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Dedicated to the memory of
Professor Yurii B. Zelinskii

Jerzy Grzybowski and Ryszard Urbański

MODELING CRYSTAL GROWTH: POLYHEDRA WITH FACES PARALLEL TO THE PLANES $x_1 \pm x_2 \pm x_3 = 0$

Summary

A growing crystal can be understood as a series of polyhedra with faces parallel to planes from a fixed finite set. Such series of polyhedra forms a curve, often broken line, in a vector space of virtual polyhedra. In this paper we apply a modified Minkowski addition to study geometry of a cone of all polyhedra with faces parallel the faces of regular octahedron.

Keywords and phrases: crystal growth, abstract cone of convex polyhedra, modified Minkowski addition

1. Introduction

A natural representation of a monocrystal is a convex polyhedron in $\mathbb{R}^3$. The family of compact convex subsets in a topological vector spaces X was given a lot of attention, e.g. [5, 3]. In particular, the family of compact convex sets with Minkowski addition $A + B = \{a + b \,|\, a \in A, B \in B\}$ and multiplication by nonnegative number $\lambda A = \{\lambda a \,|\, a \in A\}$ is an abstract convex cone ordered by inclusion. Moreover, the order law of cancellation, $A + B \subset B + C$ implies $A \subset C$, holds true.

Minkowski subtraction, $A \dot{-} B = \{x \in X \,|\, x + B \subset A\}$, enabled us to give in [1] the formula $A(u) = \frac{u-s}{t-s} A(t) \dot{-} \frac{u-t}{t-s} A(s)$, $s < t < u$ which represents the uniform growth of a crystal.

In fact, a crystal of a specific chemical compound has finite number of possible faces and planes containing these faces are determined by its crystal structure. In [2], we restricted ourselves to study a subfamily of convex polyhedra with normal vectors belonging to some fixed finite subset of Euclidean unit sphere.

In this paper we continue this approach and study in Section 4 a particular family of polyhedra with faces parallel to the planes $x_1 \pm x_2 \pm x_3 = 0$. Such polyhedra have up to eight faces parallel to the faces of regular octahedron. Such shape take many crystals, for example crystals of fluorite CaF_2, anataz TiO_2, diamond, magnetite $Fe^{2+}Fe_2^{3+}O_4$, pyrite FeS_2, alabandite MnS, spinel $MgAl_2O_4$, wardite $NaAl_3(PO_4)_2(OH)_4{\cdot}2(H_2O)$, cuprite Cu_2O and sulphohalite.

In preparatory Section 2 we present a modified Minkowski addition for convex polyhedra with normal vectors belonging to some finite set G. In Section 3 we restrict ourselves to study a simpler family of polygons with sides parallel to the lines $x_1 = 0, x_1 \pm \sqrt{3}\,x_2 = 0$. Such polygons have sides parallel to the sides of regular hexagon. We present the structure of this family of polygons due to its relative simplicity and in order to make the study of polyhedral family easier to understand.

2. Family of convex polyhedra with faces parallel to planes from fixed finite set

Let $G = \{g_1, ..., g_m\}$ be such a finite subset of the Euclidean unit sphere S^{n-1} that $0 \in \operatorname{int}\operatorname{conv}G$. We call such a set G a *grid*.

Let $\mathcal{B}(\mathbb{R}^n), n \geqslant 2$ be a family of all nonempty compact convex subsets of $\mathbb{R}^n$. The family $\mathcal{B}(\mathbb{R}^n)$ with Minkowski addition and multiplication by nonnegative numbers is an abstract convex cone satisfying the order law of cancellation. In a Cartesian product $\mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n)$ we have a relation of equivalence defined in the following way: $(A, B) \sim (C, D)$ if and only if $A+D = B+C$. A quotient class $[A, B] = [(A, B)]$ of a pair of sets (A, B) is called a *virtual body*. The cone $\mathcal{B}(\mathbb{R}^n)$ can be embeded into a vector space $\mathrm{MRH}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)^2/_\sim$ of virtual bodies.

