DOI: 10.26485/0459-6854/2017/67.2/6 PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017 Vol. LXVII

Recherches sur les déformations no. 2

pp. 61–67

Kacper Grzelakowski

SOME ESTIMATIONS OF THE ŁOJASIEWICZ EXPONENT FOR POLYNOMIAL MAPPINGS ON SEMIALGEBRAIC SETS

Summary

We strengthen some estimations of the local and global Łojasiewicz exponent for polynomial mappings on closed semialgebraic sets obtained by K. Kurdyka, S. Spodzieja and A. Szlachcińska in [4].

Keywords and phrases: Łojasiewicz exponent, semialgebraic set, semialgebraic mapping, polynomial mapping

1. Introduction

Łojasiewicz inequalities are important tools in many different areas of mathematics such as singularity theory, differential analysis or dynamical systems (for example [2], [6], [9]). They first appeared in works of Hörmander in 1958 [3] and independently in those of Łojasiewicz in 1958 [7] and 1959 [8].They were used to prove Schwartz hypothesis that a division of a distribution by a polynomial [3] and by real analytic function [7] [8] is always possible. Estimates of the Łojasiewicz exponent are nowadays widely used in real and complex algebraic geometry. Kudryka, Spodzieja and Szlachcińska in [5] have given an estimate of the Łojasiewicz exponent at a point for a continuous semialgebraic mapping on a closed semialgebraic set and an estimate of the Łojasiewicz exponent at infinity for a polynomial mapping on a semialgebraic set. In this paper we show that in case of a polynomial mapping, at a point or at infinity, it is possible to obtain slightly stronger results than they have.

2. Łojasiewicz Exponent at a point

Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set and let $F : X \to \mathbb{R}^m$ be a polynomial mapping, such that $0 \in X$ and $F(0) = 0$. Then, there exist positive constants C, η, ε such that the following Łojasiewicz inequality holds (see [7]):

$$|F(x)| \geq C \operatorname{dist}(x, F^{-1}(0) \cap X)^{\eta} \quad \text{for} \quad x \in X, |x| < \varepsilon, \tag{1}$$

where $|\cdot|$ is the Euclidean norm and $\operatorname{dist}(x, A)$ is the distance of a point x to the set A, i.e. the lower bound of $|x - a|$ for $a \in A$. By convention $\operatorname{dist}(x, \emptyset) = 1$.

Definition 2.1. The infimum of the exponents η in (1) is called the *Łojasiewicz exponent of* F *on the set* X *at* 0 and is denoted by $\mathcal{L}_0(F|X)$.

Each closed semialgebraic set $X \subset \mathbb{R}^N$ has a decomposition

$$X = X_1 \cup \cdots \cup X_k$$

into the union of closed basic semialgebraic sets

$$X_i = \{x \in \mathbb{R}^N : g_{i,1}(x) \geq 0, \ldots, g_{i,r_i}(x) \geq 0, h_{i,1}(x) = \cdots = h_{i,l_i}(x) = 0\},$$

$i = 1, \ldots, k$, where $g_{i,1}, \ldots g_{i,r_i}, h_{i,1}, \ldots, h_{i,l_i} \in \mathbb{R}[x_1, \ldots, x_N]$ (see [1]). Assume that r_i is the smallest possible number of inequalities $g_{i,j}(x) > 0$ in the definition of X_i for $i = 1, \ldots, k$. Denote by $r(X)$ the minimum of $\max\{r_1, \ldots, r_k\}$ over all decompositions into unions of sets of X. Obviously r(X) = 0 means that X is an algebraic set. Denote by $\kappa(X)$ the minimum of the numbers

$$\max\{\deg g_{1,1}, \ldots, \deg g_{k,r_k}, \deg h_{1,1}, \ldots, \deg h_{k,l_k}\}$$

over all decompositions of X into the union of sets, provided $r_i \leq r(X)$. By $\deg F$ we mean the maximum of the degrees of the components of the mapping F.

