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THE JUMP OF MILNOR NUMBER FOR LINEAR DEFORMATIONS OF HOMOGENEOUS SINGULARITIES

Summary

The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations f_s. We estimate the jump of homogeneous and semi-homogeneous singularities in the class of linear deformations.

Keywords and phrases: Enriques diagram, jump of Milnor number, homogeneous singularity, semi-homogeneous singularity

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *isolated singularity*, i.e. there exists a representative $\widehat{f_0} : U \to \mathbb{C}$ of f_0, holomorphic in an open neighbourhood U of the point $0 \in \mathbb{C}^n$ such that:

1. $\widehat{f_0}(0) = 0$,
2. $\nabla \widehat{f_0}(0) = 0$,
3. $\nabla \widehat{f_0}(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$.

In the sequel a singularity means an isolated singularity.

A *deformation of the singularity* f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

1. $f(0, z) = f_0(z)$,
2. $f(s, 0) = 0$.

The deformation $f(s,z)$ of the singularity f_0 will also be treated as a family (f_s) of functions germs, taking $f_s(z) := f(s,z)$. Since f_0 is an isolated singularity then f_s for sufficiently small s also has isolated singularities near 0 ([2] Theorem 2.6 I). By the above for sufficiently small s one can define μ_s

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n/(\nabla f_s),$$

called the *Milnor number of* f_s, where $\mathcal{O}_n$ is the ring of the holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f_0}{\partial z_1}, \ldots, \frac{\partial f_0}{\partial z_n}$.

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([2], Theorem 2.6 I), there exists an open neighbourhood S([2], Theorem 2.6 I and Proposition 2.57 II) of the point 0 such that

1. $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
2. $\mu_0 \geq \mu_s$ for $s \in S$.

The constant difference $\mu_0 - \mu_s$ (for $s \neq 0$) will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among all the jumps of deformations of the singularity f_0 will be called *the jump of the Milnor number of the singularity* f_0 and denoted by $\lambda(f_0)$.

From now we will consider only plane curve singularities $f_0 : \big(\mathbb{C}^2, 0\big) \to (\mathbb{C}, 0)$.

The first general result concerning the jump of the Milnor number was obtained by Sabir Gusein-Zade([1]), who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$. Later the same problem was consider by:

- A. Bodin ([7]) who gave a formula for $\lambda(f_0)$ for f_0 convenient with its Newton polygon reduced to one segment for non-degenerate deformations,
- J. Walewska ([6]) who generalized Bodins results to the non-convenient case for non-degenerate deformations,
- S. Brzostowski, T. Krasiski and J. Walewska ([5]) who calculated all possible Milnor numbers of all non-degenerate deformations of homogeneous singularities,
- in the same paper they proved that for the singularity $f_0^n(x,y) = x^n + y^n$ $(n \geq 2)$ we have $\lambda(f_0) = \left[\frac{n}{2}\right]$.

In this paper we consider the jump of the Milnor number of homogeneous and semi-homogeneous singularities for all *linear deformations* of f_0 i.e. deformations of the form $f_s = f_0 + sg$, where g is a holomorphic function in the neighbourhood of 0 such that $g(0) = 0$. The smallest non-zero value among all the jumps of linear deformations of the singularity f_0 will be denoted $\lambda^{lin}(f_0)$.

To get the main result the *Enriques diagrams* will be used. For any singularity we can assign its weighted Enriques diagram (D, ν) which represents the whole resolution process of this singularity ([3] Chapter 3.9). It is a finite graph with two types of edges and the weight function $\nu : D \to \mathbb{Z}$ on vertices of the diagram. Moreover M. Alberich-

Carraminñana and J. Roé ([4] Lemma 1.1) gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. It means that one singularity is a linear deformation of an another. We estimate from above the jump of Milnor number for homogeneous and semi-homogeneous singularities for linear deformations.

2. Abstract Enriques diagrams

Information about abstract Enriques diagrams can be found in [4] and [8].

