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Marta Dudek

INVARIANTS UNDER CONFORMAL RESCALING OF THE SPACE-TIME – A STUDY INCLUDING CONSEQUENCES FOR THE METRIC

Summary

The conformal transformations play crucial role in the analysis of global structure of the physical space-time. This paper shows some geometrical and physical objects which describe the space-time. There are also given transformation laws for them under conformal rescaling of the metric. The main goal of this article is to check which geometrical and physical objects are invariants under the conformal rescaling of the metric and to present the consequences of the conformal transformation of the metric like creation of the energy and momentum for the gravitional field or creation of the matter.

Keywords and phrases: conformal transformation of the metric, general theory of relativity, space-time, Riemannian manifold (pseudo-Riemannian manifold, Lorentzian manifold), Einstein's equations, Landau-Lifshitz pseudotensor of the energy-momentum

1. The information about the actual mathematical model of the physical space-time

The theory of relativity stated by Albert Einstein concerns the structure of the space-time. It consists of the special theory of relativity (SR) and the general theory of relativity (GR). The former theory is valid for inertial reference frames and does not take into account the gravitation. In the later one there are described events in the non-inertial reference frames, so it means that the gravity is taken into consideration. Both of them give some mathematical model of the physical space-time. This model is called shortly *the space-time*. It is defined as a set of all possible physical events. According to GR the space-time is a 4-dimentional connected manifold M_4 of the class C^∞, with the Hausdorff topology, orientable, having Lorentzian structure, nonextendable and time-oriented. The metric tensor of that space-time satisfies the Einstein equations.

Definition 1. *The conformal rescaling or the conformal transformation of the metric g on the Riemannian manifold (pseudoriemannian, Lorentzian) is the following transformation (in the established coordinates)*

$$\widehat{g}_{ab}(x) = \Omega^2(x) g_{ab}(x),$$

where $\Omega(x)$ is a smooth and positive-definite function called the conformal factor.

Fact 1. *Denote by g^{ab} components of the tensor $(2,\ 0)$ which is inverse to the metric tensor. Then we have*

$$\widehat{g}^{ab} = \Omega^{-2}(x) g^{ab}.$$

Proof. From the definition we have $g^{ab} g_{bd} = \delta^a_d$, where

$$\delta^a_d = \begin{cases} 1 & \text{if } a = d, \\ 0 & \text{if } a \neq d. \end{cases}$$

The symbol δ^a_d is Kronecker's delta or the unit tensor. Because

$$\widehat{g}_{bd} = \Omega^2 g_{bd}$$

and

$$\widehat{g}_{bd} \widehat{g}^{ab} = \Omega^2 g_{bd} \Omega^x g^{ab} = \delta^a_d,$$

so to preserve the equality x must be equal -2. $\square$

Fact 2. *The length of a vector changes under the conformal rescalling of the metric but the ratio of vectors is preserved.*

Proof. The length of the vector in the metric g_{ab} is given by the formula

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})} = \sqrt{g_{ik} v^i v^k}.$$

In the new metric $\widehat{g}_{ab}$ it has the following form

$$\left\|\hat{\vec{v}}\right\| = \sqrt{\widehat{g}_{ik} v^i v^k} = \sqrt{\Omega^2 g_{ik} v^i v^k} = \Omega \, \|\vec{v}\|.$$

Therefore the ratio of the vectors is not changed.

$$\frac{\|\hat{\vec{u}}\|}{\|\hat{\vec{v}}\|} = \frac{\|\vec{u}\|}{\|\vec{v}\|}. \qquad \square$$

Fact 3. *The angle between two vectors is an invariant of conformal rescaling of the metric.*

Proof. We calculate the angle between two vectors following the formula

$$\cos(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}.$$

Then in the new metric it has the form

$$\cos(\hat{\vec{u}}, \hat{\vec{v}}) = \frac{\hat{\vec{u}} \cdot \hat{\vec{v}}}{\|\hat{\vec{u}}\| \|\hat{\vec{v}}\|} = \frac{\Omega^2 \vec{u} \cdot \vec{v}}{\Omega^2 \|\vec{u}\| \|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}. \qquad \square$$

The facts 2 and 3 justify the name "the conformal transformation" or "the conformal rescaling of the metric" from the definition 1.

Fact 4. *The line element changes under conformal rescaling of the metric as follows*

$$d\hat{s}^2 = \Omega^2(x) ds^2.$$

Proof. The line element is given by the formula

$$ds^2 = g_{ik} dx^i dx^k.$$

After conformal rescaling of the metric the line element has the form

$$d\hat{s}^2 = \hat{g}_{ik}(x) dx^i dx^k = \Omega^2(x) g_{ik}(x) dx^i dx^k = \Omega^2(x) ds^2. \qquad \square$$

2. Selected geometrical aspects of conformal transformations

Definition 2. *Affine connection* (*in other words Christoffel symbols of the second kind or Christoffel's connection*) *is defined as* [2, 5] :

$$\Gamma^a_{bc} = \frac{1}{2} g^{ae} \left(\frac{\partial g_{be}}{\partial x^c} + \frac{\partial g_{ce}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^e} \right) = \frac{1}{2} g^{ae} (g_{be,c} + g_{ce,b} - g_{bc,e}).$$

Affine connection is an geometrical object, which permits us parallel transport of vectors and tensors on a manifold and develop tensor analysis on it.

Fact 5. *Christoffel symbols change under the conformal transformation of the metric as follows*

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + P^a_{bc},$$

where $P^a_{bc} = \Omega^{-1}(\delta^a_b \Omega_{,c} + \delta^a_c \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e})$.

Proof. In the proof below we will write the Christoffel symbols in the new gauge and use the transformation law for the metric tensor.

$$
\begin{aligned}
\widehat{\Gamma}^a_{bc} &= \frac{1}{2}\widehat{g}^{ae}\left(\widehat{g}_{be,c}+\widehat{g}_{ce,b}-\widehat{g}_{bc,e}\right)\\
&= \frac{1}{2}\Omega^{-2}g^{ae}\left[(\Omega^2 g_{be})_{,c}+(\Omega^2 g_{ce})_{,b}-(\Omega^2 g_{bc})_{,e}\right]\\
&= \frac{1}{2}\Omega^{-2}g^{ae}\left(2\Omega\Omega_{,c}g_{be}+\Omega^2 g_{be,c}+2\Omega\Omega_{,b}g_{ce}+\Omega^2 g_{ce,b}-2\Omega\Omega_{,e}g_{bc}-\Omega^2 g_{bc,e}\right)\\
&= \frac{1}{2}g^{ae}(g_{be,c}+g_{ce,b}-g_{bc,e})+\Omega^{-1}g^{ae}\left(\Omega_{,c}g_{be}+\Omega_{,b}g_{ce}-\Omega_{,e}g_{bc}\right)\\
&= \frac{1}{2}g^{ae}(g_{be,c}+g_{ce,b}-g_{bc,e})+\Omega^{-1}(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e})\\
&= \Gamma^a_{bc}+P^a_{bc}.
\end{aligned}
$$

If $\Omega(x) = \text{const}$, then $\widehat{\Gamma}^a_{bc} = \Gamma^a_{bc}$. □

Definition 3. *Covariant derivative is defined as* $[4, 1]$:

$$\nabla_i v^k := \partial_i v^k + \Gamma^k_{ic} v^c,$$

for the contravariant vector field i.e. for the tensor of type $(1, 0)$ *and*

$$\nabla_i v_k := \partial_i v_k - \Gamma^c_{ik} v_c$$

for the covariant vector field (*covector or the 1-form or the tensor of type* $(0, 1)$).