First aim of this paper is to prove the following theorem:

Theorem 2.2. *Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set such that $0 \in X$ and let $F : X \to \mathbb{R}^m$ be a nonzero polynomial mapping such that $F(0) = 0$. Set $r = r(X)$ and $d = \max\{\kappa(X), \deg F\}$. Then:*

$$\mathcal{L}_0(F|X) \leq d(6d-3)^{N+r+m-1}. \tag{2}$$

In [5, Corollary 2.2] Kurdyka, Spodzieja and Szlachcińska proved that:

$$\mathcal{L}_0(F|X) \leq d(6d-3)^{N+R+m-1}$$

with $R = r(X) + r(\operatorname{graph} F)$. Actually in [5] there is no m in the inequality but this should be considered a typographical error. Thus in our theorem we do improve their estimation by using $r = r(X)$ instead of $R = r(X) + r(\operatorname{graph} F)$. For this paper, to be self-contained and more clear, we will have to repeat some of the argumentation from [5] for polynomial mappings on semialgebraic sets.

In the proof of Theorem 1 we will use the result obtained in [4, Corollary 8] regarding Łojasiewicz exponent in the case of two algebraic sets. Let X and Y be algebraic subsets of $\mathbb{R}^M$ described by polynomials of degree not greater than d. Let $a \in \mathbb{R}^M$. Then there exists a positive constant C such that:

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \operatorname{dist}(x, X \cap Y)^{d(6d-3)^{M-1}} \tag{KS1}$$

in a neighbourhood $U \subset \mathbb{R}^M$ of a. We will also use another result from [4, Corollary 6]. For a real polynomial mapping $F : \mathbb{R}^N \to \mathbb{R}^m$ such that $d = \deg F$ we have

$$\mathcal{L}_0(F) \leq d(6d-3)^{M-1}. \tag{KS2}$$

Proof of Theorem 1. If $d = 1$ then the statement is obvious. Let us assume that $d \geq 2$. It suffices to consider the case when X is a basic semialgebraic set. The set X was originally in $\mathbb{R}^N$ but since we will operate in $\mathbb{R}^N \times \mathbb{R}^m$ we need the set $X \times \{0\} \subset \mathbb{R}^N \times \mathbb{R}^m$. Not to overuse the notation from now on we will use X to denote $X \times \{0\} \subset \mathbb{R}^N \times \mathbb{R}^m$. So let:

$$X := \{(x, z) \in \mathbb{R}^N \times \mathbb{R}^m : g_1(x) \geq 0, \dots, g_{r(X)}(x) \geq 0, \\ h_1(x) = \cdots = h_l(x) = 0, z = 0\}, \tag{3}$$

$$Y := \{(x, z) \in \mathbb{R}^N \times \mathbb{R}^m : g_1(x) \geq 0, \dots, g_{r(X)}(x) \geq 0, \\ h_1(x) = \cdots = h_l(x) = 0, z = F(x)\}.$$

Now, let us define a mapping $G : \mathbb{R}^N \times \mathbb{R}^r \to \mathbb{R}^r$ by:

$$G(x, y_1, \dots, y_r) := \{g_1(x) - y_1^2, \dots, g_r(x) - y_r^2\},$$

and then sets:

$$A := \{(x, z, y_1, \dots, y_r) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^r : \\ G(x, y) = 0, h_1(x) = \cdots = h_l(x) = 0, z = 0\},$$

$$B := \{(x, z, y_1, \dots, y_r) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^r : \\ G(x, y) = 0, h_1(x) = \cdots = h_l(x) = 0, z = F(X)\}.$$

Then A and B are algebraic sets and $\pi(A) = X, \pi(B) = Y$, where

$$\pi : \mathbb{R}^{N+m} \times \mathbb{R}^r \to \mathbb{R}^{N+m}, \quad \pi(x, z, y_1, \dots, y_r) = (x, z).$$

From the definitions of A and B we obtain:

$$\forall_{(x,0) \in X} \, \exists_{z \in \mathbb{R}^m} \, \exists_{y \in \mathbb{R}^r} (x, 0, y) \in A \wedge (x, z, y) \in B. \tag{4}$$