For a set $A \in \mathcal{B}(\mathbb{R}^n)$ a *support function* h_A of A is defined on $\mathbb{R}^n$ by $h_A(x) = \max_{a\in A}\langle a, x\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product. The correspondence between compact convex subsets of $\mathbb{R}^n$ and their support functions is called a *Minkowski duality.* Minkowski addition of two convex sets corresponds to pointwise addition of two support functions i.e. $h_{A+B} = h_A + h_B$. The inclusion of sets $A \subset B$ corresponds to the pointwise inequality of support functions $h_A \leqslant h_B$.

For a nonzero vector $z \in \mathbb{R}^n$ we define a *support set* $A(z) = \{a \in A \mid \langle a, z\rangle = h_A(z)\}$. Let $\mathcal{B}_G(\mathbb{R}^n) = \{A \in \mathcal{B}(\mathbb{R}^n) \mid A = \{x \in \mathbb{R}^n \mid \langle x, z_i\rangle \leqslant h_A(z_i), i = 1, ..., m\}\}$ be the family of all inersections of halfspaces determined by normal vectors belonging to the grid G. Elements of the family $\mathcal{B}_G = \mathcal{B}_G(\mathbb{R}^n)$ will be called *G-polyhedra*. G-polyhedra with nonempty interior are convex polyhedra. It can happen that G-polyhedra are polygons, line segments or singletons.

An important weakness of the family $\mathcal{B}_G$ is that Minkowski sum $A+B$ of G-polyhedra A and B may not be a G-polyhedron. This problem does not appear in case of G-polygons. Stroiński [6] characterized all grids in $\mathbb{R}^3$ for which Minkowski sum of two G-polyhedra is always a G-polyhedron. The same phenomenon happens if the cardinality m of the grid G is equal to $n+1$, because all G-polyhedra are singletons or mutually homotetic simplices. However, consider a grid of eight normal vectors to the octahedron which is a unit ball in $\mathbb{R}^3$ with norm $||\cdot||_1$ from l^1. Then segments I, J parallel respectively to x_1-axis and x_2-axis are G-polyhedra. However, $I+J$ is a rectangle parallel to x_1x_2-plane which is not a G-polyhedron.

For $A \in \mathcal{B}_G$ let $h^A = \{h_A(g_1), ..., h_A(g_m)\} \in \mathbb{R}^m$. Notice that $h_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a support function or the polyhedron A and h^A is a vector with coefficients being the values of h_A on elements of the grid G. Let $\mathcal{S}_G = \{h^A \,|\, A \in \mathcal{B}_G\} \subset \mathbb{R}^m$ be the set of all vectors h^A corresponding to some G-polyhedron. In a manner of [2] let us define G-sum of two G-polyhedra.

$$A \overset{G}{+} B := \{x \in \mathbb{R}^n \,|\, \langle x, g_i\rangle \leqslant h_i^A + h_i^B, i = 1, ..., m\}.$$

Then $A \overset{G}{+} B \in \mathcal{B}_G$, and we have $h^{A \overset{G}{+} B} = h^A + h^B \in \mathcal{S}_G$. Also $h^{tA} = th^A$ for $t \geqslant 0$. Hence the set $\mathcal{B}_G$ with the addition $\overset{G}{+}$ and multiplication by nonnegative numbers is an abstract convex cone, i.e. $(\mathcal{B}_G, \overset{G}{+})$ is a commutative semigroup with zero:

$$\begin{array}{lrcl}
\text{(i)} & (A \overset{G}{+} B) \overset{G}{+} C & = & A \overset{G}{+} (B \overset{G}{+} C), \\
\text{(ii)} & A \overset{G}{+} B & = & B \overset{G}{+} A, \\
\text{(iii)} & A \overset{G}{+} \{0\} & = & A,
\end{array}$$

for all $A, B, C \in \mathcal{B}_G$, and also

$$\begin{array}{lrcl}
\text{(iv)} & 1A & = & A, \\
\text{(v)} & t(sA) & = & (ts)A, \\
\text{(vi)} & (t+s)A & = & tA \overset{G}{+} sA, \\
\text{(vii)} & t(A \overset{G}{+} B) & = & tA \overset{G}{+} tB,
\end{array}$$

for all $A, B \in \mathcal{B}_G$ and $s, t \geqslant 0$.

Also $\mathcal{S}_G$ is a convex cone in $\mathbb{R}^m$.

