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BIFURCATION VALUES OF C^∞ FUNCTIONS

Summary

We show how one can use a trivialization of a function $f : \mathbb{R}^n \to \mathbb{R}$ on fibers of some function $g : \mathbb{R}^n \to \mathbb{R}$ to construct a trivialization of f in $\mathbb{R}^n$. Additionally we adapt a method for trivialising functions which satisfies the ρ_0-regularity condition to the case of functions defined on hypersurfaces of the form $M = g^{-1}(0)$.

Keywords and phrases: bifurcation value, trivialization, Magrange's condition, ρ_0-regularity

Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. It is well known that there exists a finite set such that f is a C^∞ fibration over the complement of this set. The smallest such a set is called the set of *bifurcation values* of f and denoted by $B(f)$. It contains the set of critical values $K_0(f)$ and the set of *bifurcation values at infinity* $B_\infty(f)$ of f. Finding an effective description of the set $B_\infty(f)$ is still an open question. However we can approximate $B_\infty(f)$ using different supersets. The most popular one uses the so-called Malgrange condition.

We say that f satisfies the *Malgrange condition* in $\lambda \in \mathbb{R}$ if there exists a neighbourhood U of λ and constans $\delta, R > 0$ such that

$$\forall_{x \in f^{-1}(U)} \quad \|\nabla f(x)\| \|x\| \geqslant \delta \quad \text{for} \quad \|x\| \geqslant R. \tag{M}$$

The set $K_\infty(f)$ of all values which do not satisfy Malgrane's condition is called the set of *asymptotic critical values* of f i.e.

$$K_\infty(f) := \{\lambda \in \mathbb{R} | \exists_{(x_k) \subset \mathbb{R}^n} \|x_k\| \to \infty, f(x_k) \to \lambda, \|x_k\| \|\nabla f(x_k)\| \to 0\}.$$

Jelonek and Kurdyka in [4] gave an effective characterization of this set. Moreover in [5] they showed how to proceed when the polynomial f is defined on an algebraic set (see also [3]).

In this paper we show how one can use a trivialization of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ on fibers of some other smooth function $g : \mathbb{R}^n \to \mathbb{R}$ to construct a desired trivialization of f in $\mathbb{R}^n$. More precisely, we introduce a notion of (g, S)-Malgrange condition; for a smooth function g and an open set $S \subset \mathbb{R}$ we say that f satisfies the (g, S)-*Malgrange condition* in λ if

1. $\nabla g(x) \neq 0$ near $f^{-1}(\lambda)$ outside some compact set
2. $g^{-1}(S)$ contains all fibers of f near λ
3. for any $s \in S$ the function $f|_{g^{-1}(s)} : g^{-1}(s) \to \mathbb{R}$ satisfies the Malgrange condition in λ

(compare to the definition in Section 3). The set of all λ that don't satisfy the (g, S)-Malgrange condition we will denote by $K_\infty^{(g,S)}(f)$. Our aim is to prove $B(f) \subset K_\infty^{(g,S)}(f) \cup K_0(f)$ and therefore

$$B(f) \subset \bigcap_{(g,S)} K_\infty^{(g,S)} \cup K_0(f)$$

(see Theorem 3.1 and Remark 3.5). It is worth noting that in (3) of the definition of (g, S)-Malgrange condition we allow different constans δ_s (in inequality(M)) for different $s \in S$. Therefore our method may be applied even if $\inf\{\delta_s |\ s \in S\} = 0$ (i.e. when the standart Malgrange condition fails). This often leads us to more sharp aproximation of $B(f)$ than in the classical method (see Example 3.6 and Example 3.7).

The other popular approach to find the bifurcation values uses a critical set $\mathcal{M}_a(f)$ of the map (f, ρ_a) where ρ_a is the Euclidean distance function from a fixed point $a \in \mathbb{R}^n$. We can define the set of asymptotic ρ_a-nonregular values as

$$S_a(f) := \{\lambda \in \mathbb{R} |\ \exists_{(x_k) \subset \mathcal{M}_a(f)} \|x_k\| \to \infty, f(x_k) \to \lambda\}.$$

It has been proven in [10], [1], [2] that $B_\infty(f) \subset S_a(f)$ for any $a \in \mathbb{R}^n$, thus in particular

$$B_\infty(f) \subset S_\infty(f) := \bigcap_{a \in \mathbb{R}^n} S_a(f).$$

In the second part of this paper we will show how to get a simillar result when f is defined on the manifold of the form $g^{-1}(0)$. We will introduce an analog of the condition used in [6] and [7] that allows us to trivialize function f (see Theorem 4.1 and Remark 4.5).

1. Auxiliary results

In this section we collect some useful facts about differential equations, which we will use later in this article.

Let M be a smooth m-dimentional manifold. We denote by T_xM the tangent space to the manifold M at a point $x \in M$ and $TM := \bigcup_{x\in M} T_xM$. Let $W : M \to TM$ be a smooth vector field on M.

Lemma 1.1. *Let $\phi : (t_0, \xi] \to M$ be a solution of the system of differential equations $x' = W(x)$. Assume that there exists a sequence $t_k \in (t_0, \xi)$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} t_k = t_0$ and $\lim_{k\to\infty} \phi(t_k) = x_0 \in M$. Then there exists $lim_{t\to t_0}\phi(t) = x_0$ and ϕ is not the maximal solution to the left.*

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_m) : U \to A \subset \mathbb{R}^m$ be a map in a neighbourhood $U \subset M$ of the point x_0. Choosing a subsequence if necessary, we may assume that $x_k := \phi(t_k) \in U$ for $k \in \mathbb{N}$. Than we can write

$$W(x) = \sum_{i=1}^{m} W_i(x)\frac{\partial}{\partial\varphi_i} \qquad \text{for } x \in U,$$

where $W_i : U \to \mathbb{R}$ for $i = 1, .., m$.

