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TOPOLOGICAL COUNTERPART OF THE NOSHIRO-WARSCHAWSKI THEOREM FOR COMPLEX-VALUED FUNCTIONS

Summary

Let $f : C \to \mathbb{C}$ be a locally one-to-one, continuous function defined on a convex domain C in the complex plain $\mathbb{C}$. In this paper we study topological properties of the image $f(B)$, where B is a suitably chosen convex subset of C, to provide certain necessary and sufficient condition for f to be globally one-to-one in C. In fact, the main result, can be seen as a generalization of the well-known univalence criterion for analytic functions due to Noshiro and Warschawski. We also obtain a sufficient condition for the univalence of a locally one-to-one, continuous function defined on an arbitrary starlike domain in the complex plane as a corollary of the main result.

Keywords and phrases: univalence criterion, Noshiro-Warschawski theorem, local homeomorphism

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and let f be a function mapping D into $\mathbb{C}$, $f : D \to \mathbb{C}$ for short. It is clear that if f is one-to-one in D, then f is locally one-to-one in D. If f is holomorphic in D, then f is locally one-to-one if and only if $f'(z) \neq 0$ for all $z \in D$. Still, even a holomorphic f that is locally one-to-one in D does not have to be one-to-one in D. However, it was proved independently by Noshiro [6] and Warschawski [10] (see also Wolff [11]) that a function $f : D \to \mathbb{C}$ holomorphic in a convex domain $D \subset \mathbb{C}$ is one-to-one in D provided $\operatorname{Re} f'(z) > 0$ for all $z \in D$.

There were several attempts to generalize the Noshiro-Warchawski theorem. Tims [9] proved that the theorem fails for every simply connected non-convex domain. Herzog and Piranian [3] pointed out that although there exist (multiply connected) non-convex domains for which the theorem holds they do not "fall far short of being convex". The possibility of weakening the condition on the derivative was also studied, see for example [2, 8]. One interesting result of this kind was given by Janiec [4] who showed that if f is holomorphic in a convex domain D and $\operatorname{Re} f'(z) + \varphi(\operatorname{Im} f(z)) \operatorname{Im} f'(z) > 0$, $z \in D$ for some continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, then f is one-to-one in D.

In this paper we replace the condition $\operatorname{Re} f'(z) > 0$, $z \in D$, by a purely topological one combined with the natural assumption that f is locally one-to-one. A precise statement of our main result is given in Theorem 2.2. Moreover, as its application we obtain a sufficient condition for a locally one-to-one continuous function defined in a starlike domain to be globally one-to-one. The idea of this paper comes from our investigations concerning local homeomorphisms (see [5, **?**]). As the main tool to prove our result we use the theorem of Ortel and Smith [7, Theorem 1].

2. Main result

Let $\mathcal{R}$ be the family of all open and non-empty rectangles in the complex plane $\mathbb{C}$. Denote by $\operatorname{diam} R$ the diameter of a rectangle $R \in \mathcal{R}$. For a fixed positive real number d we define $\mathcal{R}_d$ to be the family of all $R \in \mathcal{R}$ such that $\operatorname{diam} R = d$.

Lemma 2.1. *Fix $R \in \mathcal{R}$ and let $f : R \to \mathbb{C}$ be a locally one-to-one, continuous function such that the image of every rectangle contained in R is simply connected. Then f is one-to-one in R if and only if there exists $d \in (0, \operatorname{diam} R]$ such that f is one-to-one in each $T \in \mathcal{R}_d$ contained in R.*

Proof. If f is one-to-one in R then it is clearly one-to-one in each rectangle T contained in R.

Conversely, assume f is not one-to-one in R and there exists $d \in (0, \operatorname{diam} R]$ such that f is one-to-one in each $T \in \mathcal{R}_d$ contained in R. Observe that we can construct two rectangles P_1 and Q_1, both of the same length as R and both of the width equal to 2/3 of the width of R, such that $P_1 \cap Q_1 \neq \emptyset$ and $P_1 \cup Q_1 = R$. Moreover, if f is one-to-one in P_1 and one-to-one in Q_1 then f takes on every value in $f(P_1) \cup f(Q_1)$ once or twice and every value in $f(P_1 \cap Q_1)$ exactly once. Therefore, f is one-to-one in $P_1 \cup Q_1$ by the theorem of Ortel and Smith [7, Theorem 1]. This leads to a contradiction with the assumption that f is not one-to-one in R. Thus we deduce that f is not one-to-one in P_1 or is not one-to-one in Q_1. Without any loss of generality we can assume that f is not one-to-one in P_1. Next, we can construct two rectangles $\widetilde{P}_1$ and $\widetilde{Q}_1$, both of the same width as P_1 and both of the length equal to 2/3 of the length of P_1, such that $\widetilde{P}_1 \cap \widetilde{Q}_1 \neq \emptyset$ and $\widetilde{P}_1 \cup \widetilde{Q}_1 = P_1$. Again, by the theorem of Ortel and Smith [7, Theorem 1], f is not one-to-one in $\widetilde{P}_1$ or is not one-to-one in $\widetilde{Q}_1$. Set $R_1 := \widetilde{P}_1$ if f is not one-to-one in $\widetilde{P}_1$ and $R_1 := \widetilde{Q}_1$ otherwise.

Clearly, $R_1 \subset R$ and the length, the width and the diameter of R_1 are equal to $2/3$ of the length, $2/3$ of the width and $2/3$ of the diameter of R, respectively.

