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Dedicated to the memory of
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CONVERGENCE OF DIRICHLET SERIES ON A FINITE-DIMENSIONAL SPACE

Summary

We consider conditions for convergence of Dirichlet series on a finite-dimensional space in Stepanov's metric. Also, we obtain some applications for Stepanov's and Besicovitch's almost periodic functions.

Keywords and phrases: Dirichlet series, exponents of a Dirichlet series, Fourier series, Stepanov's metric, Besicovitch's metric, almost periodic function

Consider a Dirichlet series $\sum_k a_k e^{\lambda_k z},\ a_k \in \mathbb{C},\ \lambda_k \in \mathbb{R}$. In the paper [4] and [5], V. Stepanov obtained the following result:

Theorem S. Suppose that $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$. If $\lambda_{k+1} - \lambda_k > \alpha > 0,\ k \in \mathbb{Z}$, α does not depend on n, then the sums $S_N(x) = \sum_{k=-N}^{N} a_k e^{i\lambda_k x}$ form a Cauchy sequence with respect to the integral metric, namely

$$\sup_{y\in\mathbb{R}} \left(\int_y^{y+1} |S_M - S_N|^2 dx \right)^{\frac{1}{2}} \to 0 \quad M, N \to \infty.$$

The quantity

$$D_{S_l^p}[f(x),g(x)] = \sup_{x\in\mathbb{R}}[\frac{1}{l}\int\limits_x^{x+l}|f(y)-g(y)|^p dy]^{\frac{1}{p}}, \quad p\geq 1,$$

is called Stepanov's distance of order p ($p \geq 1$) associated with length l ($l > 0$). The corresponding metric is called Stepanov's one.

Here we assume that functions $f(x), g(x)$ are pth power integrable on each segment. Note that Stepanov's distances are equivalent for various $l > 0$; the space of functions with finite Stepanov's norm $D_{S_l^p}[f(x), 0]$ is complete (see [4]).

In our paper we prove an analogue of Theorem S on the space $\mathbb{R}^d$. In one-dimensional case our result is stronger than Theorem S.

We need some definitions and notations.

Let $B(x_0, r)$ be the open ball with center at the point $x_0 \in \mathbb{R}^d$ and radius $r > 0$, $\langle t, x\rangle$ be the scalar product on $\mathbb{R}^d$, and ω_d be the volume of a unit ball in $\mathbb{R}^d$.

Suppose that $f\colon \mathbb{R}^d \to \mathbb{C}$, $g\colon \mathbb{R}^d \to \mathbb{C}$ are measurable and L^p-integrable functions on each compact set.

Definition 1.

$$D_{S_l^p}[f(x),g(x)] = \sup_{x\in\mathbb{R}^d}\left[\frac{1}{\omega_d l^d}\int\limits_{B(x,l)}|f(y)-g(y)|^p dy\right]^{\frac{1}{p}}, \quad p\geq 1.$$

The metrics generating by these distances with different $l > 0$ are equivalent and complete, therefore we will take $l = 1$ and write D_{S^p} instead of $D_{S_1^p}$. Such distance is called Stepanov's metric.

By $SH(\mathbb{R}^d)$ denote the Schwartz space of smooth functions $f(x)$, $x \in \mathbb{R}^d$, with the following property: for any $m = (m_1, m_2, ..., m_d) \in (\mathbb{N}\cup\{0\})^d$ and for any $k \in \mathbb{N}$ the equality $\left(\frac{\partial^{m_1+m_2+\ldots+m_d}}{\partial x^{m_1}\partial x^{m_2}\ldots\partial x^{m_d}}f\right)(x) = \bar{o}\left(\frac{1}{|x|^k}\right), x \to \infty$ holds true.

Definition 2. (see [6]) The function $\widehat{f}(t) = \int\limits_{\mathbb{R}^d} f(x)e^{-i\langle t,x\rangle}dx$, $t \in \mathbb{R}^d$, is called the Fourier transform of $f(x) \in L^1(\mathbb{R}^d)$.

It is known (see, for example, [6], [8]), that the Fourier transform is the automorphism on $SH(\mathbb{R}^d)$.

