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DENSITY TOPOLOGIES ON THE PLANE BETWEEN ORDINARY AND STRONG. III

Summary

Let C_0 denote the set of all non-decreasing continuous functions $f : (0,1] \to (0,1]$ such that $\lim_{x\to 0^+} f(x) = 0$ and $f(x) \le x$ for every $x \in (0,1]$ and let A be a measurable subset of the plane. The notions of a density point of A with respect to f and the mapping D_f, defined on the family of all measurable subsets of the plane, were introduced in [13]. This mapping is a lower density, so it allowed us to introduce the topology $\mathcal{T}_f$, analogously to the density topology. In [14] the properties of the topology $\mathcal{T}_f$ and functions approximately continuous with respect to f were considered. We proved that $(\mathbb{R}^2, \mathcal{T}_f)$ is a completely regular topological space and we studied conditions under which topologies generated by two functions f and g are equal. In this note we prove that $(\mathbb{R}^2, \mathcal{T}_f)$ is a maximally resolvable and extraresolvable Baire space for which Blumberg's theorem does not hold.

Keywords and phrases: density point, density topology, Blumberg's theorem

The investigations of density-type topologies on the real line or on the plane have a long tradition. The beginning of such studies is connected with the paper of C. Goffman, C. J. Neugebauer and T. Nishiura ([4]), and the most general look, one can find in a monograph of J. Lukeš, J. Malý and L. Zajiček [9]. In this paper we are focused on the resolvability and the analogon of Blumberg's theorem for a class of topologies situated between the ordinary density topology and the strong density topology. These investigations are a continuation of [13] and [14].

Let $\mathcal{S}$ denote the family of all Lebesgue measurable subsets of the plane and m_2 – the two-dimesional Lebesgue measure. Recall that if $A \in \mathcal{S}$, then (x_0, y_0) is an ordinary density point of A if

$$\lim_{h\to 0^+} \frac{m_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h])}{4h^2} = 1$$

and it is a strong density point of A if

$$\lim_{h,k\to 0^+}\frac{m_2(A\cap([x_0-h,x_0+h]\times[y_0-k,y_0+k])}{4hk}=1$$

(compare [11], p. 106 and 129). For $A\in\mathcal{S}$ let $\Phi_o(A)$ and $\Phi_s(A)$ denote the set of all ordinary density points of A and the set of all strong density points of A, respectively. The associated ordinary density topology and strong density topology in the plane have different properties, for example the ordinary density topology is completely regular while the strong density topology is not (see [5]). This paper is devoted to study the topological properties of a class of topologies included between ordinary and strong.

If $A,B\in\mathcal{S}$ then we shall write $A\sim B$ if and only if $m_2(A\triangle B)=0$ (where $A\triangle B=(A\setminus B)\cup(B\setminus A)$).

Let C_0 denote the set of all non-decreasing continuous functions $f:(0,1]\to(0,1]$ such that $\lim_{x\to 0^+}f(x)=0$ and $f(x)\le x$ for every $x\in(0,1]$.

Let $f\in C_0$ and $A\in\mathcal{S}$.

Definition 1. [13] We say that (x_0,y_0) is a *density point of A with respect to a function f* if for each $\epsilon>0$ there exists $\delta>0$ such that for each $h,k\in(0,\delta)$ if $f(h)\le k\le h$ or $f(k)\le h\le k$, then

$$\frac{m_2(A\cap\{[x_0-h,x_0+h]\times[y_0-k,y_0+k]\})}{4hk}>1-\epsilon,$$

or, equivalently,

$$\frac{m_2(A'\cap\{[x_0-h,x_0+h]\times[y_0-k,y_0+k]\})}{4hk}<\epsilon,$$

where $A'=\mathbb{R}^2\setminus A$.

The interval $[x_0-h,x_0+h]\times[y_0-k,y_0+k]$ such that $f(h)\le k\le h$ or $f(k)\le h\le k$ will be called suitable for a function f.

Obviously, if $f(x)=x$ for $x\in(0,1]$, then the notion of a density point with respect to f coincides with the notion of the ordinary density point of a measurable subset of the plane.