Definition 2.1. An *abstract Enriques diagram* is a tree D with the root and binary relation between vertices, called proximity, which satisfies:

1. The root is proximate to no vertex.
2. Every vertex that is not the root is proximate to its immediate predecessor.
3. No vertex is proximate to more than two vertices.
4. If a vertex Q is proximate to two vertices, then one of them is the immediate predecessor of Q and it is proximate to the other.
5. Given two vertices P, Q with Q proximate to P, there is at most one vertex proximate to both of them.

The fact that Q is proximate to P we will denoted $Q \to P$. The vertices which are proximate to two points are called *satellite*, the other vertices (except the root) are called *free*.

The vertex is *final* if is has no successor.

To show graphically the proximity relation, Enriques diagrams are drawn according to the following rules:

1. If Q is a free successor of P, then the edge going from P to Q is smooth and curved and, if P is not the root, it has at P the same tangent as the edge joining P to its predecessor.
2. The sequence of edges connecting a maximal succession of vertices proximate to the same vertex P are shaped into a line segment, orthogonal to the edge joining P to the first vertex of the sequence.

Example 2.2. Let $D = \{R, S_1, ..., S_{12}\}$ be an abstract Enriques Diagram as in Figure 1 where R is the root. Proximate relation is: $S_1, S_9, S_{10} \to R$; $S_2, S_6, S_7 \to S_1$; $S_3, S_6 \to S_2$; $S_4, S_5 \to S_3$; $S_7 \to S_6$; $S_8 \to S_7$; $S_{10} \to S_9$ and $S_{11}, S_{12} \to S_{10}$.

Definition 2.3. A *system of multiplicities* is any function $\nu : D \to \mathbb{Z}$.

Definition 2.4. A pair (D, ν), where D is an abstract Enriques diagram and ν a system of multiplicities for it, is called a *weighted Enriques diagram.*

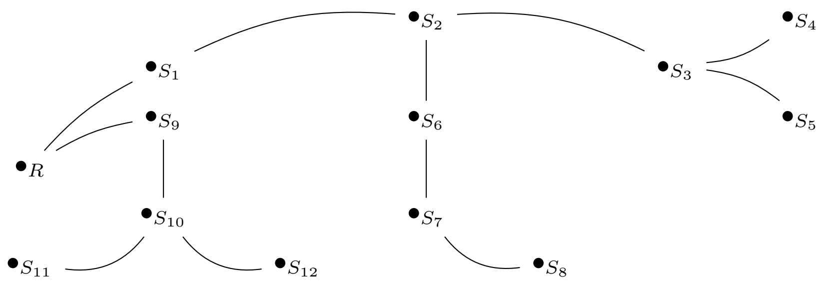


Fig. 1. Graphic presentation of an Enriques diagram D.

Definition 2.5. A *consistent Enriques diagram* is a weighted Enriques diagram such that, for all $P \in D$

$$\nu(P) \geq \sum_{Q \to P} \nu(Q).$$

Definition 2.6. To every system of multiplicities ν for a diagram D we associate a *system of values*, which is another map $\mathrm{ord}_\nu : D \to \mathbb{Z}$, defined recursively as

$$\mathrm{ord}_\nu(P) = \begin{cases} \nu(P), & P \text{ is the root,} \\ \nu(P) + \sum\limits_{P \to Q} \mathrm{ord}_\nu(Q), & \text{otherwise.} \end{cases}$$

Definition 2.7. A *subdiagram* of an abstract Enriques diagram D is a subtree $D_0 \subset D$ with the same proximity relation such that if $Q \in D_0$ then its predecessor belongs to D_0.

Definition 2.8. Let (D, ν) and (D', ν') be weighted Enriques diagrams, with (D', ν') consistent. We will write $(D', \nu') \geq (D, \nu)$ when there exists isomorphic subdiagrams $D_0 \subset D$, $D'_0 \subset D'$ and an isomorphism

$$i : D_0 \to D'_0$$

such that the new system of multiplicities $\mu : D \to \mathbb{Z}$ for D defined as

$$\mu(P) = \begin{cases} \nu'(i(P)), & P \in D_0 \\ 0, & P \notin D_0 \end{cases}$$

satisfy

$$\mathrm{ord}_\nu(P) \leq \mathrm{ord}_\mu(P)$$

for any $P \in D$.