Let $y_0 := \varphi(x_0)$ and $y_k := \varphi(x_k)$ for $k \in \mathbb{N}$.

Let $\alpha \in \mathbb{R}\cup\{-\infty\}$ be a minimal number for which $\phi((\alpha, t_1]) \subset U$. Put $I = (\alpha, t_1]$. Denote $\phi_\varphi := \varphi \circ \phi : I \to A$ and $\phi_{\varphi_i} := \varphi_i \circ \phi$ for $i = 1, .., m$. Then we have $\phi'(t) = \sum_{i=1}^m \phi'_{\varphi_i}(t)\frac{\partial}{\partial\varphi_i}$ for $t \in I$ and

$$\sum_{i=1}^{m} \phi'_{\varphi_i}(t)\frac{\partial}{\partial\varphi_i} = \sum_{i=1}^{m} W_i(\phi(t))\frac{\partial}{\partial\varphi_i}.$$

Therefore

$$\phi'_{\varphi_i}(t) = W^i_{\varphi^{-1}}(\phi_{\varphi_i}(t)) \qquad \text{for } t \in I \;\; i = 1, .., m, \tag{1}$$

where $W^i_{\varphi^{-1}}(y) := W_i \circ \varphi^{-1}(y)$ for $y \in A$.

To complete the proof we need the following well known fact.

Property 1.2. *There exists an interval J such that $t_0 \in J$ and a neighbourhood $\Gamma \subset \mathbb{R} \times A$ of (t_0, y_0) such that for any $(t', y') \in \Gamma$ every maximal solution γ of sytem (1) that passes through $(t', y') \in \Gamma$ is defined at least on J and the graph of $\gamma|_J$ is contained in some rectangle $T \subset \mathbb{R} \times A$.*

By choosing a subsequence of the sequence (t_k, y_k), we may assume that $(t_k, y_k) \in \Gamma$ for $k \in \mathbb{N}$ and $\xi > t_k > t_l > t_0$ for $k < l$.

From Property 1.2 we have that $\alpha \le t_0$. Indeed, otherwise there exists $t' \in (t_0, t_1)$ such that $\phi(t') \notin U$, hence there exists a maximal solution to the left $\widehat{\phi_\varphi} : \widehat{I} \to A$ of the system (1) such that $\widehat{\phi_\varphi}$ goes through Γ and $t_0 \notin \widehat{I}$ which contradicts Property 1.2.

Therefore $(t_0, t_1] \subset I$ and $\phi_\varphi = \phi^*_\varphi|_I$ where $\phi^*_\varphi : I^* \to A$ is a maximal solution to the left of the system (1) and $t_0 \in I^*$.

Consequently

$$\lim_{t\to t_0}\phi(t)=\lim_{t\to t_0}\varphi^{-1}(\phi^*_\varphi(t))=\lim_{t\to t_0}\varphi^{-1}(y_0)=x_0,$$

which completes the proof of Lemma 1.1. □

Lemma 1.3. *Let $\phi:(\alpha,\beta)\to M$ be a maximal solution of the system $x'=W(x)$. For every compact set $K\subset\mathbb{R}\times M$ there exist $\alpha^*,\beta^*\in\mathbb{R}$ such that the graphs of $\phi|_{(\alpha,\alpha^*)}$ and $\phi|_{(\beta^*,\beta)}$ are disjoint with K.*

Proof. Let $A:=\{t\in(\alpha,\beta)|\ (t,\phi(t))\in K\}$. If $A=\emptyset$ then as α^*,β^* we may choose arbitrary numbers from (α,β). Now assume that $A\neq\emptyset$ and let $\alpha^*=\inf A$, $\beta^*=\sup A$. Observe that $\alpha<\alpha^*$ and $\beta^*<\beta$. Indeed, if $\alpha=\alpha^*$ than there exists a sequence $(t_k)_{k=1}^\infty$ such that $(t_k,\phi(t_k))$ converges to a point in $K\subset\mathbb{R}\times M$. This contradicts Lemma 1.1. Analogously we prove that $\beta^*<\beta$. □

2. The Malgrange condition on Manifolds

Let $f,g\in C^\infty(\mathbb{R}^n)$. We will assume that $\nabla g(x)\neq 0$ for $x\in g^{-1}(0)$. Denote $M:=V(g)=g^{-1}(0)$ and $f_M:=f|_M$. Consider the following vector field $\nabla f_M:M\to\mathbb{R}^n$

$$\nabla f_M(x):=\nabla f(x)-\frac{\langle\nabla f(x),\nabla g(x)\rangle}{||\nabla g(x)||^2}\nabla g(x),\quad x\in M. \tag{2}$$

Geometrically $\nabla f_M(x)$ is the projection of $\nabla f(x)$ onto the tangent space T_xM.

A value $\lambda\in\mathbb{R}$ is called a *regular value* of f_M if $\nabla f_M(x)\neq 0$ for $x\in f_M^{-1}(\lambda)$. The set of all values λ that are not regular we will denote by $K_0(f_M)$.