The above procedure repeated with R_1 in place of R produces a rectangle $R_2 \subset R_1$ such that f is not one-to-one in R_2 and the length, the width and the diameter of R_2 are equal to $(2/3)^2$ of the length, $(2/3)^2$ of the width and $(2/3)^2$ of the diameter of R, respectively. Repeating this procedure again and again we get a descending sequence of rectangles $n \mapsto R_n \subset R$, $n \in \mathbb{N}$, such that f is not one-to-one in each R_n and $\operatorname{diam} R_n = (2/3)^n \operatorname{diam} R$. Since $\operatorname{diam} R_n$ tends to 0 as $n \to \infty$ we have a contradiction with the assumption that there exists $d \in (0, \operatorname{diam} R]$ such that f is one-to-one in each $T \in \mathcal{R}_d$ contained in R, which completes the proof. □

Theorem 2.2. *Let C be a fixed convex domain in the complex plane $\mathbb{C}$ and let $f : C \to \mathbb{C}$ be a continuous function. Then f is one-to-one in C if and only if f is locally one-to-one in C and $f(R)$ is a simply connected set for each rectangle $R \in \mathcal{R}$ contained in C.*

Proof. If f is one-to-one in C then it is clearly one-to-one in each rectangle $R \in \mathcal{R}$ contained in C and hence $f(R)$ is simply connected.

Conversly, assume that f is not one-to-one in C, that is, there exist two points $z_0, w_0 \in C$ such that $f(z_0) = f(w_0)$ and $z_0 \neq w_0$. Since C is a convex set, it is clear that there exists a rectangle $R \in \mathcal{R}$ such that the closure $\overline{R}$ of R is contained in C and $z_0, w_0 \in R$. Denote by $r := \operatorname{diam} R$. Observe that for each $n \in \mathbb{N}$ there exists a rectangle $R_n \in \mathcal{R}_{r/n}$, $R_n \subset R$, in which f is not one-to-one by Lemma 2.1. That is, for each $n \in \mathbb{N}$ there exist two points $z_n, w_n \in R_n$ such that $f(z_n) = f(w_n)$ and $z_n \neq w_n$. Consider the sequence $n \mapsto z_n$, $n \in \mathbb{N}$. Clearly, the set $\{z_n : n \in \mathbb{N}\} \subset R$ and hence the sequence $n \mapsto z_n$ is bounded, which yields that there exists a convergent subsequence $k \mapsto z_{n_k}$, $k \in \mathbb{N}$. Denote its limit by z and observe that $z \in \overline{R} \subset C$. Next consider the sequence $k \mapsto z_{n_k} - w_{n_k}$, $k \in \mathbb{N}$, which is convergent to 0 since $0 < |z_{n_k} - w_{n_k}| < r/k$. Therefore the sequence $k \mapsto w_{n_k}$, $k \in \mathbb{N}$, is also convergent to the point z. This means that in each disk centered at z and contained in C the function f is not one-to-one. But by the assumption f is locally one-to-one in C and there exists an open disk centered at z in which f is one-to-one. Thus we have a contradiction and the proof is completed. □

Corollary 2.3. *Let S be a fixed starlike domain in the complex plane $\mathbb{C}$ and $f : S \to \mathbb{C}$ be a locally one-to-one, continuous function such that $f(R)$ is a convex set for each rectangle $R \in \mathcal{R}$ contained in S. Then f is one-to-one in S.*

Proof. Assume that f is not one-to-one in S, that is, there exist two points $z, w \in S$ such that $f(z) = f(w)$. Since S is a starlike set, it is clear that there exist two rectangles $R_z, R_w \in \mathcal{R}$ contained in S and there exists a point $\zeta \in S$ such that $z, \zeta \in R_z$ and $w, \zeta \in R_w$. Obviously, $R_z \cup R_w$ is simply connected and $f(R_z) \cup f(R_w)$ is also simply connected as a union of two convex sets. Hence f is not one-to-one in

R_z or it is not one-to-one in R_w by the theorem of Ortel and Smith [7, Theorem 1]. This leads to a contradiction with Theorem 2.2. □

Remark 2.4. It should be mentioned that in the statement of Lemma 2.1, the rectangles can be replaced by another suitably chosen family of convex sets. For example the analog of Lemma 2.1 (also Theorem 2.2 and Corollary 2.3) for triangles holds true and the proof is analogous to the presented one. The main difference is a little bit more complicated procedure of constructing a sequence of triangles such that the corresponding sequence of their diameters tends to 0. However, there seems to be no analog of the proof of Lemma 2.1 in the case when rectangles are replaced by the family of disks.

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TOPOLOGICZNY ODPOWIEDNIK TWIERDZENIA NOSHIRO-WARSCHAWSKIEGO DLA FUNKCJI ZESPOLONYCH

S t r e s z c z e n i e

Niech $f : C \to \mathbb{C}$ będzie lokalnie różnowartościową oraz ciągłą funkcją określoną na pewnym obszarze wypukłym C zawartym w płaszczyźnie zespolonej $\mathbb{C}$. W niniejszej pracy badamy topologiczne własności obrazu $f(B)$, gdzie B jest pewnym szczególnym podzbiorem wypukłym zbioru C, aby uzyskać warunek konieczny i dostateczny różnowartościowości funkcji f w C. Uzyskany rezultat jest uogólnieniem znanego kryterium różnowartościowości funkcji analitycznych, które podali Noshiro i Warschawski. Dodatkowo, jako wniosek, formułujemy warunek dostateczny różnowartościowości dowolnej lokalnie różnowartościowej i ciągłej funkcji określonej na pewnym obszarze gwiaździstym zawartym w płaszczyźnie zespolonej.

Słowa kluczowe: kryterium różnowartościowości, twierdzenie Noshiro-Warschawskiego, lokalny homeomorfizm