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NO-HOLE λ-$L(k,k-1,\dots,2,1)$-LABELING FOR SQUARE GRID

Summary

Motivated by a frequency assignment problem, we demonstrate, for a fixed positive integer k, how to label an infinite square grid with a possibly small number of integer labels, ranging from 0 to $\lambda-1$, in such a way that labels of adjacent vertices differ by at least k, vertices connected by a path of length two receive values which differ by at least $k-1$, and so on. The vertices which are at least $k+1$ distance apart may receive the same label. By finding a lower bound for λ, we prove that the solution is close to optimal, with approximation ratio at most $\frac{9}{8}$. The labeling presented is a no-hole one, i.e., it uses each of the allowed labels at least once.

Keywords and phrases: graph labeling, labeling number, no-hole labeling, square grid, frequency assignment problem, approximation ratio

1. Introduction

The *frequency assignment problem* (*FAP*) is a problem of assigning frequencies to different radio transmitters so that no interference occurs [1]. This problem is also known as the *channel assignment problem* (CAP) [2,3]. Frequencies are assigned to different radio transmitters in such a way that comparatively close transmitters receive frequencies with more gap than the transmitters which are significantly apart from each other.

Motivated by this problem, a long line of researchers looked for various graph labellings with similar properties. The labelling we consider for a graph $G=(V,E)$ is a function $f: V\to\{0,\dots,\lambda-1\}$ such that $|f(x)-f(y)|+d(x,y)\geq k+1$ for $x\neq y$, where k and λ are a fixed positive integers, and $d(x,y)$ is the length of the shortest path joining x and y. The objective is, for a given k, to find a labelling with a smallest possible λ. Traditionally, the problem is called a λ-$L(k,k-1,\dots,2,1)$-labeling, and although the name is a bit clumsy, we

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stick to it for historical reasons. For example, Yeah [4] and after that Griggs and Yeh [5] proposed an $L(2,1)$-labeling for a simple graph. Various generalizations of the original problem, for diverse types of graphs, finite or infinite, has been described in the literature [6–15].

This paper, dealing with an infinite square grid, was directly inspired by that of Nandi et al. [16], who presented an $L(k,k-1,\ldots,1)$-labeling for a triangular lattice. Nevertheless, our paper differs from it in several important respects. The most important difference is the level of formalism — instead of informal considerations we present formal proofs of most of our results. After proposing a set of four formulas for the labelling function f, one for each remainder of k modulo 4, we show that they really work. The proof is conducted in one of the four cases only — the reason is that the method used is completely elementary, and it is enough to demonstrate it for one particular choice. The readers can easily check the other cases, or trust the authors that they have already done it for them. A formal proof given for the lower bound on λ allows one to avoid some inaccuracies to be found in the above-mentioned paper of Nandi et al. [16]. The approximation ratio obtained is also better.

The definition of the problem is given in Section 2. The lower bound on the value of the labeling number λ_k, i.e. the smallest value of λ for the given k, is derived in Section 3. In Section 4, a formula is given that attaches a label to any vertex of an infinite square grid for arbitrary values of k. The correctness proof of the proposed formula is given in Section 4.1. In Section 4.2, we prove that the proposed formula gives a no-hole labeling. Our λ-labeling yields immediately an upper bound on λ_k, given together with the approximation ratio implied by the proposed formula in Section 4.3. Finally, the paper is concluded in Section 5.

2. Problem Definition

Let $G = (V,E)$ be a graph with a set of vertices V and a set of edges E, and let $d(u,v)$ denote the shortest distance between vertices $u,\ v \in V$. Given a fixed $k \in \mathbb{Z}^+$ and $\lambda \in \mathbb{Z}^+$, a λ-$L(k,k-1,\ldots,2,1)$-labeling of the graph is a mapping $f : V \to \{0,\ldots,\lambda-1\}$ such that the following inequalities are satisfied:

$$|f(x)-f(y)| \geq \begin{cases} k & : d(x,y) = 1 \\ k-1 & : d(x,y) = 2 \\ \vdots & \\ 1 & : d(x,y) = k, \end{cases}$$

which can be written more compactly as

$$|f(x)-f(y)| \geq k+1-d(x,y) \text{ for } x \neq y. \tag{*}$$

We shall call any function $f : V \to \mathbb{Z}$ satisfying the inequality a *labeling function*.