We say that f_M satisfies the *Malgrange condition* over $U\subset\mathbb{R}$ if there exist constans $\delta,R>0$ such that

$$\forall_{x\in f_M^{-1}(U)}\quad \|\nabla f_M(x)\|\|x\|\geqslant\delta\quad\text{for}\quad\|x\|\geqslant R.$$

We say that f_M satisfies the *Malgrange condition* in $\lambda\in\mathbb{R}$ if there exists a neighbourhood U of λ such that f_M satisfies the Malgrange condition over U. We will denote by $K_\infty(f_M)$ the set of all λ that do not satisfy the Malgrange condition.

It is well known that the Malgrange condition allows us to integrate $\nabla f_M(x)/\|\nabla f_M(x)\|^2$ field and get the trivialization of f_M (see [9] and [3]). More precisely, we have

Theorem 2.1. *Let $\lambda\in\mathbb{R}$ be a regular value of f_M. If f_M satisfies the Malgrange condition in $\lambda\in\mathbb{R}$ than there exists a neighbourhood U of λ such that $f_{M|f_M^{-1}(U)}$ is a C^∞ trivial fibration on M.*

Immediately from Theorem 2.1 we get

Remark 2.2. $B(f_M)\subset K_\infty(f_M)\cup K_0(f_M)$.

Note that if f_M satisfies the Malgrange condition, it does not follow that $f|_{M^\varepsilon}$ satisfies it also, where $M^\varepsilon = g^{-1}((-\varepsilon, \varepsilon))$. In other words, one can not trivialise f_M using field ∇f_M in a neighbourhood M^ε of M as it is shown by the following example.

Example 2.3. Let $f, g \in \mathbb{R}[x, y]$ be polynomials defined as $g(x, y) := y$, $f(x, y) := x - x^3y^2$. Put $M = g^{-1}(0)$. Obviously $\nabla f_M = [1, 0]$ and f satisfies Malgrange condition on M. Consequently f_M is a trivial fibration on M. On the other hand, for any $\varepsilon > 0$, denoting $M_t := g^{-1}(t)$ for $t \in (-\varepsilon, \varepsilon)$ we have $\nabla f_{M_{g(x,y)}}(x, y) = (1 - 3x^2y^2, 0)$ for $(x, y) \in M^\varepsilon$. Moreover, for

$$(x_n, y_n) = \left(\frac{n\varepsilon}{2}, \frac{2}{\sqrt{3}n\varepsilon}\right), \quad n \in \mathbb{N}$$

we have $\|\nabla f_{M_{g(x,y)}}(x_n, y_n)\| = 0$, for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|(x_n, y_n)\| = \infty$, and $(x_n, y_n) \in M^\varepsilon$ as $n \to \infty$. So we can not use the trivialization method without restricting the field ∇f_M to the manifold $g^{-1}(0)$.

3. The Malgrange condition on fibers

In this section we will present a method of using fibers of some function g to construct a trivialisation of a function f.

Let $f, g \in C^\infty(\mathbb{R}^n)$. Let $D \subset \mathbb{R}^n$ be an open set and $\nabla g(x) \neq 0$ for $x \in D$. As in (2) we can define $\nabla f_{g^{-1}(g(x))}(x)$ for $x \in D$. Put

$$\nabla_g f(x) := \nabla f_{g^{-1}(g(x))}(x) \quad \text{for } x \in D.$$

Geometrically $\nabla_g f(x)$ is the projection of $\nabla f(x)$ onto the tangent space at x of the fibre $g^{-1}(g(x))$.

Denote $\Xi = \{(g, S) |\, g \in C^\infty(\mathbb{R}^n), S \text{ open in } \mathbb{R}\}$ and let $(g, S) \in \Xi$. We say that f satisfies the (g, S)-*Malgrange condition* over U if there exists a constant $R > 0$ such that

1. $\nabla g(x) \neq 0$ on $D_{(U,R)} := f^{-1}(U) \setminus \{x \in \mathbb{R}^n |\, R \geqslant \|x\|\}$
2. $D_{(U,R)} \subset g^{-1}(S)$
3. $\forall_{s\in S}\ \exists_{\delta_s>0}\ \|\nabla_g f(x)\|\|x\| \geqslant \delta_s \quad \text{for} \quad x \in D_{(U,R)} \cap g^{-1}(s)$.

We say that f satisfies the (g, S)-*Malgrange condition* in $\lambda \in \mathbb{R}$ if there exists neighbourhood U of λ such that f satisfies the (g, S)-Malgrange condition over U. We will denote by $K_\infty^{(g,S)}(f)$ the set of all λ that don't satisfy the (g, S)-Malgrange condition.

The main result of this section is the following

Theorem 3.1. *Let λ be a regular value of f. If f satisfies the (g, S)-Malgrange condition in $\lambda \in \mathbb{R}$ than there exists a neighbourhood U of λ such that $f|_{f^{-1}(U)}$ is a trivial fibration.*

The proof of the above theorem will be preceded by two properties and a lemma.

From now we will assume that $\lambda \in \mathbb{R}$ is a regular value of f and that f satisfies the (g,S)-Malgrange condition in λ.