On the other hand for $h \in \mathbb{R}^m$ we define a set $A_h := \{x \in \mathbb{R}^n \,|\, \langle x, z_i\rangle \leqslant h_i, i = 1, ..., m\} \in \mathcal{B}_G \cup \{\emptyset\}$. By $\mathcal{S}'_G = \{h \,|\, A_h \neq \emptyset\}$ we denote an *effective cone* or an *effective domain* of the correspondence $\mathbb{R}^m \ni h \longmapsto A_h \in \mathcal{B}_G \cup \{\emptyset\}$. Generally, the restricted correspondence $\mathcal{S}'_G \ni h \longmapsto A_h \in \mathcal{B}_G$ is not one-to-one. The set $\mathcal{S}_G$ is a polyhedral convex cone called a *proper cone* or a *proper domain.*

Obviously, $A_{h+g} = A_h \overset{G}{+} A_g$ and $A_{th} = tA_h$ for $h, g \in \mathcal{S}_G$ and $t \geqslant 0$.

Notice that $h^{tA} = th^A$ for negative t if and only if the set A is a singleton. In a similar way also $A_{th} = tA_h$ for negative t if and only if the set A_h is a singleton.

The mappings $\mathcal{B}_G \ni A \longmapsto h^A \in \mathcal{S}_G$ and $\mathcal{S}_G \ni h \longmapsto A_h \in \mathcal{B}_G$ are mutually inverse, and the abstract convex cones $(\mathcal{B}_G, \overset{G}{+}, \cdot)$ and $(\mathcal{S}_G, +, \cdot)$ are isomorphic. This way any polyhedron in $\mathcal{B}_G$ can be considered a point in $\mathcal{S}_G \subset \mathbb{R}^m$.

The vector h^A represents the polyhedron A in a manner of [4].

By $L_{\mathcal{S}'_G} := \mathcal{S}'_G \cap (-\mathcal{S}'_G)$ we denote the *lineality space* of the effective cone $\mathcal{S}'_G$. The lineality space $L_{\mathcal{S}'_G}$ is n-dimensional. It corresponds to the fact that if $k \in L_{\mathcal{S}'_G}$ then A_{h+k} is a translate of A_h for any given G-polyhedron A_h.

Obviously, the effective cone $\mathcal{S}'_G$ is a direct sum $L_{\mathcal{S}'_G} \oplus \mathcal{T}'_G$ of the lineality space $L_{\mathcal{S}'_G}$ and a *pointed effective cone* $\mathcal{T}'_G := \mathcal{S}'_G \cap L^{\perp}_{\mathcal{S}'_G}$, where $L^{\perp}_{\mathcal{S}'_G}$ is an orthogonal complement of the lineality space $L_{\mathcal{S}'_G}$. Let us define a *pointed proper cone* $\mathcal{T}_G := \mathcal{S}_G \cap L^{\perp}_{\mathcal{S}_G}$. Notice that the lineality spaces $L_{\mathcal{S}_G}$ and $L_{\mathcal{S}'_G}$ coincide.

3. Family of convex polygons with sides parallel to the lines $x_1 = 0, x_1 \pm \sqrt{3}\, x_2 = 0$

In this section we study a simpler case of two-dimensional G-polyhedra, convex polygons with sides parallel to sides of a regular hexagon. Let a grid G be the set of 6 vectors normal to sides of a regular hexagon, i.e. $g_1 = (0, 1)$, $g_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $g_3 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $g_4 = -g_1$, $g_5 = -g_2$ and $g_6 = -g_3$. For a given vector $h = (h_1, ..., h_6) \in \mathbb{R}^6$ we denote $A_h := \{x \in \mathbb{R}^3 | \langle x, g_i \rangle \leqslant h_i, i = 1, ..., 6\}$.

Let us see that a G-polyhedron A_h is nonempty if and only if $0 \leqslant h_i + h_{i+3}, i = 1, 2, 3$ and $0 \leqslant h_i + h_{i+1} + h_{i+2}, i = 1, 4$.

The effective cone $\mathcal{S}'_G \subset \mathbb{R}^6$ is a polyhedral subset of $\mathbb{R}^6$. The correspondence $\mathcal{S}'_G \ni h \longmapsto A_h \in \mathcal{B}_G$ is a multifunction. This correspondence is not one-to-one. For example $A_{(0,0,0,0,0,0)} = A_{(1,0,0,0,0,0)} = \{(0, 0)\}$.