If the distance between two vertices is at least $k+1$, the same label can be used for both of them. This minimum distance is known as the *reuse distance* [16]. The $L(k,k-1,\dots,2,1)$-labeling number for the graph, denoted by λ_k, is the minimum λ for which a valid λ-$L(k,k-1,\dots,2,1)$-labeling for the graph exists. Hence, our objective is to find, for each k, a no-hole λ-$L(k,k-1,\dots,2,1)$-labeling with λ as close to λ_k as possible.

We consider an infinite planar square grid $G=(V,E)$ with the set of vertices $V=\mathbb{Z}\times\mathbb{Z}$ and the set of edges $E=\{\{u,v\}\colon u=(u_1,u_2), v=(v_1,v_2),$ and either $|u_1-v_1|=1, u_2=v_2$ or $u_1=v_1, |u_2-v_2|=1\}$. It will be called *'the square grid'* in the sequel. The distance between u and v used in the sequel is the *Manhattan distance*: $d(u,v)=|u_1-v_1|+|u_2-v_2|$.

3. Lower Bound on λ_k

Theorem 3.1. *For $k\geq 1$,*

$$\lambda_k \geq \begin{cases} \frac{2}{3}p(p+1)(2p+1)+2 & \text{if } k=2p \text{ is even,} \\ \frac{2}{3}p(p+1)(2p+3)+2 & \text{if } k=2p+1 \text{ is odd.} \end{cases}$$

Proof. We start with the case of even $k=2p$. We shall write B_m for the ball $\{u\in V\colon d(0,u)\leq m\}$, and S_m for the sphere $\{u\in V\colon d(0,u)=m\}$ (here $0=(0,0)$). Note that there is just one point in S_0 and $4m$ points in S_m for $m>0$ (See Fig. 1). It is easy to calculate that there are exactly $1+4+\ldots+4m=2m^2+2m+1$ points in B_m. To obtain a lower bound on the $L(k,k-1,\dots,2,1)$-labeling number, we identify the smallest interval containing all integers needed to label the vertices in the ball B_p. To this aim, we use a labeling function $f:V\to\mathbb{Z}$. It is clear that $\lambda_k\geq \max f(B_p)-\min f(B_p)+1$.

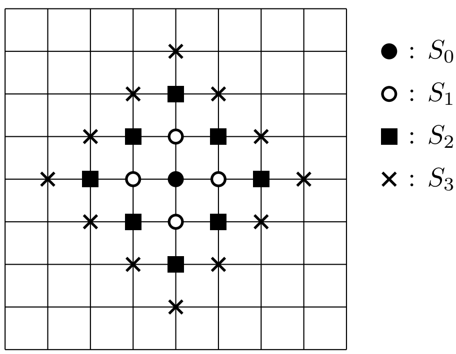

Fig. 1. S_m when $m=0,1,2,3$.

Let us put all the values of the function f on B_p in increasing order: $z_0<z_1<\ldots<z_n$. We have $\lambda_k\geq z_n-z_0+1$. Note that because of (*), the function f is injective on B_p, hence $n=2p^2+2p$ is one less than the number of points in B_p. Let $u_i=f^{-1}(z_i)$ and and let q,r be such that $u_0\in S_q, u_n\in S_r$.