The following property holds

Property 3.2. *There exists a neighbourhood U of λ such that $\nabla f(x) \neq 0$ for $x \in f^{-1}(U)$.*

Let U, R be as in the (g,S)-Malgrange condition. Shrinking the set U if necessary, we can assume that $\nabla f(x) \neq 0$ for $x \in f^{-1}(U)$. Let α, β be C^∞ functions in $\mathbb{R}^n$ such that

$$\alpha(x) = \begin{cases} 0 & \text{for } \|x\| \geqslant R+1 \\ 1 & \text{for } \|x\| \leqslant R \end{cases},$$

$$\beta(x) = \begin{cases} 1 & \text{for } \|x\| \geqslant R+1 \\ 0 & \text{for } \|x\| \leqslant R \end{cases}$$

and $0 < \alpha(x), \beta(x) < 1$ for $\|x\| \in (R, R+1)$.

We define a smooth vector field $w : f^{-1}(U) \to \mathbb{R}^n$ as

$$w(x) := \alpha(x)\nabla f(x) + \beta(x)\nabla_g f(x).$$

Here we are using convention that $\beta(x)\nabla_g f(x) = 0$ for $\|x\| \leqslant R$ (note that $\nabla_g f(x)$ might not be defined for some points x such that $\|x\| \leq R$).

From the definition of w and Property 3.2 we get

Property 3.3. *Under the above assumptions we have*

(i) $w(x) = \nabla f(x)$ *for* $\|x\| \leqslant R$ *and* $w(x) = \nabla_g f(x)$ *for* $\|x\| \geqslant R+1$

(ii) $\langle w(x), \nabla f(x)\rangle \neq 0$ *for* $x \in f^{-1}(U)$.

Let us define $u : f^{-1}(U) \to \mathbb{R}^n$ as

$$u(x) := \frac{w(x)}{\langle w(x), \nabla f(x)\rangle}.$$

From the assumptions and Property 3.3 we see that u is well defined, it is smooth and $u(x) \neq 0$ for $x \in f^{-1}(U)$.

Lemma 3.4. *Under the above assumptions we have*

(i) $\langle \nabla f(x), u(x)\rangle = 1$ *for* $x \in f^{-1}(U)$

(ii) For any $s \in S$ *there exists a constant* $\alpha_s > 0$ *such that*

$$|\langle x, u(x)\rangle| \leqslant \varepsilon_s \|x\|^2 \qquad \text{for } x \in f^{-1}(U) \cap g^{-1}(s), \|x\| > R+1.$$

Proof. (i) is obvious. We will prove (ii). From the (g,S)-Malgrange condition we have $\dfrac{1}{\|\nabla_g f\|} \leqslant \dfrac{1}{\delta_s}\|x\|$ for $x \in f^{-1}(U) \cap g^{-1}(s)$, $\|x\| > R$. Since $\langle \nabla_g f(x), \nabla f(x)\rangle =$

$\|\nabla_g f(x)\|^2$, using Schwartz inequality we get

$$|\langle x, u(x)\rangle| \leqslant \|x\|\|u(x)\| = \|x\|\frac{1}{\|\nabla_g f(x)\|} \leqslant \frac{1}{\delta_s}\|x\|^2 \text{ for } \|x\| > R+1,$$

which completes the proof. □

We are ready to prove Theorem 3.1.

Proof. Let U, R, u be defined as above. For each $\mu \in U$, consider a system of differential equations

$$x' = (\lambda - \mu)u(x) \tag{3}$$

with the right side defined in $G := \{(t,x) \in \mathbb{R} \times D |\ x \in f^{-1}(U)\}$. Denote by $\Phi_\mu : V_\mu \to D$ the general solution of system (3) and put $V_\mu := \{(\tau, \eta, t) \in \mathbb{R} \times D \times R |\ (\tau, \eta) \in G, t \in I_\mu(\tau, \eta)\}$, where $I_\mu(\tau, \eta)$ is a domain of integral solution of $t \to \Phi_\mu(\tau, \eta, t)$. From the definition of the general solution we get

$$\Phi_\mu(\tau, \eta, \tau) = \eta. \tag{4}$$

Consider the mapping

$$\Psi_1 : f^{-1}(U) \ni x \mapsto \Phi_{f(x)}(0, x, 1) \in f^{-1}(\lambda).$$

We show that the mapping Ψ_1 is well defined i.e. $1 \in I_{f(x)}(0,x)$ for each $x \in f^{-1}(U)$. Suppose the contrary that there exists $x \in f^{-1}(U)$ such that $1 \notin I_{f(x)}(0,x)$. Then the right end-point β of the interval $I_{f(x)}(0,x)$ satisfies $0 < \beta \leqslant 1$. Let φ_x be an integral solution of the system (3) with $\mu = f(x)$ satisfying the initial condition $\varphi_x(0) = x$, that is

$$\varphi(t) = \Phi_{f(x)}(0, x, t) \quad \text{ for } t \in I_{f(x)}(0, x). \tag{5}$$

We have

$$(f \circ \varphi_x)'(t) = \langle \nabla f(\varphi_x(t)), \varphi_x'(t)\rangle = \lambda - f(x) \quad \text{for} \quad t \in I_{f(x)}(0,x).$$

Therefore

$$f \circ \varphi_x(t) = (\lambda - f(x))t + f(x), \quad t \in I_{f(x)}(0,x) \tag{6}$$

and $f \circ \varphi_x(t) \in J$ for $t \in [0, \beta)$, where J is a closed interval with endpoints λ and $f(x)$.