For $A\in\mathcal{S}$ and $f\in C_0$ we denote by $D_f(A)$ the set of all points $(x,y)\in\mathbb{R}^2$ which are the density points of A with respect to f. Clearly, $\Phi_s(A)\subset D_f(A)\subset\Phi_o(A)$ for every $A\subset\mathcal{S}$. The mapping $D_f:\mathcal{S}\to 2^{\mathbb{R}^2}$ is a lower density (see [13], Theorem 5), so the family

$$\mathcal{T}_f=\{A\in\mathcal{S}:A\subset D_f(A)\}$$

is a topology on the plane, essentially stronger than the strong density topology $\mathcal{T}_s$ on the plane, and weaker than the ordinary density topology $\mathcal{T}_o$ on the plane. The properties of the topology $\mathcal{T}_f$ were studied in [13] and [14]. There was proved, among others, that $(\mathbb{R}^2,\mathcal{T}_f)$ is a completely regular ($T_{3\frac{1}{2}}$) topological space. For this purpose we proved that the theorem analogous to Lusin-Menchoff theorem holds.

Now we will prove using the method from Foran's book (see [4], page 283) that the plane with the topology $\mathcal{T}_f$ is not normal, for all $f \in C_0$. For this purpose it will be convenient for us to use a formula of a derivative in the topology $\mathcal{T}_f$ for a measurable subset of the plane. In [13], Theorem 7, it was observed that if $A \in \mathcal{S}$ then

$$A^{d\mathcal{T}_f} = \mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus A). \tag{1}$$

Theorem 2. *Let $f \in C_0$ and $G_1, G_2 \in \mathcal{T}_f$. If G_1 and G_2 are dense in the Euclidean topology on the plane then $G_1^{d\mathcal{T}_f} \cap G_2^{d\mathcal{T}_f} \neq \emptyset$.*

Proof. Let $p_1 \in G_1$, $p_1 = (x_1, y_1)$. There exists a suitable for a function f interval $I_1 = [x_1 - h_1, x_1 + h_1] \times [y_1 - k_1, y_1 + k_1]$ such that

$$\frac{m_2(G_1 \cap I_1)}{m_2(I_1)} > \frac{4}{5}.$$

Let $p_2 \in G_2 \cap \operatorname{Int} I_1$, (where $\operatorname{Int} A$ denotes the interior of A in the Euclidean topology), $p_2 = (x_2, y_2)$. There exists a suitable for a function f interval $I_2 = [x_2 - h_2, x_2 + h_2] \times [y_2 - k_2, y_2 + k_2]$, such that $I_2 \subset I_1$ and

$$\frac{m_2(G_2 \cap I_2)}{m_2(I_2)} > \frac{4}{5}.$$

Repeating preceding arguments we define a descending sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$, $I_n = [x_n - h_n, x_n + h_n] \times [y_n - k_n, y_n + k_n]$ suitable for a function f, such that for odd $n \in \mathbb{N}$

$$\frac{m_2(G_1 \cap I_n)}{m_2(I_n)} > \frac{4}{5} \tag{2}$$

and

$$\frac{m_2(G_2 \cap I_n)}{m_2(I_n)} > \frac{4}{5} \tag{3}$$

for even $n \in \mathbb{N}$.

Let $p_0 \in \bigcap_{n\in\mathbb{N}} I_n$, $p_0 = (x_0, y_0)$. Put

$$J_n = [x_0 - h_n, x_0 + h_n] \times [y_0 - k_n, y_0 + k_n]$$

for $n \in \mathbb{N}$. Clearly, $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of suitable intervals for a function f. Using (2) and (3) we obtain for odd $n \in \mathbb{N}$

$$\frac{m_2(G_1 \cap J_n)}{m_2(J_n)} \geq \frac{m_2(G_1 \cap I_n) - \frac{3}{4}m(I_n)}{m(I_n)} > \frac{1}{20}$$

and, analogously,

$$\frac{m_2(G_2 \cap J_n)}{m_2(J_n)} > \frac{1}{20}$$

for even $n \in \mathbb{N}$.