The method of obtaining the lower bound is a formalization of that used by Nandi et al. [16]. According to (*), $z_{i+1} - z_i \geq 2p+1 - \max\{d(u_i, v) \colon v \in B_p \setminus \{u_i\}\}$. If $u_i \in S_m$, then $\max\{d(u_i, v) \colon v \in B_p \setminus \{u_i\}\} = m + p$, hence $z_{i+1} - z_i \geq p + 1 - m$. Considering z_i for $i = 0, 1, \ldots, n-1$, we can already estimate that

$$\begin{aligned} z_n - z_0 &= (z_1 - z_0) + \ldots + (z_n - z_{n-1}) \\ &\geq |S_p| + 2|S_{p-1}| + \ldots + p|S_1| + (p+1)|S_0| - (p+1-r). \end{aligned}$$

Let us call the number on the RHS of the inequality c_p. Now, if a point u_i is such that $i < n$ and $u_{i+1} \in B_{p-1}$, then $z_{i+1} - z_i \geq 2p+1 - \max\{d(u_i, v) \colon v \in B_{p-1} \setminus \{u_i\}\} = p+2-m$ (instead of $p+1-m$). There are at least $|B_{p-1}|$ points like this if $q = p$, and $|B_{p-1}| - 1$ if $q \neq p$, and the RHS of the inequality above can be increased by the amount. Continuing further in this manner, we get

$$\begin{aligned} z_n - z_0 &\geq c_p + (|B_{p-1}| - 1) + \ldots + (|B_q| - 1) + |B_{q-1}| + \ldots + |B_0| \\ &= c_p + |S_{p-1}| + 2|S_{p-2}| + \ldots + (p-1)|S_1| + p|S_0| - (p-q) \\ &= 4\Big(\sum_{m=1}^{p} m(p+1-m) + \sum_{m=1}^{p-1} m(p-m)\Big) + (r+q). \end{aligned}$$

Using

$$1 \cdot p + 2 \cdot (p-1) + \ldots + (p-1) \cdot 2 + p \cdot 1 = \frac{p(p+1)(p+2)}{6},$$

and the fact that $r+q$ is at least 1, which happens if $p, q \in \{0, 1\}$ (note that they must be different, since there is only one point in S_0), we easily get $\lambda_k \geq \frac{2}{3}p(p+1)(2p+1) + 2$.

Now, if $k = 2p+1$ is odd, each of the $2p^2 + 2p$ summands $z_1 - z_0, z_2 - z_1, \ldots, z_n - z_{n-1}$ is larger by one, hence $\lambda_k \geq \frac{2}{3}p(p+1)(2p+3) + 2$. A better estimate can be obtained by considering the set $T_0 = \{(0,0), (0,1)\}$ and, for $m > 0$, the sets $T_m = \{u \in \mathbb{Z} \times \mathbb{Z} \colon d(u, T_0) = m\}$ (see Fig. 2). This, however, does not change the asymptotic behavior of λ_k.

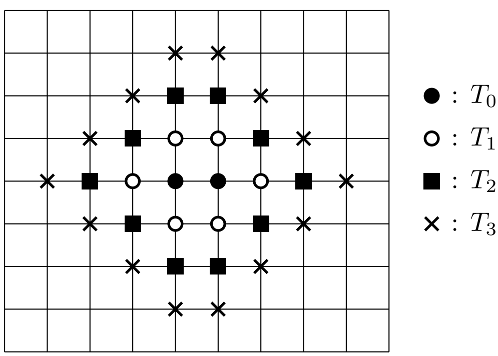

Fig. 2. T_m when $m = 0, 1, 2, 3$.

□

4. Proposed Formula

In this section a formula is given to find the label of any vertex of the square grid under $L(k, k-1, \ldots, 2, 1)$-labeling for general k. Let the label assigned to the vertex $v(x, y)$ be denoted by $L(x, y)$. Formula 1 gives the definition of $L(x, y)$.

Formula 1.

$$
L(x,y) = \begin{cases}
[(2p+3)x + (3p^2+7p+5)y] \quad \bmod \frac{1}{2}(p+1)(3p^2+5p+4), \\
\qquad\qquad\qquad\qquad \textit{if } k = 2p+1 \textit{ and } p(\geq 1) \textit{ is odd;} \\
[(2p+3)x + (3p^2+6p+3)y] \quad \bmod \frac{1}{2}(3p^3+8p^2+8p+4), \\
\qquad\qquad\qquad\qquad \textit{if } k = 2p+1 \textit{ and } p(\geq 0) \textit{ is even;} \\
[(2p+1)x + (3p^2+4p+2)y] \quad \bmod \frac{1}{2}(3p^3+5p^2+5p+1), \\
\qquad\qquad\qquad\qquad \textit{if } k = 2p \textit{ and } p(\geq 3) \textit{ is odd;} \\
[(2p+1)x + (3p^2+3p+1)y] \quad \bmod \frac{1}{2}p(3p^2+5p+4), \\
\qquad\qquad\qquad\qquad \textit{if } k = 2p \textit{ and } p(\geq 2) \textit{ is even.}
\end{cases}
$$