Fact 6. *Covariant derivative changes under the conformal transformation of the metric in the following way:*

$$\widehat{\nabla}_i v^k := \nabla_i v^k + P^k_{ic} v^c,$$

and

$$\widehat{\nabla}_i v_k := \nabla_i v_k - P^c_{ik} v_c,$$

where

$$P^c_{ik} = \Omega^{-1}(\delta^c_i \Omega_{,k} + \delta^c_k \Omega_{,i} - g_{ik} g^{ce} \Omega_{,e}).$$

Proof. Writing down the covariant derivative in the new gauge in accordance with the definition 3 and developing it we get

$$\widehat{\nabla}_i v^k = \partial_i v^k + \widehat{\Gamma}^k_{ic} v^c = \partial_i v^k + \Gamma^k_{ic} v^c + P^k_{ic} v^c = \nabla_i v^k + P^k_{ic} v^c$$

$$\widehat{\nabla}_i v^k = \nabla_i v^k \iff \Omega(x) = \text{const},$$

$$\widehat{\nabla}_i v_k = \partial_i v_k - \widehat{\Gamma}^c_{ik} v_c = \partial_i v^k - \Gamma^c_{ik} v_c - P^c_{ik} v_c = \nabla_i v_k - P^c_{ik} v_c$$

$$\widehat{\nabla}_i v_k = \nabla_i v_k \iff \Omega(x) = \text{const}.$$

□

Fact 7. *Conformal transformation preserves the metricity of the connection.*

Proof.

$$\begin{aligned}\widehat{\nabla}_c\widehat{g}_{ab} &= \partial_c\widehat{g}_{ab} - \widehat{\Gamma}^d_{ac}\widehat{g}_{db} - \widehat{\Gamma}^d_{bc}\widehat{g}_{ad}\\ &= \left(\Omega^2 g_{ab}\right)_{,c} - \left(\Gamma^d_{ac} + P^d_{ac}\right)\Omega^2 g_{db} - \left(\Gamma^d_{bc} + P^d_{bc}\right)\Omega^2 g_{ad}\\ &= \Omega^2_{,c} g_{ab} + \Omega^2 g_{ab,c} - \Gamma^d_{ac}\Omega^2 g_{db} - \Gamma^d_{bc}\Omega^2 g_{ad} - P^d_{ac}\Omega^2 g_{db} - P^d_{bc}\Omega^2 g_{ad}.\end{aligned}$$

Because $g_{ab,c} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0$ so we obtain

$$\begin{aligned}\widehat{\nabla}_c\widehat{g}_{ab} &= \Omega^2_{,c} g_{ab} - P^d_{ac}\Omega^2 g_{db} - P^d_{bc}\Omega^2 g_{ad}\\ &= 2\Omega\Omega_{,c} g_{ab} - \Omega^{-1}\left(\delta^d_a\Omega_{,c} + \delta^d_c\Omega_{,a} - g_{ac}g^{de}\Omega_{,e}\right)\Omega^2 g_{db}\\ &- \Omega^{-1}\left(\delta^d_b\Omega_{,c} + \delta^d_c\Omega_{,b} - g_{bc}g^{de}\Omega_{,e}\right)\Omega^2 g_{ad}\\ &= 2\Omega\Omega_{,c} g_{ab} - \Omega\left(g_{ab}\Omega_{,c} + g_{cb}\Omega_{,a} - g_{ac}\Omega_{,b}\right)\\ &- \Omega\left(g_{ab}\Omega_{,c} + g_{ac}\Omega_{,b} - g_{bc}\Omega_{,a}\right) = 0.\end{aligned}$$

Hence the metricity of the connection is preserved. □

Definition 4. *A geodesic is a curve on a Riemannian manifold (pseudoremannian or Lorentzian) which gives the extremum of the distance between any two neighbouring points* [1]. *Such a curve is a generalization of a straight line and satisfies the equations in so-called affine parameter u*

$$\frac{d^2x^i}{du^2} + \Gamma^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = 0.$$

Fact 8. *Under the conformal rescaling of the metric, the equations of the geodesic develop as follows*

$$\frac{d^2x^i}{du^2} + \widehat{\Gamma}^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = P^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du}.$$

Proof. In the proof below we start with the equation of the geodesic

$$\frac{d^2x^i}{du^2} + \Gamma^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = 0.$$

Then we use the Christoffel symbol after the conformal rescaling of the metric

$$\widehat{\Gamma}^i_{kl} = \Gamma^i_{kl} + P^i_{kl}.$$

Replacing Γ^i_{kl} by the expression $\widehat{\Gamma}^i_{kl} - P^i_{kl}$ in the initial equation we obtain

$$\tfrac{d^2x^i}{du^2} + \left(\widehat{\Gamma}^i_{kl} - P^i_{kl}\right)\tfrac{dx^k}{du}\tfrac{dx^l}{du} = 0.$$ □

As we can see it is not the equation of the geodesic $\widehat{\Gamma}^i_{kl}$, except the case of $\Omega(x) = \text{const}$.

Generally one gets

$$\frac{d^2x^i}{du^2} + \widehat{\Gamma}^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = \Omega^{-1}(\delta^i_k\Omega_{,l} + \delta^i_l\Omega_{,k} - g_{kl}g^{ie}\Omega_{,e})\frac{dx^k}{du}\frac{dx^l}{du}$$

$$= \Omega^{-1}\left[\Omega_{,k}\frac{dx^i}{du}\frac{dx^k}{du} + \Omega_{,k}\frac{dx^k}{du}\frac{dx^i}{du} - g_{kl}\frac{dx^k}{du}\frac{dx^l}{du}g^{ie}\Omega_{,e}\right].$$

If the geodesic was isotropic, then

$$g_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = 0,$$

and hence

$$\frac{d^2x^i}{du^2} + \widehat{\Gamma}^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = \left(2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}\right)\frac{dx^i}{du} = h(u)\frac{dx^i}{du}.$$

This is an equation of the geodesic line with non-affine parameter u. Then we have

Fact 9. *Under the conformal transformation of the metric only the null geodesics are preserved, but the affine parameter loses its affine character.*

However we can introduce a new parameter $\lambda = \lambda(u)$ on these lines, which will be affine. Namely we have

Fact 10. *The new parameter λ defined by the equation*

$$\frac{d\lambda}{du} = c\Omega^2,$$

where $c = \text{const}$, *is an affine parameter.*

Proof.

$$\frac{d^2x^i}{du^2} + \widehat{\Gamma}^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = h(u)\frac{dx^i}{du},$$

where

$$h(u) = 2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}.$$

Lets write $x^i = x^i\left[\lambda(u)\right].$

Then

$$\frac{dx^i}{du} = \frac{dx^i}{d\lambda}\frac{d\lambda}{du},$$

$$\frac{dx^i}{du^2} = \frac{d}{du}\left(\frac{dx^i}{d\lambda}\frac{d\lambda}{du}\right) = \frac{d^2x^i}{d\lambda^2}\left(\frac{d\lambda}{du}\right)^2 + \frac{dx^i}{d\lambda}\frac{d^2\lambda}{du^2}.$$

Replacing $\frac{dx^i}{du^2}$ by the above expression in the initial equation we obtain

$$\frac{d^2x^i}{d\lambda^2}\left(\frac{d\lambda}{du}\right)^2 + \frac{dx^i}{d\lambda}\frac{d^2\lambda}{du^2} + \widehat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}\left(\frac{d\lambda}{du}\right)^2 = h(u)\frac{dx^i}{d\lambda}\frac{d\lambda}{du},$$

or

$$\left(\frac{d^2x^i}{d\lambda^2}+\widehat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}\right)\left(\frac{d\lambda}{du}\right)^2=h(u))\frac{dx^i}{d\lambda}\frac{d\lambda}{du}-\frac{d^2\lambda}{du^2}\frac{dx^i}{d\lambda}. \tag{1}$$

The right side of the equation will be zero iff

$$h(u)\frac{d\lambda}{du}-\frac{d^2\lambda}{du^2}=0$$

or

$$2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}\frac{d\lambda}{du}-\frac{d^2\lambda}{du^2}=0. \tag{2}$$

On the other hand, if we differentiate

$$\frac{d\lambda}{du}=c\Omega^2$$

then we have

$$\frac{d^2\lambda}{du^2}=2c\Omega\Omega_{,k}\frac{dx^k}{du}.$$

Putting $\frac{d^2\lambda}{du^2}$ into the equation (2) we obtain (reminding that $\frac{d\lambda}{du}=c\Omega^2$)

$$2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}\frac{d\lambda}{du}-2c\Omega\Omega_{,k}\frac{dx^k}{du}=0,$$

and the equations (1) take the form

$$\frac{d^2x^i}{d\lambda^2}+\widehat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^c}{d\lambda}=0.$$

Therefore the new parameter λ which satisfies the equation

$$\frac{d\lambda}{du}=c\Omega^2,$$

where $c=\text{const}$, is the affine parameter. □

Definition 5. *The curvature tensor* (*the Riemann-Christoffel tensor or the Riemann tensor*) *is defined as follows* [1]

$$R^a_{.bcd}=\Gamma^a_{bd,c}-\Gamma^a_{bc,d}+\Gamma^a_{nc}\cdot\Gamma^n_{bd}-\Gamma^a_{nd}\cdot\Gamma^n_{bc}.$$

The curvature tensor is the tensor of type $(1,3)$ and its components are defined by the metric tensor and its derivatives. It is a basic tool used in the differential geometry bacause it is a measure of the local curvature. Lets notice that for a flat manifold the Christoffel's symbols are not already equal zero in the curvilinear coordinates but the curvature tensor $R^a_{.bcd}$ is equal zero.