We also have $\mathcal{S}_G = \{h^A = (h^A_1, ..., h^A_6) | A \in \mathcal{B}_G\}$. The correspondence $\mathcal{S}_G \ni h \longmapsto A_h \in \mathcal{B}_G$ is one-to-one. Let us notice that $h \in \mathcal{S}_G$ if and only if $h \in \mathcal{S}'_G$ and $h_1 \leqslant h_5 + h_6, h_2 \leqslant h_4 + h_6, h_3 \leqslant h_4 + h_5, h_4 \leqslant h_2 + h_3, h_5 \leqslant h_1 + h_3, h_6 \leqslant h_1 + h_2$.

The lineality space $L_{\mathcal{S}'_G} := \mathcal{S}'_G \cap (-\mathcal{S}'_G)$ of the effective cone $\mathcal{S}'_G$ is equal to $\{h | 0 = h_1 + h_2 + h_3, 0 = h_i + h_{i+3}, i = 1, 2, 3\}$. The following matrix corresponds to this system of four linear equalities.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since the rank of the matrix is four, the lineality space $L_{\mathcal{S}'_G}$ is two-dimensional.

The 3-dimensional set $\mathcal{U}'_G = \{h \in \mathcal{T}'_G | \sum_{i=1}^{6} h_i = 6\}$ is a section of the 4-dimensional pointed effective cone $\mathcal{T}'_G$ with a hyperplane defined by the equality $\sum_{i=1}^{6} h_i = 6$. Taking a hyperplane $\sum_{i=1}^{6} h_i = 1$ would be more natural, however in our case the set $\mathcal{U}'_G$ has vertices with integer coefficients, namely $\mathcal{U}'_G$ is a trigonal prism with the following 6 vertices: $h'^1 = (1, 4, 1, -1, 2, -1)$, $h'^2 = (1, 1, 4, -1, -1, 2)$, $h'^3 = (4, 1, 1, 2, -1, -1)$, $h'^4 = (-1, 2, -1, 1, 4, 1)$, $h'^5 = (-1, -1, 2, 1, 1, 4)$ and $h'^6 = (2, -1, -1, 4, 1, 1)$. A section $\mathcal{U}'_G$ and the lineality space $L_{\mathcal{S}'_G}$ fully describe the effective cone, since $\mathcal{S}'_G = L_{\mathcal{S}'_G} \oplus \bigcup_{t \geqslant 0} t\mathcal{U}'_G$.

The 3-dimensional section $\mathcal{U}_G := \{h \in \mathcal{T}_G | \sum_{i=1}^{6} h_i = 6\}$ of the pointed proper cone appears to be a trigonal bipyramid with the following 5 vertices $h^1 = (\frac{3}{2}, \frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2}, 0)$, $h^2 = (0, \frac{3}{2}, \frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2})$, $h^3 = (\frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2}, 0, \frac{3}{2})$, $h^4 = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ and $h^5 = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3})$. The following figure shows the bipyramid $\mathcal{U}_G$ contained in the trigonal prism $\mathcal{U}'_G$.

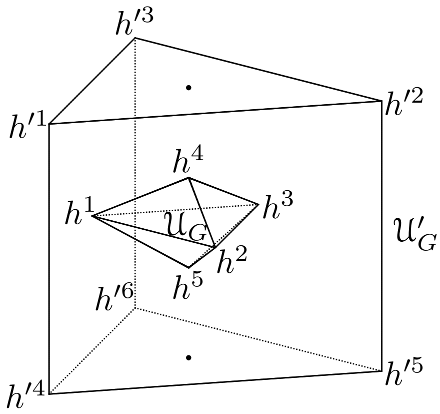

Fig. 1. The sections $\mathcal{U}'_G$ and $\mathcal{U}_G$.