Denote

$$K := \{(t, x') \in \mathbb{R} \times f^{-1}(U) |\ t \in [0,1], f(x') \in J, \|x'\| \leqslant R+1\}.$$

Obviously K is a compact set. Lemma 1.3 implies that there exists $\tau \in (0, \beta)$ such that $(t, \varphi_x(t)) \notin K$ for $t \in [\tau, \beta)$. Since $J \subset U$, we have $\|\varphi_x(t)\| > R+1$ for $t \in [\tau, \beta)$.

Consider a function $\varrho : [\tau, \beta) \to \mathbb{R}$ defined by

$$\varrho(t) := \frac{1}{2}\ln\|\varphi_x(t)\|^2.$$

Let $s_0 \in S$ be such that $\varphi_x(\tau) \in D_{s_0}$. Using Lemma 3.4 (ii) we get

$$|\varrho'(t)| = \frac{|\langle \varphi_x(t), \varphi_x'(t)\rangle|}{\|\varphi_x(t)\|^2} = \frac{|\lambda - f(x)|}{\|\varphi_x(t)\|^2}|\langle \varphi_x(t), u(\varphi_x(t))\rangle| \leqslant \varepsilon_{s_0}|\lambda - f(x)|$$

for $t \in (\tau, \beta)$. From the mean value theorem there exists $\theta_t \in (\tau, t)$ such that $\varrho(t) - \varrho(\tau) = \varrho'(\theta_t)(t-\tau)$ and therefore, by the above,

$$\varrho(t) \leqslant \varrho(\tau) + \varepsilon_{s_0}|\lambda - f(x)|(t-\tau) \leqslant \varrho(\tau) + \varepsilon_{s_0}|\lambda - f(x)|(\beta - \tau).$$

Denoting $L := \varrho(\tau) + \varepsilon_{s_0}|\lambda - f(x)|(\beta - \tau)$ we see that the solution $\varphi_{x|(\tau,\beta)}$ is contained in the compact set

$$\{(t, x') \in \mathbb{R} \times f^{-1}(U) |\, f(x') \in J, \|x'\| \leqslant \mathrm{e}^L\} \subset \mathbb{R} \times f^{-1}(U),$$

which contradicts Lemma 1.3.

Summing up we have shown that $1 \in I_{f(x)}(0, x)$ for every $x \in f^{-1}(U)$. Then from (6) we get $f(\Psi_1(x)) = f(\varphi_x(1)) = \lambda$ and the mapping Ψ_1 is defined correctly. Similarly as above we show that the mapping

$$\Theta : f^{-1}(\lambda)) \times U \ni (\xi, \mu) \mapsto \Phi_\mu(1, \xi, 0) \in f^{-1}(U)$$

is also well defined. It is easy to check that the mapping

$$\Psi : f^{-1}(U) \ni x \mapsto (\Psi_1(x), f(x)) \in f^{-1}(\lambda) \times U$$

is a C^∞ diffeomorphism and $\Psi^{-1} = \Theta$. Therefore $f|_{f^{-1}(U)}$ is a trivial fibration. □

Immediately from Theorem 3.1 we get

Remark 3.5. For every $(g, S) \in \Xi$ we have $B(f) \subset K_\infty^{(g,S)}(f) \cup K_0(f)$. Therefore $B(f) \subset \bigcap_{(g,S)\in\Xi} K_\infty^{(g,S)} \cup K_0(f)$.

In Theorem 3.1 we showed how to trivialize a function f using fibers of some function g. Now we give some examples where the assumptions of Theorem 1 are not met but using Theorem 3.1 we can deduce the triviality of f.

Example 3.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) := \frac{y}{1+x^2}.$$

It is easy to check that f does not satisfy the Malgrange condition in 0. Denote $g : \mathbb{R}^2 \to \mathbb{R}$

$$g(x, y) := x, \quad S = \mathbb{R}.$$

Put $U = \mathbb{R}$ and $R = 1$. Then

1. $\nabla g(x) = [1, 0] \neq 0$ for $D_{(U,R)} = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 |\, R \geqslant \|x\|\}$
2. $D_{(U,R)} \subset g^{-1}(\mathbb{R})$
3. for $s \in \mathbb{R}$ we have

$$\|\nabla_g f(x, y)\| = \|[0, \frac{1}{1+x^2}]\| = \frac{1}{1+s^2} \qquad \text{for } (x, y) \in D_{(U,R)} \cap g^{-1}(s).$$

Therefore f satisfies the (g, S)-Malgrange condition and using Theorem 3.1 we deduce that f is a trivial fibration.

An example of polynomials, which is a trivial fibration but does not satisfy the Malgrange condition, comes from L. Păunescu and A. Zaharia (see [8]).

Example 3.7. Let $p, g \in \mathbb{N}, f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x, y, z) := x - 3x^{2p+1}y^{2q} + 2x^{3p+1}y^{3q} + yz.$$

L. Păunescu and A. Zaharia showed that after a suitable polynomial change of coordinates, we can write $f(X, y, Z) = X$. Therefore f is a trivial fibration. Following their reasoning, we can deduce that if $p > q$ then f does not satisfy the Malgrange condition in 0. Let

$$g(x, y, z) := y \quad \text{for } (x, y, z) \in \mathbb{R}^3, S = \mathbb{R}.$$

Put $U = \mathbb{R}, R = 1$. Then

1. $\nabla g(x, y, z) = [0, 1, 0] \neq 0$ for $D_{(U,R)} = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 |\ R \geqslant \|x\|\}$
2. $D_{(U,R)} \subset g^{-1}(\mathbb{R})$
3. for $s \in \mathbb{R}$ we have

$$\begin{aligned}\nabla_g f(x, y, z) &= [1 - 3(2p + 1)x^{2p}y^{2q} + 2(3p + 1)x^{3p}y^{3q}, 0, y] = \\ &= [1 - 3(2p + 1)x^{2p}s^{2q} + 2(3p + 1)x^{3p}s^{3q}, 0, s]\end{aligned}$$

for $(x, y) \in D_{(U,R)} \cap g^{-1}(s)$. If $s \neq 0$ then $\|\nabla_g f(x, y, z)\| \geq \|s\| > 0$ and if $s = 0$ we have $\|\nabla_g f(x, y, z)\| = 1$.