Since A and B are algebraic sets defined by polynomials of degree not greater than d then by (KS1), for sets A, B there exists a positive constant C such that:

$$\operatorname{dist}((x, 0, y), A) + \operatorname{dist}((x, 0, y), B) \geq C \operatorname{dist}((x, 0, y), A \cap B)^{d(6d-3)^{N+r+m-1}} \tag{5}$$

in some neighbourhood W of $0 \in \mathbb{R}^{N+m+r}$. For any $(x, z, y) \in \mathbb{R}^{N+m+r}$ we have:

$$\operatorname{dist}((x, z, y), A \cap B) \geq \operatorname{dist}((x, z), X \cap Y). \tag{6}$$

We can now assume that $g_{i,j}(0) = 0$ for any i, j. Indeed, if $g_{i,j}(0) < 0$ for some i, j then $0 \notin X \cap Y$ which contradicts the assumption. If $g_{i,j}(0) > 0$ for some i, j then it is safe to omit this inequality in the definition of X (and Y) and the germ of 0 at X or Y will not change. If $g_{i,j}(0) > 0$ for any i, j, then we can reduce our assertion to (KS2). So, there exists a neighbourhood $V = V_1 \times V_2 \subset W$ of $0 \in \mathbb{R}^{N+m+r}$ where $V_1 \subset \mathbb{R}^{N+m}$ and $V_2 \subset \mathbb{R}^r$ such that:

$$\forall_{(x,0,y)\in A:\ (x,0)\in\mathbb{R}^{N+m},\ y\in\mathbb{R}^r} \quad (x,0) \in X \cap V_1 \Rightarrow y \in V_2, \tag{7}$$

and

$$\forall_{(x,z,y)\in B:\ (x,z)\in\mathbb{R}^{N+m},\ y\in\mathbb{R}^r} \quad (x,z) \in Y \cap V_1 \Rightarrow y \in V_2. \tag{8}$$

Note, that since A and B were defined by $N + r$ identical coordinates x, y and differ only in m of them z. This explains why in (7) and in (8) we were able to consider the same neighbourhood $V_2 \subset \mathbb{R}^r$.

Since F is a continuous mapping there exist neighbourhoods $U_1 \subset \mathbb{R}^N$ and $U_2 \subset \mathbb{R}^m$ of the origin such that $U_1 \times U_2 \subset V_1$ and for every $x \in U_1$ we have $z = F(x) \in U_2$. Then $(U_1 \times U_2) \times V_2 \subset W$. Consider some $x \in U_1$. By (4) there exist $z \in \mathbb{R}^m$ and $y \in \mathbb{R}^r$ such that $(x, 0, y) \in A$ and $(x, z, y) \in B$. Then, by (7) and (8) we see that $(x, 0, y) \in V$. Let us observe that:

$$|F(x)| = |(x, 0) - (x, z)| = |(x, 0, y) - (x, z, y)| \geq \operatorname{dist}((x, 0, y), B).$$

Since $(x, z, y) \in B$, and $(x, 0, y) \in A$ then, from the above:

$$|F(x)| \geq \operatorname{dist}((x, 0, y), A) + \operatorname{dist}((x, 0, y), B).$$

Since $A, B \in \mathbb{R}^{N+m+r}$, by (5) and by (6), we obtain :

$$\begin{aligned} |F(x)| &\geq \operatorname{dist}((x, 0, y), A) + \operatorname{dist}((x, 0, y), B) \\ &\geq C \operatorname{dist}((x, 0, y), A \cap B)^{d(6d-3)^{N+r+m-1}} \\ &\geq C \operatorname{dist}((x, 0), X \cap Y)^{d(6d-3)^{N+r+m-1}}. \end{aligned}$$

Since $X \cap Y = (F^{-1}(0) \times 0)$ we obtain the assertion. □

3. The Łojasiewicz exponent at infinity

The second result of this paper concerns the global Łojasiewicz inequality and the Łojasiewicz exponent of a polynomial mapping at infinity.

