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## APPLICATION OF JACOBI POLYNOMIALS TO APPROXIMATE SOLUTION OF A COMPLETE SINGULAR INTEGRAL EQUATION WITH CAUCHY KERNEL ON THE REAL HALF-LINE

**Summary**

Jacobi polynomials are used to derive approximate solutions of the complete singular integral equation with Cauchy–type kernel defined on the real half–line in the case of constant complex coefficients. Moreover, estimations of errors of the approximated solutions are presented and proved.

*Keywords and phrases*: Jacobi polynomials, singular integral equations, Cauchy kernel, approximate solutions

## 1. Introduction

In the theory of singular integral equations (SIEs) with Cauchy–type kernels [9, 12, 13, 26, 30] there is considered the following equation:

$$(1)\qquad a(x)\varphi(x) + \frac{1}{\pi i}\int_{-1}^{1} \frac{b(t)\varphi(t)}{t-x}dt + \frac{1}{\pi i}\int_{-1}^{1} k(x,t)\varphi(t)\,dt = f(x), \quad -1 < x < 1,$$
$$a^2(x) - b^2(x) \neq 0, \quad \forall x \in [-1,1].$$

The equation (1) plays a pivotal role in the airfoil theory, aeroelasticity, and other problems of solid and fluid mechanics [3, 4, 16, 22, 27, 28, 37]. A comprehensive survey of applications of SIEs to mechanics can be found in [5]. The first studies on the approximate solutions of this equation appeared in the beginning of the third decade of the previous century and now the number of papers concerning numerical method of solving singular integral equations with Cauchy-type kernels is very large. For the details one can consult the papers [6, 7, 10, 11, 14, 17–21, 23, 38, 41, 43], monographs [2, 8, 15] and citations therein.

In the present paper, we consider the complete singular integral equation

$$a\varphi(x) + \frac{1}{\pi i}\int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma - x}\,d\sigma + \frac{1}{\pi i}\int_0^{+\infty} k(x,\sigma)\varphi(\sigma)\,d\sigma = f(x), \quad x > 0, \tag{2}$$

where $k(x,\sigma)$, $f(x)$ are given complex-valued Hölder continuous functions, the coefficients $a$ and $b$ are given complex numbers satisfying the conditions $a^2 - b^2 \neq 0$, $b \neq 0$, and $\varphi(x)$ is the unknown function. We assume that the behavior of the kernel $k(x,\sigma)$ as $|\sigma| \to \infty$ is described by the relation $k(x,\sigma) = k_0(x,\sigma)(\sigma+1)^{-\alpha}$, $\operatorname{Re}\alpha > 1$, where $k_0(x,\sigma)$ satisfies Hölder condition with respect to both variables.

The solution $\varphi(x), x > 0$, will be sought in the class of Hölder functions, vanishing at infinity and having an integrable singularity in a neighbourhood of $x = 0$ ($h(\infty)$ class) or bounded at $x = 0$ ($h(0,\infty)$ class) (cf. [34, 35]).

The theory of singular integral equations defined on the infinite curve can be slightly different from the theory of corresponding equations on the finite curve, see, eg. the works [1, 9, 24, 29, 33, 35, 36]. Equations of this type occur i.a. in quantum mechanics [31].

The exact solution of dominant equation

$$a\varphi(x) + \frac{1}{\pi i}\int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma - x}d\sigma = f(x), \quad x > 0, \tag{3}$$

which is a special case of equation (3), was presented i.a. in [9, 34]. Moreover, in [34] Jacobi polynomials were applied to derive approximate solutions of (3).

The main objective of the present paper is to build the approximate solutions of (2) in the $h(\infty)$ and $h(0,\infty)$ function classes.

The paper is organized as follows. First we reduce the equation (2) to a Fredholm equation using Carleman-Vekua regularization and we find the conditions for the unique solvability of (2). Next, we present efficient numerical schemes for equation (2) based on Jacobi polynomials and estimate the order of accuracy of the approximate solution.

## 2. Exact solution

In this section, we reduce the singular integral equation (2) to a Fredholm equation by Carleman–Vekua regularization [12, 30, 40]. Setting

$$x = \frac{1+t}{1-t}, \ \sigma = \frac{1+\tau}{1-\tau}, \tag{4}$$

we can present the equation (2) in the form

$$\begin{gathered} a\varphi^*(t) + \frac{b}{\pi i}\int_{-1}^{1} \frac{\varphi^*(\tau)}{\tau - t}\,d\tau - \frac{b}{\pi i}\int_{-1}^{1} \frac{\varphi^*(\tau)}{\tau - 1}\,d\tau + \frac{1}{\pi i}\int_{-1}^{1} k^*(t,\tau)\varphi^*(\tau)\,d\tau = f^*(t), \\ -1 < t < 1, \ \lim_{t\to 1^-}\varphi^*(t) = \lim_{x\to\infty}\varphi(x) = 0, \end{gathered} \tag{5}$$

where

$$\varphi^*(t)=\varphi\left(\frac{1+t}{1-t}\right),\ f^*(t)=f\left(\frac{1+t}{1-t}\right),k^*(t,\tau)=2\,k\left(\frac{1+t}{1-t},\frac{1+\tau}{1-\tau}\right)(1-\tau)^{-2}.$$