Consequently, according to (1), $p_0 \in G_1^{d\mathcal{T}_f} \cap G_2^{d\mathcal{T}_f}$. □

Theorem 3. *For arbitrary function $f \in C_0$ the space $(\mathbb{R}^2, \mathcal{T}_f)$ is not normal.*

Proof. Suppose on the contrary, that $(\mathbb{R}^2, \mathcal{T}_f)$ is normal for some $f \in C_0$. Let A, B be two countable disjoint sets, dense in the Euclidean topology. Then A and B are closed in the topology $\mathcal{T}_f$, so there exist $U, V \in \mathcal{T}_f$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. Clearly, $U \cap \bar{V}^{\mathcal{T}_f} = \emptyset$. From [8, Chapter 4, page 112] it follows that there exists a set $U_1 \in \mathcal{T}_f$ such that

$$A \subset U_1 \subset \bar{U}_1^{\mathcal{T}_f} \subset U.$$

Hence $\bar{U}_1^{\mathcal{T}_f} \cap \bar{V}^{\mathcal{T}_f} = \emptyset$, which is in contradiction with the previous theorem. □

In [13] and [14] we studied also the conditions under which topologies generated by two functions f and g are equal. Clearly, if $f, g \in C_0$ and $f(x) \leq g(x)$ for every $x \in (0,1]$ then for arbitrary measurable subset A of the plane $D_f(A) \subset D_g(A)$ and, consequently, $\mathcal{T}_f \subset \mathcal{T}_g$. Also if $f \in C_0$ and $c \in (0,1]$ then $D_f(A) = D_{cf}(A)$ for $A \in \mathcal{S}$, so $\mathcal{T}_f = \mathcal{T}_{cf}$. More generally, if $f, g \in C_0$, $f(x) \leq g(x)$ for every $x \in (0,1]$ and

$$\liminf_{x \to 0^+} \frac{f(x)}{g(x)} > 0,$$

then $\mathcal{T}_f = \mathcal{T}_g$ ([13], Theorem 17). Also another condition sufficient for the equality $\mathcal{T}_f = \mathcal{T}_g$, described in porosity terms, is given in [13].

In [14] we gave some condition, which is necessary for the equality $\mathcal{T}_f = \mathcal{T}_g$ for arbitrary functions $f, g \in C_0$ such that $g(x) \leq f(x)$ for every $x \in (0,1]$. We proved also that this condition is sufficient if g is a strictly increasing function.

Now we prove that for an arbitrary function $f \in C_0$ the topological space $(\mathbb{R}^2, \mathcal{T}_f)$ is a maximally resolvable and extraresolvable Baire space for which Blumberg's theorem does not hold.

The problems connected with resolvability of some topological spaces were considered by many mathematicians. It was proved that the real line with the Euclidean topology is maximally resolvable ([3]), but not extraresolvable. The real line with the density topology has both these properties. The first property was proved by J. Luukkainen ([10]), and the second by A. Bella ([1]). Also the real line with the $\mathcal{I}$-density topology or with the ψ-density topology is maximally resolvable and extraresolvable, which was proved in [12] for $\mathcal{I}$-density topology and by G. Horbaczewska in [6] for ψ-density topology. Here we shall prove that the plane with the topology $\mathcal{T}_f$ is also simultaneously maximally resolvable and extraresolvable.

Recall the basic notions.

Let $(X, \mathcal{T})$ be a topological space. The *dispersion character of* X is the cardinal number

$$\Delta(X, \mathcal{T}) = \min\{\text{card}(U) : U \in \mathcal{T}, U \neq \emptyset\}.$$

Let α be an arbitrary cardinal number. A topological space is said to be *α-resolvable* if there exists a family of α-many pairwise disjoint subsets of X, each of which intersects each nonempty open subset of X in at least α points.

The space $(X, \mathcal{T})$ is *maximally resolvable* if it is $\Delta(X, \mathcal{T})$-resolvable.

The space $(X, \mathcal{T})$ is *extraresolvable* if there exists a family $\mathcal{D}$ of dense subsets of X such that $\operatorname{card}(\mathcal{D}) > \Delta(X, \mathcal{T})$ and the set $C \cap D$ is nowhere dense for all $C, D \in \mathcal{D}$, $C \neq D$.

As it was mentioned earlier (compare Theorem 5 and Theorem 6 in [13]), $\mathcal{T}_f$ is a topology generated by a lower density operator on $(\mathbb{R}^2, \mathcal{S}, \mathcal{I})$, where $\mathcal{I}$ is a σ-ideal of sets of two-dimensional Lebesgue measure zero. Observe that every set $A \in \mathcal{S} \setminus \mathcal{I}$ contains a nonempty perfect set, as $m_2(A) > 0$ and each Borel uncountable set contains a nonempty perfect subset. So from Theorem 7 in [7] we obtain