Definition 3.1. Assume that a closed semialgebraic set $X \subset \mathbb{R}^N$ is unbounded. By the *Łojasiewicz exponent at infinity* of a polynomial mapping $F : X \to \mathbb{R}^m$ we mean the supremum of the exponents η in the following inequality:

$$|F(x)| \geq C|x|^\eta \quad for \quad x \in X, |x| \geq c$$

for some positive constants C, c. We denote it by $\mathcal{L}_\infty(F|X)$. In case $X = \mathbb{R}^N$ we call this exponent the *Łojasiewicz exponent at infinity* and denote it by $\mathcal{L}_\infty(F)$.

In [4, Corollary 10] it is proved, that for a polynomial mapping $F = (f_1, \dots, f_m) : \mathbb{R}^N \to \mathbb{R}^m$ of degree d of a real algebraic set X we have:

$$|F(x)| \geq C\left(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^2}\right)^{d(6d-3)^{M-1}} \qquad \text{for } x \in \mathbb{R}^M. \tag{KS3}$$

Using this, in [5, Corollary 3.4] it has been shown that for a polynomial mapping F on a closed semialgebraic set X the following inequality holds:

$$|F(x)| \geq C\left(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^2}\right)^{d(6d-3)^{N+R-1}} \qquad \text{for } x \in \mathbb{R}^N,$$

where $R = 2r(X)$. We are again going to show that this estimate can be improved by substituting R with $r = r(X)$ and also by adding m.

Theorem 3.2. *Let $F : X \to \mathbb{R}^m$ be a polynomial mapping, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set. If $D = \max\{2, \kappa(X)\}, d = \max\{\deg F, D\}$ and $r = r(X)$ then:*

$$|F(x)| \geq C\left(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1+|x|^D}\right)^{d(6d-3)^{N+r+m-1}} \qquad \text{for } x \in \mathbb{R}^M. \tag{9}$$

If additionally X is unbounded set and $F^{-1}(0) \cap X$ is a compact set, then:

$$\mathcal{L}_\infty(F|X) \geq -\frac{D}{2} d(6d-3)^{N+r+m-1}. \tag{10}$$

Proof of Theorem 2. Again we shall repeat the argumentation from [5]. Also, as in the previous proof we will consider the set $X \times \{0\} \subset \mathbb{R}^N \times \mathbb{R}^m$ defined by (3), and denote it simply by X to avoid overuse of notation. Let $H : \mathbb{R}^{N+r+m} \to \mathbb{R}^{r+m+l}$ be a polynomial mapping defined by:

$$H(x, z, y) = (F(x, z), G(x, y), h_{1,1}(x), \dots, h_{1,l}(x)), \quad x \in \mathbb{R}^N, z \in \mathbb{R}^m, y \in \mathbb{R}^r$$

with G being defined as in the previous proof. Then $\deg H \leq d$. Let $V = F^{-1}(0) \cap X$ and $Z = H^{-1}(0)$. Obviously Z is an algebraic set. By (KS3) for some positive constant C we have:

$$|H(x, z, y)| \geq C\left(\frac{\operatorname{dist}((x, z, y), Z)}{1+|(x, 0, y)|^2}\right)^{d(6d-3)^{N+r+m-1}}$$

for $(x, 0, y) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^r$. Obviously $\operatorname{dist}((x, z, y), Z) \geq \operatorname{dist}((x, z), V)$ and thus:

$$|H(x, z, y)| \geq C\left(\frac{\operatorname{dist}((x, z), V)}{1+|(x, 0, y)|^2}\right)^{d(6d-3)^{N+r+m-1}} \tag{11}$$

for $(x, 0, y) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^r$. It is easy to observe that there exist constants $C_1 \geq 0, R_1 \geq 1$ such that for $(x, 0, y) \in A$ with $|(x, 0, y)| \geq R_1$ we have $C_1|y|^2 \leq |(x, 0)|^D$. Since $D \geq 2$, for a constant $C_2 > 0$ we have $|(x, 0, y)| \leq C_2|(x, 0)|^{D/2}$ for $(x, 0, y) \in$

$A, |(x,0,y)| \geq R_1$. Hence, from (11) we obtain (9) for $(x,0) \in X$, $|(x,0)| > R_1$. Again, by diminishing C, if necessary, we obtain (9) for all $(x,0) \in X$.