Passing in (5) to the new unknown function $u(t)$ by the rule $\varphi^*(t)=\frac{Z(t)}{a^2-b^2}u(t)$, and introducing notation $A=\frac{a}{a^2-b^2}$, $B=\frac{b}{a^2-b^2}$, we reduce Eq. (5) to the form

$$\begin{aligned}&AZ(t)u(t)+\frac{1}{\pi i}\int_{-1}^{1}\frac{BZ(\tau)u(\tau)}{\tau-t}\,d\tau-\frac{1}{\pi i}\int_{-1}^{1}\frac{BZ(\tau)u(\tau)}{\tau-1}\,d\tau\\ &+\frac{1}{\pi i}\int_{-1}^{1}k^*(t,\tau)\frac{Z(\tau)}{a^2-b^2}u(\tau)\,d\tau=f^*(t),\quad t\in(-1,1).\end{aligned}\tag{6}$$

Note that $k^*(t,\tau)$ as $\tau\to 1$ has the form

$$k^*(t,\tau)=2^{1-\alpha}k\left(\frac{1+t}{1-t},\frac{1+\tau}{1-\tau}\right)(1-\tau)^{\alpha-2}.\tag{7}$$

Function $Z(t)$ depends on the function class in which we are looking for its solution. If the solution $\varphi^*(t)$ belongs to $h(1)$ function class [30] (i.e. the class of Hölder continuous functions on $(-1,1)$, bounded in a neighbourhood of the point $z=1$, and admitting an integrable singularity at the point $z=-1$), then for $0<\theta<\pi$ we have (cf. [34])

$$Z(t)=\sqrt{a^2-b^2}\,(1-t)^{\alpha}(1+t)^{\beta},\quad \alpha=\omega_1+i\omega_2,\quad \beta=-\omega_1-i\omega_2,\tag{8}$$

and for $-\pi<\theta<0$ we obtain

$$Z(t)=-\sqrt{a^2-b^2}\,(1-t)^{\alpha}(1+t)^{\beta},\ \alpha=1+\omega_1+i\omega_2,\ \beta=-1-\omega_1-i\omega_2,\tag{9}$$

where

$$G=\frac{a-b}{a+b},\quad \theta=\arg G,\quad \omega_1=\frac{\theta}{2\pi}\quad\text{and}\quad \omega_2=-\frac{1}{2\pi}\ln|G|.$$

The index of the characteristic operator is equal to $\kappa=-(\alpha+\beta)=0$.

If the solution $\varphi^*(t)$ is sought in the class $h(-1,1)$ [30] (i.e. the class of Hölder continuous functions on $(-1,1)$, bounded at neighbourhoods of points $z=\pm1$), then for $0<\theta<\pi$ we have (cf. [34])

$$Z(t)=\sqrt{a^2-b^2}\,(1-t)^{\alpha}(1+t)^{\beta},\quad \alpha=\omega_1+i\omega_2,\quad \beta=1-\omega_1-i\omega_2,\tag{10}$$

and for $-\pi<\theta<0$ we obtain

$$Z(t)=-\sqrt{a^2-b^2}\,(1-t)^{\alpha}(1+t)^{\beta},\ \alpha=1+\omega_1+i\omega_2,\ \beta=-\omega_1-i\omega_2.\tag{11}$$

In this case, the index $\kappa$ is equal to $\kappa=-(\alpha+\beta)=-1$.

Moving the regular term of Eq. (6) to the right-hand side and then solving it as a dominant equation in $h(1)$ class (cf. [34]), after a few elementary transformations

Eq. (6) can be reduced to the Fredholm equation of the form

$$u(t) + \int_{-1}^{1} N(t,\tau_1) u(\tau_1)\, d\tau_1 = F(t), \tag{12}$$

where

$$N(t,\tau_1) = \frac{1}{\pi i}\frac{Z(\tau_1)}{Z(t)\,(a^2-b^2)}\left[ a k^*(t,\tau_1) - \frac{Z(t)}{\pi i}\int_{-1}^{1} \frac{b}{Z(\tau)}\frac{k^*(\tau,\tau_1)}{\tau - t}\, d\tau \right],$$

$$F(t) = \frac{a}{Z(t)} f^*(t) - \frac{1}{\pi i}\int_{-1}^{1} \frac{b}{Z(\tau)}\frac{f^*(\tau)}{\tau - t}\, d\tau + \gamma_0,$$

where $\gamma_0$ is an arbitrary complex constant. The theory of Fredholm equations is valid for Eq. (12) ([12]). Therefore, if the corresponding equation with $F(t) \equiv 0$ is unsolvable (has only the trivial solution), then the solution of the non homogeneous equation (12) is equal to

$$u(t) = F(t) - \int_{-1}^{1} \Gamma(t,\tau) F(\tau)\, d\tau, \qquad t \in (-1,1), \tag{13}$$

where $\Gamma(t,\tau)$ is the resolvent of the kernel $N(t,\tau)$.

The constant $\gamma_0$ will be uniquely determined if we supplement the equation (6) by the following orthogonality condition

$$\frac{1}{\pi i}\int_{-1}^{1} B Z(\tau)\frac{u(\tau)}{\tau - 1}\, d\tau = A_0^*, \tag{14}$$

where $A_0^*$ is a given complex number. Indeed, taking into account the Poincaré-Bertrand formula [9, 12] and the relation [30, p.140]

$$\Gamma(t,\tau) = N(t,\tau) + \int_{-1}^{1} N(t,\tau_3)\Gamma(\tau_3,\tau)\, d\tau_3,$$

by a straightforward substitution we verify that $\gamma_0 = A_0^*$.