Note that many correct labelings may exist when the coefficients of x and y are restricted to be co-prime. If this restriction is removed then correct labelings also exist with reduced λ_k. Thus, we have considered all possible combinations of the coefficients for x and y at the time of designing Formula 1 for finding a labeling with the minimum λ_k. The assignment of labeling for $k = 7$ is shown in Fig. 3 for some vertices.

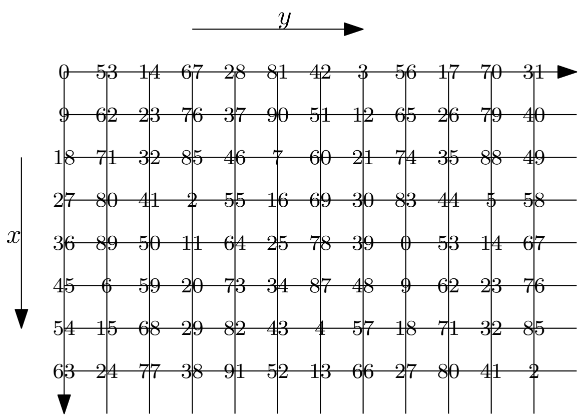

Fig. 3. Assignment of labeling for $k = 7$

4.1. Correctness Proof of the Proposed Formula

Formula 1 is said to be correct if and only if the inequality constraints of the problem mentioned in Section 2 are satisfied. The proof of Theorem 4.1 shows the correctness of Formula 1. Lemma 4.2 is needed to prove Theorem 4.1.

Theorem 4.1. *Formula 1 yields a λ-$L(k,k-1,\ldots,2,1)$-labeling of the square grid, with*

$$\lambda = \begin{cases} \frac{1}{2}(p+1)(3p^2+5p+4) & \text{if } k=2p+1 \text{ and } p(\geq 1) \text{ is odd;} \\ \frac{1}{2}(3p^3+8p^2+8p+4) & \text{if } k=2p+1 \text{ and } p(\geq 0) \text{ is even;} \\ \frac{1}{2}(3p^3+5p^2+5p+1) & \text{if } k=2p \text{ and } p(\geq 3) \text{ is odd;} \\ \frac{1}{2}p(3p^2+5p+4) & \text{if } k=2p \text{ and } p(\geq 2) \text{ is even.} \end{cases} \qquad (**)$$

More precisely, if $|x_1-x_2|+|y_1-y_2|=r$, then $|L(x_1,y_1)-L(x_2,y_2)|\geq k+1-r$, where $0<r\leq k+1$ and $L(x,y)$ is defined by Formula 1.

Lemma 4.2. *Let a, b, $c\in\mathbb{Z}^+$ and $L(x,y)=(ax+by) \mod c$. Now for any x_1, y_1, x_2, $y_2\in\mathbb{Z}$, if $L(x_1,y_1)>L(x_2,y_2)$ then $|L(x_1,y_1)-L(x_2,y_2)|=L(x_1-x_2,y_1-y_2)$.*

Proof. Clearly $0\leq L(x,y)<c$ for any $x,y\in\mathbb{Z}$. Hence, $0\leq|L(x_1,y_1)-L(x_2,y_2)|<c$. Again, for any A, $B\in\mathbb{Z}$, $(A \mod c-B \mod c) \mod c=(A-B) \mod c$. Put $A=ax_1+by_1$ and $B=ax_2+by_2$. Then $|L(x_1,y_1)-L(x_2,y_2)|=A \mod c-B \mod c=(A \mod c-B \mod c) \mod c=(A-B) \mod c=L(x_1-x_2,y_1-y_2)$. □

Proof of Theorem 4.1. We prove it for $L(x,y)=[(2p+3)x+\left(3p^2+7p+5\right)y] \mod \frac{1}{2}(p+1)(3p^2+5p+4)$ and $k=2p+1$, $p(\geq 3)$ is odd, and show the correctness for $p=1$ separately. The correctness of Formula 1 can be proved for other values of k in a similar way.

