DOI: 10.26485/0459-6854/2017/67.3/2 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017 Vol. LXVII

Recherches sur les déformations no. 3

pp. 21–32

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TOPOLOGICAL ENTROPY AND HOMOGENEOUS MEASURE FOR A SOLENOID

Summary

We consider topological and measure-theoretical approach to dynamical properties of a solenoid. In general case there is no invariant measure for a solenoid, therefore one can not say neither about a measure-theoretical entropy nor about a measure of maximal entropy of a solenoid. Following R. Bowen, we define a homogeneous measure for a solenoid and study its properties. We show that if a solenoid admits a homogeneous measure, the measure has similar properties to measure of maximal entropy in classical dynamical systems.

Keywords and phrases: entropy, local measure entropy, entropy-like quantity, solenoids, homogeneous measure

1. Introduction

In the late 1920s a solenoid was introduced to mathematics by L. Vietoris [26] as inverse limit spaces over circle maps. It was an example of a continuum for which the fundamental group, in the sense of Vietoris, depends on the base point. A solenoid can be presented either in an abstract way as an inverse limits or in a geometric way as a nested intersections of solid tori. For a given sequence of positive integers $\{k_n\}_{n\in\mathbb{N}}$, a solenoid can be described as the intersection of a sequence of tori $\{T_n\}_{n\in\mathbb{N}}$ such that T_{n+1} is wrapped around inside T_n longitudinally k_n times. Topological properties of inverse limits on intervals are relatively good understood (see [14]).

The standard construction of a solenoid, presented by L. Vietoris [26], was generalized and modified by C. McCord [20], R. Williams [29] and many other authors in different contexts. In dynamical systems a solenoid was introduced by S. Smale [24] as hyperbolic attractor of a diffeomorphism of a three-dimensional manifold. Solenoids appeared in many branches of mathematics: in geometry, dynamical systems, theory of groups, continuum theory, foliations and so on.

For example, the inverse limit of a branched covering space mappings of Riemann sphere admits an invariant subspace which is laminated by complex plane and admits transverse invariant measure. The Riemann surface laminations were studied by D. Sullivan [25], later by M. Lyubich and J. Minsky [18] and others.

In the paper we study a sequence $f_\infty = (f_n : X_n \to X_{n-1})_{n=1}^\infty$ of continuous epimorphisms of compact metric spaces X_n, called bonding maps. We assume that all spaces X_n coincide with a compact metrizable space X. By *solenoid* determined by f_∞, we mean the inverse limit

$$X_\infty = \varprojlim X_k = \{(x_k)_{k=0}^\infty : x_{k-1} = f_k(x_k)\}.$$

Clearly, X_∞ is a compact subset of the Hilbert cube ΠX_k. A distance function d_∞ on X_∞ is given by usual formula

$$d_\infty((x_k),(u_k)) = \sum_{k=0}^\infty \frac{1}{2^k} d_k(x_k,u_k).$$

Since X_∞ is uniquely determined by f_∞, we will often identify these two objects. Solenoids are compact metrizable spaces that enjoy many pathological properties. They are connected, but not locally connected or path connected. A solenoid is both a metric space and a dynamical object of a complicated structure. Its complexity yields from the dynamics of bonding maps and can be investigated from topological or ergodic point of view.

Recall that the concept of entropy arose in physics in 19th centuary to decribe the equilibria and the evolution of thermodynamics systems. In 1864 R. Clausius used the word entropy in his book to describe quantity accompanying a change from thermal to mechanical energy. Later, in 1877 L. Boltzmann introduced a concept of entropy into the probabilistic setup of statistical mechanics and in 1932 J. von Neumann generalized entropy to quantum mechanics.

C. Shannon [22] was the first who used this notion as the term in probability and information theory to describe a measure of uncertainty and complexity of the system, thus he provided foundations for information theory. Dynamical entropy in dynamical systems was introduced in 1958 by A. Kolmogorov [16] and improved by his student Y. Sinai [23], this mathematical notion is now known as Kolgomorov-Sinai entropy.