Example 2.9. Let (D, ν) and (D', ν') be weighted Enriques diagrams with roots R and Q respectively shown below (Figure 2 and Figure 3). The numbers above bullets are multiplicities of vertices.

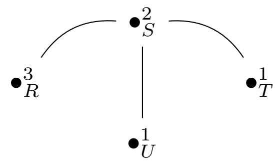


Fig. 2. Graphic presentation of an Enriques diagram (D, ν).

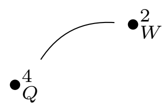


Fig. 3. Graphic presentation of an Enriques diagram (D', ν').

We can easily check that $(D', \nu') \geq (D, \nu)$. Indeed, let $D_0 = \{R, S\} \subset D$ and $D'_0 = D'$ be subdiagrams, $i : D_0 \to D'_0$ be isomorphism such that $i(R) = Q$ and $i(S) = W$. By the definition new system of multiplicities $\mu : D \to \mathbb{Z}$ for D is:

$$\mu(P) = \begin{cases} 4, & P = R \\ 2, & P = S \\ 0, & P = U \\ 0, & P = T \end{cases} .$$

Then we can check that $\text{ord}_\nu(P) \leq \text{ord}_\mu(P)$ for any $P \in D$ because:

- $\text{ord}_\nu(R) = 3 \leq 4 = \text{ord}_\mu(R)$,
- $\text{ord}_\nu(S) = 5 \leq 6 = \text{ord}_\mu(S)$,
- $\text{ord}_\nu(T) = 6 \leq 6 = \text{ord}_\mu(T)$,
- $\text{ord}_\nu(U) = 9 \leq 10 = \text{ord}_\mu(U)$.

Definition 2.10. We say that weighted diagrams (D, ν) i (D', ν') are *equivalent* if they differ at most in some free vertices of multiplicity 1 i.e. there exist subdiagrams $D_0 \subset D$ i $D'_0 \subset D'$ and an isomorphism $i : D_0 \to D'_0$ such that

1. $P \stackrel{D}{\to} Q \Leftrightarrow i(P) \stackrel{D'}{\longrightarrow} i(Q)$ for $P, Q \in D_0$,
2. for any $P \in D \setminus D_0$, $\nu(P) = 1$ and P is free,
3. for any $P \in D' \setminus D'_0$ $\nu'(P) = 1$ and P is free.

This is an equivalence relation. The *type* $[(D, \nu)]$ is the equivalence class, which representative is (D, ν).

Definition 2.11. A *minimal Enriques diagram* is a consistent Enriques diagram (D, ν) with no free vertices with multiplicity 1.

Theorem 2.12. *Let (D, ν) be a consistent weighted diagram. There exists exactly one minimal diagram which belongs to $[(D, \nu)]$.*

Proof. Let $E = \{P \in D : \nu(P) = 1 \text{ and } P \text{ is free}\}$. Then $(D_0, \nu_{|D_0})$ is an abstract Enriques diagram where $D_0 = D \setminus E$. It is obvious that $(D_0, \nu_{|D_0})$ is minimal and $(D_0, \nu_{|D_0}) \in [(D, \nu)]$. The uniqueness follows from Definition 2.10. □

The theory of Enriques diagram has its root in the theory of plane singularities. The resolution of a singularity using blow-ups can be explicitly presented as an consistent Enriques diagram. More precise description can be found in [3] Chapter 3.9. We need only the fact which easily follows from these results.

Theorem 2.13. *Let f be any singularity and (D, ν) its consistent diagram. There exists the unique minimal diagram which belongs to $[(D, \nu)]$.*

Definition 2.14 ([4])**.** Let $[(D, \nu)]$ and $[(\widetilde{D}, \widetilde{\nu})]$ be types of weighted diagrams. $[(\widetilde{D}, \widetilde{\nu})]$ is *linearly adjacent* to $[(D, \nu)]$ if there exist consistent diagram $(D', \nu') \in [(\widetilde{D}, \widetilde{\nu})]$ such that $(D', \nu') \geq (D, \nu)$.