We can change the order of (x_1,y_1) and (x_2,y_2) in such a way that $L(x_1,y_1)\geq L(x_2,y_2)$, since exchanging indices 1 and 2 does not change r. By Lemma 4.2 we have to show that for x, $y\in\mathbb{Z}$ with $|x|+|y|=r$, $L(x,y)\geq k+1-r$. Note that the inequality is always satisfied for $r=k+1$. Hence, we can assume $0<r<k+1$.

Put $a=2p+3$, $b=3p^2+7p+5$ and $c=\frac{p+1}{2}(3p^2+5p+4)$. Note that $|ax+by|<5c$ for any x, y with $|x|+|y|=r$.

Case-I Assume that $ct\leq by\leq ax+by<c(t+1)$ for some $t\in[-5,4]\cap\mathbb{Z}$. Then

$$(ax+by) \mod c=ax+by-ct\geq ax>2p+2.$$

(Since $x>0$, $ax\geq a=2p+3$.) Hence, $L(x,y)>2p+2=k+1\geq k+1-r$.

Case-II Assume that $x=0$. Let $Y_t=\{y: ct\leq by<c(t+1)\}$ and $y_t=\min(Y_t), t\in[-5,4]\cap\mathbb{Z}$ for $|y_t|\leq k$. Note that $b>0$, so that whenever $L(x,y_t)\geq k+1$, also $\forall y\in Y_t$, $L(x,y)\geq k+1$. Since $y\neq 0$ (we already have $x=0$), we have $y_0=1$ and $by_0 \mod c=b>2p+2=k+1$. Hence, we need only consider $t\neq 0$. Put $d=\frac{2p^2+3p+1}{6p^2+14p+10}=(\frac{p+1}{2})(\frac{2p+1}{b})$. Note that for each odd $p\neq 1$, $\frac{1}{4}<d<\frac{1}{3}$. Now $y_t\geq\frac{ct}{b}=t(\frac{p+1}{2}-d)$, so that $y_t=t(\frac{p+1}{2})+e$, where

$$e=\begin{cases} 0 & \text{if } t=1,2 \text{ or } 3; \\ -1 & \text{if } t=4; \\ 1 & \text{if } t=-1,-2 \text{ or } -3; \\ 2 & \text{if } t=-4 \text{ or } t=-5. \end{cases}$$

We have $L(0,y_t) = by_t - ct = t(b\frac{p+1}{2} - c) + be = \frac{t}{2}(2p^2+3p+1) + be$. The inequality $L(0,y_t) \geq 2p+2$ is obviously true if t is positive and $e = 0$. If $t = 4$, we have $L(0,y_t) = 2(2p^2+3p+1) - b = p^2 - p - 3 \geq 2p+2$ for odd $p \geq 5$, and $L(0,y_t) \geq k+1-r$ for $p = 3$. For $t = -1,-2$ or -3, it is enough to check the "worst" case, namely $t = -3$, which yields $L(0,y_t) = (5p+7)/2 \geq 2p+2$. Again, we can omit $t = -4$ and check that for $t = -5$ we get $L(0,y_t) = \frac{(2p^2+13p+15)}{2} \geq 2p+2$.