Let $\{(a_n, \lambda_n)\}_{n=1}^{\infty}$ be a set of pairs where $a_n \in \mathbb{C}$, $\lambda_n \in \mathbb{R}^d$. Let $\Lambda = \bigsqcup\limits_{j=1}^{\infty}\Lambda_j$ be a partition of the set $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with the property $\operatorname{diam}\Lambda_j < 1$, $\quad j = 1, 2, \ldots$. Denote $S_N(x) = \sum\limits_{k=1}^{N} a_k e^{i\langle\lambda_k,x\rangle}$.

Theorem 1. *Suppose* $a_n > 0, 0 < r < \infty$. *Then*

$$\sum_{j=1}^{\infty}\left(\sum_{\lambda_n\in\Lambda_j} a_n\right)^2 \le C_1 \sup_N \int\limits_{B(0;r)} |S_N(x)|^2 dx,$$

where $C_1 = C_1(r,d)$.

Proof. Let $\varphi(x) \in SH(\mathbb{R}^d)$ be an even nonnegative function such that $\operatorname{supp}\varphi(x) \subset B(0,\frac{r}{2})$. Put $\psi(x) = \frac{1}{\delta^d}(\varphi*\varphi)(\frac{x}{\delta})$ for $\delta \in (0,1)$. Clearly, $\operatorname{supp}\psi(x) \subset B(0,\delta r)$ and $\widehat{\psi}(t) = |\widehat{\varphi}(\delta t)|^2 \ge 0$, $\widehat{\psi}(0) > 0$ and

$$\widehat{\psi}(t) \ge \varepsilon > 0, \quad t \in B(0,1) \tag{1}$$

for appropriate δ.

Let $M = \sup\limits_{\mathbb{R}^d} \psi(x)$. We have the following sequence of inequalities:

$$\int\limits_{B(0;r)} |S_N(x)|^2 dx \ge$$

$$\ge M^{-1}\int\limits_{\mathbb{R}^d} \psi(x)|S_N(x)|^2 dx = M^{-1}\int\limits_{\mathbb{R}^d}\psi(x)\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l e^{i\langle\lambda_n-\lambda_l,\, x\rangle}dx =$$

$$= M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l \int\limits_{\mathbb{R}^d}\psi(x)e^{i\langle\lambda_n-\lambda_l,\, x\rangle}dx = M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l \widehat{\psi}(\lambda_l-\lambda_n).$$

Since $\widehat{\psi}(t) \ge 0$ we omit all the terms where the elements λ_n, λ_k belong to different sets Λ_j and get the following inequalities:

$$M^{-1}\sum_{n=1}^{N}\sum_{l=1}^{N} a_n a_l \widehat{\psi}(\lambda_l-\lambda_n) \ge M^{-1}\sum_j \sum_{\substack{1\le n,l\le N\\ \lambda_n,\lambda_k\in\Lambda_j}} a_n a_l \widehat{\psi}(\lambda_l-\lambda_n) \ge$$

$$\ge M^{-1}\varepsilon\sum_j \sum_{\substack{1\le n,l\le N\\ \lambda_n,\lambda_l\in\Lambda_j}} a_n a_l = M^{-1}\varepsilon\sum_j\left(\sum_{\substack{1\le n\le N\\ \lambda_n\in\Lambda_j}} a_n\right)^2.$$

Thus,

$$\sum_j\left(\sum_{\lambda_n\in\Lambda_j} a_n\right)^2 \le C_1 \sup_N \int\limits_{B(0,r)} |S_N(x)|^2 dx.$$

This completes the proof of the Theorem. □

Define $T_m = \{(j,l) : m \le \mathrm{dist}(\Lambda_j, \quad \Lambda_l) < m+1\}$. Note that $\mathbb{N}^2 = \bigsqcup_{m=0}^{\infty} T_m$.

