{0,5\cdot\sigma^2}{2\sigma\cdot\sqrt{n}} - \frac{1}{4}\sigma\cdot\frac{1}{\sqrt{n}} = \frac{\ln s_0 + r - \ln K}{2\sigma\sqrt{n}} > 0. \end{aligned}$$

From this and the Bernstein inequality, the estimate for $n \geq m$ and $s_0 > s''$ follows:

$$\begin{aligned}
s_0 \cdot P\left(\bar{S}_n \leq k_{0,n}\right) &= s_0 \cdot P\left(\frac{\bar{S}_n}{n} \leq \bar{p}_n - \left(\bar{p}_n - \frac{k_{0,n}}{n}\right)\right) \\
&\leq s_0 \cdot \exp\left(-\frac{1}{4} n \cdot \left(\bar{p}_n - \frac{k_{0,n}}{n}\right)^2\right) \\
&\leq s_0 \cdot \exp\left(-\frac{n}{4}\left(\frac{\ln s_0 + r - \ln K}{2\sigma\sqrt{n}}\right)^2\right) \qquad (9) \\
&= \exp\{\ln s_0\} \cdot \exp\left(-\frac{1}{16\sigma^2}\left(\ln s_0 + r - \ln K\right)^2\right) \\
&= \exp\left(-\frac{1}{16\sigma^2}\left[(\ln s_0 + r - \ln K)^2 - 16\sigma^2 \ln s_0\right]\right).
\end{aligned}$$

Introduce the notation:

$$h(s_0) := \left(\ln s_0 + r - \ln K\right)^2 - 16\sigma^2 \ln s_0.$$

Then

$$h(s_0) = (\ln s_0)^2 \left[\left[\frac{r - \ln K}{\ln s_0} + 1\right]^2 - \frac{16\sigma^2}{\ln s_0}\right] \underset{s_0 \to \infty}{\longrightarrow} \infty,$$

which together with (9) imply:

$$s_0 \cdot P(\bar{S}_n \leq k_{0,n}) \leq \exp\left(-\frac{1}{16\sigma^2} \cdot h(s_0)\right) \underset{s_0 \to \infty}{\longrightarrow} 0.$$

Fix an arbitrary $\varepsilon > 0$. Therefore

$$\underset{\hat{s} \in (0,\infty)}{\exists} \ \underset{\hat{s} \leq s_0}{\forall} \ s_0 \cdot P\left(\bar{S}_n \leq k_{0,n}\right) < \frac{\varepsilon}{4}, \quad \text{for each} \quad n \geq m. \qquad (10)$$

From Remark 1 and (5) we obtain

$$\begin{aligned}
&s_0 \cdot \phi(-d_+) \leq s_0 \cdot \exp\left(-0.5 \cdot d_+^2\right) \\
&= s_0 \cdot \exp\left(-0.5 \cdot \left(\frac{\ln \frac{s_0}{K} + \left(r + \frac{\sigma^2}{2}\right)}{\sigma}\right)^2\right) \\
&= s_0 \cdot \exp\left(-0.5 \cdot \left(\frac{\ln s_0}{\sigma} + C\right)^2\right) \\
&= \exp\left(-0.5 \cdot \ln s_0 \left(\frac{1}{\sigma^2} \ln s_0 + \frac{2C}{\sigma} - 2\right)\right) \cdot \exp(-0.5 \cdot C^2) \underset{s_0 \to \infty}{\longrightarrow} 0,
\end{aligned}$$

where

$$C = \frac{-\ln K + \left(r + \frac{\sigma^2}{2}\right)}{\sigma}$$

is a constant. Hence

$$\underset{\check{s}\in(0,\infty)}{\exists} \ \underset{\check{s}\le s_0}{\forall} s_0 \cdot \phi(-d_+) < \frac{\varepsilon}{4}. \tag{11}$$

Similarly as in (9), from the Bernstein inequality and (8) we obtain for $n \geq m$ and $s_0 > s''$:

$$\begin{aligned}
Ke^{-r} \cdot P(S_n^* \le k_{0,n}) &= Ke^{-r} \cdot P\left(\frac{S_n^*}{n} \le p_n^* - \left(p_n^* - \frac{k_{0,n}}{n}\right)\right) \\
&\le Ke^{-r} \cdot \exp\left(-\frac{1}{4}n \cdot \left(p_n^* - \frac{k_{0,n}}{n}\right)^2\right) \\
&\le Ke^{-r} \cdot \exp\left(-\frac{n}{4}\left(\frac{\ln s_0 + r - \ln K - \sigma^2}{2\sigma\sqrt{n}}\right)^2\right) \\
&= Ke^{-r} \cdot \exp\left(-\frac{1}{16\sigma^2}\left(\ln s_0 + r - \ln K - \sigma^2\right)^2\right) \underset{s_0\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Hence, for the $\varepsilon > 0$ fixed above, we get

$$\underset{\hat{s}\in(0,\infty)}{\exists} \ \underset{\hat{s}\le s_0}{\forall} Ke^{-r} \cdot P\left(S_n^* \le k_{0,n}\right) < \frac{\varepsilon}{4}, \quad \text{for every} \quad n \geq m. \tag{12}$$

From Remark 1 we obtain

$$\begin{aligned}
&Ke^{-r} \cdot \phi(-d_-) \le Ke^{-r} \cdot \exp\left(-0,5 \cdot d_-^2\right) \\
&= Ke^{-r} \cdot \exp\left(-0,5 \cdot \left(\frac{\ln \frac{s_0}{K} + \left(r - \frac{\sigma^2}{2}\right)}{\sigma}\right)^2\right) \\
&= Ke^{-r} \cdot \exp\left(-0,5 \cdot \left(\frac{\ln s_0}{\sigma} + \tilde{C}\right)^2\right) \\
&= Ke^{-r} \cdot \exp\left(-0,5 \cdot \ln s_0 \left(\frac{1}{\sigma^2}\ln s_0 + \frac{2\tilde{C}}{\sigma}\right)\right) \cdot (-0,5 \cdot \tilde{C}^2) \underset{s_0\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where

$$\tilde{C} = \frac{-\ln K + \left(r - \frac{\sigma^2}{2}\right)}{\sigma}$$

is a constant. Therefore

$$\underset{s'\in(0,\infty)}{\exists} \ \underset{s'\le s_0}{\forall} Ke^{-r} \cdot \phi(-d_-) < \frac{\varepsilon}{4}. \tag{13}$$

Take $\tilde{s} = \max\{\,\widehat{s}, \check{s}, \hat{s}, s', s''\}$. Then as a consequence of (6), (10)–(13) we obtain for every $n \geq m$ and every $s_0 > \tilde{s}$:

$$\begin{aligned}
&\left|C_{0,n}(s_0) - \left(s_0 \cdot \phi(d_+) - Ke^{-r} \cdot \phi(d_-)\right)\right| \\
&\le s_0 \cdot P(\bar{S}_n \le k_{0,n}) + s_0 \cdot \phi(-d_+) + Ke^{-r} \cdot P(S_n^* \le k_{0,n}) + Ke^{-r} \cdot \phi(-d_-) < \varepsilon.
\end{aligned}$$

II. $s_0 \in [0, \tilde{s}]$.

Let $C_{0,n}(0) = 0 = C_0(0)$.

Observe that for every $n \in N$, $C_{0,n}(\cdot)$ is a non-decreasing and bounded function on $[0, \tilde{s}]$, $(|C_{0,n}(s_0)| \leq s_0 \leq \tilde{s})$. Moreover,

$$C_0(s_0) = s_0\phi(d_+) - \frac{K}{e^r}\phi(d_-)$$

is a continuous function on $[0, \tilde{s}]$. From Theorem 3 we obtain

$$\lim_{n\to\infty} C_{0,n}(s_0) = s_0\phi(d_+) - \frac{K}{e^r}\phi(d_-), \quad \text{for every} \quad s_0 > 0.$$

Thus, from Dini's Theorem we have

$$\lim_{n\to\infty} C_{0,n}(s_0) = s_0\phi(d_+) - \frac{K}{e^r}\phi(d_-), \quad \text{uniformly on} \quad [0, \tilde{s}].$$

From I and II we get the conclusion of the theorem. Indeed. As in the first part of this proof, for a fixed $\varepsilon > 0$ we can choose such $m \in N$ and $\tilde{s} > 0$ that for $s_0 > \tilde{s}$ and $n \geq m$ we have

$$\big|C_{0,n}(s_0) - \big(s_0 \cdot \phi(d_+) - Ke^{-r} \cdot \phi(d_-)\big)\big| < \varepsilon. \tag{14}$$

Next, from the second part of this proof and the uniform convergence of a sequence of functions $C_{0,n}(\cdot)$ on $[0, \tilde{s}]$, we take such a number m_0 $(m_0 \geq m)$ (suitable for ε and $\tilde{s}$) that (14) is satisfied for all $n \geq m_0$ and $s \in [0, \tilde{s}]$.

Therefore, we have proved the following condition

$$\underset{\varepsilon>0}{\forall}\ \underset{m_0}{\exists}\ \underset{n\geq m_0}{\forall}\ \underset{s_0\in(0,\infty)}{\forall} \big|C_{0,n}(s_0) - \big(s_0 \cdot \phi(d_+) - Ke^{-r} \cdot \phi(d_-)\big)\big| < \varepsilon.$$

□

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JEDNOSTAJNA ZBIEŻNOŚĆ FORMUŁ COXA-ROSSA-RUBINSTEINA DO FORMUŁY BLACKA-SCHOLESA

S t r e s z c z e n i e

W pracy udowodniono, ze zbieżność skalibrowanych formuł Coxa-Rossa-Rubinsteina na wycenę opcji do wzoru Blacka-Scholesa jest jednostajna ze względu na początkową cenę akcji $s_0 \in (0, \infty)$.

Słowa kluczowe: model Coxa-Rossa-Rubinsteina, wzór Blacka-Scholesa, wycena opcji, jednostajna zbieżność

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To our best Teacher and Friend Professor Władysław Wilczyński
on the occasion of his 70th birthday

Małgorzata Filipczak and Małgorzata Terepeta

SOME REMARKS ON SIMILAR TOPOLOGIES

Summary

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies defined on a set X. Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are similar, that means the families of sets with nonempty interior in topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are equal. The aim of our paper is to check if the family of similar topologies forms a lattice and to examine the classes of continuous functions with similar topologies on the domain and range of the functions.

Keywords and phrases: similar topologies, quasicontinuous function, cliquish function, somewhat continuous function, somewhat open function

1. Introduction

It is well known that different topologies defined on the same nonempty set X can determine the same family of sets with nonempty interior. Therefore we will say that topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are *similar* (or the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are similar) if the families of sets with nonempty interior with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal. Then we will write $\mathcal{T}_1 \sim \mathcal{T}_2$. Many properties of similar topologies were examined in [1]. Some of them are rather surprising and contrary to our intuition. It is easy to check that for similar topologies the families of dense sets, the families of nowhere dense sets and the families of boundary sets are also equal. However, similar topologies need not be homeomorphic and homeomorphic spaces may not be similar. There exists nonmetrizable space which is similar to the Euclidean metric space. Similar topologies may have different separating and countability axioms, may differ with respect to Lindelöf property, compactness or connectedness.

It turned out that the notion of similarity was described several times under different names. Cranell and Kopperman (see [3]) wrote about Π-*relation*, but the authors considered mainly some extensions of topologies and, in consequence, they examine only comparable (by inclusion) topologies. Gentry and Hoyle in [5] as well as Buskirk and Nadler in [2] used the term *weak equivalence* and they presented similar approach to the problem. In particular, our Theorems 7 and 13 are formulated also in the paper [5], but without proofs.

Below we present one of their results using the notation introduced earlier and the notion of semi-open set. Let $(X, \mathcal{T})$ be a topological space. Recall that $A \subset X$ is semi-open if $A \subset \mathrm{Cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{T}} A)$. Denote by $\mathcal{T}(\mathcal{A})$ the extension of topology $\mathcal{T}$ by a family $\mathcal{A} \subset 2^X$, that means the topology for which $\mathcal{T} \cup \mathcal{A}$ is a subbase. It is obvious that $\mathcal{T}(\mathcal{A})$ is finer then $\mathcal{T}$. If $\mathcal{A} = \{A\}$ and $A \notin \mathcal{T}$ then $\mathcal{T}(A)$ is called *the simple extension of* $\mathcal{T}$.

Theorem 1. ([2], Theorem 2.1) *The topologies* $\mathcal{T}$ *and* $\mathcal{T}(\mathcal{A})$ *are similar if and only if* $\mathcal{A}$ *has property:*

> (W) *for any finite subcollection* $\mathcal{A}' \subset \mathcal{A}$ *the intersection of all its members is semi-open in* $\mathcal{T}$.

It was proved ([6], Theorem 3), that if $\mathcal{A} = \{A\}$ for a set $A \notin \mathcal{T}$ and the condition (W) is fulfilled, then the classes of continuous functions with topologies $\mathcal{T}$ and $\mathcal{T}(A)$ on the domain are equal (values of the functions belong to the same metric space (Y, ρ)). This result does not hold for arbitrary similar topologies. We will describe it more precisely in Chapter 3.

In this paper we would like to focus on other properties of similar topologies (which are not necessarily comparable, as extensions are): how much they can be different with respect to separation axioms, does the family of such topologies form a lattice, what kind of continuity is preserved under similarity of topologies on the domain (or the range) of a function.

2. Some new properties of similarity

In many papers there exist some other properties strictly connected with the openess, for example in Theorem 1 there appeared the notion of semi-openess. Remind that a set $A \subset X$ is pre-open if $A \subset \mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}} A)$ and α-open if $A \subset \mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{T}} A))$. A set $A \subset X$ is called regular open if $A = \mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}} A)$.

Observe that there exist similar topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, for which the families of regular open, semi-open, pre-open and α-open sets do not coincide. Indeed, put $X = \mathbb{R}$ and let $\mathcal{T}_S$ be a topology generated by the base consisting of intervals of the form $[a, b)$ ($a < b$, $a, b \in \mathbb{R}$). It is called Sorgenfrey line or the lower limit topology. Clearly, $\mathcal{T}_S$ is finer than natural topology $\mathcal{T}_{nat}$ and $\mathcal{T}_S \sim \mathcal{T}_{nat}$. Since $\mathrm{Cl}_{\mathcal{T}_S}([a, b)) = [a, b)$, then $[a, b)$ is $\mathcal{T}_S$-regular open and, consequently, also $\mathcal{T}_S$-pre-open and $\mathcal{T}_S$-α-open. On the other hand $[a, b)$ is neither α-open nor pre-open in $\mathcal{T}_{nat}$. Closed interval $[a, b]$ is semi-open in $\mathcal{T}_{nat}$ but not in $\mathcal{T}_S$.

The notion of regular open sets is connected with so-called semi-regularization of a topology [10]. Remind that all regular open sets in $(X, \mathcal{T})$ form a base for a new topology $\mathcal{T}_{sem}$, which is coarser then $\mathcal{T}$. It is natural to ask if $\mathcal{T}$ is similar to $\mathcal{T}_{sem}$.

Example 2. *There exists a topology $\mathcal{T}$ which semi-regularization $\mathcal{T}_{sem}$ is not similar to $\mathcal{T}$.*

Let $\mathcal{T}$ be the overlapping interval topology on $X = [-1, 1]$ [11]. It is generated by sets of the form: $[-1, b)$ if $b > 0$ and $(a, 1]$ if $a < 0$. Then all the intervals (a, b) for any $a < 0$ and $b > 0$ are $\mathcal{T}$-open and any $\mathcal{T}$-open set contains 0. Observe, that $\mathcal{T}$-closure of any set containing 0 is equal to X. Hence the only regular open sets in $\mathcal{T}$ are the empty set or the whole space. Therefore, the semi-regularization topology of $\mathcal{T}$ is equal to $\mathcal{T}_{sem} = \{\emptyset, X\}$ and $\mathcal{T}_{sem} \not\sim \mathcal{T}$. □

Levine in [7] showed that simple extensions of topologies have similar separation axioms. More precisely if a topology $\mathcal{T}$ is T_0, T_1 or T_2 so is its simple extension $\mathcal{T}(A)$ with $A \notin \mathcal{T}$. The regularity and complete regularity are preserved provided A is closed in $\mathcal{T}$. The separating axiom of similar topologies can be different, for example $\mathcal{T}_S$ is completely normal and is similar to $\mathcal{T}_{nat}$. Moreover, there exists a topology which is not T_0 and is similar to natural topology. Indeed, take different points $x_1, x_2 \in \mathbb{R}$ and put $\mathcal{T} = \{A \in \mathcal{T}_{nat} : x_1, x_2 \notin A\} \cup \mathbb{R}$. It is not difficult to check that $\mathcal{T}$ is not T_0 and $\mathcal{T} \sim \mathcal{T}_{nat}$.

The family $\mathbb{T}(X)$ of all topologies on X can be treated as a lattice with join and meet operations defined as follows: $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}(\mathcal{T}_1 \cup \mathcal{T}_2)$ for $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}(X)$. It is natural to ask if the family $\mathbb{S}(X)$ of all similar topologies forms a sublattice of $\mathbb{T}(X)$.

Theorem 3. (a) *There exist topologies $\mathcal{T}_1$, $\mathcal{T}_2$ such that $\mathcal{T}_1 \sim \mathcal{T}_2$ and $\mathcal{T}_1 \cap \mathcal{T}_2 \not\sim \mathcal{T}_1$.*

(b) *There exist topologies $\mathcal{T}_1$, $\mathcal{T}_2$ such that $\mathcal{T}_1 \sim \mathcal{T}_2$ and $\mathcal{T}(\mathcal{T}_1 \cup \mathcal{T}_2) \not\sim \mathcal{T}_1$.*

Proof. Put $X = \mathbb{R}$. Let

$$\mathcal{T}_1 = \{(2n+1, \infty) \colon n \in \mathbb{N}\} \cup \{\emptyset\} \cup \mathbb{R} \quad \text{and} \quad \mathcal{T}_2 = \{(2n, \infty) \colon n \in \mathbb{N}\} \cup \{\emptyset\} \cup \mathbb{R}.$$

Then $\mathcal{T}_1$, $\mathcal{T}_2$ are similar, but their intersection $\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset\} \cup \mathbb{R}$ is not similar to them.

To show (b) we will make use of $\langle s \rangle$-density topologies. The notion of $\langle s \rangle$-density comes from [4] and, for the convenience of the reader, we will shortly remind it. Let $\mathcal{S}$ be a family of unbounded nondecreasing sequences $\langle s \rangle = (s_n)_{n \in \mathbb{N}}$ of positive numbers. We say that $x \in \mathbb{R}$ is a density point with respect to a sequence $\langle s \rangle \in \mathcal{S}$ (shortly – $\langle s \rangle$-density point) of the set $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{|A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}]|}{\frac{2}{s_n}} = 1$$

($\mathcal{L}$ stands for the σ-algebra of Lebesgue measurable sets and $|\cdot|$ – for the Lebesgue measure). The set of all $\langle s\rangle$-density points of A is denoted by $\Phi_{\langle s\rangle}(A)$. For any sequence $\langle s\rangle \in \mathcal{S}$ the family

$$\mathcal{T}_{\langle s\rangle} = \{A \in \mathcal{L}\colon A \subset \Phi_{\langle s\rangle}(A)\}$$

is a topology $\mathcal{T}_{\langle s\rangle}$ called $\langle s\rangle$-density topology. It is finer then density topology $\mathcal{T}_d$. Let

$$\mathcal{S}_0 = \left\{\langle s\rangle \in \mathcal{S}\colon \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\right\}.$$

For any $\langle s\rangle \in \mathcal{S}_0$ it is proved [4] that $\mathcal{T}_d \subsetneq \mathcal{T}_{\langle s\rangle}$. Moreover, there exist sequences $\langle s\rangle, \langle t\rangle \in \mathcal{S}_0$ and a measurable set A ([8], Example 20) such that

$$0 \in \Phi_{\langle s\rangle}(A) \cap \Phi_{\langle t\rangle}(A').$$

Put $B = \big(\Phi_{\langle s\rangle}(A) \cap A\big) \cup \{0\}$. Then $B \in \mathcal{T}_{\langle s\rangle}$ and $\{0\} \cup B' \in \mathcal{T}_{\langle t\rangle}$. Hence $\{0\} \in \mathcal{T}(\mathcal{T}_1 \cup \mathcal{T}_2)$ and $\mathcal{T}(\mathcal{T}_1 \cup \mathcal{T}_2)$ is not similar to $\mathcal{T}_1$. $\square$

It follows that the family $\mathbb{S}(X)$ of similar topologies to a given one is not a sublattice of $\mathbb{T}(X)$.

3. Similarity and Continuity

Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces and f be a function from $(X, \mathcal{T})$ to $(Y, \mathcal{S})$. From [1] it follows that the continuity of the function f is not preserved if we exchange the topology on the domain into similar one. In the same paper there were examined some other kinds of continuity with relation to similar topologies on the domain. The notions of quasi-continuous and cliquish functions were used. In this chapter we deal also with some other notions.

From now on, we assume that $\mathrm{card}(X) \geq 2$ and $\mathrm{card}(Y) \geq 2$, because the other cases are not interesting (if $\mathrm{card}(X) = 1$, then all topologies on X are similar, if $\mathrm{card}(Y) = 1$ then we have only continuous functions from X into Y).

Definition 4. [9] *A function $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ is said to be quasi-continuous at the point $x_0 \in X$ if for each neighbourhood $U \in \mathcal{T}$ of the point x_0 and each neighbourhood $V \in \mathcal{S}$ of the point $f(x_0)$ there exists a nonempty open set $U_1 \subset U$ such that $f(U_1) \subset V$.*

We say that f is *quasi-continuous* if it is quasi-continuous at any $x \in X$. Denote by $\mathcal{Q}(\mathcal{T}, \mathcal{S})$ the family of all quasi-continuous functions $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$.

Definition 5. [9] *Let Y be a metric space with metric ρ. The function $f\colon (X, \mathcal{T}) \to (Y, \rho)$ is said to be cliquish at the point $x_0 \in X$ if for each $\varepsilon > 0$ and each neighbour-*

hood $U \subset X$ of the point x_0, there exists a nonempty open set $U_1 \subset U$ such that for each points $x_1, x_2 \in U_1$ the inequality $\rho(f(x_1), f(x_2)) < \varepsilon$ holds.

We say that f is *cliquish* if it is cliquish at any point. Denote by $\mathcal{C}_q(\mathcal{T}, \rho)$ the family of all cliquish functions $f\colon (X, \mathcal{T}) \to (Y, \rho)$. Note, that each function $f\colon (X, \mathcal{T}) \to (Y, \rho)$ quasi-continuous at $x_0 \in X$ is cliquish at this point (with the topology on Y generated by the metric ρ).

In [1] it is shown that:

$$\mathcal{Q}(\mathcal{T}_1, \mathcal{S}) = \mathcal{Q}(\mathcal{T}_2, \mathcal{S}) \quad \Longrightarrow \quad \mathcal{T}_1 \sim \mathcal{T}_2 \quad \Longrightarrow \quad \mathcal{C}_q(\mathcal{T}_1, \rho) = \mathcal{C}_q(\mathcal{T}_2, \rho)$$

and the implications can not be reversed.

We introduce two other kinds of continuity which seem to be developed especially for similarity. Denote by $\mathcal{T}^*$ the family of all nonempty open sets in $\mathcal{T}$.

Definition 6. [5] *Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces. A function $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ is said to be somewhat continuous if for any $V \in \mathcal{S}$ such that $f^{-1}(V) \neq \emptyset$ there is a set $U \in \mathcal{T}^*$ such that $f(U) \subset V$.*

Clearly, every continuous function and quasi-continuous function is somewhat continuous, but the converse is false. Indeed, let $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{b\}, \{a, c\}\}$, $\mathcal{S} = \{\emptyset, X, \{a, b\}\}$ and $f\colon (X, \mathcal{T}) \to (X, \mathcal{S})$ be identity function. Then f is somewhat continuous, but not quasi-continuous. Precisely, it is not quasi-continuous at a.

Note, that somewhat continuity and cliquishness are independent notions. The function $f : (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat})$ given by the formula

$$f(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0 \end{cases}$$

is cliquish but not somewhat continuous. Indeed, for $V = (\frac{1}{2}, \frac{3}{2})$ we have $f^{-1}(V) \neq \emptyset$, but for any $U \in \mathcal{T}^*$, $0 \in f(U)$, so $f(U) \not\subset V$.

Let the function $g : (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{nat})$ be defined as follows: $g(x) = \chi_{\mathbb{Q}}(x)$ for $x \in (0, 1,)$ ($\chi_{\mathbb{Q}}$ stands for the characteristic function of the set of rational numbers $\mathbb{Q}$), $g(x) = 1$ for $x \geq 1$ and $g(x) = 0$ for $x \leq 0$. Then g is somewhat continuous but not cliquish (at any point $x \in (0, 1)$).

By $\mathcal{S}w\mathcal{C}(\mathcal{T}, \mathcal{S})$ we denote the family of all somewhat continuous functions $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$.

Theorem 7. *If $f \in \mathcal{S}w\mathcal{C}(\mathcal{T}_1, \mathcal{S}_1)$ and $\mathcal{T}_1 \sim \mathcal{T}_2$, $\mathcal{S}_1 \sim \mathcal{S}_2$, then $f \in \mathcal{S}w\mathcal{C}(\mathcal{T}_2, \mathcal{S}_2)$.*

Proof. Let $V_2 \in \mathcal{S}_2^*$. As $\mathcal{S}_1 \sim \mathcal{S}_2$ it follows that there is a set $V_1 \subset V_2$ such that $V_1 \in \mathcal{S}_1^*$. From the assumption, for V_1 we have a set $U_1 \in \mathcal{T}_1^*$ for which $f(U_1) \subset V_1$. Using the assumption $\mathcal{T}_1 \sim \mathcal{T}_2$, we obtain a set $U_2 \in \mathcal{T}_2^*$ which is a subset of U_1. Hence $f(U_2) \subset f(U_1) \subset V_1 \subset V_2$ and the function $f\colon (X, \mathcal{T}_2) \to (Y, \mathcal{S}_2)$ is somewhat continuous. $\square$

Theorem 8. *Suppose* $(X, \mathcal{T}_1)$, $(X, \mathcal{T}_2)$ *are Hausdorff spaces and* $(Y, \mathcal{S})$ *is* T_1. *If* $\mathcal{S}w\mathcal{C}(\mathcal{T}_1, \mathcal{S}) = \mathcal{S}w\mathcal{C}(\mathcal{T}_2, \mathcal{S})$ *then* $\mathcal{T}_1 \sim \mathcal{T}_2$.

Proof. Remind that $\mathrm{card}(Y) \geq 2$, so $(Y, \mathcal{S})$ as T_1 is not indiscrete. Assume $\mathcal{T}_1 \not\sim \mathcal{T}_2$, so there is a nonempty set $A \in \mathcal{T}_1$ such that its interior in $\mathcal{T}_2$ is empty. We will show that there exists a function $f \in SwC(\mathcal{T}_1, \mathcal{S}) \setminus SwC(\mathcal{T}_2, \mathcal{S})$. Consider two cases. First, suppose that $\mathrm{Cl}_{\mathcal{T}_1}(A) \neq X$. Let

$$f(x) = \begin{cases} y_1 & \text{for } x \in A, \\ y_2 & \text{for } x \notin A. \end{cases}$$

Let $V \in \mathcal{S}^*$ be a set such that $f^{-1}(V) \neq \emptyset$. Then we have three possibilities:

- if $y_1 \in V$ and $y_2 \notin V$ then we put $U = A$ and we obtain $f(U) = f(A) = \{y_1\} \subset V$;
- if $y_2 \in V$ and $y_1 \notin V$ then we put $U = X \setminus \mathrm{Cl}_{\mathcal{T}_1}(A)$ and we obtain $f(U) = \{y_2\} \subset V$;
- if $y_1, y_2 \in V$ then $f^{-1}(V) = X$ and any open set $U \in \mathcal{T}_1^*$ fulfills the conditions $f(U) \subset V$.

Therefore, $f \in SwC(\mathcal{T}_1, \mathcal{S})$. We will show now that $f \notin SwC(\mathcal{T}_2, \mathcal{S})$. Take $V \in \mathcal{S}^*$ be a set such that $y_1 \in V$ and $y_2 \notin V$. Then $f^{-1}(V) = f^{-1}(\{y_1\}) = A$, so $f(A) = V$. Notice, that $f(U) \subset V$ for $U \in \mathcal{T}_2^*$ provided $U \subset A$. As A has the empty interior in $\mathcal{T}_2$, there is no open set $U \in \mathcal{T}_2^*$ such that $f(U) \subset V$.

Suppose now that $\mathrm{Cl}_{\mathcal{T}_1}(A) = X$. Take two different points $x_1, x_2 \in X$. Then, since $(X, \mathcal{T}_1)$ is Hausdorff, there are disjoint open sets W_1, W_2 such that $x_1 \in W_1$ and $x_2 \in W_2$. Note that

$$X = (X \setminus \mathrm{Cl}_{\mathcal{T}_1}(W_1)) \cup (\mathrm{Cl}_{\mathcal{T}_1}(W_1) \setminus W_1) \cup W_1$$

and $\mathrm{Int}_{\mathcal{T}_1}(\mathrm{Cl}_{\mathcal{T}_1}(W_1) \setminus W_1) = \emptyset$ (as it is a boundary set). Therefore, at least one of sets $W_1 \cap A$ and $(X \setminus \mathrm{Cl}_{\mathcal{T}_1}(W_1)) \cap A$ is nonempty. Putting it as A_1, we obtain a nonempty set with the properties; it is open in $\mathcal{T}_1$, $\mathrm{Int}_{\mathcal{T}_2}(A_1) = \emptyset$ and $\mathrm{Cl}_{\mathcal{T}_1}(A_1) \neq X$. Then we can repeat the previous construction. $\square$

Corollary 9. *Suppose* $(X, \mathcal{T}_1)$, $(X, \mathcal{T}_2)$ *are Hausdorff spaces and* $(Y, \mathcal{S})$ *is* T_1. *Then* $\mathcal{S}w\mathcal{C}(\mathcal{T}_1, \mathcal{S}) = \mathcal{S}w\mathcal{C}(\mathcal{T}_2, \mathcal{S})$ *if and only if* $\mathcal{T}_1 \sim \mathcal{T}_2$.

Let now consider another notion connected with the functions for which we obtain analogous result as in Theorem 7.

Definition 10. [5] *Let* $(X, \mathcal{T})$, $(Y, \mathcal{S})$ *be topological spaces. A function* $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ *is said to be somewhat open if for any* $U \in \mathcal{T}^*$ *there is a set* $V \in \mathcal{S}^*$ *such that* $V \subset f(U)$.

Obviously, any open function is somewhat open. Somewhat continuity and somewhat openess are independent notions.