In the paper [4] M. Alberich-Carramiñana and J.Roé gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. This is a key result which we will use in the sequel.

Theorem 2.15 ([4] Lemma 1.1)**.** *Let $[(D, \nu)]$ and $[(\widetilde{D}, \widetilde{\nu})]$ be types of Enriques diagrams representing singularities f i f_0. $[(\widetilde{D}, \widetilde{\nu})]$ is linearly adjacent to $[(D, \nu)]$ if and only if the singularity f is a linear deformation of f_0.*

The Milnor number of a singularity can be "read off" its Enriques diagram. The definition of Milnor number of a consistent Enriques diagram is as follows.

Definition 2.16. For any consistent (D, ν) we define *Milnor number of the diagram* (D, ν) by

$$\mu((D, \nu)) = \sum_{P \in D} \nu(P)(\nu(P) - 1) + 1 - r,$$

where

$$r = \sum_{P \in D} \left(\nu(P) - \sum_{Q \to P} \nu(Q) \right).$$

Theorem 2.17. *Let $f_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a singularity and (D, ν) corresponding consistent diagram then $\mu((D, \nu)) = \mu(f_0)$. In this case r is the number of branches (irreducible components) of f_0.*

Proof. It follows from above definition and Definition 6.4.1 in [3]. □

Example 2.18. Let $f(x, y) = (x^2 - y^3)(x^2 + y^3)(x^3 - y)(x^3 + y)(x^4 - y^3)$ be a singularity. Its Enriques diagram and minimal Enriques diagram are presented in Figure 4 and 5.

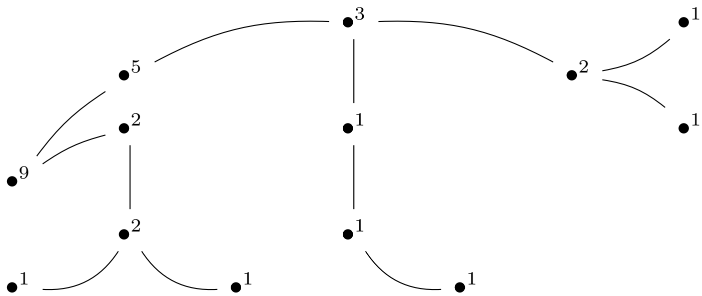


Fig. 4. The consistent Enriques diagram of singularity f (the numbers above bullets are multiplicities of vertices).

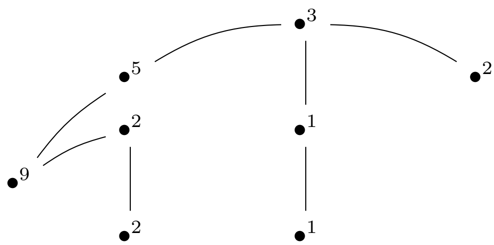


Fig. 5. The minimal Enriques diagram of singularity f.

From this minimal diagram we can compute the number of branches and the Milnor number

$$r = (9-5-2-2)+(5-3-1-1)+(3-2-1)+2+(1-1)+1+(2-2)+2 = 5,$$

$$\mu(f) = \mu((D,\nu)) = 9\cdot 8+5\cdot 4+3\cdot 2+3\cdot 2\cdot 1+1-r = 120.$$

3. Homogeneous singularities

In this Section we will first consider simple homogeneous singularities of the form $f_0(x,y) = x^n + y^n$, $n \geq 2$, and estimate the jump of its Milnor number in the class of linear deformations. The case of arbitrary homogeneous singularities and even semi-homogeneous (called also ordinary) singularities will follow from this case.