Theorem 4 (ZFC)**.** *The space $(\mathbb{R}^2, \mathcal{T}_f)$ is maximally resolvable.*

and from Theorem 8 in [7] it follows

Theorem 5 (ZFC+MA)**.** *The space $(\mathbb{R}^2, \mathcal{T}_f)$ is extraresolvable.*

H. Blumberg proved that if f is a real-valued function defined on the real line, then there exists a dense subset D of $\mathbb{R}$, such that $f_{|D}$ is continuous. J. C. Bradford and C. Goffman in [2] proved that a metric space X is a Baire space if and only if Blumberg's theorem holds. As it was observed earlier, the topological space $(\mathbb{R}^2, \mathcal{T}_f)$ is a Baire space ([13]). Here we shall prove that in our space Blumberg's theorem does not hold.

For this purpose we shall use so called G_δ-insertion property which was considered in [9].

Let τ be a topology on $\mathbb{R}^2$ and let G_δ be, as usual, the family of all countable intersections of open sets (with respect to the Euclidean topology on the plane).

Definition 6. [9] We say that τ *has G_δ-insertion property* if for each set $A \subset \mathbb{R}^2$ there exists a set $B \in G_\delta$ such that

$$\operatorname{Int}_\tau A \subset B \subset \bar{A}^\tau,$$

where $\operatorname{Int}_\tau A$ and $\bar{A}^\tau$ denote the interior and the closure of A in the topology τ, respectively.

Theorem 7. *The topology $\mathcal{T}_f$ has G_δ-insertion property.*

Proof. Let $G \in \mathcal{T}_f$. We shall prove that there exists a set B of type G_δ (with respect to the Euclidean topology) such that

$$G \subset B \subset \bar{G}^{\mathcal{T}_f}.$$

Put

$$B_{nm} = \{(x,y) \in \mathbb{R}^2 : (f(\frac{1}{n}) \leq \frac{1}{m} \leq \frac{1}{n} \text{ or } f(\frac{1}{m}) \leq \frac{1}{n} \leq \frac{1}{m}) \text{ and } m_2(G \cap \{[x-\frac{1}{n}, x+\frac{1}{n}] \times [y-\frac{1}{m}, y+\frac{1}{m}]\}) > \frac{1}{nm}\}$$

for every $n, m \in \mathbb{N}$ and

$$B = \bigcap_{k\in\mathbb{N}} \bigcup_{n,m\geq k} B_{nm}.$$

We shall prove that B_{nm} is open in the Euclidean topology for arbitrary $n, m \in \mathbb{N}$. Let $(x_0, y_0) \in B_{nm}$. We may assume $(x_0, y_0) = (0, 0)$. Then $f(1/n) \leq 1/m \leq 1/n$ or $f(1/m) \leq 1/n \leq 1/m$ and there exists a positive number α such that

$$m_2(G \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) = \frac{1}{nm} + \alpha > \frac{1}{nm}.$$

Put $\delta = \alpha nm/4(n+m)$. Let $p = (x,y) \in (-\delta, \delta) \times (-\delta, \delta)$. Then

$$m_2((G-(x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) >$$
$$> m_2(G \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) - \frac{2\delta}{m} - \frac{2\delta}{n} =$$
$$= \frac{1}{nm} + \alpha - 2\frac{\alpha nm}{4(n+m)} \cdot \frac{n+m}{nm} = \frac{1}{nm} + \frac{\alpha}{2} > \frac{1}{nm},$$

so $p \in B_{nm}$. Hence $(-\delta, \delta) \times (-\delta, \delta) \subset B_{nm}$ and B_{nm} is open. Consequently B is a set of type G_δ in the Euclidean topology.

Now we shall prove that $G \subset B$. Let $p = (x,y) \in G$. Since $G \in \mathcal{T}_f$, so $p \in D_f(G)$. From the definition of D_f it follows that there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$ if $f(1/n) \leq 1/m \leq 1/n$ or $f(1/m) \leq 1/n \leq 1/m$ then

$$\frac{m_2((G-(x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\})}{\frac{4}{nm}} > \frac{1}{4},$$

i.e.

$$m_2((G-(x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) > \frac{1}{nm}.$$

Consequently, for each $k \in \mathbb{N}$ there exist two positive integers $n, m \geq k$ (for example $n = m = \max(k, n_0)$) such that $f(1/n) \leq 1/m \leq 1/n$ or $f(1/m) \leq 1/n \leq 1/m$ and

$$m_2(G \cap \{[x-\frac{1}{n}, x+\frac{1}{n}] \times [y-\frac{1}{m}, y+\frac{1}{m}]\}) > \frac{1}{nm},$$

so $p = (x,y) \in B$.