Let X be a compact metric space and $f : X \to X$ be a continuous map or a homeomorphism. The pair (X,f) is called a topological dynamical system. Topological entropy is a main concept in topological dynamics, it is a nonnegative number which measures disorder and complexity of the system (X,f). Positive entropy of the dynamical system reflects its chaotic behaviour. For a more complete text on entropy we refer to monographs [19], [11] and survey paper [15]. The classical topological entropy of a single map was a very fruitful notion, therefore the concept of entropy was generalized to an action of algebraic structures (such as semigroups, groups and pseudogroups) on topological spaces, and to geometric objects such as distributions, laminations and foliations (see [27]). There are attempts (e.g. [17], [21]) to transfer the notion of topological entropy to generalized topological and metric spaces in the sense of Császár ([7], [8]).

In the paper we focus on dynamics of a solenoid. In general case there is no invariant measure for a solenoid, therefore it is not clear how to define its measure-theoretical entropy. Also, a notion of a measure of maximal entropy of a solenoid is not defined.

R. Bowen [4] defined a notion of homogeneous measure for a classical dynamical system determined by a continuous map $f : X \to X$ of a compact metric space X. We modify Bowen's ideas to introduce a notion of a homogeneous measure for a solenoid and we provide examples of such measures. On the other hand a local measure entropy, which was originally introduced by M. Brin and A. Katok [5] for a dynamics of a single map, is also a powerful tool for investigations of dynamics of solenoids.

In Theorem 4.6 we show that if a solenoid admits a homogeneous measure, then its local measure entropy does not depend on a point of the solenoid. In Theorem 4.8 we prove that the topological entropy of a solenoid, with a homogeneous measure, coincides with the local measure entropy. Moreover, we show that if a solenoid admits a homogeneous measure, then this measure has similar properties to the measure of maximal entropy in classical dynamical systems.

2. Measure-theoretical entropy and topological entropy of a map

In mathematics, the study of a disrete dynamical system, determined by a continuous map $f : X \to X$ of a compact metric space X, as a whole is primarily concerned with the asymptotic behavior of such systems, that is how the system evolves after repeated applications of f. Its complexity can be described by the topological entropy $h_{top}(f)$ and the measure-theoretical entropy $h_\mu(f)$ calculated with respect to an f-invariant Borel probability measure μ. For convenience of the reader, we recall briefly the basic definitions related to measure-theoretical entropy and topological entropy. For more detailed introduction to dynamical systems we recommend [28].

2.1. Measure-theoretical entropy of a map

Let (X_1, B_1, μ_1) and (X_2, B_2, μ_2) be measure spaces. A map $T : X_1 \to X_2$ is called measurable if the preimage of any measurable set is measurable. A measurable transformation $T : X_1 \to X_2$ is measure preserving if $\mu_1(T^{-1}(A_2)) = \mu_2(A_2)$ for every $A_2 \in B_2$.

Assume now that $f : X \to X$ is a continuous map defined on a compact metric space (X, d). The Krylov-Bogoliubov Theorem (see [28]) guarantees the existence of a probability f-invariant measure μ defined on Borel σ-algebra generated by the collection of open subsets of X. A partition of X is a finite family $A = \{A_1, A_2, ...A_n\}$ of pairwise disjoint measurable subsets of X such that $A_1 \cup A_2 \cup ... \cup A_n = X$. For partitions A and B of X we define the following partitions:

$$A \vee B = \{A_i \cap B_j : A_i \in A \ and\ B_j \in B\},$$

$$f^{-1}(A) = \{f^{-1}(A_i) : A_i \in A\},$$

$$A^{(n)} = A \vee f^{-1}(A) \vee ... \vee f^{-(n-1)}(A).$$

A measure entropy of a partition A of X with respect to the measure μ is defined by

$$H_\mu(A) = -\sum_{A_i \in A} \mu(A) \log(\mu(A)).$$

It is known (for details see [28]) that for any partition A of X there exists a limit

$$H_\mu(f, A) = \lim_{n\to\infty} \frac{1}{n} H_\mu(A^{(n)}).$$

Definition 2.1. Kolmogorov-Sinai or measure-theoretical entropy of a measurable map $f : X \to X$ with respect to an f-invariant measure μ is the quantity defined by

$$h_\mu(f) = \sup\{H_\mu(f, A) : A\ is\ a\ partition\ of\ X\}.$$

2.2. Topological entropy of a map

Topological entropy of a continuous map was first introduced in 1965 by R. Adler, A. Konheim and M. McAndrew [1]. In metric spaces a different definition of entropy was introduced by R. Bowen in 1971 in [2] and independently by E. Dinaburg in 1970 in [9]. Later R. Bowen [3] proved that both definitions are equivalent. Bowen's approach uses a notion of (n, ϵ)-separated points.