Example 11. *Let $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{b\}\}$, $\mathcal{S} = \{\emptyset, X, \{a, b\}\}$. Then the identity function $f : (X, \mathcal{T}) \to (X, \mathcal{S})$ is somewhat continuous but not somewhat open and the inverse function $f^{-1} : (X, \mathcal{S}) \to (X, \mathcal{T})$ is somewhat open but not somewhat continuous.*

Example 12. *The family of quasi-continuous functions and the family of somewhat open functions are not comparable. The family of cliquish functions and the family of somewhat open functions are not comparable, too.*

Let f be a function defined as follows: $f(x) = x$ for $x \neq 0$ and $f(0) = 1$. It is evident that f is somewhat open. It is not quasi-continuous at $x = 0$. For

$$U = \left(-\frac{1}{2}, \frac{1}{2}\right), \quad V = \left(\frac{3}{4}, \frac{5}{4}\right)$$

there is no open nonempty set $U_1 \subset U$ such that $f(U_1) \subset V$. Put $g(x) = 1$ for $x \in \mathbb{R}$. Then g is continuous, hence quasi-continuous, but not somewhat open.

Let $\mathcal{T}_{discr}$ be a topology generated by the discrete metric. Consider $h\colon (\mathbb{R}, \mathcal{T}_{nat}) \to (\mathbb{R}, \mathcal{T}_{discr})$ and $h = \chi_{\mathbb{Q}}$. Then h is open, hence somewhat open, but not cliquish. The function $k(x) = \operatorname{sign}(x)$ is cliquish but not somewhat open. □

By $\mathcal{S}w\mathcal{O}(\mathcal{T}, \mathcal{S})$ we denote the family of all somewhat open functions $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$. Similarly as in Theorem 7, it is easy to check:

Theorem 13. *If $f \in \mathcal{S}w\mathcal{O}(\mathcal{T}_1, \mathcal{S}_1)$ and $\mathcal{T}_1 \sim \mathcal{T}_2$, $\mathcal{S}_1 \sim \mathcal{S}_2$, then $f \in \mathcal{S}w\mathcal{O}(\mathcal{T}_2, \mathcal{S}_2)$.*

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PEWNE UWAGI O PODOBNYCH TOPOLOGIACH

S t r e s z c z e n i e

Mówimy, że topologie $\mathcal{T}_1$, $\mathcal{T}_2$ zdefiniowane na zbiorze X są podobne, jeżeli rodziny zbiorów o niepustym wnętrzu w przestrzeniach topologicznych $(X, \mathcal{T}_1)$ oraz $(X, \mathcal{T}_2)$ są równe. W pracy sprawdzamy, czy rodzina topologii podobnych tworzy kratę oraz badamy, czy różnego rodzaju ciągłość funkcji jest zachowana, gdy zmienimy topologię w dziedzinie albo zbiorze wartości funkcji na podobną do niej.

Słowa kluczowe: topologie podobne, funkcja quasi-ciągła, funkcja klikowa, funkcja nieco ciągła, funkcja nieco otwarta

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on the occasion of his 70th birthday

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EFFECTIVE ŁOJASIEWICZ GRADIENT INEQUALITY FOR GENERIC NASH FUNCTIONS WITH ISOLATED SINGULARITY

Summary

Let Ω be a closed ball in $\mathbb{R}^n$ centered at the origin and let $f : \Omega \to \mathbb{R}$ be a Nash function. Then there exists an irreducible polynomial $P \in \mathbb{R}[x, y]$, where $x = (x_1, \dots, x_n)$ is a system of variables and y is a variable such that $P(x, f(x)) = 0$. We give a D'Acunto-Kurdyka type estimation of the exponent $\varrho \in [0, 1)$ in the Łojasiewicz gradient inequality $|\nabla f(x)| \geq C|f(x)|^{\varrho}$ for $x \in \Omega$, $|f(x)| < \varepsilon$ for some constants $C, \varepsilon > 0$, in terms of degree of P, provided $f(0) = 0$ and f has an isolated singularity at the origin and $\nabla P(x, f(x)) \neq 0$ for $x \in \Omega$.

Keywords and phrases: semialgebraic function, Nash function, Łojasiewicz gradient inequality, Łojasiewicz exponent

1. Introduction

Łojasiewicz inequalities are an important tools in various branches of mathematics: differential equations, singularity theory and optimization (for more detailed references, see for example [KMP], [RS1], [RS2], [KS1], [KS2]). Quantitative aspects, like estimates (or exact computation) of these exponents are subject of intensive study in real and complex algebraic geometry (see for instance [KS1], [KS2] and [KSS]).

Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of the origin. Let $f, F : \Omega \to \mathbb{R}$ be continuous semialgebraic functions such that $F^{-1}(0) \subset f^{-1}(0)$. Then there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds:

$$|F(x)| \geq C|f(x)|^\eta \quad \text{for } x \in \Omega,\ |x| < \varepsilon. \tag{1.1}$$

The lower bound of the exponents η in (1.1) is called the *Łojasiewicz exponent* of the pair (F, f) *at* 0 and is denoted by $\mathcal{L}_0(F, f)$. It is known that $\mathcal{L}_0(F, f)$ is a rational number (see [BR]) and the inequality (1.1) holds actually with $\eta = \mathcal{L}_0(F, f)$ for some $\varepsilon, C > 0$ (see [Sp1]). An asymptotic estimate for $\mathcal{L}_0(F, f)$ was obtained by Solernó [So]:

$$\mathcal{L}_0(F, f) \leq D^{M^{ca}}, \tag{S}$$

where D is a bound for the degrees of the polynomials involved in a description of F, f and Ω; M is the number of variables in these formulas; a is the maximum number of alternating blocs of quantifiers in these formulas; and c is an unspecified universal constant.

In this paper we consider the case when F is equal to the gradient ∇f and f is a Nash function. Our main goal is to obtain an effective estimates for the *Łojasiewicz exponent* $\varrho \in [0, 1)$ in the *gradient inequality* $|\nabla f(x)| \geq C|f(x)|^\varrho$ in a neighbourhood of the origin for some constant $C > 0$, of arbitrary Nash function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity at the origin in terms of degree of a polynomial $P \in \mathbb{R}[x, y]$ describing f, provided P have no singularities on the graph of f. The above estimation is a generalization of the result by D'Acunto and Kurdyka [DK].

1.1. Łojasiewicz's gradient inequality

By $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, b)$ we denote a mapping from a neighbourhood $U \subset \mathbb{R}^n$ of the origin to $\mathbb{R}^m$ such that $F(0) = b$. We put $V(F) = \{x \in U : F(x) = 0\}$.

A function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$, is called *semialgebraic*, if the graph of f is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}$. Semialgebraic and analytic functions are called *Nash functions*.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a semialgebraic function of class C^1 in $x = (x_1, \dots, x_n)$, and let ∇f be the gradient of f, i.e.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, \nabla f(0)).$$

Then there are: a positive constant C and a constant $\varrho \in [0, 1)$ such that the following *Łojasiewicz gradient inequality* holds (see [Ł1] or [Ł2], cf., [T]):

$$|\nabla f(x)| \geq C|f(x)|^\varrho \quad \text{in a neighbourhood of } 0. \tag{Ł}$$

The smallest exponent ϱ in (Ł), denoted by $\varrho_0(f)$, is called the *Łojasiewicz exponent in the gradient inequality.*

In the case of a polynomial function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of degree d such that 0 is an isolated point of $V(f)$, J. Gwoździewicz [Gw] proved that

$$\varrho_0(f) \leq 1 - \frac{1}{(d-1)^n + 1}. \tag{G2}$$

In the general case D. D'Acunto and K. Kurdyka [DK] (cf. [Ga]) showed that

$$\text{(DK)} \qquad \varrho_0(f) \le 1 - \frac{1}{d(3d-3)^{n-1}} \quad \text{provided} \quad d \ge 2.$$

1.2. Main result

The aim of this paper is to show an effective estimates of the Łojasiewicz exponents in the gradient inequality (Ł) for arbitrary Nash function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity at 0 in terms of degree of a polynomials describing f (Theorem 1.2). If $\nabla f(0) = 0$ and $\nabla f(x) \neq 0$ for $x \neq 0$ in a neighbourhood of 0, then we will say that f has an *isolated singularity at* 0.

Let $\Omega \subset \mathbb{R}^n$ be a closed ball in $\mathbb{R}^n$ of the form

$$\Omega = \{x \in \mathbb{R}^n : |x| \le r\} \quad \text{for some} \quad r > 0.$$

By $\partial\Omega$ we denote the sphere $\{x \in \mathbb{R}^n : |x| = r\}$.

Let $f : \Omega \to \mathbb{R}$ be a Nash function such that $f(0) = 0$. We will assume that f has a singularity at 0 and that $\nabla f(x) \neq 0$ for $x \in \Omega \setminus \{0\}$.

Since f is a differentiable semialgebraic function, then its set of critical values is finite. So, we have

Fact 1.1. *There exists* $\varepsilon > 0$ *such that* f *has no critical values in the interval* $(-\varepsilon, \varepsilon)$ *except of* 0.

Let $P \in \mathbb{R}[x, y]$ be the unique irreducible polynomial such that

$$P(x, f(x)) = 0 \quad \text{for } x \in \Omega, \tag{1.2}$$

and let

$$d = \deg P.$$

We will call this number d *the degree of Nash function* f and denote it by $\deg f$. Obviously $d = \deg f > 0$. For $d = 1$, the function f has no isolated singularities, so, we will assume that $d > 1$.

Put

$$\mathcal{R}(n, d) = d(3d - 3)^n.$$

The main result of this paper is the following theorem.

Theorem 1.2. *Let* $f : \Omega \to \mathbb{R}$ *be a Nash function. Assume that in the set* $\{x \in \Omega : f(x) = 0\}$ *we have* $\nabla f(x) = 0$ *only for* $x = 0$ *and let* $d = \deg f$. *Assume that for polynomial* P *satisfying* (1.2) *holds*

$$\nabla P(x, f(x)) \neq 0 \quad \textit{for } x \in \Omega. \tag{1.3}$$

Then

$$\varrho_0(f) \le 1 - \frac{1}{\mathcal{R}(n, d)}.$$

Moreover, there exist positive constants C, ε such that

$$|\nabla f(x)| \geq C|f(x)|^{\varrho} \quad \textit{for} \quad x \in \Omega, \quad |f(x)| < \varepsilon \tag{1.4}$$

with

$$\varrho = 1 - \frac{1}{\mathcal{R}(n, d)}.$$

The assumption (1.3) holds for the generic Nash function, i.e., a Nash function satisfying (1.2) for the generic polynomial P. Then by Proposition 3.22 the assumption (1.3) is satisfied.

Theorem 1.2 is a generalization of J. Gwoździewicz (G2) and D. D'Acunto and K. Kurdyka (DK) estimates to the case of the generic Nash function, without assumptions on the set $V(f)$. It is also comparable with the Solernó estimate (S), but our estimate is explicit.

From the proof of Theorem 1.2 there follows an effective Łojasiewicz gradient inequality for a complex Nash functions with analogous formulation and the same estimation of $\varrho_0(f)$ as in Theorem 1.2.

1.3. Corollaries

Analogously as in [KS1] for polynomials, from Theorem 1.2, there follows an estimation for the Łojasiewicz exponent of Nash functions.

If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a Nash function, then there are positive constants C, η, ϵ such that the following *Łojasiewicz inequality holds*:

$$|f(x)| \geq C \operatorname{dist}(x, V(f))^{\eta} \quad \text{for} \quad |x| < \epsilon.$$

The smallest exponent η in the above inequality is called the *Łojasiewicz exponent* of f at 0 and is denoted by $\mathcal{L}_0(f)$.

Let $U \subset \mathbb{R}^n$ be a neighbourhood of Ω, and let $f : U \to \mathbb{R}$ be a Nash function such that $f(0) = 0$. Let $V = V(f)$.

Take a maximal solution (to the right) $\gamma : [0, s) \to U \setminus V$ of the equation

$$x' = -\operatorname{sign} f(x) \frac{\nabla f(x)}{|\nabla f(x)|}$$

in $U \setminus V$. By analogous argument as in [KS1, Theorem 1] we obtain

Theorem 1.3. *Let $\varrho \in [0, 1)$ and $C > 0$ be constants, such that the Łojasiewicz gradient inequality* (Ł) *holds in a neighbourhood of the origin. If $\gamma(0)$ is sufficiently close to the origin, then*

$$\operatorname{dist}(\gamma(0), V) \leq \operatorname{length} \gamma \leq \frac{1}{(1 - \varrho)C} |f(\gamma(0))|^{1-\varrho}.$$

From Theorems 1.3 and 1.2 we have (cf. [KS1, Corollary 1])

Corollary 1.4. *Under the assumptions and notations of Theorem 1.3,*

$$|f(x)| \geq (C(1-\varrho))^{1/(1-\varrho)} \operatorname{dist}(x,V)^{1/(1-\varrho)},$$

in a neighbourhood of the origin. In particular,

$$\mathcal{L}_0(f) \leq \frac{1}{1-\varrho}.$$

Moreover, if f has an isolated singularity at 0 and $d = \deg f$, and f satisfy (1.3), *then $\mathcal{L}_0(f) \leq d(3d-3)^n$.*

2. Proof of Theorem 1.2

Let Γ be a semialgebraic set defined by

$$\Gamma = \{x \in \Omega : \forall_{\zeta\in\Omega}\ f(x) = f(\zeta) \ \Rightarrow\ |\nabla f(x)| \leq |\nabla f(\zeta)|\}.$$

Let $\varepsilon > 0$ be as in Fact 1.1. Then, by definition of Γ we have

Fact 2.5. *Let $\varrho \in \mathbb{R}$ and let $C > 0$. If $|\nabla f(x)| \geq C|f(x)|^\varrho$ for $x \in \Gamma$ such that $|f(x)| < \varepsilon$, then $|\nabla f(x)| \geq C|f(x)|^\varrho$ for $x \in \Omega$, $|f(x)| < \varepsilon$.*

Let $\varrho_0 = \varrho_0(f)$ be the Łojasiewicz exponent in the gradient inequality

$$C|f(x)|^{\varrho_0} \leq |\nabla f(x)| \qquad x \in \Omega, \quad |f(x)| < \varepsilon \tag{2.1}$$

for some positive constant C. By Curve selecting lemma there exist a semialgebraic analytic curve $\varphi : [0,1) \to \Omega$ for which $f(\varphi(0)) = 0$, $f(\varphi(\xi)) \neq 0$ for $\xi \in (0,1)$ and

$$C_1|f(\varphi(\xi))|^{\varrho_0} \leq |\nabla f(\varphi(\xi))| \leq C_2|f(\varphi(\xi))|^{\varrho_0}, \quad \xi \in [0,1) \tag{2.2}$$

for some positive constants C_1, C_2.

Since 0 is unique critical point of f on $f^{-1}(0)$, then we have

Fact 2.6. $\varphi(0) = 0$.

By Fact 2.5 we may assume that

$$\varphi([0,1)) \subset \Gamma. \tag{2.3}$$

It suffices to prove that

$$|\nabla f(x)) \geq C|f(x)|^{\varrho_0} \quad \text{for} \quad x \in \Omega,\ |f(x)| < \varepsilon,$$

and more precisely

$$|\nabla f(x)) \geq C|f(x)|^{\varrho_0} \quad \text{for} \quad x \in \Gamma,\ |f(x)| < \varepsilon, \tag{2.4}$$

where C is a positive constant.

Let

$$\widetilde{\Gamma} = \{x \in \Omega : \exists_{\lambda\in\mathbb{R}}\ \nabla|\nabla f(x)|^2 - \lambda\nabla f(x) = 0\}.$$

By Lagrange multipliers theorem we see that $x_0 \in \widetilde{\Gamma}$ if and only if x_0 is a critical point of the function $x \mapsto |\nabla f(x)|^2$ on the set $f^{-1}(f(x_0)) \cap \Omega$. So, we have

Fact 2.7. $\Gamma \subset \widetilde{\Gamma}$.

Let $m_1, \dots, m_{\frac{n(n-1)}{2}}$ be the all 2×2 minors of the matrix with columns $\nabla f(x)$ and $\nabla|\nabla(f)(x)|^2$. Then

$$\widetilde{\Gamma} = \{x \in \Omega : m_1(x) = \dots = m_{\frac{n(n-1)}{2}}(x) = 0\}. \tag{2.5}$$

2.1. Critical values of norm of the gradient

Let $\mathbb{K} \subset \mathbb{C}$ be a field. By $\mathbb{K}[x]$ we denote the ring of polynomials over $\mathbb{K}$ in $x = (x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are independent variables.

Let $t = (t_1, \dots, t_n)$, $z = (z_1, \dots, z_n)$ be systems of variables and let y, u be single variables. Take a polynomial $G \in \mathbb{C}[x, y, u]$ defined by

$$G(x, y, u) = \sum_{i=1}^{n} \left(\frac{\partial P}{\partial x_i}(x, y) \right)^2 - \left(\frac{\partial P}{\partial y}(x, y) \right)^2 \cdot u.$$

Take polynomials $G_1, G_{2,i}, G_{3,i}, G_{4,i,j}, G_5 \in \mathbb{C}[x, y, u, t, z]$ defined by

$$\begin{aligned}
G_1(x, y, u, t, z) &= P(x, y)\,, \\
G_{2,i}(x, y, u, t, z) &= \frac{\partial P}{\partial x_i}(x, y) + \frac{\partial P}{\partial y}(x, y) t_i\,, && 1 \le i \le n, \\
G_{3,i}(x, y, u, t, z) &= \frac{\partial G}{\partial x_i}\,(x, y, u) + \frac{\partial G}{\partial y}\,(x, y, u)\, t_i \\
&\quad - \left(\frac{\partial P}{\partial y}\,(x, y) \right)^2 \cdot z_i\,, && 1 \le i \le n, \\
G_{4,i,j}(x, y, u, t, z) &= t_i z_j - z_i t_j\,, && 1 \le i < j \le n, \\
G_5(x, y, u, t, z) &= u - t_1^2 - \dots - t_n^2\,.
\end{aligned}$$

Since f is a C^2 class function, we have

Fact 2.8. *Take*

$$w = (x, f(x), |\nabla f(x)|^2, \nabla f(x), \nabla|\nabla f(x)|^2),$$

where $x \in \widetilde{\Gamma}$. *Then*

$$G_1(w) = 0, \tag{2.6}$$

$$G_{2,i}(w) = 0, \qquad 1 \le i \le n, \tag{2.7}$$

$$G_{3,i}(w) = 0, \qquad 1 \le i \le n, \tag{2.8}$$

$$G_{4,i,j}(w) = 0, \qquad 1 \le i < j \le n, \tag{2.9}$$

$$G_5(w) = 0, \tag{2.10}$$

and moreover,

$$G(w) = 0. \tag{2.11}$$

The equalities $G_{4,i,j}(w) = 0$ in Fact 2.8 correspond to the equations

$$m_1(x) = \cdots = m_{\frac{n(n-1)}{2}}(x) = 0$$

in (2.5).

Multiplying by $\frac{\partial P}{\partial y}$ the polynomials $G_{3,i}$, and use equations (2.7), one can take in the above the polynomials $G_{3,i}$ of the form

$$G_{3,i}(x,y,u,t,z) = \frac{\partial G}{\partial x_i}(x,y,u)\frac{\partial P}{\partial y}(x,y) - \frac{\partial G}{\partial y}(x,y,u)\frac{\partial P}{\partial x_i}(x,y) - \left(\frac{\partial P}{\partial y}(x,y)\right)^3 \cdot z_i\,,$$

as well as multiplying $G_{4,i,j}$ by $\left(\frac{\partial P}{\partial y}\right)^4$, use equations (2.7) and (2.8), one can take in the above the polynomials $G_{4,i,j}$ of the form

$$G_{4,i,j}(x,y,u,t,z) \quad = \quad \frac{\partial P}{\partial x_i}(x,y)\frac{\partial G}{\partial x_j}(x,y,u) \;-\; \frac{\partial P}{\partial x_j}(x,y)\frac{\partial G}{\partial x_i}(x,y,u). \tag{2.12}$$

Take polynomials $K_1, K_2, K_{4,i,j} \in \mathbb{C}[x,y,u]$ defined by

$$\begin{aligned} K_1(x,y,u) &= P(x,y)\,, \\ K_2(x,y,u) &= G(x,y,u)\,, \\ K_{4,i,j}(x,y,u) &= G_{4,i,j}(x,y,u,t,z)\,, \qquad\qquad 1 \le i < j \le n, \end{aligned}$$

where $G_{4,i,j}$ is of the form (2.12). In fact polynomials $G_{4,i,j}$ depends only on x, y and u.

From the above and Fact 2.8 we obtain the following fact, where the last inequality follows from the assumptions (1.3) and (2.11).

Fact 2.9. *For $x \in \widetilde{\Gamma}$, and $v = (x, f(x), |\nabla f(x)|^2)$ we have*

$$\begin{aligned} K_1(v) &= 0, \\ K_2(v) &= 0, \\ K_{4,i,j}(v) &= 0, \qquad\qquad 1 \le i < j \le n, \\ \frac{\partial P}{\partial y}(x, f(x)) &\neq 0. \end{aligned}$$

Let $\mathbb{W} \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ be an algebraic set defined by the following system of equations

$$\begin{cases} G_1(x,y,u,t,z) = 0, & \\ G_{2,i}(x,y,u,t,z) = 0, & 1 \le i \le n, \\ G_{3,i}(x,y,u,t,z) = 0, & 1 \le i \le n, \\ G_{4,i,j}(x,y,u,t,z) = 0, & 1 \le i < j \le n, \\ G_5(x,y,u,t,z) = 0, & \end{cases} \tag{2.13}$$

and let $\mathbb{V} \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ be an algebraic set defined by the following system of equations

$$\begin{cases} K_1(x,y,u)=0\,, \\ K_2(x,y,u)=0\,, \\ K_{4,i,j}(x,y,u)=0\,, 1\le i<j\le n. \end{cases} \tag{2.14}$$

Let

$$\mathbb{W}^0=\{(x,y,u,t,z)\in\mathbb{W}:\frac{\partial P}{\partial y}(x,y)\neq 0\},$$

and let

$$\mathbb{V}^0=\{(x,y,u)\in\mathbb{V}:\frac{\partial P}{\partial y}(x,y)\neq 0\}.$$

It is easy to see that

Fact 2.10. *We have either* $\mathbb{W}^0=\emptyset$ *and* $\mathbb{V}^0=\emptyset$ *or* $\dim\mathbb{W}^0\ge 1$ *and* $\dim\mathbb{V}^0\ge 1$.

Since the equations in (2.13) are of degree 1 in t and z and the coefficients of t_i and z_i are powers of $\frac{\partial P}{\partial y}$, then we have the following fact.

Fact 2.11. *The following mapping*

$$\mathbb{W}^0\ni(x,y,u,t,z)\mapsto(x,y,u)\in\mathbb{V}^0 \tag{2.15}$$

is a bijection.

Proof. Denote the mapping (2.15) by h. By the definition of $\mathbb{W}^0$ and $\mathbb{V}^0$ we see that $h(\mathbb{W}^0)\subset\mathbb{V}^0$ and the mapping is an injection. It suffices to prove that $\mathbb{V}^0\subset h(\mathbb{W}^0)$. Take any $(x_0,y_0,u_0)\in\mathbb{V}^0$. Then

$$\frac{\partial P}{\partial y}(x_0,y_0)\neq 0,$$

so, there exist uniquely determined $t_0,z_0\in\mathbb{C}^n$ such that

$$G_{2,i}(x_0,y_0,u_0,t_0,z_0)=0,\qquad 1\le i\le n, \tag{2.16}$$

$$G_{3,i}(x_0,y_0,u_0,t_0,z_0)=0,\qquad 1\le i\le n. \tag{2.17}$$

From (2.16) we have that

$$\sum_{i=1}^n\left(\frac{\partial P}{\partial x_i}(x_0,y_0)\right)^2-\left(\frac{\partial P}{\partial y}(x_0,y_0)\right)^2\sum_{i=1}^n t_{i,0}^2=0$$

and $\sum_{i=1}^n t_{i,0}^2$ is uniquely determined. Since $(x_0,y_0,u_0)\in\mathbb{V}^0$, then $G(x_0,y_0,u_0)=0$, so

$$u_0=\sum_{i=1}^n t_{i,0}^2$$

and $G_5(x_0,y_0,u_0,t_0,z_0)=0$.

We will prove that

$$t_iz_j-z_it_j=0\quad\text{for}\quad 1\le i<j\le n. \tag{2.18}$$

By the considerations after Fact 2.8 we see that the system (2.6)–(2.10) is equivalent to system (2.6), (2.7), (2.8), $K_{4,i,j}(x, y, u) = 0$, (2.10) so, the equation (2.18) holds. This gives that $w = (x_0, y_0, u_0, t_0, z_0) \in \mathbb{W}^0$ and $h(w) = (x_0, y_0, u_0)$. □

Let

$$\pi_y^0 : \mathbb{V}^0 \ni v = (x, y, u) \mapsto y \in \mathbb{C},$$

and let

$$\pi_u^0 : \mathbb{V}^0 \ni v = (x, y, u) \mapsto u \in \mathbb{C}.$$

Lemma 2.12. *For any $y_0 \in \mathbb{C}$ the function π_u^0 is constant on each connected component of $(\pi_y^0)^{-1}(y_0)$.*

Proof. Let

$$\mathbb{F} = \{y \in \mathbb{C} : \exists_{x \in \mathbb{C}^n}\ P(x, y) = 0 \ \wedge\ \frac{\partial P}{\partial y}(x, y) \neq 0 \\ \wedge \frac{\partial P}{\partial x_i}(x, y) = 0 \text{ for } i = 1, \dots, n\}.$$

At first we consider the case when $y_0 \in \mathbb{C} \setminus \mathbb{F}$. Let $\Delta_{y_0} \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ be a connected component of the set $(\pi_y^0)^{-1}(y_0)$ and let Δ^0 be the set of points $v \in \Delta_{y_0}$ at which Δ_{y_0} is smooth. Obviously Δ^0 is a dense subset of Δ_{y_0}.

By continuity of π_u^0, it suffices to prove that the function π_u^0 is constant on the set Δ^0. Take any $v_0 = (x_0, y_0, u_0) \in \Delta^0$. Then

$$\frac{\partial P}{\partial y}(x_0, y_0) \neq 0,$$

and by implicit function theorem, there exists a neighbourhood $G \subset \mathbb{C}^n \times \mathbb{C}$ of (x_0, y_0), a neighbourhood $U \subset \mathbb{C}^n$ of x_0, and a Nash function $g : U \to \mathbb{C}$ such that $P(x, y) = 0$ for $(x, y) \in G$ if and only if $y = g(x)$, $x \in U$. Moreover, we may assume that

$$\frac{\partial P}{\partial y}(x, y) \neq 0 \quad \text{for} \quad (x, y) \in G.$$

Since $y_0 \in \mathbb{C} \setminus \mathbb{F}$, then

$$\frac{\partial P}{\partial x_i}(x_0, y_0) \neq 0 \quad \text{for some} \quad i \in \{1, \dots, n\},$$

so, we may assume that

$$\frac{\partial P}{\partial x_i}(x, y) \neq 0 \quad \text{for} \quad (x, y) \in G,$$

and so,

$$\frac{\partial g}{\partial x_i}(x) \neq 0 \quad \text{for } x \in U.$$

Take a function $\lambda : U \to \mathbb{C}$ defined by

$$\lambda(x) = \frac{-1}{\frac{\partial g}{\partial x_i}(x)} \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \frac{\partial g}{\partial x_j}(x).$$

Obviously λ is a Nash function.

Put

$$\begin{aligned} u(x) &= \left(\frac{\partial g}{\partial x_1}(x)\right)^2 + \cdots + \left(\frac{\partial g}{\partial x_n}(x)\right)^2, \\ A(x) &= \left[\frac{\partial^2 g}{\partial x_i \partial x_j}(x)\right]_{1 \le i,j \le n}. \end{aligned} \tag{2.19}$$

Since

$$A(x)\nabla g(x)^T = \frac{1}{2}\nabla u(x)^T,$$

then by definition of λ we have

$$A(x)\nabla g(x)^T + \lambda(x)\nabla g(x)^T = 0 \quad \text{for } x \in U,$$

if and only if $u(x)$ is a critical value of u on the set $g^{-1}(g(x))$, where $\nabla g(x)^T$ is the transposition of the vector $\nabla g(x)$. Obviously for any point $(x, y_0, u) \in \Delta^0$ we have $g(x) = y_0$ and by Fact 2.11,

$$w(x) = (x, g(x), u(x), \nabla g(x), \nabla u(x)) \in \mathbb{W}^0.$$

Then $G_{4,i,j}(w(x)) = 0$ for any $1 \le i < j \le n$, and so, the 2×2 minors of the matrix with columns $\nabla u(x)$ and $\nabla g(x)$ are equal to zero.

Since v_0 is a smooth point of Δ_{y_0}, then diminishing neighbourhood U if necessary, there exists a neighbourhood $D \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ of v_0 such that

$$\{(x, g(x), u(x)) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : x \in X\} = \Delta_{y_0} \cap D$$

for a connected semialgebraic set $X := g^{-1}(y_0) \cap U \subset \mathbb{C}^n$. Moreover,

$$v_0 = (x_0, g(x_0), u(x_0)).$$

and v_0 is a smooth point of Δ_{y_0}, then by Lagrange multipliers theorem, $u(x)$ is a critical value of the function u on the set $g^{-1}(y_0)$. This gives that $u(x)$ is a critical value of the considered function $X \cap U \ni x \mapsto u(x) \in \mathbb{C}$. Since the set of critical values of a Nash function is a discrete set and h is a continuous function on a connected set, then π_u^0 is a constant function. This gives the assertion in the considered case.

Assume now that $y_0 \in \mathbb{F}$. For a connected component X of $(\pi_y^0)^{-1}(y_0)$ such that for any $(x, y_0, u) \in X$ we have

$$\frac{\partial P}{\partial y}(x, y_0) \neq 0 \quad \text{and} \quad \frac{\partial P}{\partial x_i}(x, y_0) \neq 0 \quad \text{for some} \quad i = 1, \ldots, n,$$

the argument is analogous as above.