Case-III Assume that $by < ct \leq ax+by$. Note that then $c(t-1) < by < ct \leq ax+by < c(t+1)$. We will show that there exist at most two y's satisfying the inequality. Let y_t =max$\{y : by < ct \ \wedge \ (\exists x : ct \leq ax+by)\}$. Thus $by_t < ct \leq ax+by_t$ for some x. Suppose $b(y_t-2) < ct \leq ax+b(y_t-2)$ for some x. Then $ax+b(y_t-1) = (ax-2b)+by_t \geq ct > by_t$. But $ax-2b \leq a(2p+2)-2b = 2[(p+1)(2p+3)-(3p^2+7p+5)] = 2(-p^2-2p-2) < 0$, which is a contradiction. If we find $x_t = \min\{x : by_t < ct \leq ax+by_t\}$ and $x_t' = \min\{x : b(y_t-1) < ct \leq ax+b(y_t-1)\}$ and if $|x_t|+|y_t| < 2p+2$ (similarly $|x_t'|+|y_t| < 2p+2$), then it is enough to check that $L(x_t,y_t) \geq k+1-r$ and $L(x_t',y_t-1) \geq k+1-r$.

Put $d = \frac{2p^2+3p+1}{6p^2+14p+10} = (\frac{p+1}{2})(\frac{2p+1}{b})$. Note that for each odd $p \neq 1$, $\frac{1}{4} < d < \frac{1}{3}$. Now $y_t < \frac{ct}{b} = t(\frac{p+1}{2} - d)$, so that $y_t = t(\frac{p+1}{2}) + e$, where

$$e = \begin{cases} -1 & \text{if } t = 1,2 \text{ or } 3; \\ -2 & \text{if } t = 4; \\ 0 & \text{if } t = -1,-2,-3 \text{ or } -4; \\ 1 & \text{if } t = -5. \end{cases}$$

Using $ct \leq ax_t + by_t \Rightarrow x_t \geq \frac{ct-by_t}{a}$, and $L(x_t,y_t) = ax_t+by_t-ct$, we construct Table 1. Whenever $|y_t|$, $|x_t|$ or r is at least $2p+2$, there is no need for further calculation, and the respective positions are filled with dashes.

Using $ct \leq ax_t' + b(y_t-1) \Rightarrow x_t' \geq \frac{ct-b(y_t-1)}{a}$, and $L(x_t',y_t-1) = ax_t'+b(y_t-1)-ct$, we construct Table 2 with the corresponding values. As above, we use dashes whenever $|y_t-1|$, $|x_t'|$ or r is at least $2p+2$, and there is no need for further calculation.

Case-IV Assume that $ax+by < ct \leq by$, where $t \in [-4,4] \cap \mathbb{Z}$. Then $c(t-1) < ax+by < ct \leq by < c(t-1)$ and $ax+by \geq ax+ct = c(t-1)+(ax+c)$. Hence, $L(x,y) = (ax+by)$ mod $c = ax+c$.

Since $ax \geq a(-2p-2) = -2(2p+3)(p+1)$, we have
$L(x,y) = \frac{(p+1)\left(3p^2+5p+4\right)}{2} - 2(2p+3)(p+1) = \frac{3}{2}p^3 - \frac{11}{2}p - 4 \geq 2p+2$, for $p \geq 3$.
Therefore, for $p \geq 3$, $L(x,y) \geq k+1-r$.

Case-V Assume that $x < 0$, $ax+by \geq ct$ and $by < c(t+1)$.

Let $Y_t = \{y : \exists\ x \text{ s.t. } ct \leq ax+by < by < c(t+1)\}$. Then it is enough to check the inequality for $y_t = \min(Y_t)$ and for y_t+1, and for them we should check if for $x_t = \min\{x : ct \leq ax+by_t < by_t < c(t+1)\}$ and $x_t' = \min\{x : ct \leq ax+b(y_t+1) < b(y_t+1) < c(t+1)\}$.

Thus we need to check $L(x_t,y_t) \geq k+1-r$ and $L(x_t',y_t+1) \geq k+1-r$.

Using $by_t < c(t+1)$, we construct Table 3.