Again, let $f : X \to X$ be a continuous map defined on a compact metric space (X, d). Following Bowen we say that a subset $E \subset X$ is (n, ϵ)-separated (where n is a positive integer and $\epsilon > 0$) if the inequality

$$\max\{d(f^i(x), f^i(y)) : i = 0, 1, ..., n-1\} \geq \epsilon$$

holds for any distinct points $x, y \in E$. Since X is a compact space the cardinality $card(A)$ of any (n, ϵ)-separated set A is finite. Let $s(n, \epsilon) = \max\{card(A) : A\ is\ (n, \epsilon) - separated\ subset\ of\ X\}$.

Definition 2.2. The *topological entropy* of a continuous map $f : X \to X$ defined on a compact metric space (X, d) is defined as

$$h_{top}(f) := \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \epsilon).$$

We recommend texbooks [28], [6] or [11] which treats properties of topological and measure-theoretical entropies.

2.3. Variational Principle

Due to the Krilov-Bogoliubov Theorem for a continuous map $f : X \to X$ the set $M(f, X)$ of f-invariant Borel probability measures on X is not empty. The topological entropy and measure-theoretical entropies of f are interrelated. The relation between them is stated in the famous Variational Principle. One inequality in the Variational principle was proved by E. Dinaburg [9], [10] and T. Goodman [12] the other by L. Goodwyn [13].

Theorem 2.3 (Variational Principle)**.** *For a continuous map $f : X \to X$ defined on a compact metric space (X, d)*

$$h_{top}(f) = \sup\{h_\mu(f) : \mu \in M(f, X)\}$$

i.e., topological entropy equals the supremum of the Kolmogorov-Sinai entropies $h_\mu(f)$ of f, where μ ranges over the set $M(f, X)$ all f-invariant Borel probability measures on X.

Remark 2.4. If $h_{top}(f) = h_\mu(f)$ then the f-invariant measure μ is called the measure of maximal entropy. In many cases a measure of maximal entropy exists.

3. Topological entropy of a solenoid

Let $f_\infty = (f_n : X_n \to X_{n-1})_{n=1}^{\infty}$ be a sequence of continuous epimorphisms of compact metric spaces X_n. We assume that all spaces X_n coincide with a compact metrizable space X. Each space X_n is equipped with a metric d_n. Recall that by *solenoid* determined by f_∞, we mean the inverse limit

$$X_\infty = \varprojlim X_k = \{(x_k)_{k=0}^{\infty} : x_{k-1} = f_k(x_k)\}.$$

In the case of a solenoid, which can be considered as a generalized dynamical system, one can define its topological entropy. In general case there is no common invariant measure and therefore it is not clear how to define a measure-theoretical entropy of a solenoid. Following Bowen [2] we define a topological entropy of a solenoid by (n, ϵ)-separated sets. For any positive integer n we define a new metric D_n on X_n by

$$D_n(x, y) = \max\{d_{i-1}(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1, ..., n\}\}$$

We say that a subset $E \subset X_n$ is $(n, \epsilon)-$ separated if for any distinct points $a_1, a_2 \in E$ the inequality $D_n(a_1, a_2) \geq \epsilon$ holds. Since (X_n, d_n) is a compact metric space, then any $(n, \epsilon)-$ separated set E is finite. Let

$$s(n, \epsilon) := \max\{card(E) : E \ is \ (n, \epsilon) - separated \ subset \ of \ X_n\}.$$

Definition 3.1. The quantity

$$h_{top}(f_\infty) := \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s(n,\epsilon)$$

is called the topological entropy of the solenoid f_∞.

Remark The topological entropy of a solenoid also can be expressed in the language of (n,ϵ)-spannings sets. A subset $F \subset X_n$ is (n,ϵ)-spanning if for any $x \in X_n$ there exists $f \in F$ such that $D_n(x,f) < \epsilon$. Let

$$r(n,\epsilon) := \min\{card(F) : F \ is \ (n,\epsilon) - spanning \ subset \ of \ X_n\}.$$

Using standard arguments (e.g. [28]) we get an estimation

$$r(n,\epsilon) \le s(n,\epsilon) \le r(n,\epsilon/2).$$

Consequently, passing to the suitable limits we obtain the equality

$$h_{top}(f_\infty) = \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log r(n,\epsilon).$$

4. Homogeneous measures

In general case for a solenoid f_∞ there exists no common f_n- invariant measure, for any $n \in \mathbb{N}$. So, it is not clear how to define a measure-theoretical entropy of f_∞. If there exists a homogeneous measure a solenoid, then we are able to provide estimation of topological entropy by local measure entropies calculated with respect to this particular homogeneous measure.