The general problem of arbitrary deformations for these singularities was considered by S. Brzostowski, T. Krasiski and J. Walewska in the paper ([5]). They proved that:

- For the singularity $f_0^n(x,y) = x^n + y^n$, $n \geq 2$

$$\lambda\left(f_0^n\right) = \left[\frac{n}{2}\right].$$

- For homogeneous singularity f_0 of degree n, $n \geq 2$

$$\lambda\left(f^0\right) \geq \left[\frac{n}{2}\right].$$

- For homogeneous singularity f_0 of degree n, $n \geq 2$, with generic coefficients

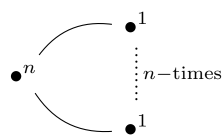
$$\lambda\left(f^0\right) > \left[\frac{n}{2}\right].$$

A. Bodin also proved ([7]) that for the singularity $f_0^n(x,y) = x^n + y^n$, $n \geq 2$,

$$\lambda^{\mathrm{nd}}\left(f_0^n\right) = n - 1,$$

where λ^{nd} is the jump of Milnor number for non-degenerated deformations.

First, we will consider the singularity $f_0^n(x,y) = x^n + y^n$, for any $n \in \mathbb{N}$. Its consistent diagram is:



and its minimal diagram is:

We will denote the latter by (D_n, ν_n), where $D_n := \{R\}$, $\nu_n(R) = n$. For this diagram we have $\mu((D,\nu_n)) = n(n-1) + 1 - n = (n-1)^2$.

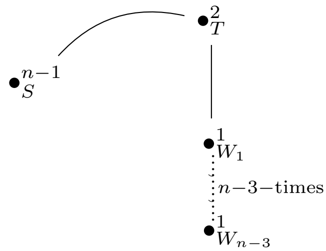
Lemma 3.1. *Let $n \in \mathbb{N}, n \geq 4$ and $[(D_n, \nu_n)]$ be the type of minimal diagram corresponding to the singularity $f_0^n(x,y) := x^n + y^n$. There exists a minimal (D,ν) such that $[(D_n,\nu_n)]$ is linearly adjacent to $[(D,\nu)]$ and*

$$\mu((D,\nu)) = (n-1)^2 - (n-2)$$

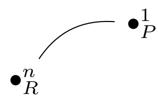
Proof. We will define (D,ν) as following:

- $D = \{S, T, W_1, \ldots, W_{n-3}\}$,
- $T \to S$, $W_1 \to S, T$ and for any $i = 2, \ldots, n-3$ $W_i \to S, W_{i-1}$,
- $\nu(S) = n-1$, $\nu(T) = 2$, $\nu(W_1) = \ldots = \nu(W_{n-3}) = 1$.

It is easy to check that this is a minimal diagram and its graph is:



We will show that $[(D_n, \nu_n)]$ is linearly adjacent to $[(D, \nu)]$. Let $D'_n = \{R, P\}$ (where P is an additional vertex), $P \to R$ and $\nu'_n(R) = n$, $\nu'_n(P) = 1$. Thus $(D'_n, \nu'_n) \in [(D_n, \nu_n)]$. Its graph is:



Moreover $(D'_n, \nu'_n) \geq (D, \nu)$. Indeed, there exist isomorphic subdiagrams $D_0 = \{S, T\} \subset D$, $D'_0 = D'_n$ and an isomorphism $i : D_0 \to D'_0$ such that $i(S) = R, i(T) = P$.

A new multiplicity system $\mu : D \to \mathbb{Z}$ in D is defined by

$$\mu(Q) = \begin{cases} \nu'_n(i(S)) = \nu'_n(R) = n, & Q = S \\ \nu'_n(i(T)) = \nu'_n(P) = 1, & Q = T \\ 0, & Q = W_i \text{ for any } i = 1, \ldots, n-3 \end{cases}$$