Let X be a connected component of $(\pi_y^0)^{-1}(y_0)$ such that for some $(x_0, y_0, u) \in X$ we have

$$\frac{\partial P}{\partial y}(x_0, y_0) \neq 0 \quad \text{and} \quad \frac{\partial P}{\partial x_i}(x_0, y_0) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Then for any point $(x_0, y_0, u) \in \mathbb{V}^0$ such that

$$\frac{\partial P}{\partial y}(x_0, y_0) \neq 0 \quad \text{and} \quad \frac{\partial P}{\partial x_i}(x_0, y_0) = 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{we have} \quad u = 0.$$

Consider a connected component X^0 of $\{(x, y_0, u) \in X : u \neq 0\}$. Then, by analogous argument as at the begining the function π_u^0 is constant on X^0. Since this function is continuous and have a zero at the border of X^0, then it vanishes on X^0. This gives that $X^0 = \emptyset$ and π_u^0 vanishes on X. □

Remark 2.13. *We are looking for a nonzero polynomial $Q \in \mathbb{C}[y, u]$ such that $Q(f(x), |\nabla f(x)|^2) = 0$ for $x \in \Gamma$ such that $|f(x)| < \varepsilon$ for some $\varepsilon > 0$. For this purpose, we are looking for an algebraic set $V \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ such that*

$$\{(\varphi(t), f(\varphi(t)), |\nabla f(\varphi(t))|^2) : t \in [0, 1)\} \subset V,$$

for the semialgebraic curve $\varphi : [0, 1) \to \Gamma$ satisfying (2.2) *and for the generic $y_0 \in \mathbb{C}$ the function*

$$h_{y_0} : \{(x, y, u) \in V : y = y_0\} \ni (x, y, u) \mapsto u \in \mathbb{C}$$

have a finite number of values. Then the desired polynomial Q one can take from the ideal of V.

By Fact 2.9 *we have that*

$$\{(\varphi(t), f(\varphi(t)), |\nabla f(\varphi(t))|^2) : t \in [0, 1)\} \subset \{(x, f(x), |\nabla f(x)|^2) : x \in \Gamma\} \subset \mathbb{V},$$

and

$$\{(\varphi(t), f(\varphi(t)), |\nabla f(\varphi(t))|^2) : t \in [0, 1)\} \subset \overline{\mathbb{V}^0}.$$

The above inclusion does not have to occur without assumption (1.3).

Let

$$\pi_u : \overline{\mathbb{V}^0} \ni (x, y, u) \mapsto u \in \mathbb{C},$$
$$\pi_y : \overline{\mathbb{V}^0} \ni (x, y, u) \mapsto y \in \mathbb{C}.$$

Lemma 2.14. *For the generic $y_0 \in \mathbb{C}$ the function π_u is constant on each connected component of $\pi_y^{-1}(y_0)$.*

Proof. Let $B(\pi_y)$ be the set of bifurcation values of π_y, i.e., π_y is a local C^0 trivial fibration over $\mathbb{C} \setminus B(\pi_y)$. It is known that the set of bifurcation values of π_y is contained in a proper algebraic subset of $\mathbb{C}$ (see [V, Corollary 5.1]), so, $B(\pi_y)$ is a finite set.

Put

$$E := \overline{\left\{x \in \mathbb{C}^n : P(x, y) = 0 \wedge \frac{\partial P}{\partial y}(x, y) = 0 \text{ for some } y \in \mathbb{C}\right\}}.$$

From the choice of polynomial P we have the following claim.

Claim 1. *The set E is a proper algebraic subsets of $\mathbb{C}^n$.*

Assume to the contrary, that $E = \mathbb{C}^n$. Then P and $\frac{\partial P}{\partial y}$ have common factor over $\mathbb{C}$ of positive degree, and consequently there exists a nonzero complex polynomial $Q \in \mathbb{C}[x, y]$ such that $0 < \deg Q < \deg P$ and $Q(x, f(x)) = 0$ for $x \in \Omega$. Let

$$Q = \sum_{i_1,\dots,i_n,j} a_{i_1,\dots,i_n,j} x_1^{i_1} \cdots x_n^{i_n} y^j,$$

and take

$$R = \sum_{i_1,\dots,i_n,j} \mathrm{Re}(a_{i_1,\dots,i_n,j}) x_1^{i_1} \cdots x_n^{i_n} y^j,$$

$$I = \sum_{i_1,\dots,i_n,j} \mathrm{Im}(a_{i_1,\dots,i_n,j}) x_1^{i_1} \cdots x_n^{i_n} y^j.$$

Then $R \neq 0$, $\deg R < \deg P$ and $R(x, f(x)) = 0$ or $I \neq 0$, $\deg I < \deg P$ and $I(x, f(x)) = 0$ for $x \in \Omega$. This contradicts the choice of polynomial P and ends the proof of Claim 1.

Let $F \subset \mathbb{C}$ be the set of numbers $y \in \mathbb{C}$ such that

$$P(x, y) = 0 \wedge \frac{\partial P}{\partial x_i}(x, y) = 0 \text{ for } i = 1, \dots, n \text{ and some } x \in \mathbb{C}^n \setminus E.$$

Claim 2. *The set F is finite.*

Take $x \in \mathbb{C}^n \setminus E$ and $y \in \mathbb{C}$ such that

$$P(x, y) = 0, \quad \text{and} \quad \frac{\partial P}{\partial x_i}(x, y) = 0 \quad \text{for} \quad i = 1, \dots, n.$$

Then y is a critical value of some Nash function g defined on a neighbourhood $U \subset \mathbb{C}^n$ of x satisfying $g(x) = y$ and $P(\zeta, g(\zeta)) = 0$ for $\zeta \in U$. Hence we deduce the assertion of Claim 2.

Let $y_0 \in \mathbb{C} \setminus (F \cup B(\pi_y))$, and let

$$H_{y_0} := \pi_y^{-1}(y_0).$$

Claim 3. *The function*

$$h : H_{y_0} \ni (x, y_0, u) \mapsto u \in \mathbb{C}$$

is constant on each irreducible component of H_{y_0}.

Take any irreducible component $\Gamma_0 \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ of H_{y_0} and let

$$\Gamma' = \{(x, y, u) \in \Gamma_0 : x \notin E\}.$$

If $\Gamma' \neq \emptyset$, then $\Gamma_0 = \overline{\Gamma'}$ and by Lemma 2.12, the function h is constant on Γ_0 as continuous and constant on Γ'. This ends the proof in this case.

Assume that $\Gamma' = \emptyset$. Take any $p_0 = (x_0, y_0, u_0) \in \Gamma_0$. Then $x_0 \in E$ and there exists a sequence $p_\nu = (x_\nu, y_\nu, u_\nu) \in \mathbb{V}^0$ such that $p_\nu \to p_0$ as $\nu \to \infty$. Sinse π_y is a C^0 trivial fibration over y_0, there exists a neighbourhood $U \subset \mathbb{C}$ of y_0 such that

$\pi_y|_{\pi_y^{-1}(U)} : \pi_y^{-1}(U) \to U$ is a C^0 trivial fibration. We may assume that $p_\nu \in \pi_y^{-1}(U)$ for $\nu \in \mathbb{N}$. Let Γ_ν be a connected component of $\pi_y^{-1}(y_\nu)$ such that $p_\nu \in \Gamma_\nu$ for $\nu \in \mathbb{N}$. By Lemma 2.12, we have that h is constant on Γ_ν, equal to u_ν, hence, by triviality of $\pi_y|_{\pi_y^{-1}(U)}$, the function h is equal to u_0 on Γ_0. This gives the assertion of Claim 3.

Since $h = \pi_u|_{H_{y_0}}$, Claim 3 gives the assertion of Lemma 2.14. □

Fact 2.15. *We have $\delta(\overline{\mathbb{V}^0}) \le d(3d-3)^n$, where $d = \deg P$ and $\delta(V)$ is the generalized degree of an algebraic set V (see Appendix).*

Proof. The assertion immediately follows from definition of $\mathbb{V}$ and Lemma 3.20, in the Appendix. □

From Lemma 2.14 we see that the set

$$\overline{\{(y,u) \in \mathbb{C}^2 : (x,y,u) \in \overline{\mathbb{V}^0} \ \text{ for some } x \in \mathbb{C}^n\}}$$

is contained in some proper algebraic subset of $\mathbb{C}^2$, so, by Fact 2.15 we obtain

Corollary 2.16. *There exists a nonzero polynomial $Q \in \mathbb{C}[y,u]$ such that*

$$\deg Q \le d(3d-3)^n$$

and

$$Q(y,u) = 0 \quad \textit{for } (x,y,u) \in \overline{\mathbb{V}^0}.$$

Proof of Theorem 1.2. From the assumption (1.3) we have

$$\{(x, f(x), |\nabla f(x)|^2) : x \in \Gamma\} \subset \mathbb{V}^0.$$

Then, by Corollary 2.16 there exists a nonzero polynomial $Q \in \mathbb{C}[y,u]$ such that

$$\deg Q \le d(3d-3)^n$$

and

$$Q(f(\varphi(\xi)), |\nabla f(\varphi(\xi))|^2) = 0 \quad \text{for } \xi \in [0,1).$$

Let $\operatorname{ord}_0(f\circ\varphi) = M$ and $\operatorname{ord}_0 |\nabla f \circ \varphi|^2 = K$. Therefore, $\operatorname{ord}_0 \nabla f \circ \varphi \le M-1$, so, $K \le 2M-2$, and so,

$$(f\circ\varphi)^{\frac{K}{2M}} \sim |\nabla f \circ \varphi|, \ \text{i.e.,} \quad \varrho_0 = \frac{K}{2M}.$$

Hence, there exist a pair of different monomials $\alpha u^N y^S$ and $\beta u^{N_1} y^{S_1}$ of polynomial Q such that $N+S \le D$, $N_1 + S_1 \le D$, where $D = d(3d-3)^n$, and

$$N\frac{K}{2} + SM = N_1\frac{K}{2} + S_1 M.$$

Hence

$$\frac{K}{2M} = \frac{S_1 - S}{N - N_1}.$$

Since $K \leq 2M-2$, then $\frac{K}{2M} < 1$. On the other hand, $|S_1 - S|, |N - N_1| \in \{0, 1, \ldots, D\}$, so

$$\varrho_0 = \frac{K}{2M} \leq \frac{D-1}{D},$$

since $\frac{D-1}{D}$ is the maximal possible rational number less than 1 with the numerator and the denominator from the set $\{0, 1, \ldots, D\}$.

As a consequance we have

$$|f(x)|^{\frac{D-1}{D}} \leq C|\nabla f(x)| \quad \text{for } x \in \Omega, \quad |f(x)| < \varepsilon.$$

This gives the assertion of Theorem 1.2. $\square$

Remark 2.17. In fact we have proved a little bit more general theorem than Theorem 1.2. Namely, we have proved that:

Let $f : \Omega \to \mathbb{R}$ be a Nash function. Assume that in the set $\{x \in \Omega : f(x) = 0\}$ we have $\nabla f(x) = 0$ only for $x = 0$ and let $d = \deg f$. Assume that for polynomial P satisfying (1.2)*, the origin is an accumulation point of the set of $x \in \Omega$ such that $\nabla P(x, f(x)) \neq 0$ and*

$$\frac{\partial P}{\partial x_i}(x, f(x))\frac{\partial G}{\partial x_j}(x, f(x), |\nabla f(x)|^2) - \frac{\partial P}{\partial x_j}(x, f(x))\frac{\partial G}{\partial x_i}(x, f(x), |\nabla f(x)|^2) = 0$$

for any $1 \leq i < j \leq n$. Then

$$\varrho_0(f) \leq 1 - \frac{1}{\mathcal{R}(n, d)}.$$

3. Appendix

3.1. Total degree of algebraic sets

Let $f = (f_1, \ldots, f_r) : \mathbb{C}^N \to \mathbb{C}^r$ be a polynomial mapping with $\deg f_i > 0$ for $i = 1, \ldots, r$. Let $V = V(f) \subset \mathbb{C}^N$.

By the total degree of V we mean the number

$$\delta(V) = \deg V_1 + \ldots + \deg V_s,$$

where $V = V_1 \cup \ldots \cup V_s$ is the decomposition of V into irreducible components (see [Ł3]).

We have the following two well-known facts (cf. [Ł3]).

Fact 3.18. $\delta(V) \leq \deg f_1 \cdots \deg f_r$*. In particular, for any irreducible component V_j of V we have*

$$\deg V_j \leq \deg f_1 \cdots \deg f_r.$$

Fact 3.19. *Let $L : \mathbb{C}^N \to \mathbb{C}^k$ be a linear mapping. Then*

$$\delta(\overline{L(V)}) \leq \delta(V).$$

We will need the following lemma (cf. [Sp2]).

Lemma 3.20. *Let V_j be an irreducible component of the set V, and let $\dim V_j \geq 1$. Then for the generic linear mapping $L = (L_1, \ldots, L_{N-1}) : \mathbb{C}^r \to \mathbb{C}^{N-1}$ the set V_j is an irreducible component of the set of common zeros of the following system of equations:*

$$L_i \circ f = 0, \quad i = 1, \ldots, N-1.$$

In particular

$$\deg V_j \leq \deg(L_1 \circ f) \cdots \deg(L_{N-1} \circ f).$$

Moreover (recall that $\deg f_1 > 0$), we can take $L_1(y_1, \ldots, y_r) = y_1$.

Proof. Let $Y = \overline{f(\mathbb{C}^N)} \subset \mathbb{C}^r$, and let $Z \subset \mathbb{C}^N \times \mathbb{C}^r$ be the graph of f. Then $\dim Z = N$. Put $k = \dim Y$. Obviously $k \leq N$.

If $r \leq N-1$, then the assertion is obvious. Assume that $r \geq N$.

For any linear mapping $L : \mathbb{C}^r \to \mathbb{C}^{N-1}$ we have

$$Z \cap \{(w, y) \in \mathbb{C}^N \times \mathbb{C}^r : L(y) = 0\} = [V \times \{0\}] \cup E_L, \tag{3.1}$$

where $E_L = \overline{\{(w, y) \in Z : y \neq 0,\, L(y) = 0\}}$.

Observe that for the generic linear mapping $L : \mathbb{C}^r \to \mathbb{C}^{N-1}$, the set E_L is empty or it is an algebraic set of dimension 1.

Indead, there exists an algebraic subset $Y_1 \subset Y$, $\dim Y_1 \leq k-1$, such that for any $y \in Y \setminus Y_1$, the fibre $f^{-1}(y)$ is an algebraic set of pure dimension $N-k$.

If $k < N-1$, then obviously $E_L = \emptyset$ for the generic linear mapping $L : \mathbb{C}^r \to \mathbb{C}^{N-1}$. So, it suffices to consider the case when $N-1 \leq k \leq N$.

If $k = N-1$, then $\dim Y_1 = N-2$ and for the generic linear mapping $L : \mathbb{C}^r \to \mathbb{C}^{N-1}$ we have that $L^{-1}(0) \cap Y_1 = \emptyset$ and $L^{-1}(0) \cap Y$ is a finite set. Then the set E_L is of pure dimension 1.

Assume that $k = N$. Then $\dim Y_1 \leq N-1$, and $\dim f^{-1}(Y_1) \leq N-1$. So, there exists an algebraic set $Y_2 \subset Y_1$ of dimension $\dim Y_2 \leq N-2$ such that $f^{-1}(y)$ is a finite set for $y \in Y_1 \setminus Y_2$. So, for any $y \in Y \setminus Y_2$, the fibre $f^{-1}(y)$ is a finite set. Hence, for the generic linear mapping $L : \mathbb{C}^r \to \mathbb{C}^{N-1}$ we have $L^{-1}(0) \cap Y_2 \subset \{0\}$, and $L^{-1}(0) \cap Y$ is an algebraic set of pure dimension 1. Moreover for any $y \in [L^{-1}(0) \cap Y] \setminus \{0\}$, the fibre $f^{-1}(y)$ is finite. This gives that E_L has dimension 1 and prove the above observation.

By (3.1) and the above observation, we see that the set

$$\overline{\{(w, y) \in Z : y \neq 0,\, L(y) = 0\}} \cap [\mathbb{C}^N \times \{0\}]$$

is finite, and consequently,

$$V_1 \not\subset \overline{\{(w,y) \in Z : y \neq 0,\, L(y) = 0\}}.$$

This gives the first part of the assertion. The particular part immediately follows from the first one and Fact 3.18. The moreover part we prove analogously as the first one, for the mapping $\tilde{f} = (f_2, \dots, f_n) : f_1^{-1}(0) \to \mathbb{C}^{r-1}$. This ends the proof. □

3.2. Singular point of generic hypersurface

Recall that $d = \deg P$.

Let $\mathcal{P} \in \mathbb{C}[a, x, y]$ denote a polynomial

$$\mathcal{P}(a, x, y) = \sum_{|\alpha|+\beta \leq d} a_{\alpha,\beta} x^\alpha y^\beta$$

with arbitrary coefficients

$$a = (a_{\alpha,\beta})_{|\alpha|+\beta\leq d} \in \mathbb{C}^{N_2}, \quad \text{where} \quad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

and

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{for} \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n.$$

Let

$$N_2 = \#\{(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N} :\ |\alpha| + \beta \leq d\},$$

and let $P_a(x, y) = \mathcal{P}(a, x, y)$ for $a \in \mathbb{C}^{N_2}$.

Fact 3.21. *$P(x, y) = \mathcal{P}_{a_0}(x, y)$ for some $a_0 \in \mathbb{C}^{N_2}$.*

Proposition 3.22. *For the generic $a \in \mathbb{C}^{N_2}$ the system*

$$\begin{cases} P_a(x, y) = 0, \\ \nabla P_a(x, y) = 0 \end{cases}$$

has no solutions.

Proof. Observe that for the generic $a \in \mathbb{C}^{N_2}$ the gradient ∇P_a has finitely many zeros.

Let $b_0 \in \mathbb{C}^{N_2}$ be a point representing the polynomial $R(x, y) = x_1^d + \ldots + x_n^d + y^d$, i.e. $R = P_{b_0}$. Then the mapping $\nabla R(x, y) = (dx_1^{d-1}, \dots, dx_n^{d-1}, dy^{d-1})$ is dominating. There exists a neighbourhood $U \subset \mathbb{C}^{N_2}$ of b_0 such that ∇P_a has no zeros at infinity for $a \in U$. Therefore, ∇P_a is a proper mapping for $a \in U$. Hence, ∇P_a has only finite number of zeros.

Let $A = \{(a, x, y) \in \mathbb{C}^{N_2} \times \mathbb{C}^n \times \mathbb{C} :\ \nabla P_a(x, y) = 0\}$, and let

$$\pi : A \ni (a, x, y) \mapsto a \in \mathbb{C}^{N_2}.$$

From the above observation $\pi(A)$ has non-empty interior. Therefore π is a dominating map. Moreover, there exists an open set $U \subset \pi(A)$ such that for every $a \in U$ we have $\#\pi^{-1}(a) < \infty$. By [M, Corollary 3.15] there exists a proper algebraic subset

$D \subset \mathbb{C}^{N_2}$ such that for every $a \in \mathbb{C}^{N_2} \setminus D$, $\dim \pi^{-1}(a)$ is constant. Hence we have $\dim \pi^{-1}(a) = 0$, which proves the observation given at the begining of the proof.

Let $A' = \overline{A \setminus \pi^{-1}(D)}$. From the above we see that

$$B = \{(a,x,y) \in A' : \ P_a(x,y) = 0\}$$

is a subset of A with dimension less than N_2. Then $\overline{\pi(B)}$ is a proper algebraic subset of C^{N_2}. This gives the assertion. □

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EFEKTYWNA NIERÓWNOŚĆ ŁOJASIEWICZA Z GRADIENTEM DLA GENERYCZNEJ FUNKCJI NASHA Z IZOLOWANĄ OSOBLIWOŚCIĄ

S t r e s z c z e n i e

Niech Ω będzie kulą domkniętą w $\mathbb{R}^n$ o środku w początku układu współrzędnych i niech $f : \Omega \to \mathbb{R}$ będzie funkcją Nasha. Wówczas istnieje nierozkładalny wielomian $P \in \mathbb{R}[x, y]$ zmiennych $x = (x_1, \dots, x_n)$ i y taki, że $P(x, f(x)) = 0$. Podajemy oszacowanie typu D'Acunto-Kurdyki wykładnika $\varrho \in [0, 1)$ w nierówności Łojasiewicza z gradientem: $|\nabla f(x)| \geq C|f(x)|^{\varrho}$ dla $x \in \Omega$, $|f(x)| < \varepsilon$, gdzie $C, \varepsilon > 0$ są pewnymi stałymi, w terminach stopnia P, przy założeniu, że $f(0) = 0$, f ma osobliwość izolowaną w zerze i $\nabla P(x, f(x)) \neq 0$ dla $x \in \Omega$.

Słowa kluczowe: funkcja semialgebraiczna, funkcja Nasha, nierówność Łojasiewicza z gradientem, wykładnik Łojasiewicza

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Rafał Zduńczyk

SIMPLE SYSTEMS AND CLOSURE OPERATORS

Summary
The connections between generalized derived-set operators and abstracted Thomson's local systems are explored.

Keywords and phrases: simple system, generalized topology, closure, derived set

Let X be a nonempty set. For an $A \subset X$ by $\mathcal{P}(A)$ we denote the power set of A, i.e. the collection of all subsets of A. A class of collections of subsets of X, $\mathcal{S} = \{S(x)\}_{x \in X}$, will be called a *simple system* [7] provided that for any $x \in X$ the following conditions hold.

S1.1. $\{x\} \notin S(x)$, *S1.2.* $S(x) \neq \varnothing$,

S2. if $S \in S(x)$, then $x \in S$,

S3. if $S_1 \in S(x)$ and $S_1 \subset S_2$, then $S_2 \in S(x)$.

By replacing *S2* in the above definition with

S2'. if $S \in S(x)$, then $S \setminus \{x\} \in S(x)$

one obtains a *vicinity system* (cf. [7]). There is an obvious one-to-one correspondence between simple and vicinity systems. Indeed, any simple system $\mathcal{S}$ can be transformed into its vicinity counterpart $\bar{\mathcal{S}} := \{\bar{S}(x)\}_{x \in X}$ by

$$S(x) \mapsto \bar{S}(x) := \{S : S \cup \{x\} \in S(x)\}$$

and any vicinity system $\mathcal{S}$ has its simple friend in $\hat{\mathcal{S}} := \{\hat{S}(x)\}_{x \in X}$, where

$$\hat{S}(x) := \{S \cup \{x\} : S \in S(x)\}.$$

Certainly, $\hat{\bar{\mathcal{S}}} = \mathcal{S}$ and $\bar{\hat{\mathcal{S}}} = \mathcal{S}$.

A simple or a vicinity system of sets will be called *local* if there is a topology $\mathcal{T} \subset \mathcal{P}(X)$ such that for any $x \in X$

S4. if $x \in U \in \mathcal{T}$ and $S \in S(x)$, then $S \cap U \in S(x)$.

Sometimes to emphasize the role of a (not necessarily unique) $\mathcal{T}$ we will say that $\mathcal{S}$ is local in $\mathcal{T}$.

Property 1. *From S1 and S4 it follows immediately that if one wishes to have any local system in $(X, \mathcal{T})$, X has to be $\mathcal{T}$-dense-in-itself.*

Property 2. *If $\mathcal{S}$ is local in τ_1 and $\tau_1 \supset \tau_2$, then $\mathcal{S}$ is local in τ_2.*

A system (vicinity or simple, not necessarily local) will be called a *filtering* one provided that

S5. if S_1, $S_2 \in S(x)$, then $S_1 \cap S_2 \in S(x)$.

We define

$$\hat{S}(x) := \begin{cases} S(x), & \text{when} \quad \{S(x)\}_{x \in X} \quad \text{is a simple system,} \\ S(x) \cap \{S \subset X : x \in S\}, & \text{when} \quad \{S(x)\}_{x \in X} \quad \text{is a vicinity system.} \end{cases}$$

One can easily check that

$$\gamma_{\mathcal{S}} := \Big\{G \subset X : \forall (x \in G)\ \exists (U \in \hat{S}(x)) \quad U \subset G\Big\} \tag{1}$$

is a topology on X and that $\mathcal{S}$ is local in $\gamma_{\mathcal{S}}$. Hence we have

Theorem 1. (cf. [7], Property 2). *Every simple filtering system and every vicinity filtering system is local.*

Example 1. Let $\mathcal{S}_0^{\mathcal{E}} := \{S_0^{\mathcal{E}}(x)\}_{x \in \mathbb{R}}$ be a Euclidean neighbourhood system in $\mathbb{R}$, i.e.

$$S_0^{\mathcal{E}}(x) := \big\{S \subset \mathbb{R} : x \in \big(a_x^S; b_x^S\big) \subset S,\ \text{for some } a_x^S, b_x^S \in \mathbb{R}\big\}$$

and $\mathcal{S}_0^{\mathcal{L}} := \{S_0^{\mathcal{L}}(x)\}_{x \in \mathbb{R}}$ be a system of Lebesgue density topology nieghbourhoods in $\mathbb{R}$, i.e.

$$S_0^{\mathcal{L}}(x) := \{S \subset \mathbb{R} : x \in S \cap \Phi_{\mathcal{L}}(S)\},$$

where $\Phi_{\mathcal{L}}(S)$ is the set of the Lebesgue density points of any Lebegue kernel of S. Both the above systems are local in $\mathbb{R}$ endowed with Euclidean topology, but $\mathcal{S}_0^{\mathcal{E}}$ is not local in Lebesgue density topology. Indeed, $U := \mathbb{R} \setminus \mathbb{Q}$ is a Lebesgue density neighbourhood of any $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{R} \in S_0^{\mathcal{E}}(x)$, while $\mathbb{R} \cap U \notin S_0^{\mathcal{E}}(x)$. $\diamond$

Any $d : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the conditions

D1. $d\,\varnothing = \varnothing$,

D2. d is surjective,

D3. $d(A \cup B) = d\,A \cup d\,B$,

will be called a *generalized derivative operator on* X and $d\,A$ a *generalized derived set of* A.

Lemma 1. ([2], Lemma 10). *For any* $d : \mathcal{P}(X) \to \mathcal{P}(X)$ *the following conditions are equivalent:*

1. $A \subset B$ *implies that* $d\,A \subset d\,B$, *for* $A, B \subset X$,
2. $d\,A \cup d\,B \subset d(A \cup B)$, *for* $A, B \subset X$,
3. $d(A \cap B) \subset d\,A \cap d\,B$, *for* $A, B \subset X$.

Lemma 2. *If* $d : \mathcal{P}(X) \to \mathcal{P}(X)$ *is such that*

$$A \subset B \quad \text{implies that} \quad d\,A \subset d\,B, \qquad \text{for } A, B \subset X,$$

then $d\,[\mathcal{P}(A)] = d\,A$, *for any* $A \subset X$.

Proof. We have $d\,[\mathcal{P}(A)] = \bigcup_{B \subset A} d\,B \subset \bigcup_{B \subset A} d\,A = d\,A \subset d\,[\mathcal{P}(A)]$. □

It follows that instead of *D2* we may equally well write

D2′. $d\,X = X$,

with no harm to definition of a generalized derivative.

Example 2. To give an evidence of existence of generalized derivatives in the mathematical world, consider w-derived set (cf. [1]) defined in a topological space $(X, \mathcal{T})$, as

$$X \supset A \mapsto A'_w := \{x \in X : (A \setminus \{x\}) \cap \mathcal{T}\text{-cl}U \neq \varnothing, \text{ when } x \in U \in \mathcal{T}\}.$$

It is an easy check that w-derived set operator satisfies *D1–D3*. ◇

Define for $d : 2^X \to 2^X$,

$$S'_d(x) := \{S \subset X : x \in d(S \setminus \{x\})\}, \qquad S''_d(x) := \{S \subset X : x \in S \cap d(S \setminus \{x\})\}$$

Note the two facts which follow straightforward from these definitions.

Fact 1. *For a generalized derivative operator* $d : \mathcal{P}(X) \to \mathcal{P}(X)$, $\mathcal{S}'_d := \{S'_d(x)\}_{x \in X}$ *and* $\mathcal{S}''_d := \{S''_d(x)\}_{x \in X}$ *are a vicinity system and a simple system, respectively.*

Fact 2. $\widehat{\mathcal{S}'_d} = \mathcal{S}''_d$ *and* $\overline{\mathcal{S}''_d} = \mathcal{S}'_d$.

Theorem 2. *If* d *is a generalized derivative operator on* X, $\mathcal{T} \subset \mathcal{P}(X)$ *a topology and* $d\,A \subset \mathcal{T}$*-cl*A *for* $A \subset X$, *then* $\mathcal{S}'_d$ *and* $\mathcal{S}''_d$ *are both local (in* $\mathcal{T}$*).*

Proof. (Compare [5], proof of (10.1) Lemma, p. 20) There is no need to distinguish cases of simple and vicinity systems, hence (here and everywhere the same applies) we use the notation $\mathcal{S}_d = \{S_d(x)\}_{x\in X}$ to embrace the both. Let $S \in S_d(x)$ and $x \in U \in \mathcal{T}$. Put

$$S_1 := S \setminus \{x\} \qquad \text{and} \qquad S_2 := (S \cap U) \setminus \{x\} = S_1 \cap U \subset S_1.$$

We have

$$x \in d\,S_1 = d(S_2 \cup (S_1 \setminus S_2)) = d\,S_2 \cup d(S_1 \setminus S_2) = d\,S_2 \cup d(S_1 \setminus U),$$

which completes the proof since $d(S_1 \setminus U) \subset \mathrm{cl}(S_1 \setminus U)$ and $x \in \mathrm{Int}\, U$. □

A generalized derivative fulfilling

D4. $\quad A \subset d\,A$

is called a *generalized closure*. To each generalized derivative there corresponds a generalized closure c_d defined for $A \subset X$ as

$$c_d A := A \cup d\,A$$

and conversely to a generalized closure c there corresponds a generalized derivative d_c defined for $A \subset X$ as

$$d_c A := \{x : x \in c(A \setminus \{x\})\}.$$

However in general neither $c_{d_c} = c$ nor $d_{c_d} = d$ must hold, the inclusions

$$c_{d_c} A \subset c\,A, \qquad d_{c_d} A \subset d\,A$$

are always satisfied.

Fact 3. (cf. [3], pp. 33, 38–39). *If $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a generalized closure additionally fulfilling*

D4′. $\quad c\{x\} = \{x\}, \qquad$ *for* $\quad x \in X,$

then $c_{d_c} = c$.

Indeed, for $x \in A$ we have $cA = c(A \setminus \{x\}) \cup c\{x\} = c(A \setminus \{x\}) \cup \{x\} = c(A \setminus \{x\})$. Hence $cA \subset A \cup \{x : x \in c(A \setminus \{x\})\} = c_{d_c} A$.

Fact 4. (cf. [3], pp. 33, 38–39). *If $d : 2^X \to 2^X$ is a generalized derivative additionally fulfilling*

D4" $\quad d\{x\} = \varnothing, \qquad$ *for* $\quad x \in X,$

then $d_{c_d} = d$.

Indeed, we have $d\,A = d(A \setminus \{x\})$. Therefore $d\,A = \{x : x \in d(A \setminus \{x\})\} = d_{c_d} A$.

Note 1. Assumptions $D4'$ and $D4''$ from the topological point of view are equivalent to T1-separation axiom, which is required when one wishes to introduce a topology by derivative operator, as without $D4''$ the derived set might not be closed in general.

For a generalized derivative or a generalized closure such that

D5. $dd\,A \subset d\,A$, for any $A \subset X$,

the word 'generalized' will be omitted. Such are fully-fledged derivative and closure operators which unambiguously determine their topologies, if only *D4′* or respectively *D4″* is assumed.