In the following picture we show two-dimensional G-polygons corresponding to the vectors $h_1, ..., h_5$. In particular, the vector $h^4 = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ corresponds to a G-polygon $A_4 = A_{h^4} = \{x = (x_1, x_2) | \langle x, g_i \rangle \leqslant h_i^4, i = 1, ..., 6\} = \{x | -x_1 \leqslant \frac{2}{3}, \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2 \leqslant \frac{2}{3}, \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \leqslant \frac{2}{3}\}$. This G-polygon is an equilateral triangle with its center in $(0, 0)$ and horizontal lower side of the length $\frac{4}{3}\sqrt{3}$. A G-polygon $A_5 = A_{h^5}$ corresponding to the vector h^5 is a similar triangle with horizontal upper side. Three

G-polygons A_1, A_2, A_3 corresponding to the vectors h^1, h^2 and h^3 are segments of the length $2\sqrt{3}$. No G-polygon corresponds to vectors $h'^1, ..., h'^6$. However, for each $i = 1, ..., 6$ we have $h'^i = h^{A'_i} - h^{B'_i}$ where A'_i is an equilateral triangle and B'_i is a segment of the lenght $4\sqrt{3}$. The vector $\frac{1}{3}(h'_1 + h'_2 + h'_3)$, the center of the upper base of trigonal prism, corresponds to a pair of a triangle $\frac{1}{3}(A'_1 + A'_2 + A'_3)$ and a hexagon $\frac{1}{3}(B'_1 + B'_2 + B'_3)$, which can be reduced to a pair of triangles $(2A_4, A_5)$.

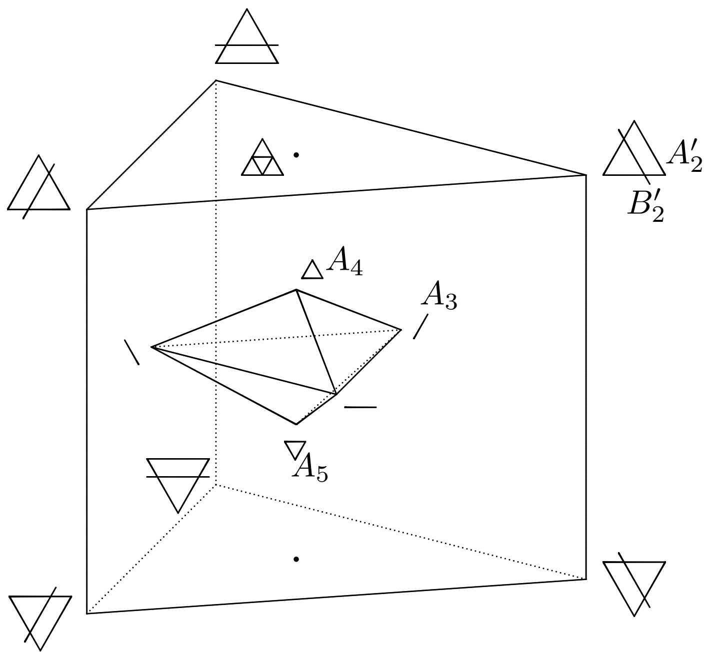

Fig. 2. The G-polygons corresponding to vectors $h^1, ..., h^5$ and pairs of G-polygons corresponding to vectors $h'^1, ..., h'^6$.

4. Family of convex polyhedra with faces parallel to planes $x_1 \pm x_2 \pm x_3 = 0$

Let a grid G be the set of 8 vectors normal to the faces of regular octahedron i.e. $g_1 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, $g_2 = (\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})$, $g_3 = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})$, $g_4 = (-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, $g_5 = -g_1$, $g_6 = -g_2$, $g_7 = -g_3$ and $g_8 = -g_4$. For a vector $h = (h_1, ..., h_8) \in \mathbb{R}^8$ we denote a convex set $A_h = \{x \in \mathbb{R}^3 | \langle x, g_i \rangle \leqslant h_i, i = 1, ..., 8\}$.

The set A_h, as an intersection of eight half-spaces, is nonempty if and only if $0 \leqslant h_i + h_{i+4}, i = 1, 2, 3, 4$, $0 \leqslant h_1 + h_2 + h_3 + h_4$ and $0 \leqslant h_5 + h_6 + h_7 + h_8$.

A cone $\mathcal{S}'_G = \{h | A_h \neq \emptyset\}$ is our effective cone. Moreover, our proper cone is $\mathcal{S}_G = \{h_A | A \in \mathcal{B}_G\} = \{h \in \mathcal{S}'_G | h_1 \leqslant h_6 + h_7 + h_8, h_2 \leqslant h_5 + h_7 + h_8, h_3 \leqslant h_5 + h_6 + h_8, h_4 \leqslant h_5 + h_6 + h_7, h_5 \leqslant h_2 + h_3 + h_4, h_6 \leqslant h_1 + h_3 + h_4, h_7 \leqslant h_1 + h_2 + h_4, h_8 \leqslant h_1 + h_2 + h_3\}$.