Therefore f satisfies the (g, S)-Malgrange condition and using Theorem 3.1 we deduce that f is a trivial fibration.

In general, finding a suitable function g can be very difficult. In the case when f is a coordinate of a mapping with non-vanishing jacobian, the natural candidates for g are other coordinates of this mapping.

Example 3.8. Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ and $Jac(F) = 1$, where $Jac(F)$ is the jacobian of F. Than we have

$$\|\nabla_{f_2} f_1(x, y)\|^2 = \frac{Jac(F)^2}{\|\nabla f_2(x, y)\|^2} = \frac{1}{\|\nabla f_2(x, y)\|^2}, \quad (x, y) \in \mathbb{R}^2.$$

Therefore f_1 satisfies the $(f_2, \mathbb{R})$-Malgrange condition over $\mathbb{R}$ if and only if there exists $R > 0$ such that

$$\|(x, y)\| > \delta_s \|\nabla f_2(x, y)\| \text{ for } (x, y) \in f_2^{-1}(s), \|(x, y)\| > R,\ s \in \mathbb{R}.$$

From Example 3.8 we get

Remark 3.9. Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth mapping with $Jac(F) = 1$ and f_2 be a polynomial, such that $\deg f_2 \leq 2$. Then the mapping F is injective.

4. ρ_0-regularity on manifolds

In this section we will consider a different condition that allows us to trivialize functions defined on the manifold of the form $g^{-1}(0)$ at infinity.

Let $f, g \in C^\infty(\mathbb{R}^n)$. Assume that $\nabla g(x) \neq 0$ for $x \in g^{-1}(0)$ and denote $M := g^{-1}(0)$, $\rho_0 := \|\cdot\|^2|_M$.

The critical set $\mathcal{M}_0(f_M)$ of the map $(f_M, \rho_0) : M \to \mathbb{R}$ we will call *the Milnor set of* f_M (with respect to ρ_0 function).

A value $\lambda \in \mathbb{R}$ is called a *ρ_0-regular value of f_M at infinity* if there exist a neighbourhood U of λ and a constant $R > 0$ such that

$$\forall_{x \in f_M^{-1}(U)} \quad x \notin \mathcal{M}_0(f_M) \quad \text{for} \quad \|x\| \geqslant R.$$

The set $S_0(f_M)$ of all values that are not a ρ_0 regular value of f_M at infinity will be called the set of *asymptotic ρ_0-nonregular values* of f_M i.e.

$$S_0(f_M) := \{\lambda \in \mathbb{R} | \ \exists_{(x_k) \subset \mathcal{M}_0(f_M)} \|x_k\| \to \infty, f_M(x_k) \to \lambda\}.$$

Our aim is to prove the following theorem

Theorem 4.1. *If λ is a ρ_0-regular value of f_M at infinity than there exists neighbourhood U of the λ and $R > 0$ such that $f_M|_{f_M^{-1}(U)}$ is a C^∞ trivial fibration on M at infinity.*

The proof of the theorem will be preceded by some technical properties.

Let $M^* := M \setminus \mathcal{M}_0(f_M)$ and consider the vector field $v : M^* \to \mathbb{R}^n$

$$v(x) := \nabla f(x) + \frac{\langle \nabla g(x), x\rangle \langle \nabla g(x), \nabla f(x)\rangle - \|\nabla g(x)\|^2 \langle x, \nabla f(x)\rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2} x +$$

$$+ \frac{\langle \nabla g(x), x\rangle \langle x, \nabla f(x)\rangle - \|x\|^2 \langle \nabla g(x), \nabla f(x)\rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2} \nabla g(x) \text{ for } x \in M^*.$$

The field v is well defined. Indeed if $\|x\|\|\nabla g(x)\| = |\langle \nabla g(x), x\rangle|$ then using the Cauchy-Schwarz inequality we deduce that x and $\nabla g(x)$ are linearly dependent. We get $\nabla \rho_0(x) = 0$ and therofore $x \in \mathcal{M}_0(f_M)$ which contradicts the assumptions.