**Remark 1.** *The same result can be retrieved by defining*

$$\varphi^*(t) = \varphi\left(\frac{1+t}{1-t}\right)\frac{1}{1-t},\ f^*(t) = f\left(\frac{1+t}{1-t}\right)\frac{1}{1-t}$$

*and considering the Muskhelishvilli class $h_0$ (index $\kappa = 1$).*

In $h(-1,1)$ class ($\kappa < 0$), Eq. (6) is solvable if and only if

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{b}{Z(\tau)}\left[ f^*(\tau) - \frac{1}{\pi i}\int_{-1}^{1} k^*(\tau,\tau_1)\frac{Z(\tau_1)}{a^2-b^2} u(\tau_1)\, d\tau_1 + A_0^* \right] d\tau = 0,$$

where $A_0^*$ is given by (14). Since the above condition is equivalent to the relation

$$\begin{aligned}\frac{1}{\pi i}\int_{-1}^{1}\frac{b}{Z(\tau)}f^*(\tau)\,d\tau &= \frac{1}{\pi i}\int_{-1}^{1}BZ(\tau)\,\frac{u(\tau)}{\tau-1}\,d\tau\\ &+\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{b}{Z(\tau)}\frac{Z(\tau_1)}{a^2-b^2}\,k^*(\tau,\tau_1)\,u(\tau_1)\,d\tau_1\,d\tau,\end{aligned}\tag{15}$$

Eq.(6) can be rewritten in the form

$$\begin{aligned}AZ(t)u(t)&+\frac{1}{\pi i}\int_{-1}^{1}\frac{BZ(\tau)u(\tau)}{\tau-t}\,d\tau+\frac{1}{\pi i}\int_{-1}^{1}k^*(t,\tau)\frac{Z(\tau)}{a^2-b^2}\,u(\tau)\,d\tau\\ &=f^*(t)+\frac{b}{\pi i}\int_{-1}^{1}\frac{f^*(\tau)\,d\tau}{Z(\tau)}-\frac{b}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{Z(\tau_1)}{Z(\tau)}\frac{u(\tau_1)}{a^2-b^2}\,k^*(\tau,\tau_1)\,d\tau_1\,d\tau.\end{aligned}\tag{16}$$

Then, in the same way as in the previous case, one can pass from Eq.(16), to Fredholm Eq.(12), in whose right-hand side one should set $\gamma_0=0$.

In the original variables $x,\sigma$, the conditions (14) and (15) acquire the following forms

$$\frac{1}{\pi i}\int_{0}^{+\infty}\frac{b\varphi(\sigma)}{\sigma+1}d\sigma=A_0^*,\tag{17}$$

where $A_0^*$ is an arbitrary number, and

$$\begin{aligned}&\frac{\operatorname{sgn}(\theta)}{\sqrt{a^2-b^2}}\frac{1}{\pi i}\int_{0}^{+\infty}\frac{f(\sigma)}{\sigma^\beta}\frac{d\sigma}{\sigma+1}\\ &=-\frac{1}{\pi i}\int_{0}^{+\infty}\varphi(\sigma)\frac{d\sigma}{\sigma+1}+\frac{\operatorname{sgn}(\theta)}{\sqrt{a^2-b^2}}\frac{1}{(\pi i)^2}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{k(\sigma,\sigma_1)}{\sigma^\beta}\frac{\varphi(\sigma)}{\sigma+1}\,d\sigma_1 d\sigma,\end{aligned}\tag{18}$$

where $\alpha,\beta$ are specified in (10) or (11) depending on $\theta=\arg\frac{a-b}{a+b}$, respectively.

We have thereby proved the following assertion.

**Theorem 2.1.** *Suppose that the coefficients $a,b$ occurring in Eq.* (6) *are given complex numbers such that $a^2-b^2\neq 0$, $b\neq 0$, the functions $f^*(t)$, $k^*(t,\tau)$ belong to the Hölder class ($k^*(t,\tau)$ with respect to both variables), and the kernel $k^*(t,\tau)$ can be represented in the form* (7) *in a neighbourhood of $\tau=1$.*

*If the index of* (6) *is equal to zero ($\kappa=0$) and the homogeneous equation* (12) *is unsolvable, then the problem* (6), (14) *has a unique solution.*

*If $\kappa=-1$, then Eq.* (6) *is equivalent (in the sense of solvability) to the Fredholm equation* (12) *with $\gamma_0\equiv 0$, supplemented with the condition* (15).