Fact 11. *The Riemann curvature changes under the conformal transformation of the metric as follows*

$$\begin{aligned}\widehat{R}^a_{.bcd} &= R^a_{.bcd} + \frac{\Omega^2}{2}(\delta^a_{[c}\Omega_{d]b} - g_{b[c}\Omega^a_{d]}) \\ &= R^a_{.bcd} + \frac{\Omega^2}{4}(\delta^a_c\Omega_{db} - \delta^a_d\Omega_{cb} - g_{bc}\Omega^a_{.d} - g_{bd}\Omega^a_{.c}).\end{aligned}$$

Proof. The idea of our proof is to show that the both sides of the equation are equal. From the definition we have

$$\begin{aligned}\widehat{R}^a_{.bcd} &= \widehat{\Gamma}^a_{bd,c} - \widehat{\Gamma}^a_{bc,d} + \widehat{\Gamma}^a_{nc}\cdot\widehat{\Gamma}^n_{bd} - \widehat{\Gamma}^a_{nd}\cdot\widehat{\Gamma}^n_{bc} \\ &= \Gamma^a_{bd,c} + P^a_{bd,c} - \Gamma^a_{bc,d} - P^a_{bc,d} + (\Gamma^a_{nc} + P^a_{nc})\cdot(\Gamma^n_{bd} + P^n_{bd}) \\ &\quad - \ (\Gamma^a_{nd} + P^a_{nd})\cdot(\Gamma^n_{bc} + P^n_{bc}) \\ &= \Gamma^a_{bd,c} + P^a_{bd,c} - \Gamma^a_{bc,d} - P^a_{bc,d} + \Gamma^a_{nc}\cdot\Gamma^n_{bd} + \Gamma^a_{nc}\cdot P^n_{bd} + P^a_{nc}\cdot\Gamma^n_{bd} \\ &\quad + P^a_{nc}\cdot P^n_{bd} - \Gamma^a_{nd}\cdot\Gamma^n_{bc} - \Gamma^a_{nd}\cdot P^n_{bc} - P^a_{nd}\cdot\Gamma^n_{bc} - P^a_{nd}\cdot P^n_{bc} \\ &= (\Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{nc}\cdot\Gamma^n_{bd} - \Gamma^a_{nd}\cdot\Gamma^n_{bc}) + (P^a_{bd,c} - P^a_{bc,d}) \\ &\quad + (\Gamma^a_{nc}\cdot P^n_{bd} + P^a_{nc}\cdot\Gamma^n_{bd} + P^a_{nc}\cdot P^n_{bd} - \Gamma^a_{nd}\cdot P^n_{bc} - P^a_{nd}\cdot\Gamma^n_{bc} - P^a_{nd}\cdot P^n_{bc}) \\ &= R^a_{.bcd} + (P^a_{bd,c} - P^a_{bc,d} + \Gamma^a_{nc}\cdot P^n_{bd} + P^a_{nc}\cdot\Gamma^n_{bd} + P^a_{nc}\cdot P^n_{bd} - \Gamma^a_{nd}\cdot P^n_{bc} \\ &\quad - P^a_{nd}\cdot\Gamma^n_{bc} - P^a_{nd}\cdot P^n_{bc}),\end{aligned}$$

where

$$R^a_{.bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{nc}\cdot\Gamma^n_{bd} - \Gamma^a_{nd}\cdot\Gamma^n_{bc}.$$

To make easier further calculations we will divide the equation into three parts. Lets denote

$$I_1 = \Gamma^a_{nc}\cdot P^n_{bd} + P^a_{nc}\cdot\Gamma^n_{bd} - \Gamma^a_{nd}\cdot P^n_{bc} - P^a_{nd}\cdot\Gamma^n_{bc},$$

$$I_2 = P^a_{nc}\cdot P^n_{bd} - P^a_{nd}\cdot P^n_{bc},$$

$$I_3 = P^a_{bd,c} - P^a_{bc,d}.$$

Then we compute I_1, I_2 and I_3.

$$\begin{aligned}I_1 &= \Omega^{-1}[\Gamma^a_{nc}(\delta^n_b\Omega_{,d} + \delta^n_d\Omega_{,b} - g^{ne}g_{bd}\Omega_{,e}) + \Gamma^n_{bd}(\delta^a_n\Omega_{,c} + \delta^a_c\Omega_{,n} - g^{ae}g_{nc}\Omega_{,e}) \\ &\quad - \Gamma^a_{nd}(\delta^n_b\Omega_{,c} + \delta^n_c\Omega_{,b} - g^{ne}g_{bc}\Omega_{,e}) - \Gamma^n_{bc}(\delta^a_n\Omega_{,d} + \delta^a_d\Omega_{,n} - g^{ae}g_{nd}\Omega_{,e})] \\ &= \Omega^{-1}[\Gamma^a_{bc}\Omega_{,d} + \Gamma^a_{cd}\Omega_{,b} - \Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e} + \Gamma^a_{bd}\Omega_{,c} + \Gamma^n_{bd}\delta^a_c\Omega_{,n} - \Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e} \\ &\quad - \Gamma^a_{bd}\Omega_{,c} - \Gamma^a_{cd}\Omega_{,b} + \Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e} - \Gamma^a_{bc}\Omega_{,d} - \Gamma^n_{bc}\delta^a_d\Omega_{,n} + \Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}] \\ &= \Omega^{-1}(\Gamma^n_{bd}\delta^a_c\Omega_{,n} - \Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e} - \Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}) \\ &\quad + \Omega^{-1}(\Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e} - \Gamma^n_{bc}\delta^a_d\Omega_{,n} + \Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}),\end{aligned}$$

$$
\begin{aligned}
I_2 &= \Omega^{-2}(\delta^a_n\Omega_{,c}+\delta^a_c\Omega_{,n}-g^{ae}g_{nc}\Omega_{,e})(\delta^n_b\Omega_{,d}+\delta^n_d\Omega_{,b}-g^{nf}g_{bd}\Omega_{,f})\\
&\quad-\Omega^{-2}(\delta^a_n\Omega_{,d}+\delta^a_d\Omega_{,n}-g^{ae}g_{nd}\Omega_{,e})(\delta^n_b\Omega_{,c}+\delta^n_c\Omega_{,b}-g^{nf}g_{bc}\Omega_{,f})\\
&= \Omega^{-2}(\delta^a_n\Omega_{,c}\delta^n_b\Omega_{,d}+\delta^a_n\Omega_{,c}\delta^n_d\Omega_{,b}-\delta^a_n\Omega_{,c}g^{ne}g_{bd}\Omega_{,e}+\delta^a_c\Omega_{,n}\delta^n_b\Omega_{,d}\\
&\quad+\delta^a_c\Omega_{,n}\delta^n_d\Omega_{,b}-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-g^{ae}g_{nc}\Omega_{,e}\delta^n_b\Omega_{,d}-g^{ae}g_{nc}\Omega_{,e}\delta^n_d\Omega_{,b}\\
&\quad+g^{ae}g_{nc}\Omega_{,e}g^{nf}g_{bd}\Omega_{,f}-\delta^a_n\Omega_{,d}\delta^n_b\Omega_{,c}-\delta^a_n\Omega_{,d}\delta^n_c\Omega_b+\delta^a_n\Omega_{,d}g^{ne}g_{bc}\Omega_{,e}\\
&\quad-\delta^a_d\Omega_{,n}\delta^n_b\Omega_{,c}-\delta^a_d\Omega_{,n}\delta^n_c\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}+g^{ae}g_{nd}\Omega_{,e}\delta^n_b\Omega_{,c}\\
&\quad+g^{ae}g_{nd}\Omega_{,e}\delta^n_c\Omega_{,b}-g^{ae}g_{nd}\Omega_{,e}g^{nf}g_{bc}\Omega_{,f})\\
&= \Omega^{-2}(\delta^a_b\Omega_{,c}\Omega_{,d}+\delta^a_d\Omega_{,c}\Omega_{,b}-g^{ae}g_{bd}\Omega_{,c}\Omega_{,e}+\delta^a_c\Omega_{,b}\Omega_{,d}+\delta^a_c\Omega_{,d}\Omega_{,b}\\
&\quad-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,e}\Omega_{,d}-g^{ae}g_{dc}\Omega_{,e}\Omega_{,b}+g^{ae}\delta^f_c g_{bd}\Omega_{,f}\Omega_{,e}-\delta^a_b\Omega_{,d}\Omega_{,c}\\
&\quad-\delta^a_c\Omega_{,d}\Omega_{,b}+g^{ae}\Omega_{,d}g_{bc}\Omega_{,e}-\delta^a_d\Omega_{,b}\Omega_{,c}-\delta^a_d\Omega_{,c}\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\\
&\quad+g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}+g^{ae}g_{cd}\Omega_{,e}\Omega_{,b}-g^{ae}\delta^f_d\Omega_{,f}\Omega_{,e}g_{bc})\\
&= \Omega^{-2}(\delta^a_c\Omega_{,b}\Omega_{,d}-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-\delta^a_d\Omega_{,c}\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\\
&\quad+g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}),
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \Omega^{-1}(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e})_{,c}-\Omega^{-1}(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e})_{,d}\\
&= \Omega^{-1}(\delta^a_b\Omega_{,dc}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec})\\
&\quad-\Omega^{-1}(\delta^a_b\Omega_{,cd}+\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{ed})\\
&\quad+\Omega^{-1}_{,c}(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e})-\Omega^{-1}_{,d}(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e})\\
&= \Omega^{-1}(\delta^a_b\Omega_{,cd}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec})\\
&\quad-\Omega^{-1}(\delta^a_b\Omega_{,cd}+\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,ed})\\
&\quad-\frac{\Omega_{,c}}{\Omega^2}(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e})+\frac{\Omega_{,d}}{\Omega^2}(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e})\\
&= \Omega^{-1}(\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}-\delta^a_c\Omega_{,bd}+g^{ae}_{,d}g_{bc}\Omega_{,e}\\
&\quad+g^{ae}g_{bc,d}\Omega_{,e}+g^{ae}g_{bc}\Omega_{,ed})-\frac{\Omega_{,c}}{\Omega^2}(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e})\\
&\quad+\frac{\Omega_{,d}}{\Omega^2}(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e}).
\end{aligned}
$$