From the definition of v we see that $v(x)$ is tangent to the manifold M^* and to the sphere $\partial B(x) := \{y \in \mathbb{R}^n | \quad \|y\| = \|x\|\}$. Namely, we have

Property 4.2. *For any $x \in M^*$ we have $\langle v(x), x\rangle = \langle v(x), \nabla g(x)\rangle = 0$.*

Proof. Take $x \in M^*$. Then

$$\langle v(x), x\rangle = \langle \nabla f(x), x\rangle + \\ + \frac{-\|\nabla g(x)\|^2 \langle x, \nabla f(x)\rangle \|x\|^2 + \langle \nabla g(x), x\rangle^2 \langle x, \nabla f(x)\rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2},$$

therefore

$$\begin{aligned}\langle v(x), x\rangle =&\langle \nabla f(x), x\rangle + \\ &+ \langle \nabla f(x), x\rangle \frac{-\|\nabla g(x)\|^2\|x\|^2 + \langle \nabla g(x), x\rangle^2}{\|x\|^2\|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2} = 0,\end{aligned}$$

which gives that $\langle v(x), x\rangle = 0$. Analogously as above we obtain that

$$\langle v(x), \nabla g(x)\rangle = 0.$$

□

Lemma 4.3. *For $x \in M^*$ we have $\langle v(x), \nabla f_M(x)\rangle \neq 0$.*

Proof. Denote

$$\Omega := \{w : M^* \to \mathbb{R}^n |\ w - smooth\},$$

$$\Omega(TM^*) := \{w \in \Omega |\ \forall_{x \in M^*}\ \langle w(x), \nabla g(x)\rangle = 0\}$$

and let $\pi_{M^*} : \Omega \to \Omega(TM^*)$ be a mapping defined by

$$\pi_{M^*}(w(x)) := w(x) - \frac{\langle w(x), \nabla g(x)\rangle}{\|\nabla g(x)\|^2}\nabla g(x) \quad \text{for } x \in M^*.$$

At first we will prove that

$$v(x) = \pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)) \text{ for } x \in M^*, \tag{7}$$

where

$$\pi_{(\pi_{M^*}(x))^\perp}(w(x)) := w(x) - \frac{\langle w(x), \pi_{M^*}(x)\rangle}{\|\pi_{M^*}(x)\|^2}\pi_{M^*}(x) \text{ for } x \in M^*, w \in \Omega.$$

From definitions and simple calculations for $x \in M^*$ we have

$$\begin{aligned}\pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)) &= \nabla f_M(x) - \frac{\langle \nabla f_M(x), \pi_{M^*}(x)\rangle}{\|\pi_{M^*}(x)\|^2}\pi_{M^*}(x) = \\ &= \nabla f(x) - \frac{\langle \nabla f(x), \nabla g(x)\rangle}{\|\nabla g(x)\|^2}\nabla g(x) - \frac{\langle \nabla f(x), \pi_{M^*}(x)\rangle}{\|\pi_{M^*}(x)\|^2}\pi_{M^*}(x) = \\ &= \nabla f(x) + \frac{\langle \nabla g(x), x\rangle\langle \nabla g(x), \nabla f(x)\rangle - \|\nabla g(x)\|^2\langle x, \nabla f(x)\rangle}{\|x\|^2\|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2}x + \\ &+ \frac{\langle \nabla f(x), \|\nabla g(x)\|^2 x - \langle \nabla g(x), x\rangle\nabla g(x)\rangle\langle \nabla g(x), x\rangle}{\|\nabla g(x)\|^2(\|x\|^2\|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2)}\nabla g(x) + \\ &- \frac{\langle \nabla f(x), \nabla g(x)\rangle(\|x\|^2\|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2)}{\|\nabla g(x)\|^2(\|x\|^2\|\nabla g(x)\|^2 - \langle \nabla g(x), x\rangle^2)}\nabla g(x) = v(x).\end{aligned}$$

which proves (7).

For $x \in M^*$ we have

$$\begin{aligned}&\langle v(x), \nabla f_M(x)\rangle = 0\\ &\Leftrightarrow \langle \pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)), \nabla f_M(x)\rangle = 0\\ &\Leftrightarrow \|\nabla f_M(x)\|^2 \|\pi_{M^*}(x)\|^2 = \langle \nabla f_M(x), \pi_{M^*}(x)\rangle^2\\ &\Rightarrow x \in \mathcal{M}_0(f_M)\end{aligned}$$

which completes the proof. □

Remark 4.4. Analogously as in the proof of Lemma 4.3 we can prove that $v(x) = \pi_{(\pi_{\partial B}(\nabla g(x)))^\perp}(\pi_{\partial B}(\nabla f(x))$, where

$$\pi_{\partial B}(w(x)) := w(x) - \frac{\langle w(x), x\rangle}{\|x\|^2} x,$$

$$\pi_{(\pi_{\partial B}(\nabla g(x)))^\perp}(w(x)) := w(x) - \frac{\langle w(x), \pi_{\partial B}(\nabla g(x))\rangle}{\|\pi_{\partial B}(\nabla g(x))\|^2} \pi_{\partial B}(\nabla g(x))$$

for $x \in M^*, w \in \Omega$.

We are ready to prove Theorem 4.1.

Proof. By the assumption that λ is a ρ_0-regular value of f_M at infinity, there exist a neighbourhood U of the λ and $R > 0$ such that

$$(\mathcal{M}_0(f_M) \cap f_M^{-1}(U)) \backslash \overline{B(R)} = \emptyset,$$

where $\overline{B(R)} := \{x \in M | \ R \geqslant \|x\|\}$. Let $w(x) := \frac{v(x)}{\langle v(x), f_M(x)\rangle}$ for $x \in f_M^{-1}(U)\backslash\overline{B(R)}$. From the assumption and Lemma 4.3, w is well defined. For each $\mu \in U$ consider the following system of differential equations

$$x' = (\lambda - \mu) w(x), \tag{8}$$

with the right hand side defined in the set $G := \{(t, x) \in \mathbb{R} \times M | \ x \in f_M^{-1}(U)\backslash\overline{B(R)}\}$. Denote by $\Phi_\mu : V_\mu \to M$ the general solution of system (8) and $V_\mu := \{(\tau, \eta, t) \in \mathbb{R} \times M \times R | \ (\tau, \eta) \in G, t \in I_\mu(\tau, \eta)\}$, where $I_\mu(\tau, \eta)$ is a domain of integral solution of $t \to \Phi_\mu(\tau, \eta, t)$. From a definition of the general solution we get

$$\Phi_\mu(\tau, \eta, \tau) = \eta. \tag{9}$$

Note that Property 4.2 implies

$$|\Phi_\mu(\tau, \eta, t)\| = \|\eta\| \text{ for } t \in I_\mu(\tau, \eta), (\tau, \eta) \in G, \mu \in U. \tag{10}$$