Thus we have for $Q \in D$

- If $Q = S$ then
$$\mathrm{ord}_\nu(S) = \nu(S) = n - 1 < n = \nu'_n(R) = \mathrm{ord}_\mu(S).$$
- If $Q = T$ then
$$\mathrm{ord}_\nu(T) = \nu(T) + \mathrm{ord}_\nu(S) = 2 + n - 1 = 1 + n = \mu(T) + \mathrm{ord}_\mu(S) = \mathrm{ord}_\mu(T).$$
- If $Q = W_1$ then
$$\mathrm{ord}_\nu(W_1) = \nu(W_1) + \mathrm{ord}_\nu(S) + \mathrm{ord}_\nu(T) = 1 + n - 1 + n + 1 = 2n + 1 =$$
$$0 + n + n + 1 = \mu(W_1) + \mathrm{ord}_\mu(S) + \mathrm{ord}_\mu(T) = \mathrm{ord}_\mu(W_1)$$
- ...
- If $Q = W_{n-3}$ then
$$\mathrm{ord}_\nu(W_{n-3}) = \nu(W_{n-3}) + \mathrm{ord}_\nu(S) + \mathrm{ord}_\nu(W_{n-4}) = 1 + n - 1 + (n-3)n + 1 =$$
$$= (n-2)n + 1 = 0 + n + (n-3)n + 1 = \mu(W_{n-3}) + \mathrm{ord}_\mu(S) + \mathrm{ord}_\mu(W_{n-4}) =$$
$$= \mathrm{ord}_\mu(W_{n-3})$$

Using definition 2.14 $[(D_n, \nu_n)]$ is linearly adjacent to $[(D, \nu)]$.

We can now compute the Milnor number for (D, ν)

$$\mu((D,\nu)) = (n-1)(n-2) + 2\cdot 1 + 1 - 2 = (n-1)(n-2) + 1 =$$
$$= (n-1)^2 - (n-2).$$

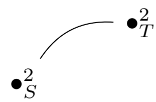
□

Lemma 3.2. *Let* $[(D_3, \nu_3)]$ *will be type of minimal diagram corresponding to the singularity* $f_0^3(x,y) := x^3 + y^3$. *There exists minimal* (D, ν) *such that* $[(D_3, \nu_3)]$ *is linearly adjacent to* $[(D, \nu)]$ *and*

$$\mu((D,\nu)) = 3$$

Proof. Let define (D, ν):

- $D = \{S, T\}$
- $T \to S$
- $\nu(S) = \nu(T) = 2$



It is easy to check that $[(D_3, \nu_3)]$ is linearly adjacent to $[(D, \nu)]$ and

$$\mu((D,\nu)) = 2 + 2 + 1 - 2 = 3.$$

□

Theorem 3.3. *Let* $n \in \mathbb{N}, n \geq 2$ *and* $[(D_n, \nu_n)]$ *be the type of minimal diagram corresponding to the singularity* $f_0^n(x,y) := x^n + y^n$. *There exists a minimal Enriques diagram* (D, ν) *such that* $[(D_n, \nu_n)]$ *is linearly adjacent to* $[(D, \nu)]$ *and*

$$\mu((D,\nu)) = \begin{cases} (n-1)^2 - (n-2), & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

Proof. For $n \geq 3$ result follow from Lemmas 3.2 and 3.1. The case $n = 2$ is trivial since for $f_0^2(x,y) = x^2 + y^2$ the only adjacent Enriques diagram is $(\{R\}, \nu)$, where $\nu(R) = 1$. □

From the above result we get an estimation of the jump of Milnor number.

Theorem 3.4. *For the singularity* $f_0^n(x,y) := x^n + y^n$ $(n \geq 3)$ *the jump of Milnor number for all linear deformations is estimated by*

$$\lambda^{lin}(f_0^n) \leq n - 2$$

We can give linear deformations of f_0^n $(n \geq 3)$ representing diagrams in Lemmas 3.2 and 3.1 in an explicit way.

Theorem 3.5. *For the singularity* $f_0^n(x,y) := x^n + y^n$ $(n \geq 3)$ *take linear deformation given by the formula:*

$$f_s(x,y) := \begin{cases} f_0^n(x,y) + s(x+y)^{n-1}, & n \notin 2\mathbb{N} \\ f_0^n(x,y) + s(x+ey)^{n-1}, & n \in 2\mathbb{N} \end{cases},$$

where $e^n = -1$.