Summing I_1, I_2 and I_3 we obtain

$$
\begin{aligned}
&\Omega^{-1}(\Gamma^n_{bd}\delta^a_c\Omega_{,n}-\Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e}-\Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}+\Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e}-\Gamma^n_{bc}\delta^a_d\Omega_{,n}\\
&+\Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}-\delta^a_c\Omega_{,bd}\\
&+g^{ae}_{,d}g_{bc}\Omega_{,e}+g^{ae}g_{bc,d}\Omega_{,e}+g^{ae}g_{bc}\Omega_{,ed})\\
&+\Omega^{-2}(\delta^a_c\Omega_{,b}\Omega_{,d}-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-\delta^a_d\Omega_{,c}\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\\
&+g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}-\delta^a_b\Omega_{,c}\Omega_{,d}-\delta^a_d\Omega_{,c}\Omega_{,b}+g^{ae}g_{bd}\Omega_{,c}\Omega_{,e}\\
&+\delta^a_b\Omega_{,d}\Omega_{,c}+\delta^a_c\Omega_{,d}\Omega_{,b}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}).
\end{aligned}
$$

Using the equations $g_{bd,c}=\Gamma^n_{bc}g_{nd}+\Gamma^n_{dc}g_{bn}$ and $g_{bc,d}=\Gamma^n_{bd}g_{nc}+\Gamma^n_{cd}g_{bn}$ we obtain

$$
\Omega^{-1}(g_{bc,d}g^{ae}\Omega_{,e}-g_{bd,c}g^{ae}\Omega_{,e}-\Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}+\Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e})=0.
$$

Lets take the equation

$$g^{ae}{}_{;c} = g^{ae}{}_{,c} + \Gamma^a_{nc} g^{ne} + \Gamma^e_{nc} g^{an},$$

and

$$g^{ae}{}_{;d} = g^{ae}{}_{,d} + \Gamma^a_{nd} g^{ne} + \Gamma^e_{nd} g^{an},$$

then the formula reduces as follows

$$\begin{aligned}
&\Omega^{-1}(\Gamma^n_{bd}\delta^a_c\Omega_{,n} + \Gamma^e_{nc}g^{an}g_{bd}\Omega_{,e} - \Gamma^n_{bc}\delta^a_d\Omega_{,n} + \delta^a_d\Omega_{,bc} - \delta^a_c\Omega_{,bd} - g^{ae}g_{bd}\Omega_{,ec}\\
&+ g^{ae}g_{bc}\Omega_{,ed} - \Gamma^e_{nd}g^{an}g_{bc}\Omega_{,e})\\
&+ \Omega^{-2}(2\delta^a_c\Omega_{,d}\Omega_{,b} - 2\delta^a_d\Omega_{,b}\Omega_{,c} + 2g^{ae}\Omega_{,e}g_{bd}\Omega_{,c} + \delta^a_d g_{bc}g^{ne}\Omega_{,n}\Omega_{,e}\\
&- 2g^{ae}g_{bc}\Omega_{,e}\Omega_{,d} - \delta^a_c g^{ne}g_{bd}\Omega_{,n}\Omega_{,e}).
\end{aligned}$$

After extending the right side of the initial equation we get the following form

$$\begin{aligned}
&R^a_{.bcd} + \frac{\Omega^2}{2}\left(\delta^a_{[c}\Omega_{d]b} - g_{b[c}\Omega^a_{.d]}\right)\\
&= R^a_{.bcd} + \frac{\Omega^2}{2}\left[\frac{1}{2}\left(\delta^a_c\Omega_{db} - \delta^a_d\Omega_{cb}\right) - \frac{1}{2}\left(g_{bc}\Omega^a_{.d} - g_{bd}\Omega^a_{.c}\right)\right]\\
&= R^a_{.bcd} + \frac{\Omega^2}{4}\left(\delta^a_c\Omega_{db} - \delta^a_d\Omega_{cb} - g_{bc}\Omega^a_{.d} + g_{bd}\Omega^a_{.c}\right).
\end{aligned}$$

Lets denote by I_4 the expression

$$\frac{\Omega^2}{4}\left(\delta^a_c\Omega_{db} - \delta^a_d\Omega_{cb} - g_{bc}\Omega^a_{.d} + g_{bd}\Omega^a_{.c}\right).$$

Putting into

$$\begin{aligned}
\Omega_{db} &= g_{ad}\Omega^a_{.b}\\
&= g_{ad}\left[4\left(\Omega^{-1}\right)\left(\Omega^{-1}\right)_{;be}g^{ae} - 2\left(\Omega^{-1}\right)_{,p}\left(\Omega^{-1}\right)_{,t}g^{pt}\delta^a_b\right]\\
&= 4\left(\Omega^{-1}\right)\left(\Omega^{-1}\right)_{;be}g_{ad}g^{ae} - 2\left(\Omega^{-1}\right)_{,p}\left(\Omega^{-1}\right)_{,t}g_{ad}g^{pt}\delta^a_b\\
&= 4\left(\Omega^{-1}\right)\left(\Omega^{-1}\right)_{;bd} - 2\left(\Omega^{-1}\right)_{,p}\left(\Omega^{-1}\right)_{,t}g_{bd}g^{pt},
\end{aligned}$$

and developing in a similar way Ω_{cb}, $\Omega^a_{.d}$ and $\Omega^a_{.c}$ we have

$$\begin{aligned}
I_4 = {}&\frac{\Omega^2}{4}\left(\frac{-4\delta^a_c\Omega_{,bd}}{\Omega^3} + \frac{8\delta^a_c\Omega_{,b}\Omega_{,d}}{\Omega^4} + \frac{4\delta^a_c\Gamma^e_{bd}\Omega_{,e}}{\Omega^3} - \frac{4\delta^a_c\Omega_{,p}\Omega_{,t}g^{pt}g_{db}}{\Omega^4}\right)\\
&+ \frac{\Omega^2}{4}\left(\frac{4\delta^a_d\Omega_{,bc}}{\Omega^3} - \frac{8\delta^a_d\Omega_{,b}\Omega_{,c}}{\Omega^4} - \frac{4\delta^a_d\Gamma^e_{bc}\Omega_{,e}}{\Omega^3} + \frac{4\delta^a_d\Omega_{,p}\Omega_{,t}g^{pt}g_{bc}}{\Omega^4}\right)\\
&+ \frac{\Omega^2}{4}\left(\frac{4\Omega_{,de}g^{ae}g_{bc}}{\Omega^3} - \frac{8\Omega_{,d}\Omega_{,e}g^{ae}g_{bc}}{\Omega^4} - \frac{4\Gamma^p_{de}\Omega_{,p}g_{bc}g^{ae}}{\Omega^3}\right)\\
&- \frac{\Omega^2}{4}\left(\frac{4\Omega_{,ce}g^{ae}g_{bd}}{\Omega^3} - \frac{8\Omega_{,c}\Omega_{,e}g^{ae}g_{bd}}{\Omega^4} - \frac{4\Gamma^n_{ce}\Omega_{,n}g_{bd}g^{ae}}{\Omega^3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Omega^{-1}(-\delta^a_c\Omega_{,bd} + \delta^a_c\Gamma^e_{bd}\Omega_{,e} + \delta^a_d\Omega_{,bc} - \delta^a_d\Gamma^e_{bc}\Omega_{,e} + \Omega_{,de}g^{ae}g_{bc} \\
&- \Gamma^p_{de}\Omega_{,p}g_{bc}g^{ae} - \Omega_{,ce}g_{bd}g^{ae} + \Gamma^n_{ce}\Omega_{,n}g_{bd}g^{ae}) \\
&+ \Omega^{-2}(2\delta^a_c\Omega_{,b}\Omega_{,d} - \delta^a_c\Omega_{,p}\Omega_{,t}g^{pt}g_{bd} - 2\delta^a_d\Omega_{,b}\Omega_{,c} \\
&+ \delta^a_d\Omega_{,p}\Omega_{,t}g^{pt}g_{bc} - 2\Omega_{,d}\Omega_{,e}g^{ae}g_{bc} + 2\Omega_{,c}\Omega_{,e}g_{bd}g^{ae}).
\end{aligned}
$$