For a vicinity sytem $\mathcal{S} = \{S(x)\}_{x\in X}$ put

$$S^*(x) := \{S^* \subset X : X \setminus S^* \notin S(x)\}, \qquad \text{for } x \in X \tag{2}$$

and $\mathcal{S}^* := \{S^*(x)\}_{x\in X}$. System $\mathcal{S}^*$ is called *dual* of $\mathcal{S}$.

Property 3. (cf. [4], 1.21 p. 285, [5], 4.2 p. 8). *If $\mathcal{S}$ is a vicinity system, then so is $\mathcal{S}^*$. Moreover, $\mathcal{S}^{**} = \mathcal{S}$ and*

$$S^*(x) = \{S^* \subset X : \forall\ (S \in S(x))\ \ (S \cap S^*) \setminus \{x\} \neq \varnothing\}\,. \tag{3}$$

Proof. First we prove that $\mathcal{S}^*$ is a vicinity system.

S1.1.* By *S1.2* and *S3* we have $X \in S(x)$. From this and from *S2′* it follows that $X \setminus \{x\} \in S(x)$, which is equivalent to $\{x\} \notin S^*(x)$.

S1.2.* We have $X \setminus X = \varnothing \notin S(x)$. Thus, $X \in S^*(x)$.

S2′.* Let $S^* \setminus \{x\} \notin S^*(x)$, i.e. $(X \setminus S^*) \cup \{x\} = X \setminus (S^* \setminus \{x\}) \in S(x)$. Hence by *S2′*, $(X \setminus S^*) \setminus \{x\} = ((X \setminus S^*) \cup \{x\}) \setminus \{x\} \in S(x)$. Therefore, by *D3*, $X \setminus S^* \in S(x)$. Thus $S^* \notin S^*(x)$.

S3.* This follows from *S3* by complementation directly.

The relation $\mathcal{S}^{**} = \mathcal{S}$ is obvious.

In order to prove that (3) defines the same notion as (2) does, suppose first that $(S \cap S^*) \setminus \{x\} = \varnothing$ for some $S \in S(x)$. Hence $S \subset (X \setminus S^*) \cup \{x\}$. Then, by assumption, $(X \setminus S^*) \cup \{x\} \in S(x)$ as well. By *S2′*, $(X \setminus S^*) \setminus \{x\} \in S(x)$ and thus $X \setminus S^* \in S(x)$. Finally, assume (3) and suppose that $S := X \setminus S^* \in S(x)$. Obviously, this contradicts (3). □

For a simple system $\mathcal{S}$ let $\mathcal{S}^* := \widehat{(\bar{\mathcal{S}})}^*$, by which the duals are now defined for both classes of the hereby considered systems.

Property 4. *If $\mathcal{S}$ is local, then so is $\mathcal{S}^*$ (with the same topology).*

Theorem 3. *Let $d : 2^X \to 2^X$ be a generalized derivative. $(\mathcal{S}_d)^*$ is filtering.*

Proof. Let S_1^*, $S_2^* \in (S_d)^*(x)$. Thus,

$$x \notin d(S_1^* \setminus \{x\}) \qquad \text{and} \qquad x \notin d(S_2^* \setminus \{x\}).$$

We infer that by *D3* and implication '*2* ⇒ *1*' from lemma 1,

$$x \notin d\,((S_1^* \cap S_1^*) \setminus \{x\}).$$

□

Theorem 4. *Let for a derivative d, $\gamma_{(\mathcal{S}''_d)^*}$ be the topology defined for $(\mathcal{S}''_d)^*$ by* (1). *$(\mathcal{S}''_d)^*$ is the $\gamma_{(\mathcal{S}''_d)^*}$-neighbourhood system, i.e.*

$$(S''_d)^*(x) = \left\{S \subset X : x \in \gamma_{\mathcal{S}''_d}\text{-Int}S\right\}.$$

In order to prove this we need the following fact.

Lemma 3. ([7], Theorem 2) *If $\mathcal{S}$ is filtering and $\gamma_{\mathcal{S}}$ is the topology defined by* (1), *then for the equality*

$$\mathcal{S} = \mathcal{S}_0^{\gamma_{\mathcal{S}}}$$

it is sufficient and necessary that $\mathcal{S}$ fulfil the following condition

$$\forall\ (S \in S(x)) \quad \exists\ (S_x \in S(x)) \quad \forall\ (y \in S_x) \quad (y \in S(y)). \tag{4}$$

Proof. [Proof of the lemma] Sufficiency of (4) is proved in [7], while necessity is obvious, since it is enough to take $S_x := \gamma_{\mathcal{S}}\text{-Int}S$. □

Proof. [Proof of Theorem 4] By the lemma and Theorem 3 it suffices to prove (4) for $S(x) = (S''_d)^*(x)$. When, according to Fact 2, $S \in (S''_d)^*(x) = \widehat{(S'_d)^*}(x)$, then

$$x \in S \setminus d\,(X \setminus (S \cup \{x\})) = S \setminus d(X \setminus S). \tag{5}$$

Put

$$S_x := S \setminus d(X \setminus S). \tag{6}$$

We shall prove that:

1. $S_x \in (S''_d)^*(x)$ and
2. $S \in (S''_d)^*(y)$, for $y \in S_x$.

Ad 1. By (5) and (6), $x \in S_x$, so it suffices to show that $X \setminus S_x \notin S'_d(x)$. We have by *D5*,

$$\begin{aligned} x \notin d(X \setminus S) &= d(X \setminus S) \cup dd(X \setminus S) = d((X \setminus S) \cup d(X \setminus S)) = \\ &= d\,(X \setminus (S \setminus d(X \setminus S))) = d(X \setminus S_x) = \\ &= d\,(X \setminus (S_x \cup \{x\}))\,. \end{aligned}$$

Therefore $X \setminus S_x \notin S'_d(x)$ as asserted.

Ad 2. Let $y \in S_x$. We will show that

$$S_x \subset S \setminus d\,(X \setminus (S \cup \{y\}))\,. \tag{7}$$

Clearly $S \subset S \cup \{y\}$. Hence $X \setminus S \supset X \setminus (S \cup \{y\})$. By implication '*2 ⇒1*' from lemma 1 it follows that $d\,(X \setminus S) \supset d\,(X \setminus (S \cup \{y\}))$. Thus, by complementation and by (6) we have (7). □

Note 2. An inspiration for the above considerations comes from $10 in [5], where Thomson introduces local systems with the aid of closure operator. Unfortunately, in Thomson's work there are two errors. The first one is a statement that 'u is finer than the usual closure operator on the real line' is a synonym for '$u(A)$ must be a subset of the points of accumulation of A'. Here, obviously, the part '$A \subset u(A)$' is missing. The second error is dropping the '$x \in S$' requirement in the definition of a local system connected with a derivative determined by a closure operator.

Note 3. As it has been already said (Example 2) examples of generalized derivatives are known, One can find such for instance in [1], the authors of which could have easily shorten their proofs of theorems 2.4 (1) and 2.7 (3) and (5), noting an obvious fact, here exposed as lemma 1.

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PROSTE UKŁADY I OPERATORY DOMYKANIA

S t r e s z c z e n i e

Badane i wykorzystywane są związki między uogólnionym wyprowadzonym zbiorem operatorów i wyabstrahowanymi lokalnymi układami Thomsona.

Słowa kluczowe: układ prosty, topologia uogólniona, domknięcie, zbiór wyprowadzony

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Aleksandra Karasińska

DUALITY PRINCIPLE FOR SOME σ-IDEALS OF SUBSETS OF THE REAL LINE

Summary

In this note there is proved that a theorem analogous to Sierpiński-Erdös duality theorem for some σ-ideals of subsets of the real line and the family of sets of the first category on the real line is valid.

Keywords and phrases: Sierpiński-Erdös duality theorem, set of the first category

Studies of σ-ideals of subsets of the real line have a long tradition. The most frequently considerations of such a type are focused on the σ-ideal $\mathcal{N}$ of sets of Lebesgue measure zero and the σ-ideal $\mathcal{K}$ of sets of the first category. The similarities and the differences between these classes are the topic of the monograph "Measure and Category" by J. C. Oxtoby. A part of these similarities is a consequence of Sierpiński-Erdös Duality Theorem. In 1934 W. Sierpiński in [7] proved (assuming CH) that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f(E)$ is a nullset if and only if E is of the first category. In 1943, P. Erdös showed that a stronger version of this theorem is also valid (compare [3]). Assuming CH it was shown that there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f = f^{-1}$ and $f(E)$ is a nullset if and only if E is a set of the first category. From Erdös result Duality Theorem, which ascertains the duality between measure and category, follows.

The analogous Duality Principle holds if the σ-ideal of Lebesgue nullsets will be substituted by the family of microscopic sets. The notion of microscopic set was introduced in [1]. We say that a set $A \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$, and $\lambda(I_n) \le \varepsilon^n$ for each $n \in \mathbb{N}$, where λ stands for the Lebesgue measure on the real line. In the paper mentioned above the authors proved that the family of all microscopic sets is a σ-ideal situated between countable sets and sets of Lebesgue zero, which is different from both these families.

In [5] it was proved that the duality between microscopic sets and sets of the first category, analogous to Sierpiński-Erdös Duality Theorem, is also valid.

In this paper we consider some family of σ-ideals $\mathcal{M}_{\{\varepsilon_n\}}$, which are connected with the sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers and prove that the theorem analogous to Duality Principle, for the σ-ideal $\mathcal{M}_{\{\varepsilon_n\}}$ and the σ-ideal $\mathcal{K}$ of sets of the first category, holds.

From now let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that the series $\sum_{i=1}^{\infty} \varepsilon_n$ is convergent.

Let $\mathcal{M}_{\{\varepsilon_n\}}$ denote the family of subsets of the real line such that for each $k \in \mathbb{N}$ there exists a sequence $\{I_n^{(k)}\}_{n\in\mathbb{N}}$ of intervals such that

$$A \subset \bigcup_{n\in\mathbb{N}} I_n^{(k)} \quad \text{and} \quad \lambda(I_n^{(k)}) \le \varepsilon_{n\cdot k}$$

for each $n \in \mathbb{N}$, i.e.

$$\mathcal{M}_{\{\varepsilon_n\}} = \left\{ A \subset \mathbb{R} : \underset{k\in\mathbb{N}}{\forall} \ \underset{\{I_n^{(k)}\}}{\exists} \left[A \subset \bigcup_{n\in\mathbb{N}} I_n^{(k)} \wedge \underset{n\in\mathbb{N}}{\forall} \lambda(I_n^{(k)}) \le \varepsilon_{n\cdot k} \right] \right\}.$$

In [4] the authors proved that each family $\mathcal{M}_{\{\varepsilon_n\}}$ is a σ-ideal of subsets of the real line situated between the family $\mathcal{S}$ of strong measure zero sets and sets of Lebesgue measure zero. There was proved also that $\mathcal{M}_{\{\varepsilon_n\}} \setminus \mathcal{S} \neq \emptyset$ for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, as each family $\mathcal{M}_{\{\varepsilon_n\}}$ contains a Cantor-type set and no perfect set is of strong measure zero (see [2]), i.e.

$$\mathcal{S} \subsetneq \mathcal{M}_{\{\varepsilon_n\}} \subset \mathcal{N}.$$

Observe, that for each sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ the σ-ideals $\mathcal{M}_{\{\varepsilon_n\}}$ and $\mathcal{K}$ are Borel orthogonal.

Theorem 1. *For any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ the real line can be represented as the union of two complementary sets A and B such that A is F_σ set of the first category and B is the set of type G_δ from the family $\mathcal{M}_{\{\varepsilon_n\}}$.*

Proof. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary non-increasing sequence of positive numbers such that the series $\sum_{n=1}^{\infty} \varepsilon_n$ is convergent. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers.

Let

$$I_n^{(j)} = (r_n - \frac{\varepsilon_{n\cdot j}}{3}, r_n + \frac{\varepsilon_{n\cdot j}}{3}), \quad n \in \mathbb{N}, \quad j \in \ N.$$

Put

$$G_j = \bigcup_{n=1}^{\infty} I_n^{(j)}, \quad \text{for} \quad j \in N,$$

$$B = \bigcap_{j=1}^{\infty} G_j \quad \text{and} \quad A = \mathbb{R} \setminus B.$$

Clearly, G_j is open dense subset of the real line for $j \in N$, so B is a residual set of type G_δ and, consequently, A is of type F_σ and of the first category. Moreover $B \in \mathcal{M}_{\{\varepsilon_n\}}$. Indeed, fix $k \in \mathbb{N}$. Obviously $B \subset G_k$, so

$$B \subset \bigcup_{n=1}^{\infty} I_n^{(k)}$$

and for each $n \in \mathbb{N}$ we have $\lambda(I_n^{(k)}) < \varepsilon_{n \cdot k}$. As k was chosen arbitrary, we obtain that $B \in \mathcal{M}_{\{\varepsilon_n\}}$. □

Corollary 2. *Every subset of the real line can be represented as the union of a set from the family $\mathcal{M}_{\{\varepsilon_n\}}$ and a set of the first category.*

From the definition of the family $\mathcal{M}_{\{\varepsilon_n\}}$ it follows immediately

Lemma 3. *For any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ every set from the family $\mathcal{M}_{\{\varepsilon_n\}}$ is contained in some set of type G_δ from this family $\mathcal{M}_{\{\varepsilon_n\}}$.*

The next theorem is a generalization of Lemma 5.1 in [6].

Lemma 4. *Every uncountable analytic subset A of $\mathbb{R}$ contains a nowhere dense perfect subset from the family $\mathcal{M}_{\{\varepsilon_n\}}$, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$.*

Proof. The proof is analogous to the construction in Theorem 2.10 in [5], but we add it for the convenience of the reader.

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary non-increasing sequence of positive numbers such that the series $\sum\limits_{n=1}^{\infty} \varepsilon_n$ is convergent and let A be an uncountable analytic set. From the Hausdorff Theorem it follows that A contains some uncountable set E of type G_δ. Let

$$E = \bigcap_{n=1}^{\infty} G_n,$$

where G_n is open for $n \in \mathbb{N}$. Let F denote the set of all condensation points of E that belong to E. The set F is non-empty and has no isolated points.

Now we shall define by induction the sequence of closed intervals $\{I(i_1, \ldots, i_n) : i_k = 0 \text{ or } i_k = 1 \text{ for } k = 1, \ldots n\}$, $n \in \mathbb{N}$, in a following way. Let $I(0)$ and $I(1)$ be two disjoint closed intervals with length less than ε_1 and ε_2 respectively, whose interiors meet F and whose union is contained in G_1. Since F has no isolated points there exist four disjoint closed intervals $I(0,0)$, $I(0,1)$, $I(1,0)$, $I(1,1)$ with length less than ε_2, ε_4, ε_6, ε_8 respectively, whose interiors meet F and such that $I(i_1,i_2) \subset I(i_1) \cap G_2$ for $i_1, i_2 \in \{0,1\}$.

Suppose that we have defined 2^k disjoint closed intervals $I(i_1,\ldots,i_k)$, $i_j \in \{0,1\}$ for $j = 1,\ldots,k$, whose interiors meet F, such that $I(i_1,\ldots,i_k) \subset G_k \cap I(i_1,\ldots,i_{k-1})$ for $i_1,\ldots,i_k \in \{0,1\}$ and $\lambda(I(i_1,\ldots,i_k)) < \varepsilon_{k\cdot(p+1)}$, where

$$p = i_k \cdot 2^0 + i_{k-1}2^1 + \cdots + i_1 2^{k-1}.$$

Let $I(i_1,\ldots,i_{k+1})$, where $i_j \in \{0,1\}$ for $j = 1,...,k+1$, be two disjoint closed intervals contained in $I(i_1,\ldots,i_k) \cap G_{k+1}$, whose interiors meet F, with length less than $\varepsilon_{(k+1)\cdot(p+1)}$, where $p = i_{k+1}2^0 + i_k \cdot 2^1 + \cdots + i_1 \cdot 2^k$. The set F has no isolated points, so a family of intervals $I(i_1,\ldots,i_k)$ having required properties can be defined. Put

$$C = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)} I(i_1,\ldots,i_n).$$

Obviously, C is nowhere dense perfect set contained in E, so also in A. Clearly, the set C belongs to the family $\mathcal{M}_{\{\varepsilon_n\}}$. □

Theorem 5. [CH] *For any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ the family $\mathcal{M}_{\{\varepsilon_n\}}$ has the following properties:*

1. *$\mathcal{M}_{\{\varepsilon_n\}}$ is a σ-ideal,*
2. *the union of $\mathcal{M}_{\{\varepsilon_n\}}$ is equal to $\mathbb{R}$,*
3. *$\mathcal{M}_{\{\varepsilon_n\}}$ has a subfamily $\mathcal{G}$ such that card$(\mathcal{G}) \leq \chi_1$ and for each $A \in \mathcal{M}_{\{\varepsilon_n\}}$ there exists $B \in \mathcal{G}$ such that $A \subset B$,*
4. *the complement of each set $A \in \mathcal{M}_{\{\varepsilon_n\}}$ contains a set with cardinality χ_1, which also belongs to $\mathcal{M}_{\{\varepsilon_n\}}$.*

Proof. The condition (a) is proved in [4], (b) is obvious, (c) follows immediately from Lemma 3. Now let $A \in \mathcal{M}_{\{\varepsilon_n\}}$. Then A is nullset, so $\mathbb{R}\setminus A$ contains some uncountable closed subset. Using Lemma 4 we obtain that $\mathbb{R} \setminus A$ contains some set from $\mathcal{M}_{\{\varepsilon_n\}}$ with cardinality χ_1. □

Theorem 6. [CH] *There exists a one-to-one mapping f of the real line onto itself such that $f = f^{-1}$ and $f(E) \in \mathcal{M}_{\{\varepsilon_n\}}$ if and only if E is of the first category.*

Proof. Our theorem is a consequence of Theorem 19.6 in [6]. Put $X = \mathbb{R}$. Let $\mathcal{K}$ denote the family of all sets of the first category and put $\mathcal{L} = \mathcal{M}_{\{\varepsilon_n\}}$. From Theorem 5 it follows that the family $\mathcal{M}_{\{\varepsilon_n\}}$ fulfills the conditions (a)–(d) of Theorem 19.5 in [6]. Moreover, for the sets M and N we may take the sets A and B from Theorem 1. □

Duality Principle (CH) Let P be any proposition involving solely the notions of sets from the family $\mathcal{M}_{\{\varepsilon_n\}}$, first category set and notions of pure set theory. Let

P^* be the proposition obtained from P by interchanging the terms "set from the family $\mathcal{M}_{\{\varepsilon_n\}}$" and "set of the first category" whenever they appear. Then each of the proposition P and P^* implies the other.

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ZASADA DUALNOŚCI DLA PEWNYCH σ-IDEAŁÓW PODZBIORÓW PROSTEJ RZECZYWISTEJ

S t r e s z c z e n i e

W obecnej nocie udowodniono twierdzenie analogiczne do twierdzenia o dualności Sierpińskiego-Erdösa, odnoszące się do pewnych σ-ideałów podzbiorów prostej rzeczywistej i rodziny zbiorów pierwszej kategorii na prostej rzeczywistej.

Słowa kluczowe: twierdzenie o dualności Sierpińskiego-Erdösa, zbiór pierwszej kategorii

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Aleksandra Karasińska and Sebastian Lindner

ON SPECIAL SATURATED SETS

Summary

In the paper the examples of sets which are 'nice' in algebraical sense and saturated non-measurable with respect to the Lebesgue measure are considered. It is investigated under what conditions on measurable space those sets remain saturated non-measurable. Obtained conditions are close but not equivalent to the Steinhaus and Smital properties.

Keywords and phrases: saturated non-measurable set, the Steinhaus property, the Smital property

1. Introduction and notations

In 1905 Vitali published his construction of a non-measurable set. Three years later E. B. Van Vleck constructed a set which has a non-empty intersection with every set of positive measure but it does not contain any such set. Sets with this property are called saturated non-measurable [11], strictly non-measurable [4] or completly non-measurable [9].

The set E constructed by Van Vleck has a beautiful structure – it is invariant with respect to translation by any binary rational number and the sets $E, -E$ and the set of binary rationals give a partition of $\mathbb{R}$.

Constructing his set Van Vleck used the axiom of choice with respect to a family of cardinality continuum. In 1964 Semadeni [10] observed that the existence of such set can be proved with a weaker assumption – the existence of Stone-Cech compactification of the space of natural numbers $\mathbb{N}$. Finally in 2000 Sardella and Ziliotti [11] presented a construction based on a countable version of the axiom of choice and a assumption that the Cantor's cube $\{0,1\}^{\mathbb{R}}$ is countably compact.

In this paper we are going to consider Van Vleck's type sets in a wider context – in reference to an abelian group equipped with a topology invariant with respect to the operations in the group, with a structure of measurable space. We show the connection between a problem of nonmeasurability of such sets and an analogue of the Steinhaus theorem.

In the second part of the paper we prove a more general result concerning so called Archimedean sets. On the real line with the Lebesgue measure they are very "neat" or saturated non-measurable. We show that in a general case this dichotomy is connected with a certain version of the Smital property.

Following the paper [3] we are going to formulate our results in quite general settings. We assume that $(X,+)$ is an abelian group.

Let $\mathcal{F} \subset 2^X$. We say that $\mathcal{F}$ is *invariant* if for any $F \in \mathcal{F}$ and any $x \in \mathbb{R}$ the set $x - F = \{x - y : y \in F\}$ belongs to the family $\mathcal{F}$.

We assume that the space X is equipped with the topology τ, which is invariant.

Let $\mathcal{A}$ be a field of subsets of X and $\mathcal{I} \subset \mathcal{A}$ be a proper ideal. The triple $(X, \mathcal{A}, \mathcal{I})$ we shall call a measurable space. We say that the measurable space $(\mathbb{R}, \mathcal{A}, \mathcal{I})$ is invariant, if $\mathcal{A}$ and $\mathcal{I}$ are invariant.

A set $A \subset X$ is *thin* if for each $E \in \mathcal{A}$, $E \subset A$ we have $E \in \mathcal{I}$. A set $A \subset X$ is $(X, \mathcal{A}, \mathcal{I})$-saturated iff both A and $X \setminus A$ are thin.

In the case of the Lebesgue measure on $\mathbb{R}$ the set is thin iff its inner measure is equall to 0. Clearly, if A is $(X, \mathcal{A}, \mathcal{I})$-saturated, then so is $X \setminus A$.

We will say that $A \subset X$ is an *Archimedean set* if there exists a dense set D such that $A + d = A$ for each $d \in D$.

We will say that the triple $(X, \mathcal{A}, \mathcal{I})$ has *the Steinhaus Property* if for any sets A, B from $\mathcal{A} \setminus \mathcal{I}$ the set $A - B$ has an interior point. We will say that the triple $(X, \mathcal{A}, \mathcal{I})$ has *the Smital Property* if for any set A from $\mathcal{A} \setminus \mathcal{I}$ and any dense set D the set $(A + D)' \in \mathcal{I}$.

In the sequel for a given $x \in \mathbb{R}$ let $\langle x \rangle_2^n$ (resp. $\langle x \rangle_3^n$) stand for n-th digit of the binary (resp. ternary) expansion of the number x, i.e.

$$x = \sum_{n=0}^{\infty} \frac{\langle x \rangle_i^n}{i^n}, \quad \text{where} \quad \langle x \rangle_i^n \in \{0, 1, \ldots, i-1\}.$$

Since binary (ternary) rationals have two binary (ternary) expansions, we choose the finite one.

2. Van Vleck's sets

In the paper [11] authors consider the functions $c_k : \mathbb{R} \to \{0,1\}$ given by the formula $c_k(x) = c(2^k x)$ for $x \in \mathbb{R}$, where

$$c(x) = \begin{cases} 0, & \text{for } x \in \bigcup\limits_{k \in \mathbb{Z}} [2k, 2k+1) \\ 1, & \text{for } x \in \bigcup\limits_{k \in \mathbb{Z}} [2k+1, 2k+2). \end{cases}$$

One can easily observe that for a given $x \in \mathbb{R}$ the function $c_k(x)$ gives the k-th digit of binary expansion of the number $x - [x]$, where $[x]$ stands for the integer part of x. Following [11] we put $G = \{c_k : k \in \mathbb{N}\}$, $D(G)$ – the set of cluster points of G with respect to the product topology in $\{0,1\}^{\mathbb{R}}$, $\mathbb{D}$ – the set of all binary rationals and $\mathcal{E} = \{E \subset \mathbb{R} : \chi_E \in D(G)\}$ In [11] it is shown that every set $E \in \mathcal{E}$ has the following properties:

(I) E is invariant under translations by any binary rational;

(II) the sets E, $-E$ and $\mathbb{D}$ give a partition of $\mathbb{R}$;

(III) E is $(\mathbb{R}, \mathcal{L}, \mathcal{N})$ saturated non-measurable.

In fact it is proved that (I) and (II) imply (III).

Remark 1. *There is another way to obtain sets having properties (I) and (II).*

Indeed. Let $\mathcal{P} \subset 2^{\mathbb{N}}$ be an arbitrary non-principal ultrafilter on $\mathbb{N}$. Consider an arbitrary point $x \in [0,1)$. Let $(\langle x\rangle_2^k)_{k\in\mathbb{N}}$ be the sequence of digits of binary expansion of x. (For $x \in \mathbb{D}$ we consider their finite expansion). Let $\tilde{E_{\mathcal{P}}} = \big\{x \in [0,1] \ : \ \{k \in \mathbb{N} : \langle x\rangle_2^k = 1\} \in \mathcal{P}\big\}$ and $E_{\mathcal{P}} = \{x \in \mathbb{R} : x - [x] \in \tilde{E_{\mathcal{P}}}\}$. Observe that for any $x \in E_{\mathcal{P}}$ and any $d \in \mathbb{D}$ the binary expansion of numbers x and $x + d$ are equal to each other except for a finite number of digits. By virtue of the fact that for any $A \in \mathcal{P}$ and $B \subset \mathbb{N}$ if $A \triangle B$ is finite then $B \in \mathcal{P}$, we have that $x + d \in E_{\mathcal{P}}$. Moreover for every $x \notin \mathbb{D}$ and every $k \in \mathbb{N}$ we have $\langle 1-x\rangle_2^k = 1 - \langle x\rangle_2^k$. Hence for every $x \notin \mathbb{D}$ exactly one element of the pair $x, -x$ belongs to $E_{\mathcal{P}}$. So the set $E_{\mathcal{P}}$ fulfills the conditions (I) and (II).

Now let $(X, +)$ be an abelian group equipped with the invariant topology τ. Let D be it's subgroup. We are now interested in finding assumptions on a measurable space $(X, \mathcal{A}, \mathcal{I})$ which imply that any set E having the properties:

(I) $E + d = E$ for each $d \in D$;

(II) the sets E, $(-E)$ and D give a partition of X;

is $(X, \mathcal{A}, \mathcal{I})$-saturated?

Theorem 2. *The following conditions are equivalent.*

(1) $(A + A) \cap D \neq \emptyset$ *for any* $A \in \mathcal{A} \setminus \mathcal{I}$.

(2) *Each set* E *satisfying the conditions* $E + D = E$ *and* $E \cap (-E) = \emptyset$ *is thin.*

Proof. Assume that the conditon (1) is satisfied and let E be such that $E + D = E$ and $E \cap (-E) = \emptyset$. Suppose that there exists $A \in \mathcal{A} \setminus \mathcal{I}$, $A \subset E$. Therefore $(A + A) \cap D \neq \emptyset$, hence there exist $a_1, a_2 \in A$ and $d \in D$ such that $a_2 - d = (-a_1)$. Since $(-a_1) \in -E$, so $a_2 - d \notin E$, contrary to the asumption.

Now assume *(2)* and suppose that $(A+A)\cap D=\emptyset$ for some $A\in\mathcal{A}\setminus\mathcal{I}$. Let $E=A+D$. It is easy to observe that E satisfies the conditions $E+D=E$ and $E\cap(-E)=\emptyset$. But $E\supset A$, so E is not thin. $\square$

The following example shows that in condition (2) the word *thin* can not be replaced by $(X,\mathcal{A},\mathcal{I})$*-saturated* even if the set D is thin.

Example 3. Let $X=Z_6=\{0,1,\ldots,5\}$. Then $G=\{0,3\}$ is a subgroup of X. Let $E=\{1,4\}$. Consider the triple $(X,\mathcal{A},\mathcal{I})$ where $\mathcal{A}$ is a field generated by the family $\{\{0,4\},\{1,5\},\{2,3\}\}$ and $\mathcal{I}=\{\emptyset\}$. Then G, E and $-E$ are thin but not saturated.

Corollary 4. *Let the triple $(X,\mathcal{A},\mathcal{I})$ be invariant and D be the dense subgroup of X such that $D\in\mathcal{I}$. If $(X,\mathcal{A},\mathcal{I})$ fulfills the Steinhaus Property, then each set having properties (I) and (II) is $(X,\mathcal{A},\mathcal{I})$-saturated.*

Proof. Let E satisfy (I) and (II) and $A\in\mathcal{A}\setminus\mathcal{I}$. Since $(X,\mathcal{A},\mathcal{I})$ is invariant the set $(-A)\in\mathcal{A}\setminus\mathcal{I}$, hence from the Steinhaus Property the set $A+A=A-(-A)$ has nonempty interior. Therefore $(A+A)\cap D\neq\emptyset$. By virtue of Theorem 2 the set E is thin. But then $-E$ is also thin. Since E fullfills the condition (II) and $D\in\mathcal{I}$ we have that $E'=(-E)\cup D$ is thin. Hence E is $(X,\mathcal{A},\mathcal{I})$-saturated. $\square$

Let $\mathcal{B}$ denote the σ-field of Borel sets in $\mathbb{R}$, and $\mathcal{K}$ stand for a σ-ideal of the first category sets. The σ-field $\mathcal{B}\triangle\mathcal{K}:=\{B\triangle I : B\in\mathcal{B} \text{ and } I\in\mathcal{K}\}$ is called the family of sets having the Baire property. By virtue of the theorem of S. Piccard the triple $(\mathbb{R},\mathcal{B}\triangle\mathcal{K},\mathcal{K})$ fulfills the Steinhaus Property, so each set fulfilling (I) and (II) is $(\mathbb{R},\mathcal{B}\triangle\mathcal{K},\mathcal{K})$-saturated.

The next example shows that assuming the Steinhaus Property in corollary 4 is essential.

Example 5. We say that a set $E\subset\mathbb{R}$ is *microscopic* if for each $\varepsilon>0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $E\subset\bigcup_{n\in\mathbb{N}}I_n$ and $|I_n|\leq\varepsilon^n$ for each $n\in\mathbb{N}$. The family of all microscopic sets forms an invariant σ-ideal (see [2]). The properties of microscopic sets were also considered in [1], [6], [7], [5].

We will show that there exists a set satisfying conditions (I) and (II) which is not $(\mathbb{R},\mathcal{B}\triangle\mathcal{M},\mathcal{M})$-saturated.