Table 1.

t	y_t	x_t	$r=\lvert x_t\rvert+\lvert y_t\rvert$	$k+1-r$	$L(x_t,y_t)$
1	$\frac{p-1}{2}$	$(p+2)$	$\frac{3}{2}(p+1)$	$\frac{1}{2}(p-3)$	$\frac{3}{2}(p+1)$
2	p	$\frac{(p+3)}{2}$	$\frac{3}{2}(p+1)$	$\frac{1}{2}(p-3)$	$\frac{1}{2}(p+1)$
3	$\frac{(3p+1)}{2}$	2	$\frac{1}{2}(3p+5)$	$\frac{1}{2}(p-1)$	$\frac{1}{2}(3p+5)$
4	$2p$	$(p+1)$	$3p+1$	–	–
-1	$-\frac{(p+1)}{2}$	$\frac{(p+1)}{2}$	$p+1$	$p+1$	$p+1$
-2	$-(p+1)$	$(p+1)$	$2(p+1)$	–	–
-3	$-\frac{3(p+1)}{2}$	$(3p+1)$	$\frac{1}{2}(9p+5)$	–	–
-4	$-2(p+1)$	–	–	–	–
-5	$-\frac{(5p+3)}{2}$	–	–	–	–

Table 2.

t	y_t-1	x'_t	$r=\lvert x'_t\rvert+\lvert y_t-1\rvert$	$k+1-r$	$L(x'_t,y_t-1)$
1	$\frac{p-3}{2}$	$\frac{(5p+7)}{2}$	$3p+2$	–	–
2	$(p-1)$	$2p+3$	–	–	–
3	$\frac{(3p-1)}{2}$	$\begin{cases}\frac{3(p+1)}{2}, & \text{if } p=3,5\\ \frac{3p+1}{2}, & \text{if } p(\geq 7)\end{cases}$	–	–	–
4	$2p-1$	$\frac{(5p+9)}{2}$	–	–	–
-1	$-\frac{(p+3)}{2}$	$2p+2$	–	–	–
-2	$-(p+2)$	$\frac{(5p+3)}{2}$	–	–	–
-3	$-\frac{(3p+5)}{2}$	$(3p+2)$	–	–	–
-4	$-(2p+3)$	–	–	–	–
-5	$-\frac{5(p+1)}{2}$	–	–	–	–

Table 3.

t	1	2	3	4	-1	-2	-3	-4	-5
y_t	p	$\frac{(3p+1)}{2}$	$2p+1$	$\frac{(5p-1)}{2}$	-1	$-\frac{(p+1)}{2}$	$-(p+1)$	$-\frac{3(p+1)}{2}$	$-(2p+1)$

If we calculate the values of x_t and x'_t from $ct \leq ax_t + by_t$ and $ct \leq ax'_t + b(y_t+1)$ respectively, then x_t and x'_t are always greater than $2p+2$. This completes the proof for $p \geq 3$.

Case $p=1$. Then $k=3$ and $L(x,y)=(5x+15y) \bmod 12$. We just need to consider different values of x and y such that $x\in\{-3,-2,-1\}$ and $y\in\{-3,-2,-1,0,1,2,3\}$. Clearly when $(x,y)\in\{(-3,-3),(-3,-2),(-3,-1),(-3,1),(-3,2),(-3,3),(-2,-3),(-2,-2),(-2,3),(-2,2),(-1,3),(-1,-3)\}$, we don't need to check anything because $r=|x|+|y|\geq 4$. When $(x,y)=(-3,0), L(x,y)=9$ and $k+1-r=1$. Similarly, when $(x,y)\in\{(-2,-1),(-2,0),(-2,1),(-1,-2),(-1,-1),(-1,0),(-1,1),(-1,2)\}$, $L(x,y)\geq k+1-r$.

Hence, we always have $L(x,y)\geq k+1-r$. □

4.2. No-hole Labeling Proof

Theorem 4.3. *Formula 1 gives no-hole labeling.*

Proof. Formula 1 is of the form $(ax+by) \mod c$, with a,b and c depending on parity of k and p. We shall show that it is enough to check that $\gcd(a,b,c)$ is 1. In fact, let $m=\gcd(a,b)$ and denote by (m) the principal ideal in $\mathbb{Z}$ generated by m. It is well known (and easy to see) that the set $\{ax+by\colon x,y\in\mathbb{Z}\}$ equals (m). Now, if $\gcd(m,c)=\gcd(a,b,c)=1$, then $mu+cv=1$ for some $u,v\in\mathbb{Z}$. If $k\in\{0,1,\dots,c-1\}$, then $kmu+kcv=k$, so that $kmu\equiv k \mod c$. But $kmu\in(m)$, which means that for some $x,y\in\mathbb{Z}, (ax+by) \mod c=k$, and all integer values from 0 up to $c-1$ are attained.