After changing some of the indexes we finally obtain that the left side of the previous equation equals the right side.

$$
\begin{aligned}
&= \Omega^{-1}(-\delta^a_c\Omega_{,bd} + \delta^a_c\Gamma^n_{bd}\Omega_{,n} + \delta^a_d\Omega_{,bc} - \delta^a_d\Gamma^n_{bc}\Omega_{,n} + \Omega_{,de}g^{ae}g_{bc} \\
&- \Gamma^e_{dn}\Omega_{,e}g_{bc}g^{an} - \Omega_{,ce}g_{bd}g^{ae} + \Gamma^e_{cn}\Omega_{,e}g_{bd}g^{an}) \\
&+ \Omega^{-2}(2\delta^a_c\Omega_{,b}\Omega_{,d} - \delta^a_c\Omega_{,n}\Omega_{,e}g^{ne}g_{bd} - 2\delta^a_d\Omega_{,b}\Omega_{,c} \\
&+ \delta^a_d\Omega_{,n}\Omega_{,e}g^{ne}g_{bc} - 2\Omega_{,d}\Omega_{,e}g^{ae}g_{bc} + 2\Omega_{,c}\Omega_{,e}g_{bd}g^{ae}).
\end{aligned}
$$

If $\Omega(x) = \text{const}$, then $\widehat{R}^a_{.bcd} = R^a_{.bcd}$. □

Definition 6. *The Ricci tensor is defined as* [2]

$$R^b_{.g} = R^{.b}_{g} = g^{ac}R^{.b..}_{a.cg} = R^{cb..}_{..cg}.$$

Fact 12. *The Ricci tensor changes under the conformal transformation of the metric as follows*

$$\widehat{R}^b_g = \Omega^{-2}R^b_g + \frac{\Omega^b_{.g}}{2} + \frac{\delta^b_g}{4}\Omega^a_{.a}.$$

Proof. In the proof we are using the transformation laws for the curvature tensor and for the metric tensor.

$$
\begin{aligned}
\widehat{R}^b_{.g} &= \widehat{g}^{al}\widehat{R}^b_{.agl} = \Omega^{-2}g^{al}\left[R^b_{.agl} + \frac{\Omega^2}{4}\left(\delta^b_g\Omega_{la} - g_{ag}\Omega^b_{.l} - \delta^b_l\Omega_{ga} + g_{al}\Omega^b_{.g}\right)\right] \\
&= \Omega^{-2}R^b_{.g} + \frac{1}{4}\left(\delta^b_g\Omega^a_{.a} - \delta^l_g\Omega^b_{.l} - g^{ab}\Omega_{ga} + \delta^a_a\Omega^b_{.g}\right) \\
&= \Omega^{-2}R^b_{.g} + \frac{1}{4}\left(\delta^b_g\Omega^a_{.a} - \Omega^b_{.g} - \Omega^b_{.g} + 4\Omega^b_{.g}\right) \\
&= \Omega^{-2}R^b_{.g} + \frac{\Omega^b_{.g}}{2} + \frac{\delta^b_g\Omega^a_{.a}}{4}.
\end{aligned}
$$

After lowering the upper index we obtain the form of the Ricci tensor as below

$$
\begin{aligned}
\widehat{R}_{ag} &= \widehat{g}_{ab}\widehat{R}^b_{.g} = \Omega^2 g_{ab}\Omega^{-2}R^b_g + \Omega^2 g_{ab}\frac{\Omega^b_{.g}}{2} + \Omega^2 g_{ab}\frac{\delta^b_g\Omega^a_a}{4} \\
&= R_{ag} + \frac{\Omega^2}{2}\Omega_{ag} + \frac{\Omega^2}{4}g_{ab}\delta^b_g\Omega^a_{.a} \\
&= R_{ag} + \frac{\Omega^2}{2}\Omega_{ag} + \frac{\Omega^2}{4}g_{ag}\Omega^a_{.a}.
\end{aligned}
$$

If $\Omega(x) = \text{const}$, then $\widehat{R}^b_g = \Omega^{-2}R^b_g$ and $\widehat{R}_{ag} = R_{ag}$. □

Definition 7. *The curvature scalar we define as follows* [2]

$$R := g^{ab} R_{ab} = R^{b}_{.b}.$$

Fact 13. *The curvature scalar changes under the conformal transformation of the metric in the following way*

$$\widehat{R} = \Omega^{-2} R + \frac{3}{2} \Omega^{a}_{.a}.$$

Proof. One has

$$R = R^{i}_{.i}$$

In our proof we are using the transformation law for the Ricci tensor.

$$\begin{aligned}
\widehat{R} = \widehat{R}^{i}_{.i} &= \Omega^{-2} R^{i}_{.i} + \frac{\Omega^{i}_{.i}}{2} + \frac{\delta^{i}_{i}}{4} \Omega^{a}_{.a} \\
&= \Omega^{-2} R^{i}_{.i} + \frac{\Omega^{i}_{.i}}{2} + \frac{4}{4} \Omega^{a}_{.a} \\
&= \Omega^{-2} R^{i}_{.i} + \frac{3}{2} \Omega^{a}_{.a}.
\end{aligned}$$

If $\Omega(x) = \text{const}$, then $\widehat{R} = \Omega^{-2} R$. □

Definition 8. *The Weyl tensor is defined as* [1]

$$\begin{aligned}
C_{abcd} : = R_{abcd} &+ \frac{1}{2} \left(g_{ad} R_{cb} - g_{ac} R_{db} + g_{bc} R_{da} - g_{bd} R_{ca} \right) \\
&+ \frac{1}{6} R \left(g_{ac} g_{db} - g_{ad} g_{cb} \right).
\end{aligned}$$

In GR the Weyl tensor describes the free gravitational field, i.e. the gravitational field independent of matter.

Fact 14. *The Weyl tensor of the conformal curvature $C^{e}_{.bcd}$ is an invariant of the conformal transformation*

$$\widehat{C}^{a}_{.bcd} = C^{a}_{.bcd}.$$

Proof. Using the formula for C_{abcd} we calculate at first the Weyl tensor $C^{a}_{.bcd}$.

$$\begin{aligned}
C_{abcd} : = R_{abcd} &+ \frac{1}{2} \left(g_{ad} R_{cb} - g_{ac} R_{db} + g_{bc} R_{da} - g_{bd} R_{ca} \right) \\
&+ \frac{1}{6} R \left(g_{ac} g_{db} - g_{ad} g_{cb} \right).
\end{aligned}$$

$$\begin{aligned}
C^{e}_{.bcd} &= g^{ae}C_{abcd}\\
&= g^{ae}R_{abcd} + \frac{1}{2}g^{ae}\left(g_{ad}R_{cb} - g_{ac}R_{db} + g_{bc}R_{da} - g_{bd}R_{ca}\right)\\
&+ \frac{1}{6}g^{ae}R\left(g_{ac}g_{db} - g_{ad}g_{cb}\right)\\
&= R^{e}_{.bcd} + \frac{1}{2}\left(\delta^{e}_{d}R_{cb} - \delta^{e}_{c}R_{db} + g_{bc}R^{e}_{.d} - g_{bd}R^{e}_{.c}\right)\\
&+ \frac{1}{6}R\left(\delta^{e}_{c}g_{db} - \delta^{e}_{d}g_{cb}\right).
\end{aligned}$$