Let $\mathcal{P}$ be any non-principal ultrafilter on $\mathbb{N}$ containing the set of all odd numbers. Let $E_{\mathcal{P}}$ be as in Remark 1. To see that $E_{\mathcal{P}}$ is not $(\mathbb{R},\mathcal{B}\triangle\mathcal{M},\mathcal{M})$-saturated we will find a $(\mathcal{B}\triangle\mathcal{M})$-measurable set $A\subset E_{\mathcal{P}}$ such that $A\notin\mathcal{M}$.

First we shall define by induction the sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ in the following way. Let $F_1=[\frac{1}{2},1]$. Next we divide F_1 into four closed intervals each of the length equal to $(\frac{1}{2})^3$. Denote by $F_{2,1}$ and $F_{2,2}$ the second and the last interval from this partition, respectively. Put $F_2=F_{2,1}\cup F_{2,2}$. Next we divide $F_{2,1}$ into four closed intervals each of the length equal to $(\frac{1}{2})^5$. Denote by $F_{3,1}$ and $F_{3,2}$ the second and the last interval from this partition, respectively. Similarly, we obtain the closed intervals $F_{3,3},F_{3,4}$ by division of the set $F_{2,2}$. Put

$$F_3 = \bigcup_{j=1}^{2^2} F_{3,j}.$$

Proceeding this way we obtain the decreasing sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ such that

$$F_n = \bigcup_{j=1}^{2^{n-1}} F_{n,j} \quad \text{for} \quad n \in \mathbb{N}, \quad \text{and} \quad F_{n,j}$$

is a closed interval of the length equal to $(\frac{1}{2})^{2n-1}$ for $n \in \mathbb{N}$ and $j = 1, ..., 2^{n-1}$.

Put

$$\tilde{A} = \bigcap_{n\in\mathbb{N}} F_n.$$

Obviously, $\tilde{A}$ consists of all numbers from the interval $[0, 1]$ which odd digits in binary expansion are 1.

Suppose that $\tilde{A} \in \mathcal{M}$. Thus for $\varepsilon = \frac{1}{4}$ one can find a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $|I_n| \leq (\frac{1}{4})^n$ for each $n \in \mathbb{N}$ and $\tilde{A} \subset \bigcup_{n\in\mathbb{N}} I_n$. The interval I_2 of the length less than $(\frac{1}{4})^2$ can meet only one of the intervals $F_{2,1}$ and $F_{2,2}$. Let us denote by B_2 that one which is disjoint with I_2 . The interval I_3 meets only one from the intervals $F_{3,j}$ for $j = 1, ..., 4$. Let B_3 be this one which is disjoint with I_3 and contained in B_2. Proceeding this way we obtain the decreasing sequence of closed intervals B_n for $n = 2, ...$ such that $B_n \cap I_n = \emptyset$ and $B_n \subset F_n$ for each $n \in \mathbb{N} \setminus \{1\}$. Thus

$$\bigcap_{n=2}^{\infty} B_n \cap \bigcup_{n=2}^{\infty} I_n = \emptyset, \quad \text{but} \quad \bigcap_{n=2}^{\infty} B_n \neq \emptyset \quad \text{and} \quad \bigcap_{n=2}^{\infty} B_n \subset \tilde{A},$$

which contradicts the assumption that $\tilde{A} \subset \bigcup_{n\in\mathbb{N}} I_n$. Finally we obtain that $\tilde{A} \notin \mathcal{M}$. The method using here is analogous to the considerations in [2], Example 2, concerning Cantor set.

Let $x \in \tilde{A} \setminus \mathbb{D}$. Then x has odd digits of it's unique binary expansion equal to 1. Hence $x \in E_{\mathcal{P}}$. Since $\mathbb{D} \in \mathcal{M}$ the set $A = \tilde{A} \setminus \mathbb{D}$ is also not microscopic and $A \subset E_{\mathcal{P}}$. Moreover, $A \in \mathcal{B}\triangle\mathcal{M}$ as a Borel set.

By virtue of Corollary 4 the triple $(\mathbb{R}, \mathcal{B}\triangle\mathcal{M}, \mathcal{M})$ hasn't got the Steinhaus Property. This fact was already known (see [6] Theorem 20.18).

3. Archimedean sets

Now we consider again $(X, +)$ – an abelian group equipped with an invariant topology τ. Let $(X, \mathcal{A}, \mathcal{I})$ be a measurable space. We introduce a property a little bit weaker than the Smital Property:

Definition 6. *We say that the triple $(X, \mathcal{A}, \mathcal{I})$ satisfies the (wS) property iff for every set $A \in \mathcal{A} \setminus \mathcal{I}$ and for every dense subgroup D we have $(A + D)' \in \mathcal{I}$.*

Theorem 7. *Let a measurable space $(X,\mathcal{A},\mathcal{I})$ be invariant. The following statements are equivalent:*

1. *$(X,\mathcal{A},\mathcal{I})$ fulfills (wS);*
2. *each Archimedean set A fulfills exactly one of the three conditions:*
   (a) $A \in \mathcal{I}$.
   (b) $X \setminus A \ \in \mathcal{I}$.
   (c) A is $(X,\mathcal{A},\mathcal{I})$-saturated.

Proof. (1) $\Rightarrow$ (2) Let A be an arbitrary Archimedean set. Let $D = \{d \in X : d + A = A\}$. It is easy to observe that D is a subgroup of X. Since A is Archimedean, D is dense. Assume that A is not $(X,\mathcal{A},\mathcal{I})$-saturated. Thus there exists $B \in \mathcal{A} \setminus \mathcal{I}$ such that $B \subset A$ or $B \subset X \setminus A$. In the first case by virtue of (wS) we obtain

$$X \setminus A = X \setminus (A + D) \ \subset \ X \setminus (B + D) \in \mathcal{I}.$$

The second case is analogous.

(2) $\Rightarrow$ (1) Let $A \in \mathcal{A}\setminus\mathcal{I}$ and let D be a dense subgroup of X. Observe that $A+D$ is Archimedean. In fact, let $b \in A + D$. Then $b = a + d$ for some $a \in A$ and $d \in D$. Let $d_1 \in D$. Then $b + d_1 = a + (d + d_1) \in A + D$, and similarly $b - d_1 \in A - D$. Therefore $(A+D)+d_1 = A+D$. From the asumption *2.* we have that $A+D$ satisfies one of conditions *(a)*, *(b)* or *(c)*. Let $d \in D$, then $A + d \in \mathcal{A} \setminus \mathcal{I}$ and $A + d \subset A + D$. Hence the set $A + D$ does not satisfy the conditions (a) nor *(c)*. □

In the paper [3] authors give the example (Example 1.15) of measurable space $(\mathbb{R},\mathcal{A},\mathcal{I})$ which is invariant, which has the Steinhaus Property but it does not have the Smital Property. Let C be the classical "middle third" Cantor set and

$$\mathcal{F} = \left\{ \frac{C}{3^n} + t : n \in \mathbb{N}, t \in \mathbb{R} \right\}.$$

Let

$$\mathcal{A} = S(\mathcal{F}) = \{A \subset \mathbb{R}; \forall_{F_1 \in \mathcal{F}} \exists_{F_2 \in \mathcal{F}} [F_2 \subset (F_1 \cap A) \vee F_2 \subset F_1 \setminus A]\}$$

and

$$\mathcal{I} = S_0(\mathcal{F}) = \{A \subset \mathbb{R}; \forall_{F_1 \in \mathcal{F}} \exists_{F_2 \in \mathcal{F}} F_2 \subset F_1 \setminus A\}.$$

Let us observe that the triple $(\mathbb{R},\mathcal{A},\mathcal{I})$ does not have the (wS) property. Indeed, let T be the set of all ternary numbers (i.e. the numbers of the form $\frac{n}{3^k}$ for $n, k \in \mathbb{N}$). Then T is a dense group. Let $A = C + T$. Then A is Archimedean, $C \subset A$, hence A does not fullfill conditions (a) nor (c). Observe that for $x \in A$ we have $\langle x \rangle_3^n \neq 1$ for almost every $n \in \mathbb{N}$. To show that the set A does not satisfy the condition (b) consider the set $F = C + \frac{1}{2} \in \mathcal{F}$. Suppose that there exists $F_0 \in \mathcal{F}$, $F_0 \subset F \cap A$. Let $x_0 = \min(F_0)$. Since $F_0 \subset C + \frac{1}{2}$, x_0 is of the form $x_0 = \frac{1}{2} + t_0$, for some $t_0 \in T$. Hence $\langle x_0 \rangle_3^n = 1$ for almost all $n \in \mathbb{N}$. Hence $x_0 \notin A$.

The next example shows that the property (wS) is essentially weaker than the Smital Property.

Example 8. Let us consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. It is isomorphic to the vector space $\mathbb{R}^2$ over $\mathbb{Q}$. (Both spaces have Hamel bases of the cardinality $\mathfrak{c}$, compare [4].) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be such an isomorphism. In the paper [4] Cichoń and Szczepaniak show that for every set $A \subset \mathbb{R}^2$ such that $\mathrm{Int(A)} \neq \emptyset$ and $\mathrm{Int}(\mathbb{R}^2 \setminus \mathrm{A}) \neq \emptyset$ the set $\Phi(A)$ is Lebesgue saturated non-measurable, hence dense. Let $||\mathbf{x}||$ be the standard norm in $\mathbb{R}^2$. Let

$$P = \left\{ \mathbf{x} \in \mathbb{R}^2 \ : \ \left[\sqrt{||\mathbf{x}||} \right] \text{ is odd} \right\}$$

where $[a]$ denote the integer part of the number a.

Then every non-constant arithmetic sequence in $\mathbb{R}^2$ meets P and $\mathbb{R}^2 \setminus P$.

Let $\mathcal{A} = \{\emptyset, \mathbb{R}, \Phi(P), \Phi(\mathbb{R}^2 \setminus P)\}$, $\mathcal{I} = \{\emptyset\}$. Let $H \subset \mathbb{R}$ be a non-trivial group and let $h \in H$, $h \neq 0$. Let $x \in \mathbb{R}$. Then $(\Phi^{-1}(x - nh))_{n \in N}$ forms an arithmetic sequence in $\mathbb{R}^2$. Hence $x - nh \in \Phi(P)$ for some $n \in \mathbb{N}$, so $x \in \Phi(P) + H$. Similarly $x \in (\mathbb{R}^2 \setminus P) + H$. Therefore the triple $(\mathbb{R}, \mathcal{A}, \mathcal{I})$ satisfies the property (wS).

Now let $A = \{\mathbf{x} \in \mathbb{R}^2 \ : \ ||x|| < 1\}$. Then $\Phi(A)$ is a dense subset of $\mathbb{R}$ by virtue of the result of Cichoń and Szczepaniak. We have that $A + P \neq \mathbb{R}^2$, hence $\Phi(A + P) = \Phi(A) + \Phi(P) \neq \mathbb{R}$ – so the Smital property is not satisfied.

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O SPECJALNYCH ZBIORACH NASYCONYCH

S t r e s z c z e n i e

W pracy zbadano zbiory, które są bardzo regularne w sensie algebraicznym i całkowicie niemierzalne względem miary Lebesgue'a. Znaleziono warunki, jakie musi spełniać przestrzeń mierzalna, aby zbiory tego typu pozostały całkowicie niemierzalne. Otrzymane warunki okazały się bliskie, lecz nierównoważne własnościom Steinhausa i Smitala.

Słowa kluczowe: zbiór niemierzalny nasycony, własność Steinhausa, własność Smitala

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Dedicated to Professor Władysław Wiczyński
on the occasion of his 70th birthday

Agnieszka Sibelska

ON THE CLASS $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES OF THE BAVRIN TYPE

Summary

The paper concerns functions of several complex variables, holomorphic in bounded complete n-circular domains $\mathcal{G}$ that fulfill some geometric conditions. The classes $\mathcal{X}_G$ of such type of functions were considered among others by Bavrin [1], Dobrowolska and Liczberski [4], Fukui [5], Jakubowski and Kamiński [6] and Stankiewicz [8]. The above functions were applied to research of on some families of locally biholomorphic mappings in $\mathcal{C}^n$ (see for instance Pfaltzgraff and Suffridge [9]). R. Długosz and E. Leś in [3] defined the family $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ that corresponds to the class of classically normalized univalent functions, starlike with respect to two symmetric points. In this paper we define the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ using the Temljakov operator and the evenness of functions. We prove the Alexander type theorem for the families $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ and the inclusion $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ (analogously as for the Bavrin classes $\mathcal{N}_{\mathcal{G}}$ and $\mathcal{M}_{\mathcal{G}}$ corresponding to the well-known families of convex and starlike univalent functions of one variable, respectively). We show the sharp estimates for $\mathcal{G}$-balances of k-homogeneous polynomials, which appear in the Taylor series development of any function of the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. We also give a topological property of the family $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

Keywords and phrases: holomorphic function of several complex variables, complete n-circular domain, the Minkowski function, the Temljakov operator, $\mathcal{G}$-balance of k-homogeneous polynomials

AMS Subject Classifications: MSC2010: 32A30, 30C45

1. Introduction

A domain $\mathcal{G} \subset \mathbb{C}^n$, $n \geq 2$, containing the origin is called complete n-circular domain if $z\Lambda = (z_1\lambda_1, ..., z_n\lambda_n) \in \mathcal{G}$ for any $z = (z_1, ..., z_n) \in \mathcal{G}$ and for each $\Lambda = (\lambda_1, ..., \lambda_n) \in \overline{U^n}$, where U is the unit disc $\{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$.

In the paper we consider the bounded complete circular domains, which play the same role for Taylor series in $\mathbb{C}^n$ as open discs in one dimensional case.

Let us note that every bounded complete n-circular domain $\mathcal{G} \in \mathbb{C}$ is symmetric $(-1\mathcal{G} = \mathcal{G})$.

By $H_\mathcal{G}$, let us denote the family of all holomorphic functions $f : \mathcal{G} \longrightarrow \mathbb{C}$, $\mathcal{G} \in \mathbb{C}^n$ and by $\mathcal{H}_\mathcal{G}\,(1)$ the subclass of $H_\mathcal{G}$ of functions with the normalization $f(0) = 1$.

It is known that each function $f \in \mathcal{H}_\mathcal{G}\,(1)$ can be developed into series of k-homogeneous polynomials $Q_{f,k},\ k \in \mathbb{N}$ of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} Q_{f,k}(z), \quad z \in \mathcal{G}, \tag{1}$$

where

$$Q_{f,k} = \sum_{\alpha_1+\ldots+\alpha_n=k} c_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

and the coefficients $c_{\alpha_1\ldots\alpha_n}$, $\alpha_j \in \mathbb{N} \cup \{0\}, j = 1, \ldots, n$ are defined by the partial derivatives

$$c_{\alpha_1\ldots\alpha_n} = \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}(0).$$

Every bounded complete n-circular domain $\mathcal{G}$ and its boundary $\partial\mathcal{G}$ can be determined by using the Minkowski function of the domain $\mathcal{G}$, $\mu_\mathcal{G} : \mathbb{C}^n \longrightarrow [0, +\infty)$ of the form

$$\mu_\mathcal{G}(z) = inf\{t > 0 : \frac{1}{t} z \in \mathcal{G}\} \qquad z \in \mathcal{C}^n.$$

Namely,

$$\mathcal{G} = \{z \in \mathbb{C}^n : \mu_\mathcal{G}(z) < 1\}, \quad \partial\mathcal{G} = \{z \in \mathbb{C}^n : \mu_\mathcal{G}(z) = 1\}.$$

Bearing in mind that for the considered domains $\mu_\mathcal{G}$ is a seminorm in $\mathbb{C}^n$ and it is a norm in $\mathbb{C}^n$ in the case if $\mathcal{G}$ is also convex, we will use a generalization $\mu_\mathcal{G}(Q_{f,k})$ of the norm of homogeneous polynomials $Q_{f,k}$. Putting for $k \in \mathbb{N}$

$$\mu_\mathcal{G}(Q_{f,k}) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|Q_{f,\,k}(w)|}{(\mu_\mathcal{G}(w))^k}$$

and using the k-homogeneity of $Q_{f,k}$ and the maximum principle for modulus of holomorphic functions of several variables we have

$$\mu_\mathcal{G}(Q_{f,\,k}) = \sup_{\nu \in \partial\mathcal{G}} |Q_{f,k}(\nu)| = \sup_{u \in \mathcal{G}} |Q_{f,k}(u)|.$$

It is easy to see that

$$|Q_{f,k}(w)| \leq \mu_{\mathcal{G}}(Q_{f,k})(\mu_{\mathcal{G}}(w))^k, \quad w \in \mathbb{C}^n,\ k \in \mathbb{N}, \tag{2}$$

and the above estimate generalizes the well-known inequality

$$|Q_{f,k}(w)| \leq \| Q_{f,k} \| \cdot \| w \|^k, \quad w \in \mathbb{C}^n,\ k \in \mathbb{N}.$$

By the above considerations and in view of the fact that every complete n-circular domain is balanced, the quantities $\mu_{\mathcal{G}}(z)$ and $\mu_{\mathcal{G}}(Q_{f,k})$ will be called *$\mathcal{G}$-balance of the point z* and *$\mathcal{G}$-balance of k-homogeneous polynomials $Q_{f,k}$*, respectively.

I. I. Bavrin in [1] gave the analytic conditions which define some families of $\mathcal{X}_{\mathcal{G}}$ type by the Temljakov operator $\mathcal{L} : H_{\mathcal{G}} \longrightarrow H_{\mathcal{G}}$ of the form

$$\mathcal{L}f(z) = f(z) + Df(z)(z), \quad z \in \mathcal{G},$$

where $Df(z)$ is the Frechet derivative of the function f at the point $z \in \mathcal{G}$.

It is known that the inverse operator to $\mathcal{L}$ has the form

$$\mathcal{L}^{-1}f(z) = \int_0^1 f(tz)dt, \ \ z \in \mathcal{G}.$$

In [1] there have been defined among others the following classes:

$$\mathcal{M}_{\mathcal{G}} = \{f \in H_{\mathcal{G}}(1) : \mathrm{Re}\frac{\mathcal{L}f(z)}{f\,(z)} > 0,\ z \in \mathcal{G}\},$$

$$\mathcal{N}_{\mathcal{G}} = \{f \in H_{\mathcal{G}}(1) : \mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f\,(z)} > 0,\ z \in \mathcal{G}\},$$

$$\mathcal{C}_{\mathcal{G}} = \{f \in H_{\mathcal{G}}(1) : \mathrm{Re}f(z) > 0,\ z \in \mathcal{G}\}\,,$$

$$\mathcal{R}_{\mathcal{G}} = \{f \in H_{\mathcal{G}}(1) : \exists \varphi \in \mathcal{N}_{\mathcal{G}}\ \ \mathrm{Re}\ \frac{\mathcal{L}f(z)}{\mathcal{L}\varphi\,(z)} > 0,\ z \in \mathcal{G}\}.$$

The prototypes of the above families in the geometric theory of functions of one complex variable are well-known families of starlike, convex, with a positive real part and close-to convex functions, respectively.

I. I. Bavrin [1] showed among others that the inclusion

$$\mathcal{N}_{\mathcal{G}} \subsetneq \mathcal{M}_{\mathcal{G}} \tag{3}$$

holds and he proved the Alexander type theorem for these classes i.e. if $f \in \mathcal{N}_{\mathcal{G}}$ then $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ and conversely, if $f \in \mathcal{M}_{\mathcal{G}}$ than $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}$.

R. Długosz and E. Leś using the even parts of functions of the class $H_{\mathcal{G}}$ defined in [3] the class $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ by the formula

$$\mathcal{M}_{\mathcal{G}}^{\mathcal{S}} = \left\{f \in H_{\mathcal{G}}(1) : \mathrm{Re}\frac{\mathcal{L}f(z)}{f_E\,(z)} > 0,\ z \in \mathcal{G}\right\}, \tag{4}$$

where f_E is the even part of f.

It is known that for any complex-valued function f on a symmetric set $G \in \mathbb{C}^n$ there exist exactly one function f_E from the class $\mathcal{F}_E(G)$ of even functions and exactly one function f_O from the class $\mathcal{F}_O(G)$ of odd functions such that $f = f_E + f_O$ and

$$f_E(z) = \frac{1}{2}\left[f(z) + f(-z)\right], \quad f_O(z) = \frac{1}{2}\left[f(z) - f(-z)\right] \quad z \in G.$$

Let us observe that if $\mathcal{G}$ is a bounded complete n-circular domain then for any function $f \in H_{\mathcal{G}}(1)$ the condition

$$(\mathcal{L}f)_E = \mathcal{L}\left(f_E\right) \tag{5}$$

holds.

It is reasonable to ask, if defining the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ in the following way:

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} = \left\{ f \in H_{\mathcal{G}}(1) : \mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_E\left(z\right)} > 0,\ z \in \mathcal{G} \right\}, \tag{6}$$

where f_E is the even part of f, we will obtain for the classes $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ and $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ the inclusion of the same type as (3)? Whether for such classes of the Alexander type relationship is true? The answers to these questions give theorems contained in the rest of the paper.

2. The even functions of the class $\mathcal{N}_{\mathcal{G}}$

Before we present the main results of this paper, we will prove a few theorems concerning the subclass of the even functions of the class $\mathcal{N}_{\mathcal{G}}$

Theorem 1. *Let $\mathcal{G}$ be a bounded complete n-circular domain in $\mathbb{C}^n$, $f \in H_{\mathcal{G}}(1)$. A function $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ iff there exists a function $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$, such that*

$$\mathcal{L}\mathcal{L}f(z) = h(z)\mathcal{L}f(z), \quad z \in \mathcal{G}. \tag{7}$$

Proof. "$\Rightarrow$" Let f be a function of the class $\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. It means that $f \in \mathcal{N}_{\mathcal{G}}$ and $f = f_E$. Hence $g = \mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ and $g = \mathcal{L}f = \mathcal{L}\left(f_E\right) \overset{(5)}{=} \left(\mathcal{L}f\right)_E = g_E$, so $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. In view of (see [3], Thm 1) there exists a function $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ such that (7) holds.

"$\Leftarrow$" Let us suppose that there exists a function $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ such that (7) holds. Hence $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ (see [3], Thm 1), therefore $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ and $\mathcal{L}f = \left(\mathcal{L}f\right)_E$. In view of the Alexander type theorem [1] $\mathcal{L}^{-1}\left(\mathcal{L}f\right) = f \in \mathcal{N}_{\mathcal{G}}$ and

$$f = \mathcal{L}^{-1}\left(\mathcal{L}f\right) = \mathcal{L}^{-1}\left(\left(\mathcal{L}f\right)_E\right) = \mathcal{L}^{-1}\left(\mathcal{L}\left(f_E\right)\right) = f_E,$$

therefore $f \in \mathcal{F}_E(\mathcal{G})$. Hence $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. □

Let us observe that if $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ then $f = f_E$ and $\mathcal{L}f = \left(\mathcal{L}f\right)_E = \mathcal{L}(f_E)$. The condition (7) can be written in one of the following forms:

$$\mathcal{L}\mathcal{L}(f_E)(z) = h(z)\mathcal{L}f_E(z), \quad z \in \mathcal{G} \tag{8}$$

or

$$\mathcal{L}\mathcal{L}(f_E)(z) = h(z)\mathcal{L}f(z), \quad z \in \mathcal{G} \tag{9}$$

or

$$\mathcal{L}\mathcal{L}(f)(z) = h(z)\mathcal{L}f_E(z), \quad z \in \mathcal{G}, \tag{10}$$

where $h \in \mathcal{C} \cap \mathcal{F}_E(\mathcal{G})$ in all equations (8)–(10).

In view of Theorem 1 a function $f \in H_{\mathcal{G}}(1)$ fulfilling (7) with $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ belongs to the class $\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. Some question comes to mind, in the cases of the equations (8)–(10) will f still belong to this partition?

The answer is negative in the case of equation (8). Indeed, let us consider the function of the form

$$f(z) = 1 - \frac{1}{\Delta}\sum_{j=1}^{n} z_j, \quad z = (z_1, z_2, \dots, z_n) \in \mathcal{G}, \tag{11}$$

where

$$\Delta = \Delta(\mathcal{G}) = \sup_{z=(z_1,z_2,\dots,z_n)\in\mathcal{G}} \left|\sum_{j=1}^{n} z_j\right|.$$

The function f of the form (11) is holomorphic in $\mathcal{G}$, $f(0) = 1$, therefore $f \in H_{\mathcal{G}}(1)$. Moreover

$$f_E(z) = \frac{1}{2}\left(f(z) + f(-z)\right) = 1 \quad \text{and} \quad \mathcal{L}f_E(z) = 1,\ \mathcal{L}\mathcal{L}\left(f_E\right)(z) = 1.$$

So the function f fulfills (8) with $h(z) \equiv 1$, $z \in \mathcal{G}$, wherein $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. On the other hand $f \notin \mathcal{M}_{\mathcal{G}}$ (see [3], p. 888), therefore $f \notin \mathcal{N}_{\mathcal{G}}$.

The following result gives the answer to the above question in the case (9) or (10).

Theorem 2. *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain and $f \in H_{\mathcal{G}}(1)$. If there exists a function $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ such that the equality* (9) *or* (10) *holds then $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$.*

Proof. Let $\mathcal{G}$ satisfies the assumptions of the Theorem 2, $f \in H_{\mathcal{G}}(1)$ and there exists a function $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ such that the equation (9) holds. Then [3] $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. Since $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ then $\mathcal{L}^{-1}(\mathcal{L}f) = f \in \mathcal{N}_{\mathcal{G}}$ and the fact that $\mathcal{L}f \in \mathcal{F}_E(\mathcal{G})$ implies the equality $\mathcal{L}f = (\mathcal{L}f)_E$. Hence $f = \mathcal{L}^{-1}(\mathcal{L}f) = \mathcal{L}^{-1}\left((\mathcal{L}f)_E\right) \overset{(5)}{=} \mathcal{L}^{-1}\left(\mathcal{L}f_E\right) = f_E$, therefore $f \in \mathcal{F}_E(\mathcal{G})$. Finnaly $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$.

Now let $h \in \mathcal{C}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ be such that (10) holds. Then $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ and proceeding analogously to the first part of the proof we obtain $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$. □

Let us note that the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ defined by (6) is the class of functions fulfilling the condition (10) with some function $h \in \mathcal{C}_{\mathcal{G}}$ without the assumption that $h \in \mathcal{F}_E(\mathcal{G})$.

3. The class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$

First we will prove the Alexander type relationship between the classes $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. This theorem will be very useful in further investigation of the properties of the family $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

Theorem 3. *If $f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ then $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and conversely, if $f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ then $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.*

Proof. Let us assume that $f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. It means that $f \in H_{\mathcal{G}}(1)$ and there exists $h \in \mathcal{C}_{\mathcal{G}}$ such that

$$\mathcal{L}\mathcal{L}f(z) = h(z)\mathcal{L}f_E(z), \quad z \in \mathcal{G}. \tag{12}$$

In view of (5) the condition (12) can be written in the form

$$\mathcal{L}\left(\mathcal{L}f(z)\right) = h(z)\left(\mathcal{L}f\right)_E(z), \quad z \in \mathcal{G},$$

therefore $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$.

Now let f belong to the class $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ i.e. $f \in H_{\mathcal{G}}(1)$ and there exists $h \in \mathcal{C}_{\mathcal{G}}$ such that

$$\mathcal{L}f(z) = h(z)f_E(z), \quad z \in \mathcal{G}.$$

Hence

$$\mathcal{L}\left(\mathcal{L}\mathcal{L}^{-1}f(z)\right) = h(z)\mathcal{L}\mathcal{L}^{-1}f_E(z), \quad z \in \mathcal{G},$$

$$\mathcal{L}\mathcal{L}\left(\mathcal{L}^{-1}f(z)\right) = h(z)\mathcal{L}\left((\mathcal{L}^{-1}f)_E(z)\right), \quad z \in \mathcal{G}.$$

Therefore $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. $\square$

Next result shows dependencies between the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ and the subclass of the class $\mathcal{N}_{\mathcal{G}}$ of even functions.

Theorem 4. *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain. The following inclusions hold*

$$\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G}) \subsetneq \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}, \tag{13}$$

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \cap \mathcal{F}_E(\mathcal{G}) \subsetneq \mathcal{N}_{\mathcal{G}}. \tag{14}$$

Proof. Let us observe that $\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G}) \subset \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. Indeed, a membership of function f in the class $\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_E(\mathcal{G})$ means that

$$f \in H_{\mathcal{G}}(1), \quad \mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f(z)} > 0,\ z \in \mathcal{G} \quad \text{and} \quad f = f_E.$$

Hence f fulfills the condition

$$\mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_E(z)} > 0,\ z \in \mathcal{G}$$

which defines the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

To prove that equality in (13) does not hold, let us consider the functions of the form

(15)

$$\bar{f}_a(z)=\begin{cases} \frac{\Delta}{2\sum_{j=1}^n z_j}\left[(1-a)\log\left(1+\frac{\sum_{j=1}^n z_j}{\Delta}\right)-(1+a)\log\left(1-\frac{\sum_{j=1}^n z_j}{\Delta}\right)\right] & z\in\mathcal{G}\setminus\mathcal{A},\\ 1 & z\in\mathcal{A},\end{cases}$$

where $\log 1=0$, $a\in(0,1)$ and $\mathcal{A}=\{z\in\mathcal{G}:\sum_{j=1}^n z_j=0\}$.

The functions $\bar{f}_a$ of the form (15) extend holomorphicaly from $\mathcal{G}\setminus\mathcal{A}$ to $\mathcal{G}$ ($\mathcal{A}$ is closed and anywhere dense set in $\mathcal{G}$), therefore $\bar{f}_a\in H_{\mathcal{G}}(1),\ a(0,1)$. In view of Theorem 3 and the fact that the functions

$$\tilde{f}_a=\Delta\frac{\Delta+a\sum_{j=1}^n z_j}{\Delta^2-\sum_{j=1}^n z_j},\ z\in\mathcal{G},\ a\in(0,1)$$

belong to the class $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ (see [2]) we obtain that $\bar{f}_a\in\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

Moreover

$$\bar{f}_{aE}(z)=\frac{\Delta}{2\sum_{j=1}^n z_j}\log\frac{\Delta+\sum_{j=1}^n z_j}{\Delta-\sum_{j=1}^n z_j},\quad z\in\mathcal{G},\quad a\in(0,1),\quad \log 1=0,$$

therefore $\bar{f}_{aE}\neq\bar{f}_a$ at an arbitrarily fixed $a\in(0,1)$. Hence $\bar{f}_a\notin\mathcal{F}_E(\mathcal{G})$, so $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}\neq\mathcal{N}_{\mathcal{G}}\cap\mathcal{F}_E(\mathcal{G})$.

Analogously as in the case of inclusion (13) it is easy to check that $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}\cap\mathcal{F}_E(\mathcal{G})\subset\mathcal{N}_{\mathcal{G}}$. We will show that there exists a function $f\in\mathcal{N}_{\mathcal{G}}$ such that $f\notin\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}\cap\mathcal{F}_E(\mathcal{G})$.