The lineality space $L_{\mathcal{S}'_G} = L_{\mathcal{S}_G}$ simultaneously of the effective cone $\mathcal{S}'_G$ and the proper cone $\mathcal{S}_G$ is the set $\{h \in \mathbb{R}^8 | 0 = h_1 + h_2 + h_3 + h_4, 0 = h_i + h_{i+4}, i = 1, 2, 3, 4\}$. The lineality space is three-dimensional.

As we know the cone $\mathcal{S}'_G$ is a direct sum of the lineality space $L_{\mathcal{S}}{}'$ and a pointed effective cone $\mathcal{T}'_G = \mathcal{S}'_G \cap L^{\perp}_{\mathcal{S}'_G}$. The pointed effective cone is 5-dimensional.

A 4-dimensional section of the effective cone $\mathcal{U}'_G = \{h \in \mathcal{T}'_G | \sum_{i=1}^{8} h_i = 8\}$ is a 4-dimensional prism with 3-dimensional tetrahedral bases having the following 8 vertices:

$$
\begin{aligned}
&h'^1 = (5,1,1,1,3,-1,-1,-1),\ h'^2 = (1,5,1,1,-1,3,-1,-1),\\
&h'^3 = (1,1,5,1,-1,-1,3,-1),\ h'^4 = (1,1,1,5,-1,-1,-1,3),\\
&h'^5 = (3,-1,-1,-1,5,1,1,1),\ h'^6 = (-1,3,-1,-1,1,5,1,1),\\
&h'^7 = (-1,-1,3,-1,1,1,5,1),\ h'^8 = (-1,-1,-1,3,1,1,1,5).
\end{aligned}
$$

The number 8 was chosen in such a way that coefficients of the vertices of the section $\mathcal{U}'_G$ are integer.

The 4-dimensional section $\mathcal{U}_G = \{h \in \mathcal{T}_G | \sum_{i=1}^{8} h_i = 8\}$ of the proper pointed cone $\mathcal{T}_G$ is a 4-dimensional polyhedron with the following 16 vertices:

$$
\begin{array}{ll}
h^1 = (2,2,0,0,2,2,0,0), & h^2 = (2,0,2,0,2,0,2,0),\\
h^3 = (2,0,0,2,2,0,0,2), & h^4 = (0,0,2,2,0,0,2,2),\\
h^5 = (0,2,0,2,0,2,0,2), & h^6 = (0,2,2,0,0,2,2,0),\\
h^7 = (\frac{5}{3},\frac{5}{3},\frac{5}{3},\frac{1}{3},1,1,1,-\frac{1}{3}), & h^8 = (\frac{5}{3},\frac{5}{3},\frac{1}{3},\frac{5}{3},1,1,-\frac{1}{3},1),\\
h^9 = (\frac{5}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3},1,-\frac{1}{3},1,1), & h^{10} = (\frac{1}{3},\frac{5}{3},\frac{5}{3},\frac{5}{3},-\frac{1}{3},1,1,1),\\
h^{11} = (1,1,1,-\frac{1}{3},\frac{5}{3},\frac{5}{3},\frac{5}{3},\frac{1}{3}), & h^{12} = (1,1,-\frac{1}{3},1,\frac{5}{3},\frac{5}{3},\frac{1}{3},\frac{5}{3}),\\
h^{13} = (1,-\frac{1}{3},1,1,\frac{5}{3},\frac{1}{3},\frac{5}{3},\frac{5}{3}), & h^{14} = (-\frac{1}{3},1,1,1,\frac{1}{3},\frac{5}{3},\frac{5}{3},\frac{5}{3}),\\
h^{15} = (\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}), & h^{16} = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}).
\end{array}
$$