The sequence of metrics D_n on X_n given by

$$D_n(x,y) := \max\{d_{i-1}(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1,...,n\}\}$$

determine a sequence of n-balls

$$B_n(x,r) := \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}[B_{d_{i-1}}(f_i \circ f_{i+1} \circ ... \circ f_n(x), r)],$$

where $B_{d_i}(y,r) = \{z \in X_i : d_i(z,y) < r\}$ is a standard ball in (X_i, d_i) centered at y and of radius r.

Definition 4.1. We say that a Borel measure μ on a metric space X is f_∞-homogeneous measure for a solenoid f_∞ if:

(1) $\mu(K) < \infty$ for any compact $K \subset X$,
(2) there exists $K_0 \subset X$ with $\mu(K_0) > 0$ and
(3) for any $\epsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that the inequality

$$\mu(B_n(y,\delta)) \le c \cdot \mu(B_n(x,\epsilon))$$

holds for all $n \in \mathbb{N}$ and all $x, y \in X$.

4.1. Examples of f_∞-homogeneous measure

We provide two examples of homogeneous measures for solenoids.

Example 4.2. Choose a closed compact and oriented Riemannian manifold (M, d) with volume form dV. Let $(X_n, d_n) = (M, d)$, for any $n \in N$, and $f_\infty = (f_n : X_n \to X_{n-1})_{n=1}^{\infty}$ be a sequence of isometries of M. The volume form induces a natural measure μ on M :

$$\mu(A) = \int_A 1 \cdot dV,$$

where A is a Borel subset of M. Notice that in this case

$$B_n(x, r) = \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}[B_{d_{i-1}}(f_i \circ f_{i+1} \circ ... \circ f_n(x), r)] = B_d(x, r).$$

Since $(X_n, d_n) = (M, d)$ is compact space we get that for any $\epsilon > 0$ the quantity

$$C(\epsilon) = \frac{\sup\{\mu(B_n(z, \epsilon)) : z \in M\}}{\inf\{\mu(B_n(z, \epsilon)) : z \in M\}} = \frac{\sup\{\mu(B_d(z, \epsilon)) : z \in M\}}{\inf\{\mu(B_d(z, \epsilon)) : z \in M\}} < \infty,$$

so for any $0 < \delta < \epsilon$ and arbitrary $x, y \in M$ we obtain

$$\mu(B_n(y, \delta)) \leq C(\epsilon) \cdot \mu(B_n(y, \delta)).$$

Example 4.3. Let G be a compact topological group with a right invariant Haar measure μ, then G admits a right invariant metric d. Fix an isomorphism (so a homeomorphism and homomorphism) $H : G \to G$ of the topological group and infinite sequence $\{g_n\}_{n \in N}$ of elements of G. Define $f_i := R_{g_i} \circ H$, where $R_{g_i}(x) = x \cdot g_i$ is a right multiplication for any $x \in G$. The sequence $f_\infty = \{f_n : X_n \to X_{n-1}\}_{n \in \mathbb{N}}$, where $X_n = G$, determines a solenoid.

Example 4.3 and the proof of Proposition 4.4 is based on Example 8 in [2], which was written for a single map.

Proposition 4.4. *The solenoid described in Example 4.3 admits a homogeneous measure.*

Proof. Let $B(x, r)$ be a standard ball in G (with respect to metric d) and denote by e the identity element of G.

First we claim that for for any $x \in X$ and $r > 0$

$$f_i^{-1}[B(f_i(x), r)] = H^{-1}[B(e, r)] \cdot x.$$

Indeed, for any $x, y \in X$ and $y_1 = H^{-1}(y)$ we get

$$H^{-1}[y \cdot H(x)] = H^{-1}[H(y_1 \cdot x)] = H^{-1}(y) \cdot x,$$

using the right invariance of metric d we obtain

$$f_i^{-1}[B(f_i(x), r)] = H^{-1}\{R_{g_i}^{-1}[B(H(x) \cdot g_i, r)]\} = H^{-1}[B(H(x), r)]$$

and

$$H^{-1}[B(H(x),r)] = H^{-1}[B(H(x)\cdot H(e),r)] = H^{-1}[B(H(e),r)\cdot H(x)] = \\ = H^{-1}[B(H(e),r)]\cdot x$$

which completes the proof of the first claim.