Let $\{B(x_j, 1)\}$ be a set of balls such that multiplicities of their intersections do not exceed h and $\Lambda_j \subset B(x_j, 1)$ for all $j \in \mathbb{N}$. Note that for a fixed k and any j such that $B(x_k, 2) \cap B(x_j, 2) \ne \emptyset$ we have $|x_j - x_k| < 4$ and $B(x_j, 1) \subset B(x_k, 5)$. Let M be a number of such balls $B(x_j, 1)$. The sum of volumes of these balls is at most $M\omega_d$. Clearly, $M\omega_d \le h5^d\omega_d$, therefore multiplicities of the system of the balls $B(x_j, 2)$ bound by $H = h5^d$. Replace each ball $B(x_j, 1)$ by some ball $B(x'_j, 1)$ with $x'_j \in \Lambda_j \subset B(x_j, 1)$. Note that $\Lambda_j \subset B(x'_j, 1)$. Since $B(x'_j, 1) \subset B(x_j, 2)$, we see that multiplicities of intersections of the system $\{B(x'_j, 1)\}$ are bounded by H. Hence we may suppose that $x_j \in \Lambda_j$.

Lemma. *For any $l, m \in \mathbb{N}$ the number of elements of the set $\{k \in \mathbb{N}\colon (k,l) \in T_m\}$ does not exceed $C_2 H m^{d-1}$, $C_2 = C_2(d)$.*

Proof. Let $(k,l) \in T_m$. We have $m \le \mathrm{dist}(\Lambda_k, \Lambda_l) \le |x_k - x_l| \le \mathrm{dist}(\Lambda_k, \Lambda_l) + 2 \le m+3$. Therefore, all balls $B(x_k, 1)$ with $(k,l) \in T_m$ are contained in the spherical layer $\{x : m-1 \le |x - x_l| \le m+4\}$. The volume of this spherical layer is $\omega_d((m+4)^d - (m-1)^d) \le C_2\omega_d m^{d-1}$, where C_2 depends on d only.

Hence a common value of the set T_m of balls $B(x_k, 1)$ with $(l,k) \in T_m$ does not exceed $C_2 H m^{d-1}$. □

Theorem 2. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$, $\Lambda = \bigsqcup_{j=1}^{\infty} \Lambda_j$, $\mathrm{diam}\,\Lambda_j < 1, j = 1, 2, \ldots$. Suppose that $\Lambda_j \subset B(x_j, 1)$, $x_j \in \Lambda_j$ and the multiplicities of intersections of the balls $B(x_j, 1)$ do not exceed h, also suppose that $\sum_{j=1}^{\infty} \left(\sum_{\lambda_n \in \Lambda_j} |a_n| \right)^2 = K^2 < \infty$ for some $a_n \in \mathbb{C}$.*

Then the following conditions are fulfilled:

$$a)\ \ D_{S^2}[S_N(x), 0] \le C_3 K,$$

where $S_N(x) = \sum_{k=1}^{N} a_k e^{i\langle \lambda_k, x\rangle}$, C_3 does not depend on N.

$$b)\ \ \lim_{M,N\to\infty} D_{S^2}[S_N(x), S_M(x)] = 0,$$

therefore the series $\sum_k a_k e^{i\langle\lambda_k, x\rangle}$ converges in the metric D_{S^2}.

Proof. Let $\varphi(x) \in \ SH(\mathbb{R}^d)$ be a function such that $\varphi(x) = 1,\ x \in B(0;1)$ and $\mathrm{supp}\varphi(x) \subset B(0,2),\ 0 \le \varphi(x) \le 1$.

Then

$$\int\limits_{B(y;1)} |S_N(x)|^2 dx \le \int\limits_{\mathbb{R}^d} \varphi(x-y) \sum_{1\le k\le N} \sum_{1\le l\le N} a_k \overline{a}_l e^{i\langle \lambda_k - \lambda_l,\, x\rangle} dx =$$

$$= \sum_{1\le k\le N}\sum_{1\le l\le N} a_k\overline{a}_l \int\limits_{\mathbb{R}^d} \varphi(x)e^{i\langle\lambda_k-\lambda_l,\, x+y\rangle}dx \le$$

$$\le \sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||\overline{a}_l| \left|\int\limits_{\mathbb{R}^d} \varphi(x)e^{i\langle\lambda_k-\lambda_l,\, x+y\rangle}dx\right| =$$

$$= \sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_l-\lambda_k)|.$$

Since $\widehat{\varphi} \in \; SH(\mathbb{R}^d)$, we get $|\widehat{\varphi}(x)| \le C_4 \min\{1, \frac{1}{|x|^{d+1}}\}$. After appropriate rearrangement of the summands

$$\sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)|$$

we get:

$$\sum_{1\le k\le N}\sum_{1\le l\le N} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| =$$

$$= \sum_{j}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k,\, \lambda_l\in\Lambda_j}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| +$$

$$+ \sum_{m=1}^{\infty}\sum_{(j,\,p)\in T_m}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k\in\Lambda_j,\, \lambda_l\in\Lambda_p}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| = \Sigma_1+\Sigma_2.$$

We estimate the sums Σ_1 and Σ_2 separately.