After the conformal rescaling of the metric one has

$$\begin{aligned}
\widehat{C}^{a}_{.bcd} &= \widehat{R}^{a}_{.bcd} + \frac{1}{2}\left(\delta^{a}_{d}\widehat{R}_{cb} - \delta^{a}_{c}\widehat{R}_{db} + \widehat{g}_{bc}\widehat{R}^{a}_{.d} - \widehat{g}_{bd}\widehat{R}^{a}_{.c}\right) + \frac{1}{6}\widehat{R}\left(\delta^{a}_{c}\widehat{g}_{db} - \delta^{a}_{d}\widehat{g}_{cb}\right)\\
&= R^{a}_{.bcd} + \frac{\Omega^2}{4}\left(\delta^{a}_{c}\Omega_{db} - \delta^{a}_{d}\Omega_{cb} - g_{bc}\Omega^{a}_{.d} + g_{bd}\Omega^{a}_{.c}\right)\\
&+ \frac{1}{2}\delta^{a}_{d}\left[R_{cb} + \frac{\Omega^2}{2}\left(\Omega_{cb} + \frac{g_{cb}}{2}\Omega^{e}_{.e}\right)\right] - \frac{1}{2}\delta^{a}_{c}\left[R_{db} + \frac{\Omega^2}{2}\left(\Omega_{db} + \frac{g_{db}}{2}\Omega^{e}_{.e}\right)\right]\\
&+ \frac{1}{2}\Omega^2 g_{bc}\left(\Omega^{-2}R^{a}_{.d} + \frac{\Omega^{a}_{.d}}{2} + \frac{\delta^{a}_{d}}{4}\Omega^{e}_{.e}\right) - \frac{1}{2}\Omega^2 g_{bd}\left(\Omega^{-2}R^{a}_{.c} + \frac{\Omega^{a}_{.c}}{2} + \frac{\delta^{a}_{c}}{4}\Omega^{e}_{.e}\right)\\
&+ \frac{1}{6}\left(\Omega^{-2}R + \frac{3}{2}\Omega^{e}_{.e}\right)\left(\delta^{a}_{c}\Omega^2 g_{bd} - \delta^{a}_{d}\Omega^2 g_{cb}\right)\\
&= R^{a}_{.bcd} + \frac{\Omega^2}{4}\left(\delta^{a}_{c}\Omega_{db} - \delta^{a}_{d}\Omega_{cb} - g_{bc}\Omega^{a}_{.d} + g_{bd}\Omega^{a}_{.c}\right)\\
&+ \frac{1}{2}\left(\delta^{a}_{d}R_{cb} + \frac{\Omega^2}{2}\delta^{a}_{d}\Omega_{cb} + \frac{\Omega^2}{4}g_{cb}\delta^{a}_{d}\Omega^{e}_{.e} - \delta^{a}_{c}R_{db} - \frac{\Omega^2}{2}\delta^{a}_{c}\Omega_{db} - \frac{\Omega^2}{4}\delta^{a}_{c}g_{db}\Omega^{e}_{.e}\right)\\
&+ \frac{1}{2}\left(g_{bc}R^{a}_{.d} + \frac{\Omega^2}{2}g_{bc}\Omega^{a}_{.d} + \frac{\Omega^2}{4}g_{bc}\delta^{a}_{d}\Omega^{e}_{.e} - g_{bd}R^{a}_{.c} - \frac{\Omega^2}{2}g_{bd}\Omega^{a}_{.c} - \frac{\Omega^2}{4}\delta^{a}_{c}g_{bd}\Omega^{e}_{.e}\right)\\
&+ \frac{1}{6}R\left(\delta^{a}_{c}g_{bd} - \delta^{a}_{d}g_{cb}\right) + \frac{1}{4}\delta^{a}_{c}\Omega^2 g_{bd}\Omega^{e}_{.e} - \frac{1}{4}\delta^{a}_{d}\Omega^2 g_{cb}\Omega^{e}_{.e}\\
&= R^{a}_{.bcd} + \frac{1}{2}\left(\delta^{a}_{d}R_{cb} - \delta^{a}_{c}R_{db} + g_{bc}R^{a}_{.d} - g_{bd}R^{a}_{.c}\right) + \frac{1}{6}R\left(\delta^{a}_{c}g_{db} - \delta^{a}_{d}g_{cb}\right)\\
&= C^{a}_{.bcd}.
\end{aligned}$$

If we change the location of the indexes in the Weyl tensor, then the above invariance is missed, e.g. $\widehat{C}^{ab}_{..cd} = \widehat{g}^{be}\widehat{C}^{a}_{.ecd} = \Omega^{-2}g^{be}C^{a}_{.ecd} = \Omega^{-2}C^{ab}_{cd}.$ □

Definition 9. *The Einstein equations* (*in the other words the field equations in GR or the equations of the gravitational field*) *have the following form* [1]

$$G^{.k}_{i.} = \kappa T^{.k}_{i.},$$

where $\kappa := \frac{8\pi G}{c^4}$.

If we use the geometrized units $G = c = 1$, then $\kappa = 8\pi$ and the Einstein's equation has the simpler form $G^{.k}_{i.} = 8\pi T^{.k}_{i.}$. The left side of the equation represents

the geometry of the space-time. On the other hand the right side describes matter and energy which fills the space-time. The Einstein's equation can be summarized as:
the distribution of the matter and the energy in the space-time determines its local geometry.
The tensor $G_{i.}^{.k}$ has the following structure

$$G_{i.}^{.k} = R_{i.}^{.k} - \frac{1}{2}\delta_i^k R,$$

and it is called the Einstein tensor. Raising or lowering one of the indexes we obtain the Einstein tensor in the covariant or contravariant form.

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R,$$
$$G^{ik} = R^{ik} - \frac{1}{2}g^{ik}R.$$

Fact 15. *The Einstein tensor changes under the conformal transformation of metric in the following way*

$$\widehat{G}_{i.}^{.k} = \Omega^{-2}G_{i.}^{.k} + \frac{1}{2}\left(\Omega_{i.}^{.k} - \delta_i^k \Omega_{.a}^{a}\right).$$

Proof.

$$\begin{aligned}
\widehat{G}_{i.}^{.k} &= \widehat{R}_{i.}^{.k} - \frac{1}{2}\delta_i^k \widehat{R} \\
&= \Omega^{-2}R_{i.}^{.k} + \frac{\Omega_{i.}^{.k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^{a} - \frac{1}{2}\delta_i^k\left(\Omega^{-2}R + \frac{3}{2}\Omega_{.a}^{a}\right) \\
&= \Omega^{-2}R_{i.}^{.k} - \frac{1}{2}\Omega^{-2}\delta_i^k R + \frac{\Omega_{i.}^{.k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^{a} - \frac{3}{4}\delta_i^k\Omega_{.a}^{a} \\
&= \Omega^{-2}G_{i.}^{.k} + \frac{\Omega_{i.}^{.k}}{2} - \frac{1}{2}\delta_i^k\Omega_{.a}^{a} \\
&= \Omega^{-2}G_{i.}^{.k} + \frac{1}{2}\left(\Omega_{i.}^{.k} - \delta_i^k\Omega_{.a}^{a}\right).
\end{aligned}$$

As we can see the Einstein tensor is not conformally invariant. □

3. Selected physical aspects of conformal transformations

In the previous chapter we have calculated the formula for the Einstein tensor under the conformal rescaling of the metric. Now we will show the transformation law for the right side of the Einstein equations, the matter tensor $T_{i.}^{.k}$, which is responsible for the curvature of the space-time.

Definition 10. *The matter tensor of the macroscopic matter (the matter energy-momentum tensor or the stress tonsor) we define as follows* [1]

$$T_{i.}^{.k} = (\rho + p)u_i u^k - p\delta_i^k,$$

where ρ denotes a density of the matter and p pressure.

Fact 16. *The matter tensor changes under the conformal transformation of the metric in the following way*

$$\widehat{T}_{i.}^{.k} = \Omega^{-4} T_{i.}^{.k}.$$

Proof. In the proof we will use the formula for the conformal rescaling of the metric g_{ik} and also the formula for mass of the matter contained in the volume element $d^4\Omega = \sqrt{|g|}d^4x$ of 4-dimentional space-time, where $g = det[g_{ik}]$. Hence

$$M = \sqrt{|g|}\rho d^4x,$$

where ρ is proper density of matter. After the conformal rescaling of the metric $d^4\Omega$ transforms into $d^4\widehat{\Omega} = \sqrt{|\widehat{g}|}d^4x$. The conformal rescaling of the metric does not change the mass included in the volume element $d^4\Omega$. It changes only into its density $\rho \to \widehat{\rho}$. As a consequence one has

$$M = \widehat{\rho} d^4\widehat{\Omega} = \sqrt{|\widehat{g}|}\widehat{\rho} d^4x,$$

where M indicates the mass of the matter contained in the element $\sqrt{|\widehat{g}|}d^4x$. Because this mass does not change under the conformal rescaling of the metric, we can compare both values and hence we get

$$\sqrt{|\widehat{g}|}\widehat{\rho} d^4x = \sqrt{|g|}\rho d^4x. \tag{3}$$

Using the formula for $\widehat{g} = \Omega^8 g$ and putting the expression $\sqrt{|\widehat{g}|} = \Omega^4\sqrt{|g|}$ in (3) we obtain the transformation law for the density of the matter under the conformal rescaling of the metric.

$$\Omega^4\sqrt{|g|}\widehat{\rho} d^4x = \sqrt{|g|}\rho d^4x$$

$$\widehat{\rho} = \Omega^{-4}\rho.$$

Returning to the initial equation which describes the matter tensor and using the transformtion formula for the density of the matter we obtain

$$\widehat{T}_{i.}^{.k} = \Omega^{-4}\rho u_i u^k + \widehat{p}\widehat{u}_i\widehat{u}^k - \widehat{p}\delta_i^k.$$