Our second claim is as follows: for any $i \in N$

$$(f_{i-1}\circ f_i)^{-1}B(f_{i-1}\circ f_i(x),r) = H^{-2}[B(e,r)]\cdot x.$$

Indeed, due to the first claim we may write

$$(f_{i-1}\circ f_i)^{-1}B(f_{i-1}\circ f_i(x),r) = (f_i^{-1}\circ f_{i-1}^{-1})B(f_{i-1}(f_i(x)),r) = \\ = f_i^{-1}\{H^{-1}[B(e,r)]\cdot f_i(x)\} = (H^{-1}\circ R_{g_i}^{-1})\{H^{-1}[B(e,r)]\cdot H(x)\cdot g_i\} = \\ = H^{-1}\{H^{-1}[B(e,r)]\cdot H(x)\} = H^{-2}[B(e,r)]x.$$

The proof of the second claim is done. By simple induction for any $k \in \mathbb{N}$ with $k \leq i$ we arrive at the equality, which is our third claim

$$(f_k\circ f_{k+1}\circ ...\circ f_{i-1}\circ f_i)^{-1}B(f_k\circ f_{k+1}\circ ...\circ f_{i-1}\circ f_i(x),r) = \\ = H^{-[(i-k)+1]}[B(e,r)]\cdot x.$$

Now, using the third claim we are able to calculate the n-ball

$$B_n(x,r) = \bigcap_{i=1}^{n}(f_i\circ f_{i+1}\circ ...\circ f_n)^{-1}[B_{d_{i-1}}(f_i\circ f_{i+1}\circ ...\circ f_n(x),r)] =$$

$$= \bigcap_{i=1}^{n} H^{-[(n-i)+1]}[B(e,r)]\cdot x$$

Therefore, the right invariance of the Haar measure μ yields

$$\mu[B_n(x,r)] = \mu\{\bigcap_{i=1}^{n} H^{-[(n-i)+1]}[B(e,r)]\cdot x\} = \mu\{\bigcap_{i=1}^{n} H^{-[(n-i)+1]}[B(e,r)]\}.$$

So, for any $x, y \in G$ we obtain equality $\mu[B_n(x,r)] = \mu[B_n(y,r)]$, which completes the proof. □

4.2. f_∞-homogeneous measures and topological entropy

M.Brin and A.Katok [5] introduced a notation of the local measure entropy for a single continuous map $f : X \to X$. We adapt this notion of the local measure entropy to a solenoid determined by $f_\infty = \{f_n : X_n \to X_{n-1}\}_{n\in\mathbb{N}}$ in the following way:

Definition 4.5. For any $x \in X$ and a Borel probability measure μ on X the quantity

$$h^{\mu}_{f_\infty}(x) = \lim_{\epsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon))$$

is called a local upper μ-measure entropy at the point x, with respect to f_∞, while the quantity

$$h_{\mu,f_\infty}(x) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon))$$

is called a local lower μ-measure entropy at the point x, with respect to f_∞.

Theorem 4.6. *If μ is a f_∞-homogeneous measure on X, then the equalities $h^\mu_{f_\infty}(x) = h^\mu_{f_\infty}(y)$ and $h_{\mu,f_\infty}(x) = h_{\mu,f_\infty}(y)$ hold for any $x, y \in X$.*

Proof. By definition of a f_∞-homogeneous measure, for $\epsilon > 0$ there exist $0 < \delta(\epsilon) < \epsilon$ and $c > 0$ such that

$$\mu(B_n(y,\delta(\epsilon))) \le c\cdot\mu(B_n(x,\epsilon)).$$

Thus

$$\frac{1}{n}\log\mu(B_n(y,\delta(\epsilon))) \le \frac{\log(c)}{n} + \frac{1}{n}\log\mu(B_n(x,\epsilon)),$$

so

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(y,\delta(\epsilon))) \ge \limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon))$$

and

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(y,\delta(\epsilon))) \ge \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon)).$$