We have $|\widehat{\varphi}(\lambda_k-\lambda_l)| \le C_4$ for any j under the condition $\lambda_k, \lambda_l \in \Lambda_j$. Hence the next bound for Σ_1 holds:

$$\sum_{\substack{1<k,l<N\\ \lambda_k,\lambda_l\in\Lambda_j}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| \le C_4 \sum_{\lambda_k\in\Lambda_j}|a_k| \sum_{\lambda_l\in\Lambda_j}|a_l| = C_4\left(\sum_{\lambda_k\in\Lambda_j}|a_k|\right)^2,$$

Therefore,

$$\Sigma_1 \le C_4 K^2. \tag{2}$$

Further, for each fixed $m \ge 1$:

$$\sum_{(j,p)\in T_m}\sum_{\substack{1\le k,\, l\le N\\ \lambda_k\in\Lambda_j,\, \lambda_l\in\Lambda_p}} |a_k||a_l||\widehat{\varphi}(\lambda_k-\lambda_l)| \le C_4\frac{1}{m^{d+1}} \sum_{(j,p)\in T_m}\sum_{\substack{1\le k\le N\\ \lambda_k\in\Lambda_j}}|a_k| \sum_{\substack{1\le l\le N\\ \lambda_l\in\Lambda_p}}|a_l| \le$$

$$\leq \frac{1}{2} C_4 \frac{1}{m^{d+1}} \sum_{(j,p)\in T_m} \left(\left(\sum_{\lambda_k \in \Lambda_j} |a_k| \right)^2 + \left(\sum_{\lambda_l \in \Lambda_p} |a_l| \right)^2 \right) \tag{3}$$

Using Lemma and replacing the summation over p such that $(j,p) \in T_m$ by the summation over all $s \in \mathbb{N}$, we obtain the following estimate for (3):

$$\frac{C_2 C_4}{2} \frac{m^{d-1}}{m^{d+1}} \sum_s \left(\left(\sum_{\lambda_k \in \Lambda_s} |a_k| \right)^2 + \left(\sum_{\lambda_l \in \Lambda_s} |a_l| \right)^2 \right) = \frac{C_2 C_4}{m^2} \sum_s \left(\sum_{\lambda_l \in \Lambda_s} |a_l| \right)^2 .$$

Therefore,

$$\Sigma_2 \leq C_5 K^2. \tag{4}$$

Finally, taking into account (2) and (4), we obtain

$$\int\limits_{B(y;1)} |S_N(x)|^2 dx \leq C_6 \cdot K^2,$$

where C_6 does not depend on N. Hence, $D_{S^2}[S_N(x)] \leq C_3 \cdot K$, where C_3 does not depend on N, so the proposition a) is proved.

Prove the proposition b). Let $K_N^2 = \sum\limits_j \left(\sum\limits_{\substack{1\leq k\leq N \\ \lambda_k \in \Lambda_j}} |a_k| \right)^2$. Actually we have just proved the inequality

$$\sup_y \int\limits_{B(y,1)} |S_N(x)|^2 dx \leq (C_3 K_N)^2. \tag{5}$$

Substituting the sum $S_N(x) - S_M(x)$ for $S_N(x)$ in inequality (5), we get

$$D_{S^2}[S_N(x), S_M(x)] \leq C_3^2 (K_N^2 - K_M^2),$$

here $K_N^2 - K_M^2 = \sum\limits_j (\sum\limits_{\substack{M\leq n\leq N \\ \lambda_n \in \Lambda_j}} |a_k|)^2$.