In order $\widehat{T}_{i.}^{.k}$ to be connected with $T_{i.}^{.k}$ by the transformation rule $\widehat{T}_{i.}^{.k} = \Omega^x T_{i.}^{.k}$, the pressure must change under the conformal rescaling of the metric following the formula

$$\widehat{p} = \Omega^{-4}p,$$

because

$$u_i u^k = \widehat{u}_i\widehat{u}^k, \quad \delta_i^k = \widehat{\delta}_i^k.$$

If this condition is satisfied, then the transformation formula has the following form

$$\widehat{T}_{i.}^{.k} = \Omega^{-4} T_{i.}^{.k}$$

Raising or lowering one of the indexes we obtain the other useful transformation rules

$$\widehat{T}^{ik} = \widehat{g}^{il} \widehat{T}_{i.}^{.k} = \Omega^{-6} T^{ik}$$

and

$$\widehat{T}_{ik} = \widehat{g}_{kl} \widehat{T}_{i.}^{.k} = \Omega^{-2} T_{ik}. \qquad \square$$

Fact 17. *Einstein's equations are not conformally invariant.*

Proof. In the proof we use the formula for the Einstein tensor after the conformal transformation of the metric, which we have proved in the Fact 15. Using the Einstein's equations in the metric g_{ik} and replacing the Einstein tensor by the matter tensor we get

$$\widehat{G}_{i.}^{.k} = \kappa \Omega^{-2} T_{i.}^{.k} + \frac{1}{2} \left(\Omega_{i.}^{.k} - \delta_i^k \Omega_{.a}^{a} \right). \tag{4}$$

Converting the relation between the matter tensor in the initial gauge and the new one we obtain

$$\widehat{T}_{i.}^{.k} = \Omega^{-4} T_{i.}^{.k} \implies T_{i.}^{.k} = \Omega^{4} \widehat{T}_{i.}^{.k}. \tag{5}$$

Replacing $T_{i.}^{.k}$ in (4) by (5) one has

$$\widehat{G}_{i.}^{.k} = \kappa \Omega^{2} \widehat{T}_{i.}^{.k} + \frac{1}{2} \left(\Omega_{i.}^{.k} - \delta_i^k \Omega_{.a}^{a} \right)$$

or

$$\widehat{G}_{i.}^{.k} = \kappa \Omega^{2} \widehat{T}_{i.}^{.k} + \kappa \widetilde{T}_{i.}^{.k},$$

where $\widetilde{T}_{i.}^{.k} = \frac{1}{\kappa} \frac{1}{2} \left(\Omega_{i.}^{.k} - \delta_i^k \Omega_{.a}^{a} \right)$. $\square$

The final formula shows that Einstein's equations are not conformally invariant, i.e. we do not have the formula of the form $\widehat{G}_{i.}^{.k} = \kappa \widehat{T}_{i.}^{.k}$. As a consequence there is a new additional matter described by the tensor $\widetilde{T}_{i.}^{.k}$, apart from the matter which had already existed before, the conformal rescaling of the metric, and satisfied Einstein's equations $G_{i.}^{.k} = \kappa T_{i.}^{.k}$.

This fact was used in the articles “An interesting property of the Friedman universes” [6], “On Energy of the Friedman Universes in Conformally Flat Coordinates” [7] and “Superenergy, conformal transformations, and Friedman universes” [8] written by J. Garecki. Namely, he used the conformal rescaling of the metric to create the Friedman dust universes from the empty Minkowskian space-time. One should emphasize in this context that Friedman universes are actually the best mathematical models of the real universe.

As a consequence of Einstein's Equivalence Principle the gravitional field does not have an energy-momentum tensor. It has only the energy-momentum pseudotensors. One such pseudotensor is given in the book “Teoria pola” by L. Landau i E. M. Lifshitz. It is described in the definition 11.

Definition 11. *The Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field is defined as* [1]

$$\begin{aligned}
t^{ik} =& \frac{1}{16\pi}\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km} - g^{ik}g^{lm}\right) \\
&+ \frac{1}{16\pi} g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}\right) \\
&+ \frac{1}{16\pi} g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right) \\
&+ \frac{1}{16\pi} g^{lm}g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp} - \Gamma^i_{lm}\Gamma^k_{np}\right).
\end{aligned}$$

Fact 18. *The Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field changes under the conformal transformation of the metric in the following way*

$$\begin{aligned}
\widehat{t}^{ik} =& \frac{1}{16\pi}\left(2\widehat{\Gamma}^n_{lm}\widehat{\Gamma}^p_{np} - \widehat{\Gamma}^n_{lp}\widehat{\Gamma}^p_{mn} - \widehat{\Gamma}^n_{ln}\widehat{\Gamma}^p_{mp}\right)\left(\widehat{g}^{il}\widehat{g}^{km} - \widehat{g}^{ik}\widehat{g}^{lm}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{il}\widehat{g}^{mn}\left(\widehat{\Gamma}^k_{lp}\widehat{\Gamma}^p_{mn} + \widehat{\Gamma}^k_{mn}\widehat{\Gamma}^p_{lp} - \widehat{\Gamma}^k_{np}\widehat{\Gamma}^p_{lm} - \widehat{\Gamma}^k_{lm}\widehat{\Gamma}^p_{np}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{kl}\widehat{g}^{mn}\left(\widehat{\Gamma}^i_{lp}\widehat{\Gamma}^p_{mn} + \widehat{\Gamma}^i_{mn}\widehat{\Gamma}^p_{lp} - \widehat{\Gamma}^i_{np}\widehat{\Gamma}^p_{lm} - \widehat{\Gamma}^i_{lm}\widehat{\Gamma}^p_{np}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{lm}\widehat{g}^{np}\left(\widehat{\Gamma}^i_{ln}\widehat{\Gamma}^k_{mp} - \widehat{\Gamma}^i_{lm}\widehat{\Gamma}^k_{np}\right) \\
&= \Omega^{-4}t^{ik} + \Omega^{-5}\frac{1}{16\pi}\left(I^{ik}_1 + I^{ik}_2\right) + \Omega^{-6}\frac{1}{16\pi}I^{ik}_3.
\end{aligned}$$

The analytical structures of I^{ik}_1, I^{ik}_2 and I^{ik}_3 are given in the proof.

Proof.

$$\begin{aligned}
\widehat{t}^{ik} =& \frac{1}{16\pi}\left(2\widehat{\Gamma}^n_{lm}\widehat{\Gamma}^p_{np} - \widehat{\Gamma}^n_{lp}\widehat{\Gamma}^p_{mn} - \widehat{\Gamma}^n_{ln}\widehat{\Gamma}^p_{mp}\right)\left(\widehat{g}^{il}\widehat{g}^{km} - \widehat{g}^{ik}\widehat{g}^{lm}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{il}\widehat{g}^{mn}\left(\widehat{\Gamma}^k_{lp}\widehat{\Gamma}^p_{mn} + \widehat{\Gamma}^k_{mn}\widehat{\Gamma}^p_{lp} - \widehat{\Gamma}^k_{np}\widehat{\Gamma}^p_{lm} - \widehat{\Gamma}^k_{lm}\widehat{\Gamma}^p_{np}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{kl}\widehat{g}^{mn}\left(\widehat{\Gamma}^i_{lp}\widehat{\Gamma}^p_{mn} + \widehat{\Gamma}^i_{mn}\widehat{\Gamma}^p_{lp} - \widehat{\Gamma}^i_{np}\widehat{\Gamma}^p_{lm} - \widehat{\Gamma}^i_{lm}\widehat{\Gamma}^p_{np}\right) \\
&+ \frac{1}{16\pi} \widehat{g}^{lm}\widehat{g}^{np}\left(\widehat{\Gamma}^i_{ln}\widehat{\Gamma}^k_{mp} - \widehat{\Gamma}^i_{lm}\widehat{\Gamma}^k_{np}\right).
\end{aligned}$$