## 3. Approximate solution

Now, we will find an approximate solution $u_n(t)$ of the problem (6), (14) in $h(1)$ class ($\kappa = 0$). For this purpose, we interpolate the function $f^*(t)$ at Chebyshev nodes

$$t_k = \cos\frac{(2k-1)\,\pi}{2\,(n+1)}, \quad k = 1, 2, ..., n+1, \tag{19}$$

by polynomial $f_n^*(t)$ of degree $n$ of the form (cf. [32])

$$f_n^*(t) = \frac{2}{n+1}\sum_{j=0}^{n}\delta_j\left(\sum_{k=1}^{n+1}T_j(t_k)\,f^*(t_k)\right)T_j(t), \quad \delta_j = \begin{cases} 1, & j=0,\\ 2, & j>0,\end{cases} \tag{20}$$

where $T_j(t) = \cos(j\arccos t)$ are Chebyshev polynomials of the first kind. By expressing Chebyshev polynomials $T_j(t)$ in terms of Jacobi polynomials [42], we obtain

$$T_j(t) = \sum_{l=0}^{j}\rho_{jl}P_l^{(-\alpha,-\beta)}(t), \tag{21}$$

where

$$\rho_{jl} = \frac{1}{h_l^{(-\alpha,-\beta)}}\frac{1}{\pi}\int_{-1}^{1}q(t)\,T_j(t)\,P_l^{(-\alpha,-\beta)}(t)\,dt$$

$$= (-1)\frac{1}{h_l^{(-\alpha,-\beta)}}\frac{1}{\sin\pi\alpha}\operatorname*{Res}_{z=\infty}\left\{(z-1)^{-\alpha}(z+1)^{-\beta}T_j(z)\,P_l^{(-\alpha,-\beta)}(z)\right\}, \tag{22}$$

$$q(t) = (1-t)^{-\alpha}(1+t)^{-\beta}, \qquad \alpha+\beta = 0,$$

$$h_l^{(-\alpha,-\beta)} = \frac{1}{\pi}\int_{-1}^{1}q(t)\left[P_l^{(-\alpha,-\beta)}(t)\right]^2 dt = \frac{2\Gamma(l-\alpha+1)\,\Gamma(l-\beta+1)}{(2l+1)\pi l!\Gamma(l+1)}. \tag{23}$$

Using (21), the interpolation polynomial (20) takes the form

$$f_n^*(t) = \sum_{k=0}^{n}f_k^*P_k^{(-\alpha,-\beta)}(t), \tag{24}$$

where

$$f_k^* = \frac{2}{n+1}\sum_{j=k}^{n}\delta_j\left(\sum_{i=1}^{n+1}T_j(t_i)\,f^*(t_i)\right)\rho_{jk}, \quad k = 0, 2, ..., n.$$

Next, we interpolate the regular kernel $k(t,\tau)$ at Chebyshev nodes (19) with the polynomial $k_{nn}^*(x,t)$ of the form

$$k_{nn}^*(t,\tau) = \sum_{j=0}^{n}\sum_{p=0}^{n}k_{jp}^*\,P_j^{(-\alpha,-\beta)}(t)P_p^{(\alpha,\beta)}(\tau), \tag{25}$$

where

$$k_{jp}^* = \sum_{l=j}^{n}\sum_{m=p}^{n}\alpha_{lm}\,\rho_{lj}^{(-\alpha,-\beta)}\rho_{mp}^{(\alpha,\beta)},$$

coefficients $\rho_{lj}^{(-\alpha,-\beta)}$, $\rho_{mp}^{(\alpha,\beta)}$ are given by (22), and $\alpha_{lm}$ are defined as follows

$$\alpha_{lm} = \frac{\delta_l}{n+1}\sum_{k=1}^{n+1} T_l(t_k)\left\{\frac{\delta_m}{n+1}\sum_{r=1}^{n+1} T_m(\tau_r)k^*(t_k,\tau_r)\right\}.$$

As in this case, by (6) we have

$$\frac{1}{\pi i}\int_{-1}^{1} k^*(1,\tau)\frac{Z(\tau)}{a^2-b^2}\,u(\tau)\,d\tau = f^*(1). \tag{26}$$

Therefore the approximate solution

$$u_n(t) = \sum_{k=0}^{n} c_k P_k^{(\alpha,\beta)}(t), \tag{27}$$

of the problem (6), (14) can be defined as the solution of the following equation

$$\begin{aligned} &AZ(t)u_n(t) + \frac{1}{\pi i}\int_{-1}^{1}\frac{BZ(\tau)u_n(\tau)d\tau}{\tau - t} \\ &+ \frac{1}{\pi i}\int_{-1}^{1}(k_{nn}^*(t,\tau) - k_{nn}^*(1,\tau))\,\frac{Z(\tau)u_n(\tau)\,d\tau}{a^2-b^2} = f_n^*(t) - f_n^*(1) + A_0^*, \end{aligned} \tag{28}$$

where $A_0^*$ is a given complex number, coefficients $\alpha$, $\beta$ are defined in (8) for $0<\theta<\pi$ and in (9) for $-\pi<\theta<0$.