Let f be the function of the form

(16) $$f(z)=\frac{\Delta}{\Delta-\sum_{j=1}^n z_j},\quad z\in\mathcal{G}.$$

It is known [1] that $f\in\mathcal{N}_{\mathcal{G}}$. Moreover

$$f_E(z)=\frac{\Delta^2}{\Delta^2-\left(\sum_{j=1}^n z_j\right)^2},$$

therefore $f\neq f_E$ and consequently $f\notin\mathcal{F}_E(\mathcal{G})$. Hence $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}\cap\mathcal{F}_E(\mathcal{G})\neq\mathcal{N}_{\mathcal{G}}$. □

Directly Theorem 4 implies

Corollary 1. *Let $\mathcal{G}\subset\mathbb{C}^n$ be a bounded complete n-circular domain, then*

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}\cap\mathcal{F}_E(\mathcal{G})=\mathcal{N}_{\mathcal{G}}\cap\mathcal{F}_E(\mathcal{G}).$$

In the proof of next theorem we will use the following result of E. Leś and P. Liczberski:

Theorem A. [7] *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain. Let us assume that a function $F \in H_{\mathcal{G}}(1)$, $H \in \mathcal{M}_{\mathcal{G}}$ and ρ is a relation defined as follows:*

$$F \rho H \Longleftrightarrow \mathrm{Re}\frac{F(z)}{H(z)} > 0, \quad z \in \mathcal{G}. \tag{17}$$

If $(\mathcal{L}F)\rho(\mathcal{L}H)$, then $F\rho H$.

Theorem 5. *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain. The following inclusion holds:*

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}.$$

Proof. Let f belong to $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. It means that $f \in H_{\mathcal{G}}(1)$ and $\mathcal{L}f \in H_{\mathcal{G}}(1)$. Let us note that $f_E \in \mathcal{N}_{\mathcal{G}}$. Indeed, we have

$$\mathrm{Re}\frac{\mathcal{L}\mathcal{L}f_E(z)}{\mathcal{L}f_E(z)} = \frac{1}{2}\left[\mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_E(z)} + \mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(-z)}{\mathcal{L}f_E(-z)}\right] > 0.$$

From (3) it follows that $f_E \in \mathcal{M}_{\mathcal{G}}$. Putting $F = \mathcal{L}f$ and $H = f_E$ and using the definition of the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ and (17) we have $\mathcal{L}F\rho\mathcal{L}H$. In view of Theorem A we obtain $F\rho H$ ie.:

$$\mathrm{Re}\frac{\mathcal{L}f(z)}{f_E(z)} > 0 \; z \in \mathcal{G}.$$

Finally, $f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$.

The function of the form (16) belongs to the class $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ (see [3]). Moreover

$$\mathcal{L}f(z) = \frac{\Delta}{\left(\Delta - \sum_{j=1}^n z_j\right)^2}, \qquad \mathcal{L}\mathcal{L}f(z) = \frac{\Delta^2\left(\Delta + \sum_{j=1}^n z_j\right)}{\left(\Delta - \sum_{j=1}^n z_j\right)^3},$$

$$f_E(z) = \frac{\Delta^2}{\Delta^2 - \left(\sum_{j=1}^n z_j\right)^2}, \qquad \mathcal{L}f_E(z) = \frac{\Delta^2\left(\Delta^2 + (\sum_{j=1}^n z_j)^2\right)}{\left(\Delta^2 - (\sum_{j=1}^n z_j)^2\right)^2}, \quad z \in \mathcal{G}$$

and consequently

$$\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_E(z)} = \frac{\left(\Delta + \sum_{j=1}^n z_j\right)^3}{\left(\Delta - \sum_{j=1}^n z_j\right)\left(\Delta^2 + (\sum_{j=1}^n z_j)^2\right)}, \quad z \in \mathcal{G}$$

and at every point $\overset{o}{z} \in \mathcal{G}$ such that

$$\sum_{j=1}^n \overset{o}{z}_j = ri\Delta \quad \text{and} \quad r \in (\sqrt{3 - 2\sqrt{2}}, 1)$$

we have

$$\mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(\overset{o}{z})}{\mathcal{L}f_E(\overset{o}{z})} = \frac{r^4 - 6r^2 + 1}{1 - r^4} < 0,$$

therefore the function f does not belong to the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. $\square$

In [3] there was proved the veracity of the inclusions

$$\mathcal{N}_{\mathcal{G}} \subsetneq \mathcal{M}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{R}_{\mathcal{G}}.$$

Theorem 5 implies

Corollary 2. *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain. The inclusions*

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{M}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{R}_{\mathcal{G}}$$

hold.

Theorem 6. *The inclusions*

$$\mathcal{N}_{\mathcal{G}} \subset \mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \tag{18}$$

and

$$\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \subset \mathcal{N}_{\mathcal{G}} \tag{19}$$

do not hold.

Proof. In the case of the inclusion (18) it suffices to consider the function of the form (16). It is known [1] that $f \in \mathcal{N}_{\mathcal{G}}$. In the proof of Theorem 5 we showed that $f \notin \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

To prove the statement (19) we consider the function $\bar{f}_{\frac{1}{2}}$ of the form (15). It is known that $\bar{f}_{\frac{1}{2}} \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ (see proof of Theorem 4).

Moreover, we have

$$\mathcal{L}\bar{f}_{\frac{1}{2}}(z) = \Delta \frac{\Delta + \frac{1}{2}\sum_{j=1}^{n} z_j}{\Delta^2 - \sum_{j=1}^{n} z_j}, \quad \mathcal{L}\mathcal{L}\bar{f}_{\frac{1}{2}}(z) = \Delta^2 \frac{\Delta^2 + \Delta \sum_{j=1}^{n} z_j + \left(\sum_{j=1}^{n} z_j\right)^2}{\left(\Delta^2 - \left(\sum_{j=1}^{n} z_j\right)^2\right)^2}, \ z \in \mathcal{G}.$$

The condition $\mathcal{L}\mathcal{L}\bar{f}_{\frac{1}{2}}(z) = h(z)\mathcal{L}\bar{f}_{\frac{1}{2}}(z)$ holds in $\mathcal{G}$ with

$$h(z) = \Delta \frac{\Delta^2 + \Delta \sum_{j=1}^{n} z_j + \left(\sum_{j=1}^{n} z_j\right)^2}{\left(\Delta^2 - \left(\sum_{j=1}^{n} z_j\right)^2\right)\left(\Delta + \frac{1}{2}\sum_{j=1}^{n} z_j\right)} \in H_{\mathcal{G}}(1).$$

The function h extends continuously to all boundary points $z \in \partial\mathcal{G}$ such that

$$\sum_{j=1}^{n} z_j = \Delta e^{i\theta}, \ \theta \in (0, 2\pi) \setminus \{\pi\}.$$

Putting $\overset{\star}{z} = (\overset{\star}{z}_1, \ldots, \overset{\star}{z}_n)$ such that

$$\sum_{j=1}^{n} \overset{\star}{z}_j = \Delta e^{i\frac{5\pi}{6}}$$

we have

$$\mathrm{Re}h(\overset{\star}{z}) < 0,$$

thus there exist points $z \in \mathcal{G}$ such that $\mathrm{Re}h(z) < 0$, so $h \notin \mathcal{C}_{\mathcal{G}}$. Hence $\bar{f}_{\frac{1}{2}} \notin \mathcal{N}_{\mathcal{G}}$. $\square$

The above considerations show that

Corollary 3. *The class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ does not separate the classes $\mathcal{N}_{\mathcal{G}}$ and $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and also the class $\mathcal{N}_{\mathcal{G}}$ does not separate the classes $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ and $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$.*

In the next theorem we give the sharp estimates $\mathcal{G}$-balance of k-homogeneous polynomials which appear in the Taylor series development of the form (1).

Theorem 7. *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain and let $f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ develop in the Taylor series of the form (1). Then the following sharp estimates hold*

$$\mu_{\mathcal{G}}(Q_{f,k}) \le \frac{1}{k+1}, \quad k \in \mathbb{N}. \tag{20}$$

Proof. Let $f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ develops into the Taylor series of the form (1). From Theorem 3 we have $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and

$$\mathcal{L}f(z) = \sum_{k=0}^{\infty}(k+1)Q_{f,k}, \quad z \in \mathcal{G}, \quad Q_{f,0} = 1.$$

Hence

$$Q_{\mathcal{L}f,k}(z) = (k+1)Q_{f,k}(z), \quad z \in \mathcal{G}, \quad k \in \mathbb{N}.$$

It is known [3] that for a function $g \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ the sharp estimates $\mu_{\mathcal{G}}(Q_{g,k}) \le 1$, $k \in \mathbb{N}$ hold. Therefore we have

$$1 \ge \mu_{\mathcal{G}}(Q_{\mathcal{L}f,k}) = \sup_{z\in\mathcal{G}} |Q_{\mathcal{L}f,k}(z)| = \sup_{z\in\mathcal{G}} |(k+1)Q_{f,k}(z)|,$$

hence

$$|Q_{f,k}(z)| \le \frac{1}{k+1}, \quad z \in \mathcal{G}, \quad k \in \mathbb{N}.$$

Finally

$$\mu_{\mathcal{G}}(Q_{f,k}) = \sup_{z\in\mathcal{G}} |Q_{f,k}(z)| \le \frac{1}{k+1}, \quad k \in \mathbb{N}.$$

The equality in (20) is realized by the function $\tilde{f}$ of the form

$$\tilde{f}(z) = \begin{cases} \frac{-\Delta}{\sum_{j=1}^n z_j} \log\left(1 - \frac{\sum_{j=1}^n z_j}{\Delta}\right) & z \in \mathcal{G} \setminus \mathcal{A}, \\ 1 & z \in \mathcal{A}, \end{cases} \tag{21}$$

where $log 1 = 0$, $\mathcal{A} = \{z \in \mathcal{G} : \sum_{j=1}^n z_j = 0\}$.

The function $\tilde{f} \in H_{\mathcal{G}}(1)$ (it extends holomorphicaly on $\mathcal{G}$). In view of Theorem 3 this function belongs to the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$, because of

$$\tilde{f} = \mathcal{L}^{-1}f(z),$$

where f is the function of the form (16) and $f \in \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$.

Let us observe that $\mathcal{G}$-balance $\mu_{\mathcal{G}}(Q_{\tilde{f},k})$ it is enough to consider on the set $\mathcal{G} \setminus \mathcal{A}$.

The function $\tilde{f}$ develops in the series of the form (1), where

$$Q_{\tilde{f},k}(z) = \frac{1}{k+1}\left(\frac{\sum_{j=1}^{n} z_j}{\Delta}\right)^k, \quad z \in \mathbb{C}^n, \quad k \in \mathbb{N},$$

therefore

$$\mu_{\mathcal{G}}(Q_{\tilde{f},k}) = \sup_{z\in\mathcal{G}} \left| \frac{1}{k+1}\left(\frac{\sum_{j=1}^{n} z_j}{\Delta}\right)^k \right| =$$

$$= \sup_{z\in\mathcal{G}} \frac{1}{k+1}\left|\frac{\sum_{j=1}^{n} z_j}{\Delta}\right|^k = \frac{1}{k+1}\left(\frac{\Delta}{\Delta}\right)^k = \frac{1}{k+1}.$$

□

Next results concern the topological properties of the family $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

Observation *The family $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ is not convex.*

Proof. Let us consider the function $f = \frac{1}{2}(g+h)$, where

$$g(z) = \frac{\Delta}{2\sum_{j=1}^{n} z_j} \log\left(\frac{\Delta + \sum_{j=1}^{n} z_j}{\Delta - \sum_{j=1}^{n} z_j}\right),$$

$$h(z) = -\frac{i\Delta}{2\sum_{j=1}^{n} z_j} \log\left(\frac{\Delta + i\sum_{j=1}^{n} z_j}{\Delta - i\sum_{j=1}^{n} z_j}\right), \quad z \in \mathcal{G}.$$

The functions $g, h \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$, because of

$$\mathcal{L}g(z) = \frac{\Delta^2}{\Delta^2 - \left(\sum_{j=1}^{n} z_j\right)^2} \quad \text{and} \quad \mathcal{L}h(z) = \frac{\Delta^2}{\Delta^2 + \left(\sum_{j=1}^{n} z_j\right)^2}, \quad z \in \mathcal{G}$$

belong to the class $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ and Theorem 3 holds. Moreover

$$f(z) = \frac{\Delta}{4\sum_{j=1}^{n} z_j}\left(\log\left(\frac{\Delta + \sum_{j=1}^{n} z_j}{\Delta - \sum_{j=1}^{n} z_j}\right) - i\log\left(\frac{\Delta + i\sum_{j=1}^{n} z_j}{\Delta - i\sum_{j=1}^{n} z_j}\right)\right),$$

$$\mathcal{L}f(z) = \frac{\Delta^4}{\Delta^4 - \left(\sum_{j=1}^{n} z_j\right)^4}, \qquad \mathcal{L}\mathcal{L}f(z) = \frac{\Delta^8 + 3\Delta^4\left(\sum_{j=1}^{n} z_j\right)^4}{\left(\Delta^4 - \left(\sum_{j=1}^{n} z_j\right)^4\right)^2}, \qquad z \in \mathcal{G}$$

and $f_E(z) = f(z), \quad z \in \mathcal{G}$.

Hence

$$\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_E(z)} = \frac{\Delta^4 + 3\left(\sum_{j=1}^{n} z_j\right)^4}{\Delta^4 - \left(\sum_{j=1}^{n} z_j\right)^4}, \quad z \in \mathcal{G}.$$

Choosing $\overset{0}{z} \in \mathcal{G}$ such that

$$\mu_{\mathcal{G}}(\overset{0}{z}) = r \in (\frac{1}{3}, 1) \quad \text{and} \quad \sum_{j=1}^{n} \overset{0}{z}_j = \Delta \sqrt[4]{-r}$$

we have

$$\frac{\mathcal{L}\mathcal{L}f(\overset{0}{z})}{\mathcal{L}f_E(\overset{0}{z})} = \frac{1-3r}{1+r} < 0,$$

therefore $f \notin \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. Hence the family $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ is not convex. $\square$

However the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ is a path connected set in the space $H_{\mathcal{G}}(1)$ with the topology introduced by the closure operator in the following way that for any $\mathcal{B} \subset H_{\mathcal{G}}(1)$ we denote by $\bar{\mathcal{B}}$ a set of functions $f \in H_{\mathcal{G}}(1)$ for which there exists a sequence $\{f_\nu\}$ of functions $f_\nu \in \mathcal{B} \subset H_{\mathcal{G}}(1)$, almost uniformly convergent to f on the set $\mathcal{G}$. Then the family $\tau = \{H_{\mathcal{G}}(1) \setminus \mathcal{B} : \mathcal{B} = \bar{\mathcal{B}}\}$ is the mentioned topology.

In the proof of path connectivity of the class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ there will be used

Lemma 8. *Let $f_\nu,\ \nu \in [0,1]$ belong to a one-parameter family of functions of the form*

$$f_\nu(z) = f(\nu z), \quad z \in \mathcal{G}, \quad f \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}. \tag{22}$$

The functions from this family are equicontinuous with respect to ν in

$$\overline{r\mathcal{G}} := \{z \in \mathbb{C}^n : \mu_{\mathcal{G}}(z) \le r\},$$

it means

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{\nu_1,\nu_2\in[0,1]}\ \forall_{z\in\overline{r\mathcal{G}}}\ \ (|\nu_2-\nu_1| < \delta \Rightarrow |f_{\nu_2}(z) - f_{\nu_1}(z)| < \epsilon). \tag{23}$$

Proof. Let us observe that from the definition (22) of f_ν and by the form (1) of the development of the function f into the series of k-homogeneous polynomials, we have

$$|f_{\nu_2}(z) - f_{\nu_1}(z)| = |f(\nu_2 z) - f(\nu_1 z)| = |\sum_{k=1}^{\infty}(\nu_2^k - \nu_1^k)Q_{f,k}(z)| \le$$

$$\le \sum_{k=1}^{\infty}|\nu_2^k - \nu_1^k||Q_{f,k}(z)|$$

In view of (2) and Theorem 6 we obtain

$$|f_{\nu_2}(z) - f_{\nu_1}(z)| \le \sum_{k=1}^{\infty} k|\nu_2-\nu_1|\mu_{\mathcal{G}}(Q_{f,k})\,(\mu_{\mathcal{G}}(z))^k \le |\nu_2-\nu_1|\sum_{k=1}^{\infty}\frac{k}{k+1}r^k \le$$

$$\le |\nu_2-\nu_1|\sum_{k=1}^{\infty} r^k = |\nu_2-\nu_1|\frac{r}{1-r}.$$

Putting $\delta = \frac{\epsilon(1-r)}{r}$ we obtain (23). $\square$

Theorem 9. *The class $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ is a path connected set, so connected set in the space $H_{\mathcal{G}}(1)$ with the topology introduced by the closure operator.*

Proof. We will show that for every two functions $\check{f}, \hat{f} \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ there exists a one-parameter family of functions $f_\nu \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$, $\nu \in [0,1]$ such that $f_0 = \check{f}$, $f_1 = \hat{f}$ and for any sequence of numbers $\{\nu_k\}_{k\in\mathbb{N}}$, where $\nu_1 = 0$, $\lim_{k\to\infty} \nu_k = 1$, a corresponding sequence $\{f_{\nu_k}\}_{k\in\mathbb{N}}$, $f_{\nu_k} \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$, $k \in \mathbb{N}$ converges almost uniformly to $\hat{f}$ on the set $\mathcal{G}$.

Let us consider an arbitrarily fixed compact set $K \subset \mathcal{G}$. Then there exists $r \in (0,1)$ such that $K \subset \overline{r\mathcal{G}} \subset \mathcal{G}$. In view of Lemma 8 there exists on the set K the limit $\lim_{k\to\to\infty} f_{\nu_k}(z) = f_1(z) = f(z), z \in \mathcal{G}$ and the convergence is uniform. In view of the arbitrariness of K we have almost uniform convergence of sequence $\{f_{\nu_k}\}$ to $f_1 = f$ in $\mathcal{G}$.

Similarly, for every sequence $\{\nu_k\}_{k\in\mathbb{N}}$, $\nu_k \in [0,1]$ $\nu_1 = 1$, $\lim_{k\to\infty} \nu_k = 0$ an adequate sequence $\{f_{\nu_k}\}_{k\in\mathbb{N}}$ is almost uniformly convergent on $\mathcal{G}$ to $f_0 = 1$.

Ultimately, for every two functions $\check{f}, \hat{f} \in \mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ the searched family is the sum of the following families

$$\check{f}_\nu(z) = \check{f}\left((1-2\nu)z\right), \quad \nu \in \left[0, \frac{1}{2}\right], \ \check{f}_0 = \check{f}, \ \check{f}_{\frac{1}{2}} = 1$$

and

$$\hat{f}_\nu(z) = \hat{f}\left((2\nu-1)z\right), \quad \nu \in \left[\frac{1}{2}, 1\right], \ \hat{f}_{\frac{1}{2}} = 1, \ \hat{f}_1 = \hat{f}$$

and this family joins the functions $\check{f}, \hat{f}$ through the function $f = 1$ (in the sense of almost unfiormly convergence). □

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O KLASIE $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ TYPU BAVRINOWSKIEGO FUNKCJI HOLOMORFICZNYCH WIELU ZMIENNYCH ZESPOLONYH

S t r e s z c z e n i e

W pracy rozważane są funkcje wielu zmiennych zespolonych, holomorficzne w ograniczonym pełnym obszarze n-kołowym $\mathcal{G}$ i spełniające w nim pewne warunki geometryczne. Klasy $\mathcal{X}_G$ tego typu były rozważane między innymi przez Bavrina [1], Dobrowolską i Liczberskiego [4], Fukui [5], Jakubowskiego i Kamińskiego [6] oraz Stankiewicza [8]. Wspomniane wyżej funkcje są wykorzystywane do badania pewnych rodzin odwzorowań biholomorficznych w $\mathcal{C}^n$ (np. Pfaltzgraff and Suffridge [9]). Długosz i Leś zdefiniowały w [3] rodzinę $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ będącą swego rodzaju odpowiednikiem rodziny klasycznie znormalizowanych funkcji jednolistnych, gwiaździstych względem dwóch punktów symetrycznych. W tej pracy, wykorzystując operator Temljakova i parzystość funkcji, zdefiniowano klasę $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$. Dla rodzin $\mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ oraz $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ udowodnione zostało twierdzenie typu Alexander'a oraz inkluzja $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}} \subsetneq \mathcal{M}_{\mathcal{G}}^{\mathcal{S}}$ (analogiczna jak dla Bavrinowskich klas $\mathcal{N}_{\mathcal{G}}$ i $\mathcal{M}_{\mathcal{G}}$ będących odpowiednikami znanych rodzin funkcji wypukłych oraz gwiaździstych i jednolistnych jednej zmiennej). Pokazane zostały dokładne oszacowania dla $\mathcal{G}$-balansów wielomianów k-homogenicznych, pojawiających się w rozwinięciu każdej funkcji klasy $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$ w szereg Taylora. Podano także pewną własność topologiczną klasy $\mathcal{N}_{\mathcal{G}}^{\mathcal{S}}$.

Słowa kluczowe: funkcja holomorficzna wielu zmiennych zespolonych, obszar n-kołowy zupełny, funkcja Minkowskiego, operator Temljakova, $\mathcal{G}$-równowaga wielomianów k-jednorodnych

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Arezki Touzaline

ANALYSIS OF A VISCOELASTIC FRICTIONLESS CONTACT PROBLEM WITH UNILATERAL CONSTRAINT

Summary

We consider a quasistatic problem which models the contact between a deformable body and a foundation. The material is assumed to have a viscoelastic behavior that we model with a constitutive law with long-term memory; thus, at each moment of time, the stress tensor depends not only on the present strain tensor, but also on its whole history. The contact is frictionless and is modeled by a normal compliance condition with unilateral constraint. We derive a weak formulation of the problem and, under appropriate regularity hypotheses, we prove its unique solvability by using arguments of time-dependent variational inequalities and Banach fixed point theorem. Moreover, using compactness properties we study a penalized contact problem which has a unique solution which converges to the solution of the original model by passing to the limit as the penalization parameter converges to zero.

Keywords and phrases: viscoelastic, normal compliance, frictionless, contact, weak solution

AMS and Subject Classifications: 74H20, 74M15, 47J20, 49J40

1. Introduction

Mathematical models describing the frictionless contact between a deformable body and a foundation, have been studied by several authors. The construction of mathematical models depends on several ingrendients such that the contact boundary conditions, the frictionless law and the constitutive law. The study of frictionless problems represents a first step to study more complicated frictional contact problems. Recently a book [16] appeared which introduces the reader the theory of variational inequalities with applications in contact mechanics, and, specifically, with study of antiplane frictional contact problems. Until now, in the literature of contact mechanics, we find that various contact boundary conditions have been used to model contact phenomena. One of the most popular the *Signorini condition* introduced in [17], which describes the contact with a perfectly rigid obstacle. A more general contact condition, called the *normal compliance condition with unilateral constraint*, was introduced in [12]. It models the contact with an elastic-rigid obstacle and it contains, as particular cases, both the Signorini contact condition and the normal compliance condition. Recall that the unilateral contact models take an important place in the theory of variational inequalities, see for instance [1–8, 10–15, 18, 19] and the references therein. The mathematical analysis of unilateral contact problems, including existence and uniqueness results, was widely developed in [10]. Numerical studies of contact problems with Signorini conditions were made, see for instance [3–6, 13–15] and the references therein. A numerical study of a frictional contact problem with normal compliance and unilateral constraint was made in [2]. The mathematical and the numerical state-of-the-art can be found in [13]. The analysis of elastic models involving the contact condition with normal compliance and unilateral constraint can be found in [2, 18].

The current paper represents a continuation of [18]. Its aim is to describe a new quasistatic process of frictionless contact between a viscoelastic with long term-memory, within the framework of small deformation theory. The contact is modeled with normal compliance of such a type that the penetration is restricted with unilateral constraint. We prove the existence and uniqueness result of solution and establish a convergence result. The rest of the paper is structured as follows. In Section 2 the mechanical problem (Problem P) is formulated, some notation are presented and the variational formulation (Problem P_1) is established. In Section 3 we give an existence and uniqueness result. In Section 4 we study a penalized problem. We prove that it admits a unique solution which converges strongly to the solution of Problem P_1 when the penalization parameter converges to zero.

2. Mechanical problem and its variational formulation

Let $\Omega \subset \mathbb{R}^d; (d = 2, 3)$, be a domain occupied by a viscoelastic body with long-term memory. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. We assume that Γ is composed of three sets $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, and $\bar{\Gamma}_3$, with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3, such that $meas\,(\Gamma_1) > 0$. The body is acted upon by a volume force of density f_1 in Ω and a surface traction of density f_2 on Γ_2. On Γ_3 the body is in frictionless contact with normal compliance and unilateral contraint with a foundation. Thus, the classical formulation of the mechanical problem is written as follows.

Problem P. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ such that for all $t \in [0,T]$:

$$div\sigma\left(u\left(t\right)\right) = -f_1\left(t\right) \text{ in } \Omega, \tag{2.1}$$

$$\sigma\left(u\left(t\right)\right) = F\varepsilon\left(u\left(t\right)\right) + \int_0^t \mathcal{F}\left(t-s\right)\varepsilon\left(u\left(s\right)\right)ds \text{ in } \Omega, \tag{2.2}$$

$$u\left(t\right) = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\left(u\left(t\right)\right)\nu = f_2\left(t\right) \quad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu\left(t\right) \le g,\, \sigma_\nu\left(t\right) + p\left(u_\nu\left(t\right)\right) \le 0 \\ \\ \left(\sigma_\nu\left(t\right) + p\left(u_\nu\left(t\right)\right)\right)\left(u_\nu\left(t\right) - g\right) = 0,\, \sigma_\tau\left(t\right) = 0 \end{array}\right\} \text{ on } \Gamma_3. \tag{2.5}$$

Equation (2.1) represents the equilibrium equation in which we denote $\sigma\left(t\right) = \sigma\left(u\left(t\right)\right)$ the strain tensor at time t; (2.2) is the viscoelastic constitutive law of the material in which F and $\mathcal{F}$ denote the elasticity operator and the relaxation fourth-order respectively while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. (2.5) represents the unilateral constraint where p is a normal compliance function which satisfies the assumption (2.15); g denotes the maximum value of the penetration which satisfies $g \ge 0$.

Next, we denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $\|.\|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $\|v\| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2\left(\Omega\right)\right)^d,\ H_1 = \left(H^1\left(\Omega\right)\right)^d,\ Q = \left\{\tau = \left(\tau_{ij}\right);\ \tau_{ij} = \tau_{ji} \in L^2\left(\Omega\right)\right\},$$

$$Q_1 = \left\{\sigma \in Q;\ div\sigma \in H\right\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon\left(u\right) = \left(\varepsilon_{ij}\left(u\right)\right), \text{ where } \varepsilon_{ij}\left(u\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right));$$

$div\sigma = \left(\sigma_{ij,j}\right)$ is the divergence of σ. For every $v \in H_1$ we also denote by v the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu) .\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

and the following Green's formula holds:

$$\langle \sigma, \varepsilon (v) \rangle_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where da is the surface measure element. Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\, v = 0 \text{ on } \Gamma_1\}$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V;\ v_\nu \leq g \ \ a.e. \text{ on } \Gamma_3\}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds [9],

$$\|\varepsilon (v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We endow V with the inner product

$$(u, v)_V = \langle \varepsilon (u), \varepsilon (v) \rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.10}$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X .$$

We suppose that the body forces and surface tractions have the regularity

$$f_1 \in C([0, T]; H), \qquad f_2 \in C\left([0, T]; \left(L^2 (\Gamma_2)\right)^d\right). \tag{2.11}$$

Next, we define the function $f : [0, T] \to V$ by

$$(f(t), v)_V = \int_\Omega f_1 (t) .vdx + \int_{\Gamma_2} f_2 (t) .vda \quad \forall v \in V,\, t \in [0, T] \tag{2.12}$$

and we note that (2.11) and (2.12) imply

$$f \in C([0, T]; V).$$

In the study of the mechanical problem P_1 we assume that the elasticity operator $F : \Omega \times S_d \to S_d$, satisfies

$$(2.13)\qquad \left.\begin{array}{l}(a) \text{ there exists } M>0 \text{ such that}\\ \quad \left\|F(x,\varepsilon_1)-F(x,\varepsilon_2)\right\| \le M\left\|\varepsilon_1-\varepsilon_2\right\| \quad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\\ \quad a.e.\ x\ \in\ \Omega;\\ (b) \text{ there exists } m>0 \text{ such that}\\ \qquad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m\left\|\varepsilon_1-\varepsilon_2\right\|^2,\\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x\ \in\ \Omega;\\ (c) \text{ the mapping } x\to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega\\ \ \text{for any } \varepsilon\in\ S_d;\\ (d)\ F(x,0_{S_d})=0_{S_d}, \text{ for } a.e.\ x \text{ in } \Omega.\end{array}\right\}$$

Also we need to introduce the space of the tensors of fourth order defined by

$$Q_\infty=\left\{\mathcal{E}=(\mathcal{E}_{ijkl});\ \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega)\right\},$$

which is the real Banach space with the norm

$$\|\mathcal{E}\|_{Q_\infty}=\max_{0\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

$$(2.14)\qquad \mathcal{F}\in C([0,T];Q_\infty).$$

Next, we define the functional $j: V\times V\to\mathbb{R}$ by

$$j(u,v)=\int_{\Gamma_3} p(u_\nu)\,v_\nu da \qquad \forall u,v\in V.$$

We assume that the normal compliance function p satisfies

$$(2.15)\qquad \left\{\begin{array}{l}(a)\ \ p:\mathbb{R}\to\mathbb{R};\\ (b)\ \text{ there exists } L_p>0 \text{ such that}\\ \qquad |p(r_1)-p(r_2)|\le L_p|r_1-r_2|,\ \text{ for all } r_1,\ r_2\in\mathbb{R};\\ (c)\ \ (p(r_1)-p(r_2))(r_1-r_2)\ge 0, \text{ for all } r_1,\ r_2\in\mathbb{R};\\ (d)\ \ p(r)=0 \text{ for all } r\le 0.\end{array}\right.$$

Now, we use Green's formula to obtain the following variational formulation of Problem P_1.

Problem P_1. Find a displacement field $u:[0,T]\to V$ such that

$$(2.16)\qquad \begin{array}{l} u(t)\in K,\ \langle F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t))\rangle_Q\\[1ex] +\left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u(s))\,ds,\varepsilon(v)-\varepsilon(u(t))\right\rangle_Q\\[1ex] +j(u(t),v-u(t))\ge(f(t),v-u(t))_V \quad \forall\ v\in K,\ t\in[0,T].\end{array}$$

3. Existence and uniqueness of solution

The main result in this section is the following existence and uniqueness theorem.