We show that $B \subset \bar{G}^{\mathcal{T}_f}$. From Theorem 7 in [13]

$$\bar{G}^{\mathcal{T}_f} = G \cup (\mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus G)).$$

Observe that $B \subset \mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus G)$. Let $p = (x, y) \in B$. Then for each $k \in \mathbb{N}$ there exist two positive integers $n, m \geq k$ such that

$$f(\frac{1}{n}) \leq \frac{1}{m} \leq \frac{1}{n} \qquad \text{or} \quad f(\frac{1}{m}) \leq \frac{1}{n} \leq \frac{1}{m}$$

and

$$m_2(G \cap \{[x - \frac{1}{n}, x + \frac{1}{n}] \times [y - \frac{1}{m}, y + \frac{1}{m}]\}) > \frac{1}{nm}.$$

Hence

$$\frac{m_2((\mathbb{R}^2 \setminus G) \cap \{[x - \frac{1}{n}, x + \frac{1}{n}] \times [y - \frac{1}{m}, y + \frac{1}{m}]\})}{\frac{4}{n \cdot m}} < \frac{3}{4},$$

so $p = (x, y) \notin D_f(\mathbb{R}^2 \setminus G)$. Consequently $p \in \mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus G)$. □

Theorem 8 (CH). *The space $(\mathbb{R}^2, \mathcal{T}_f)$ is not a Blumberg space.*

Proof. From (CH) it follows that each subset of the plane dense in the topology $\mathcal{T}_f$ is of cardinality $\mathfrak{c} = 2^{\aleph_0}$. The topology $\mathcal{T}_f$ has G_δ-insertion property and $\text{card}(G_\delta) = \mathfrak{c}$. Using Theorem 4.7 in [9] we obtain that $(\mathbb{R}^2, \mathcal{T}_f)$ is not a Blumberg space. □

Finally, the space $(\mathbb{R}^2, \mathcal{T}_f)$ is a completely regular Baire space, which is both maximally resolvable, extraresolvable and for which Blumberg's theorem does not hold.

In the next theorem we shall prove that the derivative in the topology $\mathcal{T}_f$ of each subset of the plane is a Borel set (exactly of type $G_{\delta\sigma}$) with respect to the Euclidean topology.

Now we shall prove that the result analogous to (1) holds for an arbitrary subset of the plane.

Let $\widetilde{A} \subset A$ be an arbitrary measurable set with the inner measure of $A \setminus \widetilde{A}$ equal to zero. Such a set is usually called a measurable kernel of A.

Lemma 9. *For an arbitrary set $A \subset \mathbb{R}^2$*

$$A^{d_{\mathcal{T}_f}} = \mathbb{R}^2 \setminus D_f(\widetilde{\mathbb{R}^2 \setminus A}).$$

Proof. Let $p \in D_f(\widetilde{\mathbb{R}^2 \setminus A})$. Put

$$U = (\widetilde{\mathbb{R}^2 \setminus A}) \cap D_f(\widetilde{\mathbb{R}^2 \setminus A}) \cup \{p\}.$$

We have $U \sim (\widetilde{\mathbb{R}^2 \setminus A}) \cap D_f(\widetilde{\mathbb{R}^2 \setminus A}) \sim (\widetilde{\mathbb{R}^2 \setminus A})$, so $D_f(U) = D_f(\widetilde{\mathbb{R}^2 \setminus A})$. Consequently $p \in D_f(U)$ and $U \in \mathcal{T}_f$. Simultaneously $(U \cap A) \setminus \{p\} = \emptyset$, so $p \notin A^{d_{\mathcal{T}_f}}$.