Prove that $(K_N^2 - K_M^2) \to 0$ as $N, M \to \infty$. Assume that M is sufficiently large. By the condition $\sum\limits_j \left(\sum\limits_{\lambda_n \in \Lambda_j} |a_n| \right)^2 = K^2$, for each $\varepsilon > 0$ there exists $q \in \mathbb{N}$ (q does not depend on M and on N) such that $\sum\limits_{j=q+1}^{\infty} \left(\sum\limits_{\substack{M\leq n\leq N \\ \lambda_n \in \Lambda_j}} |a_n| \right)^2 \leq \frac{\varepsilon}{2}$.

Next, for each fixed $1 \le j \le q$ there exists M such that the inequality

$$\left(\sum_{\lambda_n \in \Lambda_j} |a_n|\right)^2 \le \frac{\varepsilon}{2q}$$

is satisfied for $n > M$. Then $\sum\limits_{j=1}^{q} \left(\sum\limits_{\substack{M \le n \le N \\ \lambda_n \in \Lambda_j}} |a_n|\right)^2 \le q \cdot \frac{\varepsilon}{2q} = \frac{\varepsilon}{2}$. Hence, for each $\varepsilon > 0$ we obtain $(K_N^2 - K_M^2) \le \varepsilon$. This completes the proof. □

Remark 1. Theorem 2 is true for $\operatorname{diam} \Lambda_j \le r, j = 1, 2, ...,$ and for the balls of radius $R \ge r$.

Suppose that there exists a set of balls $\{B(x_j, R)\}$ such that multiplicities of intersections of the balls do not exceed h, and the numbers of points $\lambda \in \Lambda$ contained in $B(x_j, R)$ are uniformly bounded.

Put $\Lambda_1 = \Lambda \cap B(x_1, R),\ \Lambda_2 = \Lambda \cap B(x_1, R) \setminus \Lambda_1,\ \Lambda_j = (\Lambda \cap B(x_1, R)) \setminus \bigcup\limits_{k=1}^{j-1} \Lambda_k$. The sets Λ_j satisfy all the conditions of Theorem 2 and for any j the number of elements Λ_j does not exceed some bound $s < \infty$.

Clearly, $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$ implies $\sum\limits_{j=1} \left(\sum\limits_{\lambda_n \in \Lambda_j} |a_n|\right)^2 \le \sum\limits_{j=1} s \sum\limits_{\lambda_n \in \Lambda_j} |a_n|^2 < \infty$.

We get the following consequence of Theorem 2:

Theorem 3. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\{B(x_j, R)\}$ be a set of balls such that multiplicities of intersections of the balls do not exceed h. Suppose that numbers of elements of the sets $\Lambda \cap B(x_j, R)$ are uniformly bounded for all $j \in \mathbb{N}$. If for some $a_n \in \mathbb{C}$ $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$, then the following conditions are fulfilled:*

$$a) \sup_N S_N(x) < \infty,$$

here $S_N(x) = \sum\limits_{k=1}^{N} a_k e^{i\langle \lambda_k, x\rangle}$.

$$b) \lim_{M,N \to \infty} D_{S^2}[S_N(x), S_M(x)] = 0.$$

Consider some applications of the obtained results.

Definition 3. (see [2] for the case d=1). Function $f(x) \colon \mathbb{R}^d \to \mathbb{C}$ is called Stepanov's almost periodic function of order p (S^p-almost periodic function) if there exists a sequence of finite exponential sums $S_n(x) = \sum\limits_j c_j e^{i\langle \lambda_j, x\rangle},\ c_j \in \mathbb{C},\ \lambda_j \in \mathbb{R}^d$, such that

$$\lim_{n \to \infty} D_{S^p}[f(x), S_n(x)] = 0.$$

To each S^p-almost periodic function $f(x),\ x \in \mathbb{R}^d$, we associate the Fourier series

$$f(x) \sim \sum_{\lambda \in \mathbb{R}^d} a(\lambda, f) e^{i\langle \lambda, x\rangle},$$

where $a(\lambda, f) = \lim\limits_{T\to\infty} \frac{1}{\omega_d T^d} \int\limits_{B(0,\,T)} f(x) e^{-i\langle\lambda,x\rangle} dx.$

Definition 4. (see [2] for the case $d = 1$ and [3] for the case $d > 1$) The spectrum of function $f(x)$ is the set $spf = \{\lambda \in \mathbb{R}^d : a(\lambda, f) \neq 0\}$.