Theorem 3.1. *Let* (2.11) , (2.13), (2.14) *and* (2.15) *hold. Then Problem* P_1 *has a unique solution which satisfies*

$$u \in C([0,T];V).$$

To prove this theorem, for $\eta \in C([0,T];Q)$, we consider the following auxiliary problem.

Problem $P_{1\eta}$. Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{aligned} &u_\eta(t) \in K, \ \langle F\varepsilon(u_\eta(t)), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q + \langle \eta, \varepsilon(v - u_{\beta\eta}(t))\rangle_Q \\ &+ j(u_\eta(t), v - u_\eta(t)) \geq (f(t), v - u_\eta(t))_V \quad \forall v \in K, \ t \in [0,T]. \end{aligned} \tag{3.1}$$

We have the following result.

Lemma 3.2. *Problem* $P_{1\eta}$ *has a unique solution which satisfies* $u_\eta \in C([0,T];V)$.

Proof. Riesz's representation theorem leads to the existence of an element $f_\eta \in C([0,T];V)$ such that

$$(f_\eta(t), v)_V = (f(t), v)_V - \langle \eta, \varepsilon(v)\rangle_Q.$$

Let $t \in [0,T]$ and let $B : V \to V$ be the operator defined by

$$(Bu, v)_V = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + j(u,v) \quad \forall u, v \in V.$$

Using (2.13) and (2.15), we see that B is strongly monotone and Lipschitz continuous. Then from [16], since K is a nonempty closed convex subset of V, using the standard results for elliptic variational inequalities, we deduce that there exists a unique element $u_\eta(t) \in K$ which satisfies the inequality

$$\begin{aligned} &\langle F\varepsilon(u_\eta(t)), \varepsilon(v - u_\eta(t))\rangle_Q + j(u_\eta(t), v - u_\eta(t)) \\ &\geq (f_\eta(t), v - u_\eta(t))_V \quad \forall v \in K, \ t \in [0,T]. \end{aligned} \tag{3.2}$$

To show that $u_\eta \in C([0,T];V)$, it suffices to see that for t_1, $t_2 \in [0,T]$ we have

$$\|u_\eta(t_1) - u_\eta(t_2)\|_V \leq \|f_\eta(t_1) - f_\eta(t_2)\|_V / m$$

and use that $f_\eta \in C([0,T];V)$. □

Now, to end the proof we need to introduce the operator

$$\Lambda : C([0,T];Q) \to C([0,T];Q)$$

defined by

$$\Lambda\eta(t) = \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\eta(s))\,ds \quad \forall \eta \in C([0,T];Q),\ t \in [0,T]. \tag{3.3}$$

Lemma 3.3. *The operator* Λ *has a unique fixed point* η^*.

Proof. Let η_1, $\eta_2 \in C([0,T];Q)$. Relations (3.2), (3.3) and the assumption (2.14) imply that there exists a constant $c > 0$ such that

$$\left\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\right\|_Q \le c\int_0^t \left\|\eta_1(s)-\eta_2(s)\right\|_Q ds \quad \forall t\in[0,T].$$

We deduce that for a positive integer m, Λ^m is a contraction; then it admits a unique fixed point η^* which is a unique fixed point of Λ i.e.

$$\Lambda\eta^*(t)=\eta^*(t) \quad \forall t\in[0,T]. \tag{3.4}$$

Proof of Theorem 3.1. It suffices now to use (3.2) and (3.4) to see that u_{η^*} is the unique solution of the inequality (3.1).

4. A convergence result

In what follows we study the frictionless contact problem by using the penalization method. To this end, for each $\delta > 0$ we consider the following penalized contact problem.

Problem P_δ. Find a displacement field $u_\delta : \Omega\times[0,T]\to\mathbb{R}^d$ such that for all $t\in[0,T]$:

$$div\sigma(u_\delta(t)) = -f_1(t) \text{ in } \Omega, \tag{4.1}$$

$$\sigma(u_\delta(t)) = F\varepsilon(u_\delta(t)) + \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds \text{ in } \Omega, \tag{4.2}$$

$$u_\delta(t)=0 \quad \text{on } \Gamma_1, \tag{4.3}$$

$$\sigma(u_\delta(t))\,\nu = f_2(t) \quad \text{on } \Gamma_2, \tag{4.4}$$

$$-\sigma_{\delta\nu}(t) = p(u_{\delta\nu}(t)) + p_\delta(u_{\delta\nu}(t)) \text{ on } \Gamma_3. \tag{4.5}$$

Here and below, $\sigma(u_\delta)=\sigma_\delta$ denotes the stress tensor, $u_{\delta\nu}$ and $\sigma_{\delta\nu}$ represent the normal components of the functions u_δ and σ_δ, respectively. In this problem, we replace the contact condition (2.5) by condition (4.5), in which the unilateral constraint $u_\nu\le g$ is penalized and where the penalization function $p_\delta:\mathbb{R}\to\mathbb{R}$ is defined by

$$p_\delta(r)=\left(\frac{r-g}{\delta}\right)_+.$$

The penalization parameter δ may be interpreted as the deformability coefficient of the foundation, and $1/\delta$ is the stiffness coefficient; for $r\in\mathbb{R}$, $r_+=\max(r,0)$. Problem P_δ represents a frictionless contact problem with normal compliance, without unilateral constraint. However, the penetration is allowed but penalized. Next, we study the behavior of the solution as $\delta\to 0$ and prove that in the limit we obtain the solution of frictionless contact problem with normal compliance and unilateral constraint. For thus, we define the functional $j_\delta : V\times V\to\mathbb{R}$ by

$$j_\delta(u,v)=\int_{\Gamma_3}(p_\delta(u_\nu)+p(u_\nu))v_\nu da \;\forall u,v\in V.$$

Using these notations, the variational formulation of the penalized frictionless contact problem with normal compliance is the following.

Problem $P_{1\delta}$. Find a displacement field $u_\delta : [0,T] \to V$ such that

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(v)\rangle_Q + \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(v)\right\rangle_Q \\ &+ j_\delta(u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T].\end{aligned} \tag{4.6}$$

We have the following result.

Theorem 4.1. *Problem $P_{1\delta}$ has a unique solution which satisfies $u_\delta \in C([0,T];V)$.*

Proof. We consider the operator $B_\delta : V \to V$ defined by

$$(B_\delta u, v)_V = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + j_\delta(u,v), \quad \forall u, v \in V.$$

We use (2.13), (2.15) and that for $a, b \in \mathbb{R}$, we have the properties:
$(a-b)(a_+ - b_+) \geq (a_+ - b_+)^2$ and $|a_+ - b_+| \leq |a-b|$, to see that the operator B_δ is strongly monotone and Lipschitz continuous. For the rest of the proof, it suffices to invoke the same reasoning used in the proof of Theorem 3.1.
Next, the behavior of the solution u_δ as $\delta \to 0$ is given by the following theorem.

Theorem 4.2. *Assume that* (2.11), (2.13), (2.14) *and* (2.15) *hold. Then the solution u_δ of Problem $P_{1\delta}$ converges to the solution u of Problem P_1, that is,*

$$\lim_{\delta \to 0} \|u_\delta(t) - u(t)\|_V = 0, \textit{ for all } t \in [0,T]. \tag{4.7}$$

The proof is carried out in several steps. In the first step, we show the following lemma.

Lemma 4.3. *For each $t \in [0,T]$, there exists $\bar{u}(t) \in K$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$u_\delta(t) \to \bar{u}(t) \textit{ weakly in } V. \tag{4.8}$$

Proof. Let $t \in [0,T]$. Take $v = u_\delta(t)$ in (4.6) then

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t))\rangle_Q + j_\delta(u_\delta(t), u_\delta(t)) \\ &+ \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(u_\delta(t))\right\rangle_Q = (f(t), u_\delta(t))_V.\end{aligned} \tag{4.9}$$

Since we have

$$j_\delta(u_\delta(t), u_\delta(t)) \geq 0,$$

it follows from (4.9) that

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t))\rangle_Q \\ &\quad\leq (f(t), u_\delta(t))_V - \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(u_\delta(t))\right\rangle_Q.\end{aligned} \tag{4.10}$$

We have

$$\left|\left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(u_\delta(t))\right\rangle_Q\right| \leq c(\int_0^t \|u_\delta(s)\|_V\,ds)\,\|u_\delta(t)\|_V\,,$$

where $c = \|\mathcal{F}\|_{C([0,T];Q_\infty)}$. This above inequality and (4.10) imply that there exists a constant $c > 0$ such that

$$\|u_\delta(t)\|_V \leq c(\int_0^t \|u_\delta(s)\|_V\,ds + \|f(t)\|_V)/m.$$

Then by Gronwall's inequality, it follows that the sequence $(u_\delta(t))$ is bounded in V. Then there exists an element $\bar{u}(t) \in V$ and a subsequence again denoted $(u_\delta(t))$ such that (4.8) holds. Also as the sequence $(u_\delta(t))$ is bounded in V, it follows that there exists a constant $c_1 > 0$ such that

$$\left|\left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(u_\delta(t))\right\rangle_Q\right| \leq c_1.$$

Then, from (4.9) we get

$$j_\delta(u_\delta(t), u_\delta(t)) \leq (f(t), u_\delta(t))_V + c_1.$$

On the other hand, we have

$$j_\delta(u_\delta(t), u_\delta(t)) = \int_{\Gamma_3} (p_\delta(u_{\delta\nu}(t)) + p(u_{\delta\nu}(t)))\,u_{\delta\nu}(t)\,da.$$

Then, as $\int_{\Gamma_3} p(u_{\delta\nu}(t))\,u_{\delta\nu}(t)\,da \geq 0$, it follows from the above equality that

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))\,u_{\delta\nu}(t)\,da \leq (f(t), u_\delta(t))_V + c_1.$$

Now, we write the left hand side of the previous inequaliy as

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))\,u_{\delta\nu}(t)\,da = \int_{\Gamma_3}\left(\frac{u_{\delta\nu}(t)-g}{\delta}\right)_+ (u_{\delta\nu}(t)-g)\,da$$

$$+\int_{\Gamma_3}\frac{g}{\delta}(u_{\delta\nu}(t)-g)_+\,da,$$

which implies

$$\int_{\Gamma_3}\left(\frac{u_{\delta\nu}(t)-g}{\delta}\right)_+ (u_{\delta\nu}(t)-g)\,da \leq \int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))\,u_{\delta\nu}(t)\,da$$

and then we deduce that

$$\int_{\Gamma_3}\left(\frac{u_{\delta\nu}(t)-g}{\delta}\right)_+ (u_{\delta\nu}(t)-g)\,da \leq (f(t), u_\delta(t))_V + c_1.$$

This inequality implies that there a is constant $c > 0$ such that

$$\left\|(u_{\delta\nu}(t)-g)_+\right\|^2_{L^2(\Gamma_3)} \leq \delta c.$$

Hence, using (4.8), we deduce that

$$\left\|(\bar{u}_\nu(t)-g)_+\right\|_{L^2(\Gamma_3)} \leq \liminf_{\delta\to 0}\left\|(u_{\delta\nu}(t)-g)_+\right\|_{L^2(\Gamma_3)} = 0. \tag{4.11}$$

Therefore, it follows from (4.11) that $(\bar{u}_\nu(t) - g)_+ = 0$ *a.e.* on Γ_3, i.e. $\bar{u}_\nu(t) \le g$ *a.e.* on Γ_3 and then $\bar{u}(t) \in K$. $\square$

Next, we prove the following lemma.

Lemma 4.4. *We have* $\bar{u}(t) = u(t)$ *for each* $t \in [0,T]$.

Proof. Let $t \in [0,T]$. For $v \in K$, we take $v - u_\delta(t)$ in (4.6) to get

$$\begin{aligned}
&\langle F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t))\rangle_Q + j_\delta(u_\delta(t), v - u_\delta(t)) \\
&+ \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds, \varepsilon(v - u_\delta(t))\right\rangle_Q = (f(t), v - u_\delta(t))_V.
\end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}
&j_\delta(u_\delta(t), v - u_\delta(t)) \\
&= \int_{\Gamma_3} p(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\,da + \int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\,da
\end{aligned}$$

and we see that we have also

$$\begin{aligned}
&\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\,da \\
&= \int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ ((v_\nu - g) - (u_{\delta\nu}(t) - g))da \\
&= \int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ (v_\nu - g))da - \int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ (u_{\delta\nu}(t) - g)da.
\end{aligned}$$

Now, we use that

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ (v_\nu - g))da \le 0 \text{ since } v \in K$$

and

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ (u_{\delta\nu}(t) - g)da \ge 0$$

to obtain

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\,da \le 0.$$

Then we deduce from the above inequality that

$$j_\delta(u_\delta(t), v - u_\delta(t)) \le \int_{\Gamma_3} p(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\,da. \tag{4.13}$$

Now, take $v = \bar{u}(t)$ in (4.12) to obtain by using (2.13) (b) that

$$
\begin{aligned}
& m\left\|u_\delta(t)-\bar{u}(t)\right\|_V^2 \leq \langle F\varepsilon(\bar{u}(t)), \varepsilon(\bar{u}(t))-\varepsilon(u_\delta(t))\rangle_Q \\
& + \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_\delta(s))-\varepsilon(\bar{u}(s))\right) ds, \varepsilon(u_\delta(t))-\varepsilon(\bar{u}(t))\right\rangle_Q \\
& + \textstyle\int_{\Gamma_3} \left(p(u_{\delta\nu}(t))\right)\left(\bar{u}_\nu(t)-u_{\delta\nu}(t)\right) da \qquad (4.14) \\
& + \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\, \varepsilon(\bar{u}(s))\, ds, \varepsilon(u_\delta(t))-\varepsilon(\bar{u}(t))\right\rangle_Q \\
& + \left(f(t), u_\delta(t)-\bar{u}(t)\right)_V .
\end{aligned}
$$

Next, we denote

$$
\begin{aligned}
R_\delta(t) = & \textstyle\int_{\Gamma_3} p(u_{\delta\nu}(t))\left(\bar{u}_\nu(t)-u_{\delta\nu}(t)\right) da \\
& + \langle F\varepsilon(\bar{u}(t)), \varepsilon(\bar{u}(t))-\varepsilon(u_\delta(t))\rangle_Q \\
& + \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\, \varepsilon(\bar{u}(s))\, ds, \varepsilon(u_\delta(t))-\varepsilon(\bar{u}(t))\right\rangle_Q \\
& + \left(f(t), u_\delta(t)-\bar{u}(t)\right)_V .
\end{aligned}
$$

In the sequel, we have

$$
\begin{aligned}
& \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_\delta(s)-\bar{u}(s)\right) ds, \varepsilon(u_\delta(t)-\bar{u}(t))\right\rangle_Q \\
& \leq c\left(\textstyle\int_0^t \left\|u_\delta(s)-\bar{u}(s)\right\|_V ds\right)\left\|u_\delta(t)-\bar{u}(t)\right\|_V ,
\end{aligned}
$$

where $c = \|\mathcal{F}\|_{C([0,T];Q_\infty)}$. Using the elementary inequality

$$cab \leq c^2\frac{a^2}{2m}+\frac{m}{2}b^2,$$

we find that

$$
\begin{aligned}
& \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u(s))-\varepsilon(u_\delta(s))\right) ds, \varepsilon(u_\delta(t)-u(t))\right\rangle_Q \\
& \leq \frac{c^2}{2m}\left(\textstyle\int_0^t \left\|u_\delta(s)-u(s)\right\|_V ds\right)^2+\frac{m}{2}\left\|u_\delta(t)-u(t)\right\|_V^2 . \qquad (4.15)
\end{aligned}
$$

Now, we combine inequalities (4.14) and (4.15) to obtain

$$
\begin{aligned}
& \frac{m}{2}\left\|u_\delta(t)-\bar{u}(t)\right\|_V^2 \\
& \qquad \leq \frac{c^2}{2m}\left(\textstyle\int_0^t \left\|u_\delta(s)-\bar{u}(s)\right\|_V ds\right)^2+R_\delta(t),
\end{aligned}
$$

which yields

$$\|u_\delta(t)-\bar u(t)\|_V^2 \le c^2/m^2\left(\int_0^t \|u_\delta(s)-\bar u(s)\|_V^2\,ds + R_\delta(t)\right).$$

The Gronwall argument implies that there exists a constant $c>0$ such that

$$\|u_\delta(t)-\bar u(t)\|_V \le c\sqrt{|R_\delta(t)|}. \tag{4.16}$$

Now using Lemma 4.3, we get the following convergences as $\delta \to 0$:

$$\int_{\Gamma_3} p(u_{\delta\nu}(t))(\bar u_\nu(t)-u_{\delta\nu}(t))\,da \to 0,$$

$$\langle F\varepsilon(\bar u(t)),\varepsilon(\bar u(t)-u_\delta(t))\rangle_Q$$

$$+\left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(\bar u(s))\,ds, \varepsilon(u_\delta(t))-\varepsilon(\bar u(t))\right\rangle_Q \to 0,$$

$$(f(t),u_\delta(t)-\bar u(t))_V \to 0,$$

which imply

$$R_\delta(t)\to 0, \text{ as } \delta\to 0.$$

Hence we obtain

$$\|u_\delta(t)-\bar u(t)\|_V \to 0, \text{ as } \delta\to 0, \text{ for all } t\in[0,T]. \tag{4.17}$$

Now from (4.12), (4.13) and (4.17) we deduce the inequality below

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)),\varepsilon(v-u_\delta(t))\rangle_Q\\ &+\int_{\Gamma_3} p(u_{\delta\nu}(t))(v_\nu-u_{\delta\nu}(t))\,da\\ &+\left\langle\int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\delta(s))\,ds,\varepsilon(v-u_\delta(t))\right\rangle_Q \ge (f(t),v-u_\delta(t))_V.\end{aligned} \tag{4.18}$$

Therefore, using (4.17), (2.14) and (2.15) (b), we obtain by passing to the limit as $\delta\to 0$ in inequality (4.18) that

(4.19)

$$\begin{aligned}&\langle F\varepsilon(\bar u(t)),\varepsilon(v)-\varepsilon(\bar u(t))\rangle_Q + \left\langle\int_0^t \mathcal{F}(t-s)\,\varepsilon(\bar u(s))\,ds,\varepsilon(v)-\varepsilon(\bar u(t))\right\rangle_Q\\ &+j(\bar u(t),v-\bar u(t)) \ge (f(t),v-\bar u(t))_V.\end{aligned}$$

Hence we combine Lemma 4.3 and (4.19) to see that $\bar u$ is a solution of Problem P_1. Then since Problem P_1 has a unique solution, thus we deduce that $\bar u(t)=u(t)$. □

Proof of Theorem 4.2. Let $t\in[0,T]$. We conclude by the previous equality that $u(t)$ is the unique weak limit in V of any weak convergent subsequence of the sequence $u_\delta(t)$, and therefore it follows that the whole sequence $u_\delta(t)$ converges weakly to the element $u(t)$ in V and therefore (4.7) is a consequence of (4.17).

Remark. We have

$$\|\sigma_\delta(t) - \sigma(t)\|_Q \leq M \|u_\delta(t) - u(t)\|_V .$$

Then we deduce that $\sigma_\delta(t)$ converges strongly to $\sigma(t)$ in Q for all $t \in [0,T]$.

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ANALIZA ELASTYCZNEGO KONTAKTU Z UWZGLĘDNIENIEM LEPKOŚCI W ASPEKCIE JEDNOSTRONNYCH WIĘZÓW

S t r e s z c z e n i e

Przedstawiony jest matematyczny model zagadnienia sformułowanego w tytule pracy.

Słowa kluczowe: elastyczność, lepkość, odkształcenie normalne, kontakt bez tarcia, słabe rozwiązanie

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B U L L E T I N

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Dedicated to Professor Władysław Wilczyński
on the occasion of his 70th birthday

Osamu Suzuki, Julian Ławrynowicz, and Agnieszka Niemczynowicz

BINARY AND TERNARY STRUCTURES IN PHYSICS II

BINARY AND TERNARY STRUCTURES IN ELEMENTARY PARTICLE PHYSICS VS. THOSE IN THE PHYSICS OF CONDENSED MATTER

Summary

We recall the common understanding on the history of the universe and give the descriptions of quarks by Turing machine. Then we proceed to the basic problems by use of Turing machine. Finally, we discuss the quark confinement and generation problem of quarks. The theory appears in some sense dual to that within the physics of condensed matter; the both join topology and physics.

Keywords and phrases: binary structure, ternary structure, Turing machine, systems of elementary particles, systems of alloy structures

1. Binary and ternary structures in elementary particles physics

In this section we describe the evolution of the universe in terms of the evolution of Turing machines and show that the push-down automatons can describe baryons. Moreover leptons can be described by the additional tapes of l.b.a. (linear bounded automatons; cf. [16], p. 59) which promotes its operations into action.

Here we will describe the common understanding on the evolution of the universe and will try each process to the corresponding automatons.

(stage 1): The first stage is at the Big-Bang and the universe is filled with photons.

(stage 2): Quarks and anti-quarks are created and they behave as free particles.

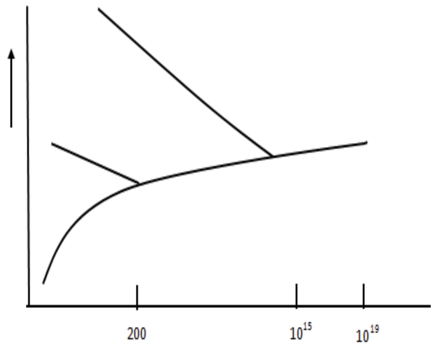


Fig. 1: The six stage evolution of the universe.

(stage 3): A quark and an anti-quark constitutes a meson.

(stage 4): A concept of colours, G, R, Y are born and 3-quarks (3-anti-quarks) constitute a baryon (res. anti-baryons).

(stage 5): The interactions between baryons give rises to the generations of baryons and leptons.

(stage 6): At this stage no more particles are created and the universe evolutes itself to the present universe.

We give a description of the evolution by Turing machine.

(1) The stage 1 (The Big-Bang)

At the begining of the universe, which has began with the Big-Bang, photons, which are described by γ are created and the universe is filled with photons:

$$\circledast \to \gamma \to \gamma + \gamma \to ... \to \gamma + \gamma + ... + \gamma \to ..., \tag{1}$$

where $\circledast$ is the "origin of the universe" or the singularity of Penrose and Hawking. Then we see easily that we can describe this process in terms of finite automaton by use of the process in (1) in the previous section.

(2) The stage 2 (The quark and anti-quark)

After the Big-Bang, the photons create quarks and anti-quarks:

$$\gamma + \gamma \to q_i + \bar{q}_i, \quad i = 1, 2, ...M, \tag{2}$$

where q_i and $\bar{q}_i$ behave as free particles and they have no interactions, because of the high temperature. This stage can be described by the push-down automaton associating

$$q_i \Rightarrow \curlywedge_i, \quad \bar{q}_i \Rightarrow \curlywedge_i.$$

Also, the vaccuum $\langle 0|$ and the anti-vaccuum $|0\rangle$ are created.

(3) The stage 4 (The creation of coloured quarks)

When the production of quarks finished and the number of quarks is fixed, then the evolution of the Turing machine starts, which is l.b.a. The simplest l.b.a. is generated by $\mathcal{L}^{(3)}$:

$$\mathcal{L}^{(3)} = \{a_1^n a_2^n a_3^n | n > 0\}.$$

We can associate the particles with colours G.R.Y:

$$\psi_q(R) \Rightarrow a_1, \ \ \psi_q(G) \Rightarrow a_, \ \ \psi_q(Y) \Rightarrow a_3$$

for any quark. Here we want to make stress on the following fact:

$$\mathcal{L}^{(4)} = \{a_1^n a_2^n a_3^n a_4^n | n > 0\}$$

is also l.b.a. The corresponding fields are possible, but they can be generated by $\mathcal{L}^{(3)}$ and they can exist only in the form "resonance" and they are decoupled into a sum od baryons. This will be discussed in the next section.

(4) The stage 3 (The creation of mesons)

After the temperature goes down, a quark and anti-quark combine and constitute a meson:

$$q_i \bar{q}_j, \ (i, j = 1, 2, ..., M).$$

These generations can be realised by the suitable choices of the stack operation. This stack operation describes the roles of leptons for the description of the interaction. For example:

$$b\bar{b} \to d\bar{b} + b\bar{d} + X,$$

where X is a set of leptons.

(5) The stage 5 (The baryons with flavours)

Next we proceed to the generation of baryons with flavours. We consider the following process:

$$(q(R), q(G), q(Y)) \Rightarrow (q(R), q^{'}(G), q(Y)).$$

The typical example is $p \to n + e^- + \bar{\nu}$.

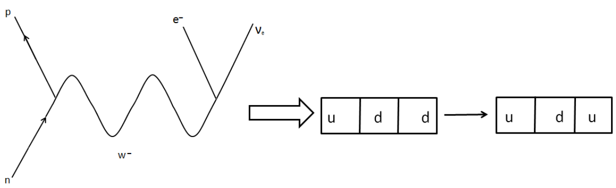
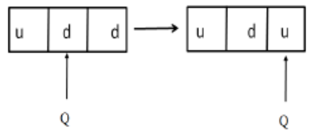


Fig. 2: Generation of baryon with flavours.

This operation can be described by:



Hence we can describe the Q operation by contribution of leptons:

$$Q \Leftrightarrow W \Leftrightarrow e^- + \nu_e.$$

The other decay process, composition process can be described in an analogous manner.

(6) The stage 6 (At present)

After the repetition of the operations of l.b.a. we can obtain the generations of quarks and leptons, we can have the complete set of particles (Table 1).

Table 1.

Generation	I	II	III
Quarks	$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}$
Leptons	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$

In part III of the paper, we shall discuss the generations in terms of binary and ternary Galois extension.

2. A counterpart in the physics on condensed matter

Following [11–14] J. Ławrynowicz, O. Suzuki, A. Niemczynowicz and M. Nowak-Kępczyk [15] proposed the following six-stages evolution related to the physics of condensed matter.

(stage 1'): Jordan-von Neumann-Wigner (JWN) procedure for related fractals including their duality.

(stage 2'): JWN-procedure adjusted for binary (ternary) alloys.

(stage 3'): Ising-Onsager approach related to perfect 15-elements systems of objects.

(stage 4'): Perfect JWN-systems.

(stage 5'): Complete and self-dual perfect JNW systems.

(stage 6'): Forming algebra of self-dual perfect JNW systems.

Nobelists 2016 in physics: David J. Thouless, F. Duncan M. Haldane, and J. Michael Kosterlitz [17] observed that studying of phase transitions in elementary particles physics and physics of condensed matter is merely a topological problem easier when two branches of physics quoted are discussed parallely. Hence we can deduce the following six stages:

(stage 1”): long range order and metastability in solids and superfluids; ordering, metastability and phase transition [9, 10];

(stage 2”): critical properties of two dimensional xy model [8];

(stage 3”): quantized Hall conductance in periodic potential [18];

(stage 4”): continuum dynamics of the 1-D Heisenberg antiferromagnet: identification with the $\mathcal{O}(3)$ nonlinear sigma model [3];

(stage 5”): nonlinear field theory of large-spin Heisenberg antiferromagnets: semiclassically quantized solitons of the 1-D easy-axis Néel state [4];

(stage 6”): model for a quantum Hall effect without Landau levels: condensed-matter realization of the “parity anomaly” [5].

3. Physics described by Turing machine I (Quark confinement)

In this section we shall discuss some basic problems on elementary physics in the terms of Turing machine. The following problems will be discussed:

- How can we describe quark confinement by Turing machine?
- Can we find m, $(m \geq 4)$ kinds of quarks?

3.1. The universal diagram

We shall consider these problems by use of the concept of the universal diagram. We associate the graph structure to the generation of quarks, which is called universal diagram and discuss decay, composite states. We associate the composite particles which are constituted with m-quarks and n-anti-quarks with the following diagram which is called the universal diagram.
We assume that the universal diagram contains the possible diagrams. We give an example for $m = n = 2$.

We make some comments on the universal diagram:

(1) All the diagrams can exist at the Big-Bang time. After the time passes, some of them remain and some of them vanish.

(2) When a push-down automaton happens, the diagram of (m, o)-type and (o, m)-type is typical one, because the particle and anti-particle are created and they behave as free particles. Hence we have the diagram shown in Fig. 4. With the universal diagram we shall discuss the problem:

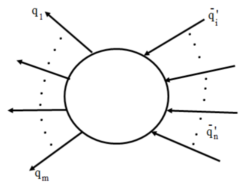


Fig. 3: The universal diagram for quarks.

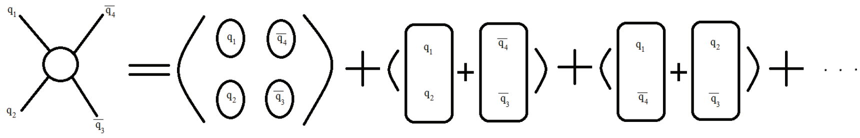


Fig. 4: An example of the universal quark diagram with $m = n = 2$.

(a) Quark confinement

The knowledge on physics tells us that there can not be observed single quark and double pure quarks. Moreover m-pure quarks ($m \geq 4$) are not observed except ($m = 3m'$). The universal diagram becomes as follows.

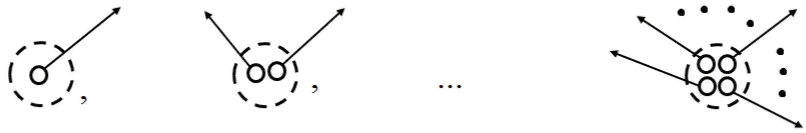


Fig. 5: Universal diagram for quark confinement.

This fact can be explained by the Turing machine as follows: at the very early time of the universe quarks or anti-quarks behave as free particles. Hence quarks are not confined. After some times passed, the Turing machine evaluated and the push-down automaton or l.b.a. started to operate. The finite automaton has evaluated and $q_i\bar{q}_j$ or $q_iq_jq_k$ have appeared. Because of this, a single quark and double quarks can not be produced. hence it can not be observed.

(b) Multi-quarks as resonance

We shall show that multi-quarks which are bigger then 3 can be realised as resonance particles. For this we shall understand the combined particles can be created by m-quarks and m-anti-quarks.

This equality implies that m-quarks and m-anti-quarks confine composite particles when the temperature went down. The right side configuration implies the universal diagram. We may expect that we can describe these process in terms of

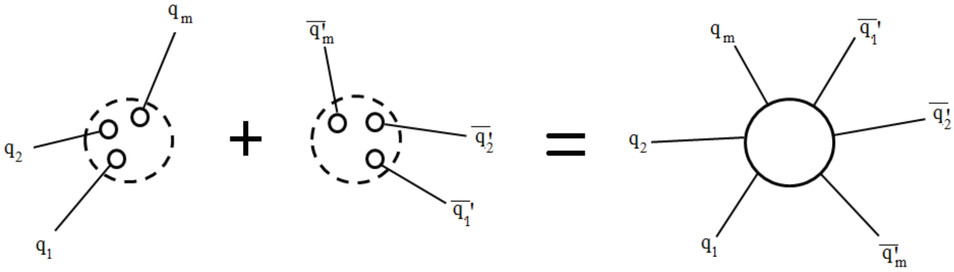


Fig. 6: Multi-quarks as resonance.

l.b.a. Turing machine, because the total number is not changed in the process. We shall discuss this later.