Now, let us prove the second assertion of Theorem 2. To do this, we will need yet another result from [4], namely [Corollary 11]. The authors have shown that if $F = (f_1, \ldots, f_m) : \mathbb{R}^N \to \mathbb{R}^m$ is a polynomial mapping of degree $d \geq 1$, and $F^{-1}(0)$ is a compact set then:

$$\mathcal{L}_\infty(F) \geq -d(6d-3)^{n-1}. \tag{KS3}$$

Since X is unbounded we may assume that so is A. Since V is compact, so is $H^{-1}(0)$. By (KS3) we have $\mathcal{L}_\infty(H) \geq -d(6d-3)^{N+r+m-1}$, in particular, for some constants $C, R > 0$,

$$|H(x,0,y)| \geq C|(x,0,y)^{-d(6d-3)^{N+r+m-1}} \quad \text{for } (x,0,y) \in A, |(x,0,y)| \geq R.$$

Since $|(x,0,y)| \leq C_2|(x,0)|^{D/2}$ for $(x,0,y) \in A$, $|(x,0,y)| \geq R_1$, then for some constant $C_3 > 0$:

$$|F(x,0)| = |H(x,0,y)| \geq C_3|x|^{-\frac{D}{2}d(6d-3)^{N+r+m-1}} \quad \text{for } (x,0,y) \in A, |(x,0,y)| \geq R$$

and also $\mathcal{L}^{\mathbb{R}}_\infty(F|X) \geq -\frac{D}{2}d(6d-3)^{N+r+m-1}$, which ends the proof. □

Acknowledgement

The author wishes to thank Stanisław Spodzieja who has provided both inspiration and support in the process of writing this paper and Tadeusz Krasiński whose insight has been more than helpful.

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Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22, PL-90-238 Łódź
Poland
E-mail: kacper.grzelakowski@wp.pl

Presented by Andrzej Łuczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 1, 2016.

PEWNE SZACOWANIA WYKŁADNIKA ŁOJASIEWICZA DLA ODWZOROWAŃ WIELOMIANOWYCH NA ZBIORACH SEMIALGEBRAICZNYCH

S t r e s z c z e n i e

Nierówności Łojasiewicza są ważnymi narzędziami w wielu gałęziach matematyki takich jak teoria osobliwości, analiza różniczkowa czy układy dynamiczne (patrz na przykład [2], [6], [9]). Po raz pierwszy pojawiły się w pracach Hörmandera w 1958 [3] i niezależnie w pracach Łojasiewicza w 1958 [7] i 1959 [8]. Zostały użyte do udowodnienia hipotezy Schwartz że dzielenie dystrybucji przez wielomian [3] i przez rzeczywistą funkcję analityczną [7], [8] jest zawsze możliwe. Oszacowania wykładnika Łojasiewicza są szeroko używane w rzeczywistej i zespolonej geometrii. K. Kudryka, S. Spodzieja i A. Szlachcińska w [5] podali oszacowanie wykładnika Łojasiewicza w punkcie dla ciągłego odwzorowania semialgebraicznego na semialgebraicznym zbiorze domkniętym i wykładnika Łojasiewicza w nieskończoności dla odwzorowania wielomianowego. W tej pracy wykazano, że w przypadku odwzorowania wielomianowego, czy to w punkcie czy nieskończoności można uzyskać trochę dokładniejsze szacunki.

Słowa kluczowe: wykładnik Łojasiewicza, zbiór semialgenraiczny, odwzorowanie semialgebraiczne, odwzorowanie wielomianowe