Putting $\widehat{\Gamma}^n_{lm} = \Gamma^n_{lm} + P^n_{lm}$ in the equation, where

$$P^n_{lm} = \Omega^{-1}\left(\delta^n_l\Omega_{,m} + \delta^n_m\Omega_{,l} - g_{lm}g^{ne}\Omega_{,e}\right)$$

and $\widehat{g}^{mn} = \Omega^{-2}g^{mn}$ we obtain

$$
\begin{aligned}
&= \Omega^{-4}\frac{1}{16\pi}\left(2\Gamma^n_{lm}\Gamma^p_{np}-\Gamma^n_{lp}\Gamma^p_{mn}-\Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn}+\Gamma^k_{mn}\Gamma^p_{lp}-\Gamma^k_{np}\Gamma^p_{lm}-\Gamma^k_{lm}\Gamma^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn}+\Gamma^i_{mn}\Gamma^p_{lp}-\Gamma^i_{np}\Gamma^p_{lm}-\Gamma^i_{lm}\Gamma^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp}-\Gamma^i_{lm}\Gamma^k_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}\left(2\Gamma^n_{lm}P^p_{np}+2P^n_{lm}\Gamma^p_{np}-\Gamma^n_{lp}P^p_{mn}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\
&\quad-\Omega^{-4}\frac{1}{16\pi}\left(P^n_{lp}\Gamma^p_{mn}+\Gamma^n_{ln}P^p_{mp}+P^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{lp}P^p_{mn}+P^k_{lp}\Gamma^p_{mn}+\Gamma^k_{mn}P^p_{lp}+P^k_{mn}\Gamma^p_{lp}\right)\\
&\quad-\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{np}P^p_{lm}+P^k_{np}\Gamma^p_{lm}-\Gamma^k_{lm}P^p_{np}-P^k_{lm}\Gamma^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{lp}P^p_{mn}+P^i_{lp}\Gamma^p_{mn}+\Gamma^i_{mn}P^p_{lp}+P^i_{mn}\Gamma^p_{lp}\right)\\
&\quad-\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{np}P^p_{lm}+P^i_{np}\Gamma^p_{lm}+\Gamma^i_{lm}P^p_{np}+P^i_{lm}\Gamma^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(\Gamma^i_{ln}P^k_{mp}+P^i_{ln}\Gamma^k_{mp}-\Gamma^i_{lm}P^k_{np}-P^i_{lm}\Gamma^k_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}\left(2P^n_{lm}P^p_{np}-P^n_{lp}P^p_{mn}-P^n_{ln}P^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(P^k_{lp}P^p_{mn}+P^k_{mn}P^p_{lp}-P^k_{np}P^p_{lm}-P^k_{lm}P^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(P^i_{lp}P^p_{mn}+P^i_{mn}P^p_{lp}-P^i_{np}P^p_{lm}-P^i_{lm}P^p_{np}\right)\\
&\quad+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(P^i_{ln}P^k_{mp}-P^i_{lm}P^k_{np}\right).
\end{aligned}
$$

Hence the above equation we can write in the form

$$
\widehat{t}^{ik}=\Omega^{-4}t^{ik}+\Omega^{-5}\frac{1}{16\pi}\left(I^{ik}_1+I^{ik}_2\right)+\Omega^{-6}\frac{1}{16\pi}I^{ik}_3,
$$

where

$$
\begin{aligned}
I^{ik}_1 &= \left(2\Gamma^n_{lm}\widetilde{P}^p_{np}-\Gamma^n_{lp}\widetilde{P}^p_{mn}-\Gamma^n_{ln}\widetilde{P}^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\
&\quad+g^{il}g^{mn}\left(\Gamma^k_{lp}\widetilde{P}^p_{mn}+\Gamma^k_{mn}\widetilde{P}^p_{lp}-\Gamma^k_{np}\widetilde{P}^p_{lm}-\Gamma^k_{lm}\widetilde{P}^p_{np}\right)\\
&\quad+g^{kl}g^{mn}\left(\Gamma^i_{lp}\widetilde{P}^p_{mn}+\Gamma^i_{mn}\widetilde{P}^p_{lp}-\Gamma^i_{np}\widetilde{P}^p_{lm}-\Gamma^i_{lm}\widetilde{P}^p_{np}\right)\\
&\quad+g^{lm}g^{np}\left(\Gamma^i_{ln}\widetilde{P}^k_{mp}-\Gamma^i_{lm}\widetilde{P}^k_{np}\right),
\end{aligned}
$$

$$
\begin{aligned}
I_2^{ik} &= \left(2\widetilde{P}^n_{lm}\Gamma^p_{np} - \widetilde{P}^n_{lp}\Gamma^p_{mn} - \widetilde{P}^n_{ln}\Gamma^p_{mp}\right)(g^{il}g^{km} - g^{ik}g^{lm}) \\
&\quad + g^{il}g^{mn}\left(\widetilde{P}^k_{lp}\Gamma^p_{mn} + \widetilde{P}^k_{mn}\Gamma^p_{lp} - \widetilde{P}^k_{np}\Gamma^p_{lm} - \widetilde{P}^k_{lm}\Gamma^p_{np}\right) \\
&\quad + g^{kl}g^{mn}\left(\widetilde{P}^i_{lp}\Gamma^p_{mn} + \widetilde{P}^i_{mn}\Gamma^p_{lp} - \widetilde{P}^i_{np}\Gamma^p_{lm} - \widetilde{P}^i_{lm}\Gamma^p_{np}\right) \\
&\quad + g^{lm}g^{np}\left(\widetilde{P}^i_{ln}\Gamma^k_{mp} - \widetilde{P}^i_{lm}\Gamma^k_{np}\right), \\
I_3^{ik} &= \left(2\widetilde{P}^n_{lm}\widetilde{P}^p_{np} - \widetilde{P}^n_{lp}\widetilde{P}^p_{mn} - \widetilde{P}^n_{ln}\widetilde{P}^p_{mp}\right)(g^{il}g^{km} - g^{ik}g^{lm}) \\
&\quad + g^{il}g^{mn}\left(\widetilde{P}^k_{lp}\widetilde{P}^p_{mn} + \widetilde{P}^k_{mn}\widetilde{P}^p_{lp} - \widetilde{P}^k_{np}\widetilde{P}^p_{lm} - \widetilde{P}^k_{lm}\widetilde{P}^p_{np}\right) \\
&\quad + g^{kl}g^{mn}\left(\widetilde{P}^i_{lp}\widetilde{P}^p_{mn} + \widetilde{P}^i_{mn}\widetilde{P}^p_{lp} - \widetilde{P}^i_{np}\widetilde{P}^p_{lm} - \widetilde{P}^i_{lm}\widetilde{P}^p_{np}\right) \\
&\quad + g^{lm}g^{np}\left(\widetilde{P}^i_{ln}\widetilde{P}^k_{mp} - \widetilde{P}^i_{lm}\widetilde{P}^k_{np}\right)
\end{aligned}
$$

and

$$\widetilde{P}^p_{mn} = (\delta^p_m\Omega_{,n} + \delta^p_n\Omega_{,m} - g_{mn}g^{pe}\Omega_{,e}).$$

As we can see, the structures of I_1^{ik}, I_2^{ik} and I_3^{ik} are the same as the structure of t^{ik} with the change $\Gamma\Gamma \to \Gamma P$ in I_1^{ik}, $\Gamma\Gamma \to P\Gamma$ in I_2^{ik} and $\Gamma\Gamma \to PP$ in I_3^{ik}. □

Conformal rescaling of the metric creates an additional energy and momentum for the gravitation. It is represented by expressions with Ω^{-5} and Ω^{-6} in the transformation law of the Landau-Lifshitz pseudotensor $t^{ik} = t^{ki}$.

4. Conclusion

- I have proved several transformation formulas for various geometrical objects under the conformal rescaling of the metric on the Riemannian (or pseudoriemannian) manifold.
- Some of the formulas I have obtained have simpler form than the formulas given in standard books. One can easily check this fact by comparing our formulas with the ones given e.g. in "The Large Structure of Space-Time" by S. Hawking and G. F. R. Ellis.
- The basic equations of the general theory of relativity the Einstein equations are not conformally invariant. This fact means that there is a creation of the new matter and there is also possibility of creation of the Friedman universes from the vaccum.
- A new result of this paper is the transformation law for the Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field under conformal transformation of the metric. It is easily seen from this law that a conformal rescaling of the metric creates an additional energy and momentum for gravitation.

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Institute of Mathematics
University of Szczecin
Wielkopolska 15, PL-70-451 Szczecin, Poland
e-mail: marta.dudek@vp.pl

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NIEZMIENNIKI KONFOREMNEGO PRZESKALOWANIA CZASOPRZESTRZENI – BADANIE OBEJMUJĄCE KONSEKWENCJE DLA METRYKI

S t r e s z c z e n i e

Transformacje konforemne odgrywają istotną rolę w analizie globalnej struktury czasoprzestrzeni. Głównym celem tego artykułu jest sprawdzenie, które z geometrycznych i fizycznych obiektów są niezmiennikami konforemnego przeskalowania metryki oraz przedstawienie konsekwencji konforemnego przeskalowania metryki takich jak kreacja energii i pędu dla pola grawitacyjnego oraz kreacja materii.

Słowa kluczowe: transformacja konforemna metryki, ogólna teoria względności, rozmaitość riemannowska pseudoriemmanowska, lorentzowska, czasoprzestrzeń, równania Einsteina, pseudotensor energii-pędu Landaua-Lifszyca