The Milnor number of f_s *is* $\mu(f_s) = (n-1)^2 - (n-2)$ *and its minimal diagram is* (D,ν) *from the proof of Lemmas 3.2 and 3.1.*

Let us pass to the general case of arbitrary homogeneous singularities. Notice that any homogeneous singularity has the form

$$f_0(x,y) = a_0x^n + a_1x^{n-1}y + \ldots + a_{n-1}xy^{n-1} + a_ny^n, \quad a_0,\ldots,a_n \in \mathbb{C}.$$

Its Enriques diagram is the same as for the singularity $f_0^n(x,y) = x^n + y^n$. Hence the same reasoning gives us the theorem:

Theorem 3.6. *For the homogeneous singularity* f_0 $(n \geq 2)$ *the jump of Milnor number for all linear deformations is estimated by*

$$\lambda^{lin}(f_0^n) \leq \begin{cases} n-2, & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

If we consider more general class of singularities - semi-homogeneous singularities i.e. singularities of the form

$$f_0 = f_0' + g,$$

where f_0' is a homogeneous singularity and $\mathrm{ord} g > \mathrm{ord} f_0'$, then we also easily notice that their Enriques diagrams are the same as in the case of homogeneous singularities. Hence we obtain:

Theorem 3.7. *For the semi-homogeneous singularity* f_0 $(n \geq 2)$ *the jump of Milnor number for all linear deformations is estimated by*

$$\lambda^{lin}(f_0^n) \leq \begin{cases} n-2, & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

At the end we pose the conjecture:

Conjecture 3.8. *In theorem 3.4 the estimation can be replaced by the equality i.e.: for the singularity* $f_0^n(x,y) := x^n + y^n$ $(n \geq 2)$ *the jump of Milnor number for all linear deformations equals*

$$\lambda^{lin}(f_0^n) = \begin{cases} n-2, & n \geq 3 \\ 1, & n = 2 \end{cases}.$$

This conjecture seems to be true because for $n < 10$ it was checked by a computer program.

References

[1] S. Gusein-Zade, *On singularities from which an A_1 can be split off*, Funct. Anal. Appl. **27** no. 1, (1993), 57–60.

[2] G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to Singularities and Deformation*, Springer Verlag, 2006.

[3] E. Casas-Alvero, *Singularities of Plane Curve*, Cambridge University Press, 2000.

[4] M. Alberich-Carramiñana, J. Roé, *Enriques Diagrams and Adjacency of Planar Curve Singularities*, Canad. J. Math. **57** no. 1 (2005), 3–16.

[5] S. Brzostowski, T. Krasiński, J. Walewska, *Milnor numbers in deformations of homogeneous singularities* (2014), ArXiv 1104.7704.

[6] J. Walewska, *The jump of Milnor numbers in families of non-degenerate and non-convenient singularities*, Analytic and Algebraic Geometry, University of d Press (2013), 141–153.

[7] A. Bodin, *Jump of Milnor numbers*, Bull. Braz. Math. Soc. (N.S.) **38** no. 3 (2007), 389–396.

[8] S. Kleiman, R. Piene, *Enumerating singular curves on surfaces*, Proc. Conference on Algebraic Geometry: Hirzebruch **70** (Warsaw 1998) A.M.S. Contemp. Math. **241** (1999), 209–238.

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SKOK LICZBY MILNORA LINIOWYCH DEFORMACJI JEDNORODNYCH OSOBLIWOŚCI KRZYWYCH

S t r e s z c z e n i e

Skokiem liczby Milnora osobliwości izolowanej f_0 nazywamy najmniejszą niezerową różnicę między liczbą Milnora f_0 a jedną z jej deformacji f_s. W pracy został oszacowany skok liczby Milnora dla osobliwości jednorodnych i semi-jednorodnych w klasie deformacji liniowych.

Słowa kluczowe: diagram Enriquesa, skok liczby Milnora, osobliwość jednorodna, osobliwość semi-jednorodna