Substituting (27), (25), and (24) into (28) and using formula [23, 34]

$$AZ(x)P_k^{(\alpha,\beta)}(x) + \frac{B}{\pi i}\int_{-1}^{1} Z(t)\frac{P_k^{(\alpha,\beta)}(t)}{t-x}dt = P_k^{(-\alpha,-\beta)}(x),\ \alpha+\beta=0,\ k\geq 0,$$

for the computation of the singular integrals, we obtain

$$\begin{aligned} \sum_{k=0}^{n} c_k P_k^{(-\alpha,-\beta)}(t) + \sum_{l=0}^{n}\left(\sum_{m=0}^{n} k_{lm}^* h_m\right)\left(P_l^{(-\alpha,-\beta)}(t) - P_l^{(-\alpha,-\beta)}(1)\right) \\ = \sum_{k=0}^{n} f_k^* P_k^{(-\alpha,-\beta)}(t) - f_n^*(1) + A_0^*, \end{aligned}$$

where

$$\begin{aligned} h_m &= \frac{1}{\pi i}\int_{-1}^{1}\frac{Z(\tau)}{a^2-b^2}\left(P_m^{(\alpha,\beta)}(\tau)\right)^2 d\tau = \frac{\operatorname{sgn}(\arg(\theta))}{i\sqrt{a^2-b^2}}\,h_l^{(\alpha,\beta)} \\ &= \frac{\operatorname{sgn}(\arg(\theta))}{i\sqrt{a^2-b^2}\sin\alpha\pi}\operatorname*{Res}_{z=\infty}\left((z-1)^{\alpha}(z+1)^{\beta}\left(P_m^{(\alpha,\beta)}(z)\right)^2\right). \end{aligned} \tag{29}$$

Thus, coefficients $c_k$, $k = 0, 1, \dots, n$, of the solution $u_n(t)$ can be determined from the following system of linear equations

$$\begin{gathered} c_i + \sum_{j=0}^{n} c_j\, k^*_{ij}\, h_j = f^*_i, \quad i = 1, 2, \dots, n, \\ c_0 + \sum_{j=0}^{n} c_j\, k^*_{0j}\, h_j - \sum_{l=0}^{n} \sum_{j=0}^{n} c_j k^*_{lj} h_j P_l^{(-\alpha,-\beta)}(1) = f^*_0 + A_0 - f^*_n(1). \end{gathered} \tag{30}$$

If $\kappa < 0$, then the approximate solution

$$u_{n-1}(t) = \sum_{k=0}^{n-1} c_k P_k^{(\alpha,\beta)}(t), \tag{31}$$

of (16) in the class $h(-1,1)$ may be found as a solution of the following problem

$$\begin{gathered} AZ(t)u_{n-1}(t) + \frac{1}{\pi i}\int_{-1}^{1} BZ(\tau)\frac{u_{n-1}(\tau)}{\tau - t}d\tau + \frac{1}{\pi i}\int_{-1}^{1} \frac{k^*_{n,n-1}(t,\tau)}{a^2 - b^2} Z(\tau)u_{n-1}(\tau)d\tau \\ = f^*_n\,(t) + \frac{b}{\pi i}\int_{-1}^{1}\frac{f^*_n\,(\tau)}{Z\,(\tau)}d\tau - \frac{b}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{Z(\tau_1)}{a^2 - b^2}\frac{k^*_{n,n-1}(\tau,\tau_1)}{Z\,(\tau)}\,u_{n-1}(\tau_1)d\tau_1 d\tau, \end{gathered} \tag{32}$$

where $f^*_n(t)$ is defined by the rule (24). Moreover, $k^*_{n,n-1}(t,\tau)$ is the interpolation polynomial of the kernel $k^*(t,\tau)$ with Chebyshev nodes

$$\tau_l^{'} = \cos\frac{(2l-1)\pi}{2n},\ l = 1, \dots, n,\ \ t_k = \cos\frac{(2k-1)\,\pi}{2\,(n+1)},\ k = 1, \dots, n+1, \tag{33}$$

and it has the form

$$k^*_{n,n-1}(t,\tau) = \sum_{j=0}^{n}\sum_{p=0}^{n-1} k^*_{jp}\, P_j^{(-\alpha,-\beta)}(t) P_p^{(\alpha,\beta)}(\tau), \tag{34}$$

where

$$k^*_{jp} = \sum_{l=j}^{n}\sum_{m=p}^{n-1} \alpha_{lm}\, \rho_{lj}^{(-\alpha,-\beta)}\, \rho_{mp}^{(\alpha,\beta)},$$

$$\alpha_{lm} = \frac{\delta_l}{n+1}\sum_{k=1}^{n+1} T_l(t_k)\left\{\frac{\delta_m}{n}\sum_{r=1}^{n} T_m(\tau_r^{'})k^*(t_k,\tau_r^{'})\right\},\ \delta_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, ..., \end{cases}$$

and $\rho_{lj}^{(-\alpha,-\beta)}, \rho_{mp}^{(\alpha,\beta)}$ are defined in the same way as in (22). Let us stress that in this case taking into account the relation (26) doesn't have an influence on final algorithm.

Substituting (31), (34) and (24) into (32), and using formula [23, 34]

$$AZ\,(x)P_k^{(\alpha,\beta)}(x) + \frac{B}{\pi i}\int_{-1}^{1} Z\,(t)\,\frac{P_k^{(\alpha,\beta)}(t)}{t - x}dt = 2P_{k+1}^{(-\alpha,-\beta)}(x)\,,\ \alpha+\beta = 1,\ k \geq 0,$$

we obtain

$$2\sum_{k=0}^{n-1} c_k P_{k+1}^{(-\alpha,-\beta)}(t) + \sum_{l=1}^{n}\left(\sum_{m=0}^{n-1} k_{lm}^{*}\, c_m\, h_m\right) P_l^{(-\alpha,-\beta)}(t) = \sum_{k=1}^{n} f_k^{*} P_k^{(-\alpha,-\beta)}(t),$$

where $h_m$ are defined in the same way as in (29). The coefficients $\alpha$, $\beta$ and the function $Z(t)$ occurring in $h_m$ are given by (10) for $0 < \theta = \arg \frac{a-b}{a+b} < \pi$ and by (11) for $-\pi < \theta < 0$. Comparing the coefficients at $P_k^{(-\alpha,-\beta)}(t)$, $k = 1, \ldots, n$, we derive the system

$$2c_{i-1} + \sum_{j=0}^{n-1} c_j\, k_{ij}^{*} h_j = f_i^{*}, \quad i = 1, \ldots, n. \tag{35}$$