Taking the limit as $\epsilon\to 0$ we arrive at $h^\mu_{f_\infty}(y) \ge h^\mu_{f_\infty}(x)$ and $h_{\mu,f_\infty}(y) \ge h_{\mu,f_\infty}(x)$. Similarly, for $\epsilon' > 0$ there exist $\delta'(\epsilon') > 0$ and $c' > 0$ such that

$$\mu(B_n(x,\delta'(\epsilon'))) \le c'\cdot\mu(B_n(y,\epsilon')).$$

Applying the same arguments, we obtain the inequalities $h^\mu_{f_\infty}(x) \ge h^\mu_{f_\infty}(y)$ and $h_{\mu,f_\infty}(x) \ge h_{\mu,f_\infty}(y)$, which completes the proof. $\square$

Definition 4.7. If μ is a f_∞-homogeneous measure on X, then the common value of local upper measure entropies is denoted by $h^\mu_{f_\infty}$.

Theorem 4.8. *For a solenoid $f_\infty = \{f_n : X_n \to X_{n-1}\}_{n=1}^\infty$ admitting a f_∞-homogeneous measure μ on X, we have*

$$h_{top}(f_\infty) = h^\mu_{f_\infty}.$$

Proof. Take an (n,ϵ)-separated subset $E \subset X$ with maximal cardinality equal to $s(n,\epsilon)$. Then

$$B_n(x,\epsilon/2)\cap B_n(y,\epsilon/2) = \emptyset,$$

for any distinct points $x, y \in E$. So

$$s(n,\epsilon)\cdot\mu(B_n(x,\epsilon/2)) \le \mu(X).$$

The f_∞-homogenity of the measure μ allows us to choose $0 < \delta(\epsilon) < \epsilon/2$ and $c > 0$ so that the inequality

$$\mu(B_n(y,\delta(\epsilon))) \le c\cdot\mu(B_n(x,\epsilon/2))$$

holds for any $n \in \mathbb{N}$ and all $x, y \in X$. Thus

$$s(n,\epsilon) \cdot \mu(B_n(y,\delta(\epsilon))) \leq c \cdot \mu(X)$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log s(n,\epsilon) \leq \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(y,\delta(\epsilon))).$$

Taking the limit as $\epsilon \to 0$ we obtain

$$h_{top}(f_\infty) \leq h^{\mu}_{f_\infty}(y) = h^{\mu}_{f_\infty}.$$

Now take an (n,δ)-spanning subset $F \subset X$, with minimal cardinality equal to $r(n,\delta)$. Notice that $X \subset \bigcup_{x\in F} B_n(x, 2\delta)$. Given $\epsilon > 0$ choose $0 < \delta(\epsilon) < \epsilon$ and $c > 0$ so that

$$\mu(B_n(x, 2\delta(\epsilon))) \leq c \cdot \mu(B_n(y,\epsilon))$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Then inequality

$$c \cdot \mu(B_n(y,\epsilon)) \cdot r(n,\delta(\epsilon)) \geq \mu(X) > 0$$

yields that

$$\limsup_{n\to\infty} \frac{1}{n} \log r(n,\delta(\epsilon)) \geq \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(y,\epsilon)).$$

Finally, as $\epsilon \to 0$ we obtain

$$h_{top}(f_\infty) \geq h^{\mu}_{f_\infty}(y) = h^{\mu}_{f_\infty}.$$

The proof is complete. □

Acknowledgement

The research of the first author was supported by the National Science Centre (NCN) under grant Maestro 2013/08/A/ST1/00275.

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Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 22, 2017.

ENTROPIA TOPOLOGICZNA I MIARA JEDNORODNA DLA SOLENOIDU

S t r e s z c z e n i e

Rozpatrujemy topologiczne i miarowe podejście do opisu dynamicznych własności solenoidu. W ogólnym przypadku solenoid nie posiada miary miezmienniczej, nie wiadomo jak zdefiniować entropię solenoidu względem miary ani tym bardziej jego miarę o maksymalnej entropii. Uogólniamy definicje miary jednorodnej podanej przez R. Bowena dla pojedyńczego odwzorowania na przypadek solenoidu. Podajemy przykłady miar jednorodnych dla solenoidu i badamy ich własności.

Słowa kluczowe: entropia, lokalna entropia miarowa, solenoid, miara jednorodna