We note the values of $\gcd(a,b)$ for different values of k.

$$\gcd(a,b)=\begin{cases}1 \text{ or } 5 & \text{if } k=2p+1 \text{ and } p(\geq 1) \text{ is odd;}\\ 1 \text{ or } 3 & \text{if } k=2p+1 \text{ and } p(\geq 0) \text{ is even;}\\ 1 \text{ or } 3 & \text{if } k=2p \text{ and } p(\geq 3) \text{ is odd;}\\ 1 & \text{if } k=2p+1 \text{ and } p(\geq 2) \text{ is even.}\end{cases}$$

Consider the case when $k=2p+1$ and $p(\geq 1)$ is odd. In this case $a=2p+3, b=3p^2+7p+5$ and $c=\frac{1}{2}(p+1)(3p^2+5p+4)$. If $\gcd(a,b)=1$, $\gcd(a,b,c)=1$, and there is nothing to prove. If $\gcd(a,b)=5$, then p is congruent to 1 modulo 5, and c is congruent to 2 modulo 5. So, c is not divisible by 5, and hence $\gcd(a,b,c)=1$. The proof will be similar for other values of k. □

4.3. Upper Bound on λ_k and approximation ratio

Theorem 4.4. *We have $\lambda_k\leq\lambda$, with λ given by (**). Consequently, the approximation ratio for the problem is not greater than $\frac{9}{8}$.*

Proof. The first statement follows directly from Theorem 4.1: $\lambda_k\leq\lambda$ for any λ-labeling. The approximation ratio is the ratio between the upper bound (UB), given by λ from (**), and the lower bound (LB), given in Theorem 3.1. Note that for all the cases mentioned in Formula 1, $\lim_{p\to\infty}\dfrac{UB}{LB}=\dfrac{9}{8}$. □

5. Conclusion

In this paper λ-$L(k,k-1,\dots,2,1)$-labeling for square grid is proposed and the lower bound on λ_k, the $L(k,k-1,\dots,2,1)$-labeling number, is computed. A formula for a no-hole λ-$L(k,k-1,\dots,2,1)$-labeling of square grid is given, implying at most $\frac{9}{8}$ approximation ratio. The correctness proof of the proposed formula is given and it is also proved that the proposed formula gives a no-hole labeling.

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BEZLUKOWE λ-$L(K,K-1,\ldots,2,1)$-ETYKIETOWANIE KWADRATOWEJ KRATY

S t r e s z c z e n i e

Motywowani Problemem Przypisania Częstotliwości (*Frequency Allocation Problem*), pokazujemy, dla ustalonej liczby naturalnej k, w jaki sposób przypisać wartości wierzchołkom nieskończonej kwadratowej kraty, używając możliwie małej liczby całkowitych etykiet, zmieniających się od 0 do $\lambda-1$, w taki sposób, by etykiety wierzchołków przylegających różniły się przynajmniej o k, wierzchołki połączone drogą o długości dwa otrzymały wartości różniące się przynajmniej o $k-1$, itd.. Wierzchołki, których odległość wynosi przynajmniej $k+1$ mogą być oznaczone tą samą etykietą. Znalezione ograniczenie dolne dla λ pozwala pokazać, że przedstawione rozwiązanie jest bliskie optymalnemu, ze stosunkiem aproksymacji $\frac{9}{8}$. Etykietowanie to nie posiada luk, to znaczy używa każdej z dopuszczalnych etykiet przynajmniej raz.

Słowa kluczowe: etykietowanie grafu, stała etykietowania, etykietowanie bezlukowe, Problem Przypisania Częstotliwości, stosunek aproksymacji