Now let $p \notin A^{d_{\mathcal{T}_f}}$. Then there exists the set $U \in \mathcal{T}_f$ such that $p \in U$ and $(U \cap A) \setminus \{p\} = \emptyset$. Then $p \in D_f(U)$ and $U \subset (\mathbb{R}^2 \setminus A) \cup \{p\}$. We shall prove that $p \in D_f(\widetilde{\mathbb{R}^2 \setminus A})$. We have

$$U \subset (\widetilde{\mathbb{R}^2 \setminus A}) \cup (U \setminus (\widetilde{\mathbb{R}^2 \setminus A})) \cup \{p\}.$$

Obviously $U \setminus (\widetilde{\mathbb{R}^2 \setminus A}) \subset [(\mathbb{R}^2 \setminus A) \setminus (\widetilde{\mathbb{R}^2 \setminus A})] \cup \{p\}$, so from the definition of a measurable kernel $m_2(U \setminus (\widetilde{\mathbb{R}^2 \setminus A})) = 0$. Consequently $D_f(U) \subset D_f(\widetilde{\mathbb{R}^2 \setminus A})$ and $p \in D_f(\widetilde{\mathbb{R}^2 \setminus A})$. □

Theorem 10. *If $A \subset \mathbb{R}^2$ then $A^{d\mathcal{T}_f}$ is a set of type $G_{\delta\sigma}$.*

Proof. Let $A \in \mathcal{S}$. From Theorem 2 in [13] it follows that the point $p = (x, y)$ belongs to $D_f(\mathbb{R}^2 \setminus A)$ if and only if for each $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$, such that for each $n, m \geq n_0$ if $f(1/n) \leq 1/m \leq 1/n$ or $f(1/m) \leq 1/n \leq 1/m$, then

$$\frac{m_2((A - (x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\})}{\frac{4}{nm}} \leq \frac{1}{k}.$$

Observe that $p = (x, y) \notin D_f(\mathbb{R}^2 \setminus A)$ if and only if there exists $k \in \mathbb{N}$, such that for each $r \in \mathbb{N}$ there exist $n, m \geq r$, such that $f(1/n) \leq 1/m \leq 1/n$ or $f(1/m) \leq 1/n \leq 1/m$ and

$$m_2((A - (x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) > \frac{4}{knm}.$$

Put

$$B^k_{nm} = \{(x,y) \in \mathbb{R}^2 : (f(\frac{1}{n}) \leq \frac{1}{m} \leq \frac{1}{n} \text{ or } f(\frac{1}{m}) \leq \frac{1}{n} \leq \frac{1}{m}) \\ \text{and } m_2((A - (x,y)) \cap \{[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]\}) > \frac{4}{knm}\}$$

for $k, n, m \in \mathbb{N}$. Analogously as in the proof of Theorem 5 one can prove that B^k_{nm} is open in the Euclidean topology. Simultaneously, from (1) and the above observation we obtain

$$A^{d\mathcal{T}_f} = \bigcup_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcup_{n \geq r} \bigcup_{m \geq r} B^k_{nm},$$

so $A^{d\mathcal{T}_f}$ is a set of type $G_{\delta\sigma}$ for arbitrary measurable subset A of the plane.

Now let $A \notin \mathcal{S}$. From Lemma 9 and (1) we obtain

$$A^{d\mathcal{T}_f} = \mathbb{R}^2 \setminus D_f(\widetilde{\mathbb{R}^2 \setminus A}) = (\mathbb{R}^2 \setminus (\widetilde{\mathbb{R}^2 \setminus A}))^{d\mathcal{T}_f}.$$

Obviously, $\mathbb{R}^2 \setminus (\widetilde{\mathbb{R}^2 \setminus A}) \in \mathcal{S}$. From the first part of the proof $A^{d\mathcal{T}_f}$ is a set of type $G_{\delta\sigma}$. □

Using the last theorem and Exercise 4.B.1 in [9], p. 141, we can again deduce that $(\mathbb{R}^2, \mathcal{T}_f)$ is not a Blumberg space.

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TOPOLOGIE GĘSTOŚCI NA PŁASZCZYŹNIE USYTUOWANE MIĘDZY ZWYCZAJNĄ I SILNĄ TOPOLOGIĄ GĘSTOŚCI. III

S t r e s z c z e n i e

Praca dotyczy rodziny topologii gęstości na płaszczyźnie, które zawierają w sobie silne topologie gęstości i zawarte są w zwyczajnej topologii gęstości oraz generowane są przez pewną podrodzinę rodziny funkcji ciągłych. Przedstawione tu wyniki stanowią kontynuację wcześniejszych badań. W pracy dowodzi się, że płaszczyzna z topologią z rozważanej rodziny jest przestrzenią Baire'a maksymalnie rozkładalną i ekstrarozkładalną, dla której twierdzenie Blumberga nie zachodzi.

Słowa kluczowe: punkt gęstości, topologia gęstości, twierdzenie Blumberga