## 4. Numerical examples

Let us find the solution the following problem

$$\begin{gathered}(1+i)\varphi(x) + \frac{1-i}{\pi i}\int_0^{+\infty} \frac{\varphi(\sigma)\, d\sigma}{\sigma - x} - \frac{2}{\pi i}\int_0^{+\infty} \frac{\varphi(\sigma)\, d\sigma}{(x+1)(\sigma+1)(\sigma+2)} = \frac{1}{2x+3},\\ \frac{1}{\pi i}\int_0^{+\infty} \frac{(1-i)\,\varphi(\sigma)}{\sigma+1}\, d\sigma = -1,\end{gathered} \tag{36}$$

in the $h(\infty)$ class ($\kappa = 0$). Using the above substitutions (4) the problem (36) acquires the form

$$\frac{1-i}{4} Z(t)u(t) - \frac{1+i}{4\pi i}\int_{-1}^{1} \frac{Z(\tau)u(\tau)}{\tau - t}\, d\tau = \frac{t-1}{t-5} - \frac{1}{\pi i}\int_{-1}^{1} \frac{Z(\tau)}{a^2-b^2}\,\frac{1-t}{\tau-3}\, u(\tau)\, d\tau - 1,$$

where

$$Z(t) = \sqrt{2}(1+i)(1-t)^{1/4}(1+t)^{-1/4}.$$

By solving the above equation in the $h(1)$ class (cf. [34]), we obtain

$$\Gamma(t,\tau) = \frac{1}{\pi i}\,\frac{1-2t}{2\left(\frac{5(1+i)}{4\sqrt[4]{2}} - i\right)}\,\frac{Z(\tau)}{4i}\,\frac{1}{\tau-3}, \qquad Z(t) = (\sqrt{2} + i\sqrt{2})\sqrt[4]{\frac{1-t}{1+t}}.$$

Since

$$u(t) = \frac{2t-1}{2}\,\frac{\sqrt[4]{3}-\sqrt{2}}{\sqrt{2}}\,\frac{(1+i)\sqrt[4]{2}}{5 + i\left(1 - 4\sqrt[4]{2}\right)} + 4\left(\frac{3}{2}\right)^{\frac{1}{4}}\frac{1}{t-5},$$

or, which is the same,

$$\varphi(x) = -i\frac{e^{\frac{i\pi}{4}}}{4\sqrt[4]{x}}\,\frac{x-3}{x+1}\,\frac{\sqrt[4]{3}-\sqrt{2}}{\sqrt{2}}\,\frac{(1+i)\sqrt[4]{2}}{5 + i\left(1 - 4\sqrt[4]{2}\right)} + i\frac{e^{\frac{i\pi}{4}}}{\sqrt[4]{x}}\left(\frac{3}{2}\right)^{\frac{1}{4}}\frac{x+1}{2x+3}.$$

The values of $u(t)$, $u_n(t)$ for $n = 10$ are shown in Table 1. In Table 2 we tabulate the values of the exact and approximated solutions of the problem (36) in the $h(\infty)$ class at the points $x$ corresponding to $t$ from the previous table.

Tab. 1: Comparison of the values of $u(t)$, $u_n(t)$

| $t$ | $u(t)$ | $u_n(t)$ |
|---|---|---|
| -0.717 | -0.690070196863+0.076314953186i | -0.690070196857+0.076314953184i |
| 0.0016 | -0.851205627507+0.031230292033i | -0.851205627521+0.031230292033i |
| -0.956 | -0.642559577943+0.091272157657i | -0.642559577944+0.091272157655i |
| 0.185 | -0.897813651028+0.019688117490i | -0.897813651035+0.019688117489i |

Tab. 2: Comparison of the values of exact and approximated solutions of (36)

| $x$ | $\varphi(x)$ | $\varphi_n(x)$ |
|---|---|---|
| 41.37 | -0.160791967205+0.152881347804i | -0.160791967212+0.152881347811i |
| 7.916 | -0.228530579779+0.221248009243i | -0.228530579772+0.221248009237i |
| 4.787 | -0.248345980190+0.243720960213i | -0.248345980195+0.243720960218i |
| 0.020 | -0.513057865584+0.683492017265i | -0.513057865587+0.683492017263i |

## 5. Estimation of errors

In this section, we determine the estimates of errors of the approximate solutions of (5). In order to investigate proposed algorithms, we need the following theorem presented in [40].