It is well known (for the case $d = 1$ see [2], the proof for the case $d > 1$ can be treated in the same way) that spectrum of S^p-a.p.function is at most countable. The properties of the spectrum of the almost periodic functions in various metrics were considered in [7]. There were considered Stepanov's, Weil's and Besicovitch's almost periodic functions on $\mathbb{R}^d$.

Theorem 4. *For any set of pairs $\{(a_n, \lambda_n)\}_{n=1}^{\infty}$ that satisfy the conditions of Theorem 2 there exists S^2– almost periodic function $f(x)$ with Fourier series $\sum\limits_n a_n e^{i\langle\lambda_n,x\rangle}$.*

Proof. It follows from the completeness of the metric D_{S^2} and Theorem 2 that the sums $\sum\limits_{n\le N} a_n e^{i\langle\lambda_n,x\rangle}$ converge to $f(x)$ with respect to the metric D_{S^2}. □

Also we get

Theorem 5. *For any set of pairs $\{(a_n, \lambda_n)\}_{n=1}^{\infty}$ that satisfy the conditions of Theorem 3 there exists S^2– almost periodic function $f(x)$ with Fourier series $\sum\limits_n a_n e^{i\langle\lambda_n,x\rangle}$.*

Let the functions $f\colon \mathbb{R}^d \to \mathbb{C}$, $g\colon \mathbb{R}^d \to \mathbb{C}$ be measurable and L^p-integrable on each compact in $\mathbb{R}^d$.

Generalizing the definition of Besikovitch's distance (see [1]) for the function on $\mathbb{R}^d$ we have the following definition.

Definition 5. Put

$$D_{B^p}[f(x), g(x)] = \left\{ \overline{\lim_{T\to\infty}} \frac{1}{\omega_d T^n} \int\limits_{B(0,T)} |f(y) - g(y)|^p dy \right\}^{\frac{1}{p}}, \quad p \ge 1,$$

the metric generated by this distance is called Besicovitch's distance of order p.

Definition 6. (see [1] for the case d=1) Function $f(x)\colon \mathbb{R}^d \to \mathbb{C}$ is called Besicovitch's almost periodic function of order p (B^p–almost periodic function) if there exists a sequence of finite exponential sums $S_n(x) = \sum\limits_j c_j e^{i\langle\lambda_j,x\rangle}, c_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^d$, such that

$$\lim_{n\to\infty} D_{B^p}[f(x), S_n(x)] = 0.$$

Each B^p–almost periodic function $f(x),\ x \in \mathbb{R}^d$, has at most countable spectrum

$$\mathrm{sp}f = \{\lambda\colon a(\lambda, f) = \lim_{T\to\infty} \frac{1}{\omega_d T^d} \int\limits_{B(0,\,T)} f(x) e^{-i\langle\lambda,x\rangle} dx \neq 0\}.$$

Moreover, for each B^2 – almost periodic function f we have

$$\sum_{\lambda_n \in \mathrm{sp}f} |a(\lambda_n, f)|^2 < \infty.$$

The proof is similarly to the case $d = 1$.

Hence we obtain

Theorem 6. *Let $f(x),\ x \in \mathbb{R}^d$, be B^2– almost periodic function with the spectrum $\Lambda = \{\lambda_k\}_{k=1}^\infty$. Suppose that there exists a set of balls $\{B(x_j, R)\}$ such that the multiplicities of intersections do not exceed h, and numbers of elements $\lambda \in \Lambda \cap B(x_j, R)$ is uniformly bounded. Then the function $f(x)$ is S^2 – almost periodic.*

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O ZBIEŻNOŚCI SZEREGU DIRICHLETA W PRZESTRZENI SKOŃCZENIE WYMIAROWEJ

S t r e s z c z e n i e

Rozważamy warunki zbieżności szeregów Dirichleta w przestrzeni skończenie wymiarowej przy metryce Stepanova. Uzyskujemy też pewne zastosowania dla funkcji prawie okresowych Stepanova i Besicovitcha.

Słowa kluczowe: szereg Dirichleta, wykładniki w szeregu Dirichleta, szereg Fouriera, metryka Stepanova, metryka Besicovitcha, funkcje prawie okresowe