Consider the mapping

$$\Psi_1 : f_M^{-1}(U)\backslash\overline{B(R)} \ni x \mapsto \Phi_{f(x)}(0, x, 1) \in f_M^{-1}(\lambda)\backslash\overline{B(R)}.$$

We show the mapping Ψ_1 is well defined that is $1 \in I_{f(x)}(0, x)$ for each $x \in f_M^{-1}(U)\backslash\overline{B(R)}$.

Suppose the contrary that there exists $x \in f_M^{-1}(U)\backslash\overline{B(R)}$ such that $1 \notin I_{f(x)}(0,x)$ and denote $\varphi_x(t) := \Phi_{f(x)}(0,x,t)$ for $t \in I_{f(x)}(0,x)$. From (8) and the initial condition (9) we get

$$f_M \circ \varphi_x(t) = (\lambda - f(x))t + f(x), \quad t \in I_{f(x)}(0,x). \tag{11}$$

Denoting J as closed interval with endpoints λ and $f(x)$ we see that the set

$$F := \{(t,x') \in \mathbb{R} \times f_M^{-1}(U)\backslash\overline{B(R)} \mid t \in [0,1], f_M(x') \in J, \|x'\| = \|x\|\}$$

is a compact subset of G such that the graph of $\varphi_x|_{[0,\beta)}$ is contained in F. This contradicts Lemma 1.3 and proves $[0,1] \subset I_{f(x)}(0,x)$.

Using (11) we get $f(\Psi_1(x)) = f(\varphi_x(1)) = \lambda$ and from (10) we have $\Psi_1(x) \in f_M^{-1}(\lambda)\backslash\overline{B(R)}$. Summing up we show that the mapping Ψ_1 is defined correctly. Similarly we can show that the mapping

$$\Theta : f_M^{-1}(\lambda)\backslash\overline{B(R)} \times U \ni (\xi,\mu) \mapsto \Phi_\mu(1,\xi,0) \in f_M^{-1}(U)\backslash\overline{B(R)}$$

is well defined. It is easy to check that the mapping

$$\Psi : f_M^{-1}(U)\backslash\overline{B(R)} \ni x \mapsto (\Psi_1(x), f_M(x)) \in f_M^{-1}(\lambda)\backslash\overline{B(R)} \times U$$

is a C^∞ diffeomorphism and $\Psi^{-1} = \Theta$. Therefore $f_{M|f_M^{-1}(U)\backslash\overline{B(R)}}$ is a C^∞ trivial fibration on M. □

Remark 4.5. From Theorem 4.1 we have $B_\infty(f) \subset S_0(f)$.

It is worth noting that unlike in a flat case we need to take into account the set $V := \{x \in M | \exists_{t\in\mathbb{R}} \nabla g(x) = tx\} \subset \mathcal{M}_0(f_M)$. In general the field $v : M\backslash V \to \mathbb{R}^n$ can not be continuously extended on the set V. The following example illustrates the fact

Example 4.6. Let

$$g(x,y,z) := \frac{1}{2}x^2 + y^2 - \frac{1}{2}, \quad f(x,y,z) = y \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

We have

$$\begin{aligned} v(x,y,z) &= [v^1(x,y,z), v^2(x,y,z), v^3(x,y,z)] = \\ &= \left[\frac{-2xyz^2}{x^2y^2+x^2z^2+4y^2z^2}, \frac{x^2z^2}{x^2y^2+x^2z^2+4y^2z^2}, \frac{x^2yz}{x^2y^2+x^2z^2+4y^2z^2}\right] \end{aligned}$$

for $(x,y,z) \in M\backslash V$. Note that $(1,0,0) \in V$ and

$$v^2(1,0,z) = 1 \text{ for } z \neq 0 \text{ and } v^2(x,y,0) = 0 \text{ for } x \neq 0, y \neq 0,$$

therefore the limit $\lim_{(x,y,z)\to(1,0,0)} v^2(x,y,z)$ does not exist.

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WARTOŚCI BIFURKACYJNE FUNKCJI KLASY C^∞

S t r e s z c z e n i e

W pracy pokazujemy jak można używać trywializacji funkcji $f:\mathbb{R}^n\to\mathbb{R}$ na poziomicach pewnej funkcji $g:\mathbb{R}^n\to\mathbb{R}$ do skonstruowania trywializacji funkcji f na całej przestrzeni $\mathbb{R}^n$.

Dodatkowo przenosimy metodę trywializacji funkcji spełniających warunek ρ_0-regularności na przypadek funkcji zdefiniowanych na hiperpowierzchniach postaci $M = g^{-1}(0)$.

Słowa kluczowe: wartości bifurkacyjne, trywializacja, warunek Malgrange'a, ρ_0-regularność