**Theorem 5.1.** *Let us consider two Fredholm equations, namely the exact*

$$K(\varphi;x) \equiv \varphi(x) + \int_{-1}^{1} k(x,t)\varphi(t)dt = f(x), \quad -1 \le x \le 1 \tag{37}$$

*and the approximate*

$$K_n(\varphi_n;x) \equiv \varphi_n(x) + \int_{-1}^{1} k_n(x,t)\varphi_n(t)dt = f_n(x), \quad -1 \le x \le 1, \tag{38}$$

*where $k(x,t), k_n(x,t), f(x)$ and $f_n(x)$ are given continuous functions (the first two can have an integrable singularity at $\pm 1$).*

*In addition, suppose that the homogeneous equation (37) is not solvable (i.e. it has only the zero solution) and $\max\limits_{-1\le x\le 1} \int_{-1}^{1} |\gamma(x,t)|\,dt \le \rho$, where $\gamma(x,t)$ is the resolvent of the kernel $k(x,t)$.*

*If the following condition $\varepsilon_1 B < 1$ is satisfied, where*

$$\varepsilon_1 = \max_x \int_{-1}^{1}\int_{-1}^{1} |k(x,t) - k_n(x,t)|\,|k_n(t,\tau)|\,d\tau dt, \quad B = 1 + \rho,$$

*then the non homogeneous equation* (38) *is solvable and*

$$\left\|K_n^{-1}\right\|_\infty \le \frac{1+BK_1}{1-\varepsilon_1 B}, \quad where \quad K_1 = \max_x \int_{-1}^{1} |k_{n,n}(x,t)|\,dt.$$

*Besides,*

$$\|\varphi - \varphi_n\|_\infty \le \frac{1+BK_1}{1-\varepsilon_1 B}\left[\varepsilon_2 B\,\|f\|_\infty + \varepsilon_3\right],$$

*where*

$$\varepsilon_2 = \max_x \int_{-1}^{1} |k(x,t) - k_n(x,t)|\,dt, \quad \varepsilon_3 = \|f - f_n\|_\infty.$$

**Theorem 5.2.** *Let us suppose that the Hölder continuous functions* $f^*(t)$, $k^*(t,\tau)$ *occurring in Eq.* (6) *are approximated by the polynomials* $f_n^*(t)$ *and* $k_{n,n}^*(t,\tau)$ *defined as in* (24) *and* (25), *respectively, and that the homogeneous equation corresponding to* (12) *is not solvable.*

*Then the system of linear equations* (30) *is solvable for sufficiently large* $n$, *and the estimate*

$$\|(1-t)^\alpha(u(t) - u_n(t))\|_\infty \le M\frac{\ln^3 n}{n^\mu}, \quad \operatorname{Re}\alpha > 0, \tag{39}$$

*holds, where* $M$ *is a constant independent of* $n$.

*Proof.* The system (30) and the problem (28) are simultaneously solvable or unsolvable. As follows from the above-performed considerations, the latter is equivalent to the solvability of the equation

$$(1-t)^\alpha u_n(t) + \int_{-1}^{1} N_n(t,\tau)(1-\tau)^\alpha u_n(\tau)\,d\tau = F_n(t), \quad \operatorname{Re}\alpha > 0, \tag{40}$$

where

$$N_n(t,\tau) = \frac{(1-t)^\alpha}{(1-\tau)^\alpha}\frac{1}{\pi i}\frac{Z(\tau)}{(a^2-b^2)}\left[\frac{ak_{nn}^*(t,\tau)}{Z(t)} - \frac{b}{\pi i}\int_{-1}^{1}\frac{k_{nn}^*(\tau_1,\tau)\,d\tau_1}{Z(\tau_1)(\tau_1 - t)}\right],$$

$$F_n(t) = (1-t)^\alpha\left\{\frac{a}{Z(t)}f_n^*(t) - \frac{1}{\pi i}\int_{-1}^{1}\frac{b}{Z(\tau)}\frac{f_n^*(\tau)}{\tau - t}\,d\tau + \gamma_0\right\}.$$

By Theorem 5.1, to establish the solvability of Eq.(40), it suffices to estimate

$$\varepsilon_1 = \max_{-1<t<1}\int_{-1}^{1}\int_{-1}^{1}|N(t,\tau) - N_n(t,\tau)|\,|N_n(\tau,\tau_1)|\,d\tau d\tau_1.$$

Since in the $h(1)$ class we have

$$Z(t) = \pm\sqrt{a^2-b^2}\,(1-t)^{\alpha}\,(1+t)^{\beta}, \quad 0<\operatorname{Re}\alpha<1, \ \ -1<\operatorname{Re}\beta<0,$$

one can apply the estimates from [39] in order to get

$$\|k(t,\tau)-k_{nn}(t,\tau)\|_{\infty} \le M_1\frac{\ln^2 n}{n^{\mu}},$$

$$\left|\frac{1}{\pi i}\int_{-1}^{1}\frac{b}{Z(\tau_1)}\left[k(\tau_1,\tau)-k_{nn}(\tau_1,\tau)\right]\frac{d\tau_1}{\tau_1-t}\right| \le \frac{M_2}{(1-t)^{\operatorname{Re}\alpha}}\,\frac{\ln^3 n}{n^{\mu}}.$$

Consequently, we have

$$|N(t,\tau)-N_n(t,\tau)| = \left|\frac{(1-t)^{\alpha}Z(\tau)}{\pi i\,(a^2-b^2)\,Z(t)}\left\{a\left[k(t,\tau)-k_{nn}(t,\tau)\right]\right.\right.$$
$$\left.\left.-\frac{Z(t)}{\pi i}\int_{-1}^{1}\frac{b}{Z(\tau_1)}\left[k(\tau_1,\tau)-k_{nn}(\tau_1,\tau)\right]\frac{d\tau_1}{\tau_1-t}\right\}\right| \le M(1+\tau)^{\operatorname{Re}\beta}\,\frac{\ln^3 n}{n^{\mu}},$$

where $M, M_1, M_2$ are constants independent of $n$.