(2.2)-quarks as dual resonance

At first we observe (2.2)-quarks (2-quarks and 2-anti-quarks combined) that can be observed as the resonance of 4-quarks.

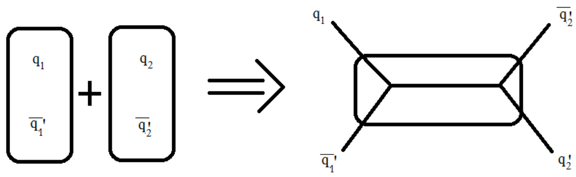


Fig. 7: Two-two quarks as a dual resonance.

Hence we see that the resonance particle behaves as a single particle.Also from the duality in the resonance theory, we have another decomposition.

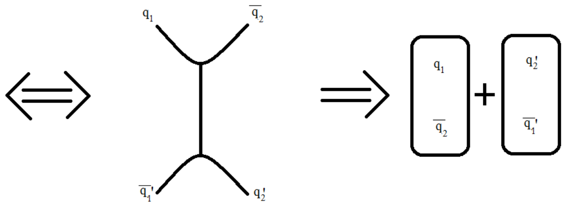


Fig. 8: Another decomposition for two-two quarks.

We can understand the resonance and the duality in terms of the universal diagram and its evolution to change the resonance state:

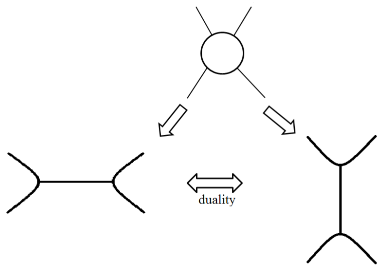


Fig. 9: Another duality for quarks.

(4.0)-quarks in the decay process

We consider the following decay process:

$$J/\Phi \to \pi^+ + \pi^- + \pi^0.$$

The diagram is as follows.

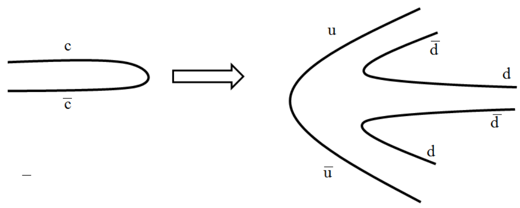


Fig. 10: A (4.0)-quark in the decay process.

Then we may find a (4.0)-quark and its conjugate (0.4)-quark, which produced the decay.

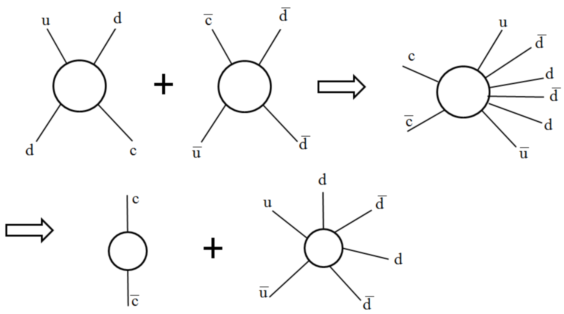


Fig. 11: The conjugate (0.4) quark which produced the decay.

We may expect another system of particles considering the other decomposition.

3.2. 3-generation of quark physics described by Turing machine

At first we give the geometric description on 3-generation and see that m-generations ($m \geq 4$) can be reduced to 3-generation. Here we shall see that the geometric description can happen in elementary particles physics. We begin with the geometric construction for quarks. Geometric construction for quarks maybe described as follows:

(1) We denote photon γ by $\circledcirc$ Fig. 12.

(2) A quark (or anti-quark) q_i (resp. $\bar{q}_i$) is denoted as in Fig. 13.

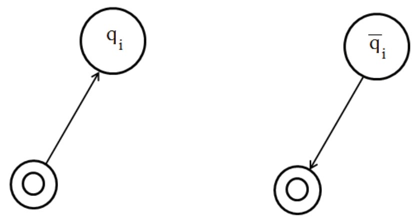


Fig. 12: A quark and anti-quark.

Then we can see a quark composed with its anti-quark as visualized in Fig. 13:

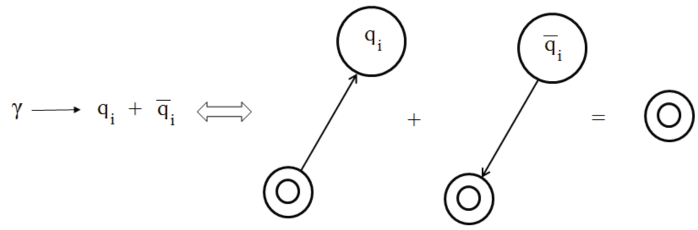


Fig. 13: A quark composed with its anti-quark.

(3) A meson $q_i\bar{q}_i$ is expressed by Fig. 14 (in our convention):

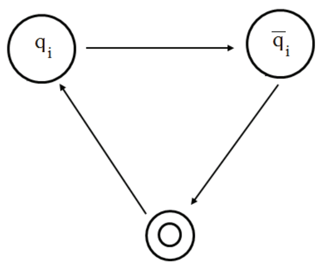


Fig. 14: A meson.

(4) A baryon $q_iq_jq_k$ is expressed as shown in Fig. 15.

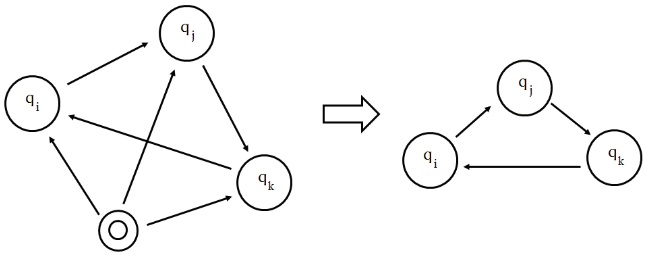


Fig. 15: A baryon.

The triangle can be constructed if and only if the colours of q_i, q_j, q_k are different from each other. Next we proceed to the generation and reduction for quarks. For this we shall show that the following diagram holds on the **base of physical observation.**

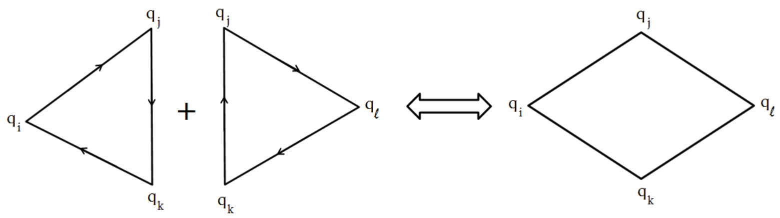


Fig. 16: Triangles involved in geometrical visualisation of baryon.

For this we put the following conditions:

- The diagram X exists in physics, then $\bar{X}$ also exists.
- X exists, then $X + \bar{X}$ exists.

Then we have the following diagrams shown in Figs. 17–19 following Fig. 16 we may call them ternary analogues of diagrams X, $\bar{X}$, $X + \bar{X}$.

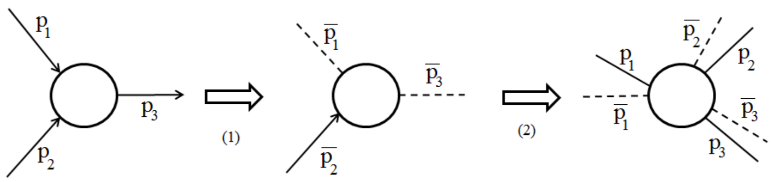


Fig. 17: Ternary analysis of the diagram X.

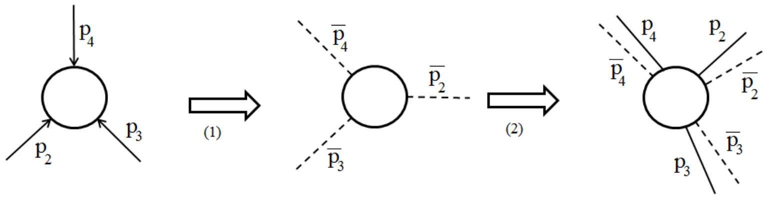


Fig. 18: Ternary analysis of the diagram $\bar{X}$.

Hence we have

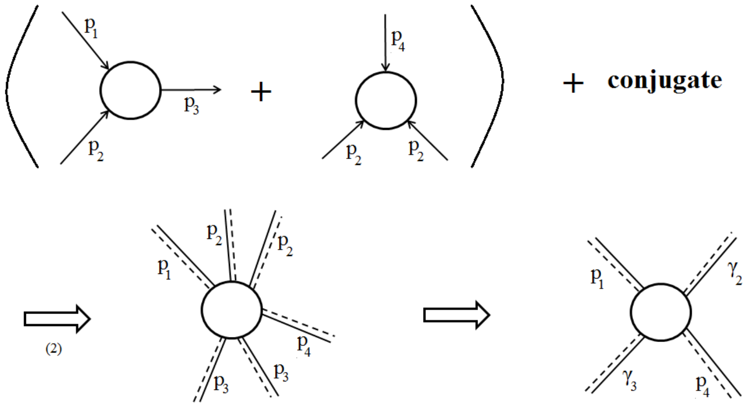


Fig. 19: Ternary analysis of the diagram $X + \bar{X}$.

The above diagrams have the following decay-process consequences, as shown in Figs. 20–21. Here we assume the following processes, for example:

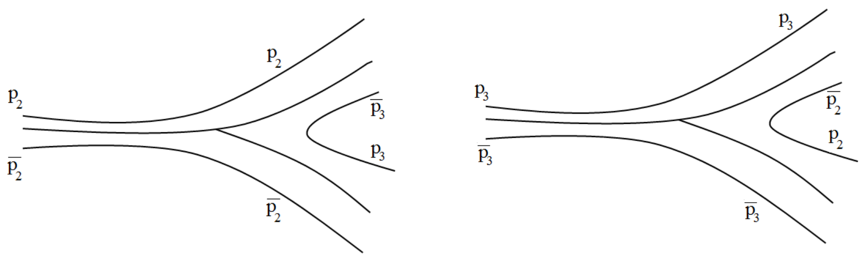


Fig. 20: Simple decay processes.

This process can be observed in meson resonance.

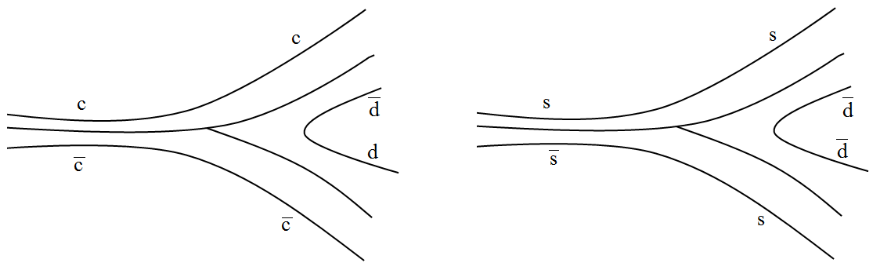


Fig. 21: Meson resonance J/Φ-decay and Ψ-decay.

Finally, considering quark-parts, we can obtain Fig. 22 what can be understood that 4-quarks can exist as resonance.

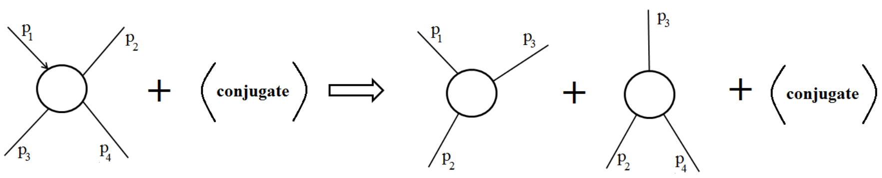


Fig. 22: Four quarks can exist as resonance.

4. A counterpart in the physics of condensed matter. Ternary approach

The 3-generation of quark physics described above, in particular the geometric construction for quarks, mesons and baryons, acts quite paralelly in our ternary approach given in [11–15]. The stages of our procedure described in those papers and corresponding to the ideas of Figs. 12–22 may be formulated as fellows (the formulated stages **1”’–6”’** are a more general version of stages **1’–6’**):

(stage 1”’): ternary approach applied to fractals and chaos related to Ising-Onsager lattices.

(stage 2”’): construction of perfect, complete, and dual Jordan-von Neumann-Wigner system, especially a perfect 15-element system.

(stage 3”’): fractal structures of Jordan-von Neumann-Wigner procedure.

(stage 4”’): some composition algebras and their geometric cubic or nonion significance.

(stage 5”’): two ternaries vs. three binaries for crystallographic lattices.

(stage 6"'): construction of the collection of two ternaries and the related Dirac-like operators in connection with the cubic or nonion algebras.

5. Physics described by Turing machine II [16] (3-generation problem)

In this section we are concerned with the basic problem in quark physics. We know that we have the three generations of quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}.$$

The basic problems are:

- How the generations arise?
- Are there no other generations?

It seems to the authors, we have still no answers to the problems. Here we shall try to treat them from the Turing machine [16]. The basic idea arises from the transcription mechanism in molecular biology. Still we cannot give answers to the problems but we may say that our method will give the basic foundations to the problem.

5.1. The transcription mechanism in DNA

Here we recall some basic facts in the transcription mechanism in DNA.We can find binary structure in DNA and find ternary structure in transcription mechanism.

(1) When we choose RNA world, then we can start from a sequence of nucleotide A,T,G C(U).

Fig. 23: Entering the RNA world.

(2) Then we have coiled two strips of sequences of nucleotides, which is called DNA-sequence.

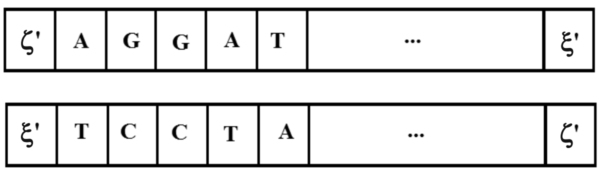


Fig. 24: Entering the DNA world.

Between these sequences, we have a kind of duality

$$A \leftrightarrow T, \quad G \leftrightarrow C.$$

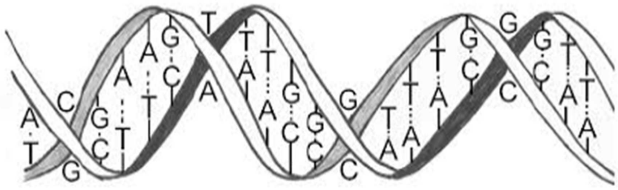


Fig. 25: Internal duality in DNA.

(3) Promoters act on DNA, then the transcription mechanism is initiated and proteins are created. Proteins are determined by the following codon tables.

Table 2. The universal genetic code chart.

First base	Second base U	Second base C	Second base A	Second base G	Third base
U	UUU, UUC } PHE UUA, UUG } LEU	UCU, UCC, UCA, UCG } SER	UAU, UAC } TYR UAA, UAG } STOP	UGU, UGC } CYS UGA } STOP UGG } TRP	U C A G
C	CUU, CUC, CUA, CUG } LEU	CCU, CCC, CCA, CCG } PRO	CAU, CAC } HIS CAA, CAG } GLN	CGU, CGC, CGA, CGG } ARG	U C A G
A	AUU, AUC, AUA } ILE AUG } MET or START	ACU, ACC, ACA, ACG } THR	AAU, AAC } ASN AAA, AAG } LYS	AGU, AGC } SER AGA, AGG } ARG	U C A G
G	GUU, GUC, GUA, GUG } VAL	GCU, GCC, GCA, GCG } ALA	GAU, GAC } ASP GAA, GAG } GLU	GGU, GGC, GGA, GGG } GLY	U C A G

The process is described by the following way as shown in Fig. 26.

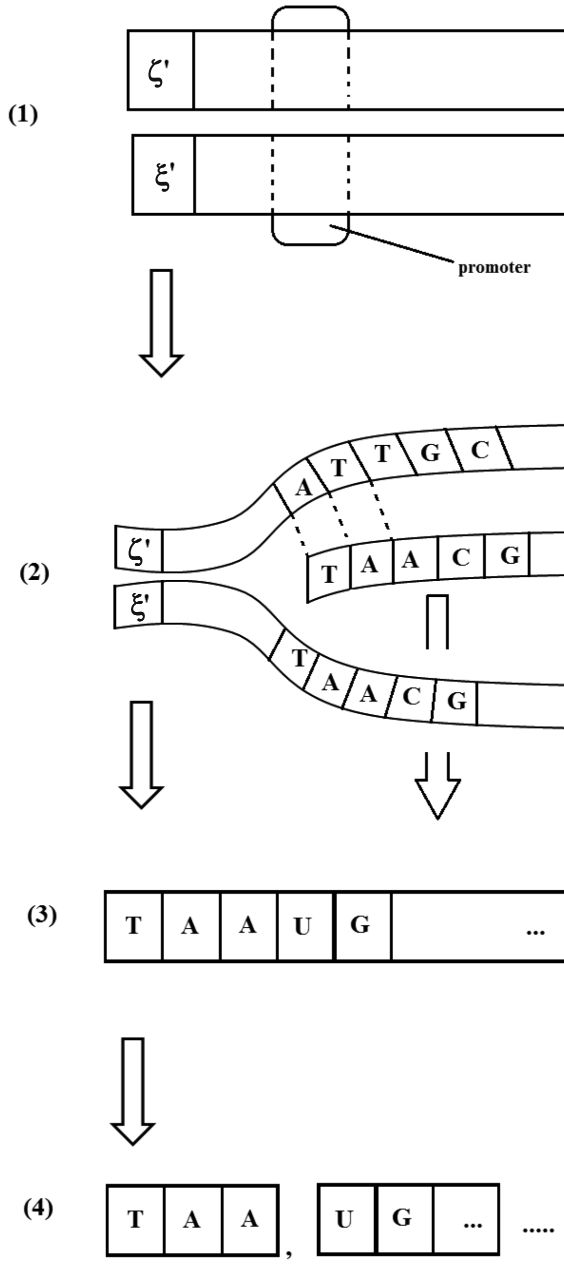


Fig. 26: DNA-type generation.

5.2. Quark generation problem

Next we proceed to the quark generation problem. We shall discuss it form the view point of the the evolution of Turing machines. For this we consider a pair of quarks x and y which is denoted by $\binom{x}{y}$. We may understand the pair as the element of two tables of the Turing machine (see Fig. 27).

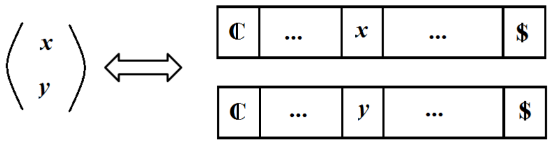


Fig. 27: Quark generation with help of Turing machine (TM).

(1) Developing the evolution of Turing machine, we start from $\binom{t}{b}$.

Then we can generate the copies of this element. The generation is performed, keeping the conjugation

$$t \leftrightarrow b.$$

Following the generation in DNA, we have the scheme showed in Fig. 28.

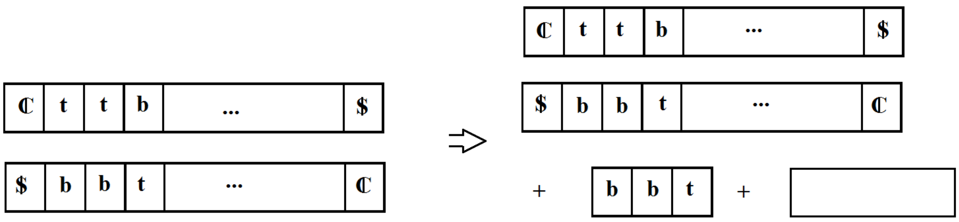


Fig. 28: Baryon generation with the help of TM.

Hence we can obtain the following baryons.

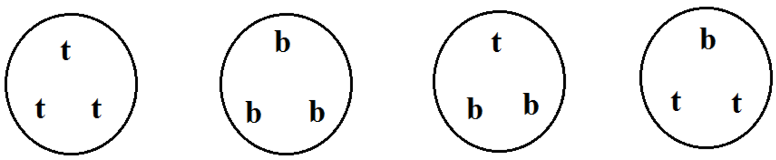


Fig. 29: The resulting baryons.

(2) Soon later we have a push-down structure and obtain its conjugate

$$\begin{pmatrix} t \\ b \end{pmatrix} \Leftrightarrow \begin{pmatrix} \bar{t} \\ \bar{b} \end{pmatrix}.$$

Then we can generate mesons.

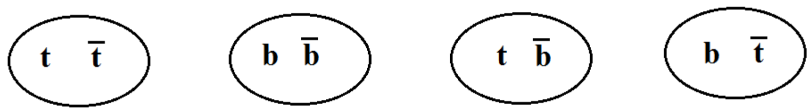


Fig. 30: The resulting mesons.

From the energy levels of each generation, we have

$$\Upsilon(4s)(= b\bar{b}) \rightarrow \begin{cases} \begin{array}{c} B^0 + \bar{B}^0 \\ (d\bar{b})\ (b\bar{d}) \end{array} \rightarrow ... \\ \\ \begin{array}{c} B^+ + B^- \\ (u\bar{b})\ (\bar{u}b) \end{array} \rightarrow ... \end{cases}$$

$$J/\Phi)(= c\bar{c}) \rightarrow \begin{cases} \begin{array}{c} \pi^+ + \pi^- \\ (u\bar{d})\ (\bar{u}d) \end{array} \rightarrow ... \\ \\ \begin{array}{c} D^+ + D^- \\ (c\bar{d})\ (\bar{c}d) \end{array} \rightarrow ... \end{cases}$$

We have then the following flow chart (Fig. 31).

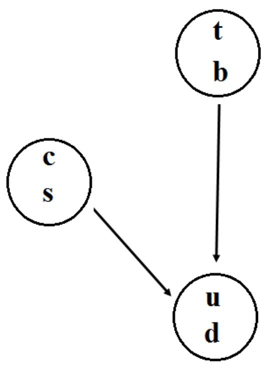


Fig. 31: Flow chart for the mesons in question.

Hence we have new pairs (c, s), (u, d) with the following conjugation:

$$c \leftrightarrow s,\ u \leftrightarrow d.$$

Hence we have new tapes and obtain the following baryons:

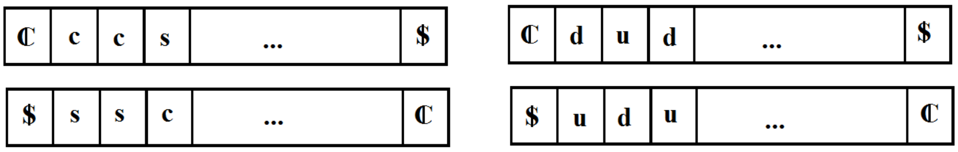


Fig. 32: Additional types caused by the baryons.

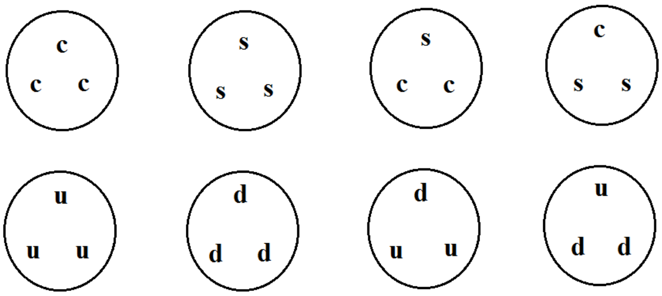


Fig. 33: Baryons resulting for the types of Fig. 32.

We notice that the final two quarks are proton and neutron which are stable baryons. These baryons are generated independently. Also we know that each generation is not independent and generations are mixed. We have 2-generation mix and 3-generation mix.

- 2-generation mixture

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = U \begin{bmatrix} d \\ s \end{bmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{bmatrix} d \\ s \end{bmatrix},$$

  where θ_c is the Cabibbo angle.

- 3-generation mixture

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = U_{CKM} \begin{bmatrix} d \\ s \\ b \end{bmatrix},$$

where U_{CKM} is the Cabibbo-Kobayashi-Maskawa matrix. This implies that there exists a kind of evolution in quark world. Hence we have the following generation diagram (Fig. 34):

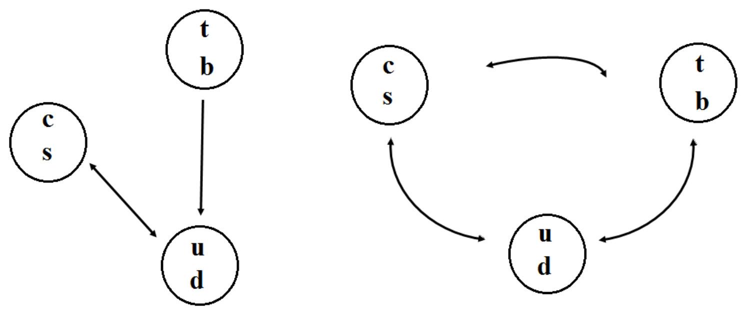


Fig. 34: Two generation diagrams for quarks.

When a mixture happens, we may mix tapes: for the decay type, we make the mixture, for the case of 3-generation mixture type, we can make the 3-double tables into 1-double tape with words (u, d), (c, s), (t, b) in a completely analogous manner.

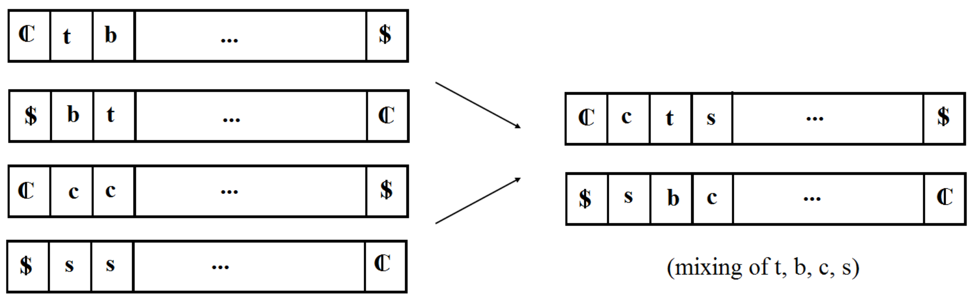


Fig. 35: Example of mixture type of the decay-type.

5.3. Summary and discussions

We have proposed a method of molecular biology to the quark generation problem and discussed the generation and mixture. We may give a partial answer to the first problem, but it is not enough. As for the second problem we can find the evidence of 3-generations.

One of the difficulties lies in the following facts:

(1) In the case of atom generation, we have 1-generation (Fig. 36).

(2) In the case of the transcription mechanism (Fig. 37), since $T \leftrightarrow A$, $G \leftrightarrow C$, we have 2-generation.

Thus we have no evidence for 3-generation problem from Turing machine method. It is possible 4-generation, 5-generation for higher temperature (Fig. 38).

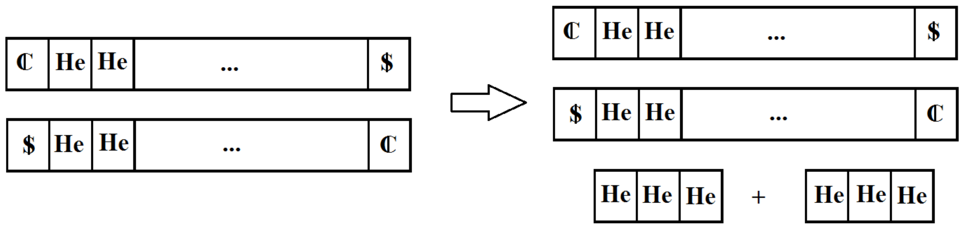


Fig. 36: An example in the case of 1-generation.

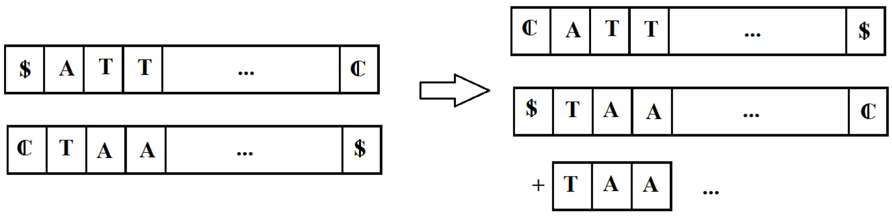


Fig. 37: An example in the case of transcription mechanism.

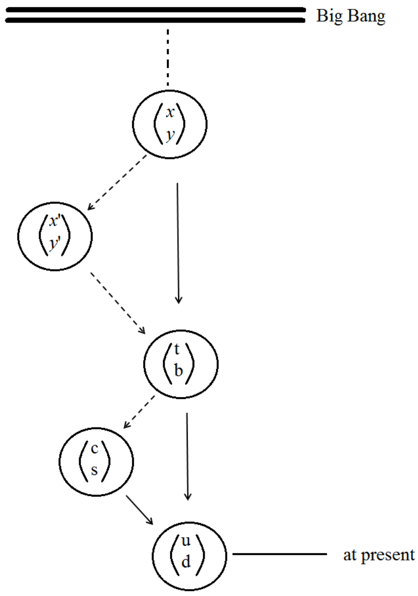


Fig. 38: Example of a higher generation of quarks.

6. Conclusions and discussion

From the discussions done above, we see that interaction, decay, compositions can be described in terms of the Turing machine operations:

1. At the beginning of the universe, quarks are created and they behave as free particles.
2. Then quarks and anti-quarks are created and (m, o)-system and (o, m)-system constitute an unified system by the push-down automaton structure.
3. Then the l.b. automaton starts, because the system in (2) does not change the total number of quarks. Then we have the decompositions of the diagram, so that we can obtain mesons and baryons.
4. The existence of leptons and W-boson contributes to the operations in l.b. automatons.
5. We have discussed on 3-generation problem motivated from molecular biology and shown that we may have m-generation ($m \geq 4$) in quarks.

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STRUKTURY BINARNE I TERNARNE FIZYKI CZĄSTEK ELEMENTARNYCH A FIZYKA FAZY SKONDENSOWANEJ

S t r e s z c z e n i e

Przypominamy fakty dotyczące powstania Wszechświata oraz opisujemy model fizyki kwarków w terminach maszyny Turinga. Następnie rozważamy podstawowe problemy fizyki kwarków przy użyciu maszyny Turinga. Teoria wydaje się być w pewnym sensie dualna do swego odpowiednika w zakresie fizyki fazy skondensowanej i obie łączą jednocześnie topologię i fizykę.

Słowa kluczowe: struktura binarna, struktura ternarna, maszyna Turinga, układy cząstek elementarnych, układy struktur w stopie

Professor Władysław Wilczyński (first from the right)
during his jubilee celebration.

Rapporteurs – Referees

CONTENU DU VOLUME LXVII, no. 1