Hence it follows that $\varepsilon_1 \le M\frac{\ln^3 n}{n^{\mu}}$. Thus, the Eq. (40) is solvable for sufficiently large $n$.

Using the following estimations

$$K_1 = \max_t\int_{-1}^{1}|N_n(t,\tau)|\,d\tau = \max_t\int_{-1}^{1}|[N_n(t,\tau)-N(t,\tau)]+N(t,\tau)|\,d\tau = O(1),$$

$$\varepsilon_2 = \max_t\int_{-1}^{1}|N(t,\tau)-N_n(t,\tau)|\,d\tau \le M\frac{\ln^3 n}{n^{\mu}},$$

$$\varepsilon_3 = \|F(t)-F_n(t)\|_{\infty} \leqslant M\frac{\ln^2 n}{n^{\mu}},$$

we finish the proof of (39). □

**Theorem 5.3.** *Let the Hölder continuous functions* $f^*(t), k^*(t,\tau)$ *be approximated by polynomials* $f_n^*(t)$ *and* $k_{n,n-1}^*(t,\tau)$ *defined as in* (24) *and* (34), *respectively, and the let the homogeneous equation* (12) *with* $\gamma_0 = 0$ *be not solvable.*

*Then the system of linear equations* (35) *is solvable for sufficiently large* $n$ *and the estimate*

$$\|Z(t)(u(t)-u_{n-1}(t))\|_{\infty} \le M\frac{\ln^3 n}{n^{\mu}}, \tag{41}$$

*holds, where* $M$ *is a constant independent of* $n$.

*Proof.* We have to estimate the value

$$\varepsilon_1^* = \max_{-1<t<1} \int_{-1}^{1} \int_{-1}^{1} \left| N^*(t,\tau) - N_{n-1}^*(t,\tau) \right| \left| N_{n-1}^*(\tau,\tau_1) \right| d\tau d\tau_1,$$

where

$$N_{n-1}^*(t,\tau) = \frac{1}{\pi i} \frac{1}{(a^2-b^2)} \left[ a\, k_{n,n-1}^*(t,\tau) - Z(t) \frac{b}{\pi i} \int_{-1}^{1} \frac{k_{n,n-1}^*(\tau_1,\tau)\, d\tau_1}{Z(\tau_1)(\tau_1 - t)} \right].$$

Since $Z(t) = \pm\sqrt{a^2-b^2}\,(1-t)^{\alpha}\,(1+t)^{\beta}$, $0 < \mathrm{Re}\alpha, \mathrm{Re}\beta < 1$, in the $h(-1,1)$ class, it follows from [39] that

$$\left| N^*(t,\tau) - N_{n-1}^*(t,\tau) \right| \le M_2 \frac{\ln^3 n}{n^{\mu}}.$$

Thus, for sufficiently large $n$, the equation

$$Z(t)u_{n-1}(t) + \int_{-1}^{1} N_{n-1}^*(t,\tau) Z(\tau) u_{n-1}(\tau)\, d\tau = F_n^*(t), \quad -1 < t < 1,$$

where $F_n^*(t) = a\, f_n^*(t) - \frac{Z(t)}{\pi i} \int_{-1}^{1} \frac{b}{Z(\tau)} \frac{f_n^*(\tau)}{\tau - t}\, d\tau$, is solvable.

Further, using the estimates

$$K_1 = \max_t \int_{-1}^{1} \left| N_{n-1}^*(t,\tau) \right| d\tau = O(1),$$

$$\varepsilon_2 = \max_t \int_{-1}^{1} \left| N^*(t,\tau) - N_{n-1}^*(t,\tau) \right| d\tau \le M \frac{\ln^3 n}{n^{\mu}},$$

$$\varepsilon_3 = \left\| F^*(t) - F_n^*(t) \right\|_{\infty} \leqslant M \frac{\ln^2 n}{n^{\mu}},$$

we finish the proof of estimate (41). □

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## ZASTOSOWANIE WIELOMIANÓW JACOBIEGO DO ROZWIĄZANIA ZUPEŁNEGO MOCNO OSOBLIWEGO RÓWNANIA CAŁKOWEGO Z JĄDREM CAUCHY'EGO NA NIEUJEMNEJ PÓŁOSI RZECZYWISTEJ

S t r e s z c z e n i e

W pracy skonstruowane zostały algorytmy przybliżonego rozwiązania zupełnego mocno osobliwego równania całkowego z jądrem Cauchy'ego na nieujemnej półosi rzeczywistej o stałych współczynnikach zespolonych z użyciem wielomianów Jacobiego. Ponadto wyznaczone zostały oszacowania błędów przedstawionych rozwiązań przybliżonych.

*Słowa kluczowe*: wielomiany Jacobiego, mocno osobliwe równania całkowe, jądro Cauchy'ego, rozwiązanie przybliżone