The following nine figures are parallel three-dimensional slices of the four-dimensional polyhedron $\mathcal{U}'_G$ and contained in it polyhedron $\mathcal{U}_G$. And these two polyhedra are in turn bounded slices of the five-dimensional cones, respectively, of the pointed effective cone $\mathcal{T}'_G$ and the pointed proper cone $\mathcal{T}_G$. Figure 3 shows a base of the prism $\mathcal{U}'_G$. This base contains the four vertices $h'^i, i = 1, ..., 4$ and no points from the polyhedron $\mathcal{U}_G$. Comparing Figures 3,4 and 5 or by simple calculation we see that the vertices h'^1, h^{15} and h^{10} are colinear. Since $h'^1 - h^{15} = 3(h^{15} - h^{10})$, we have $h'^1 + 3h^{10} = 4h^{15}$. Hence the pair of a tetrahedron and a triangle $(4A_{15}, 3A_{10})$ where $A_i = A_{h^i}, i = 1, ..., 16$ represents the vertex h'^1. This pair is minimal, but we suspect that it is not the only minimal representation of h'^1. Figure 4 shows a parallel slice of $\mathcal{U}'_G$ which contains the vertex h^{15}. The corresponding G-polyhedron is the tetrahedron A_{15}.

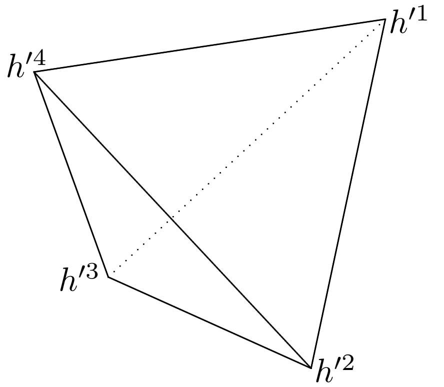

Fig. 3. A base of the prism $\mathcal{U}'_G$ contained in the hyperplane $\{h|\sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = 8\}$.

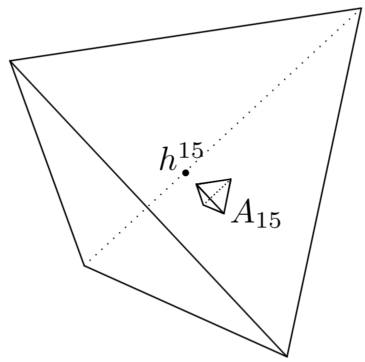

Fig. 4. Intersection of $\mathcal{U}'_G$ and the hyperplane $\{h|\sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = 4\}$.

Another parallel slice of the prism $\mathcal{U}'_G$ shown in Figure 5 containes a slice of polyhedron $\mathcal{U}_G$ which is a tetrahedron with vertices $h^i, i = 7, ..., 10$. G-polyhedra $A_i, i = 7, ..., 10$ corresponding to these vertices are triangles homothetic with the facets of the tetrahedron A_{15}. The tetrahedral slice of $\mathcal{U}_G$ is inscribed in the tetrahedral slice of $\mathcal{U}'_G$ with vertices of the former one being centers of the facets of the latter one.

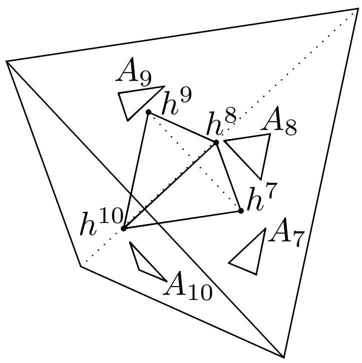

Fig. 5. Intersection of $\mathcal{U}'_G$ and the hyperplane $\{h|\sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = \frac{8}{3}\}$

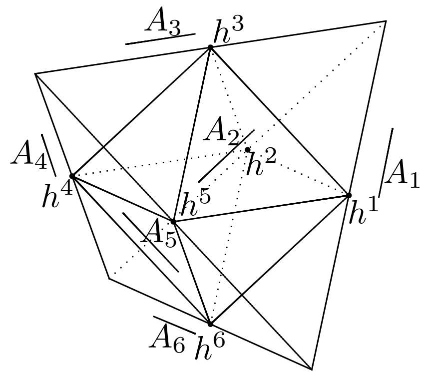

Fig. 6. Intersection of $\mathcal{U}'_G$ and the hyperplane $\{h|\sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = 0\}$.

In Figure 6, a middle slice of $\mathcal{U}'_G$ contains an octahedral slice of $\mathcal{U}_G$. G-polyhedra $A_i, i = 1, ..., 6$ corresponding to the vertices $h^i, i = 1, ..., 6$ are six line segments parallel to the edges of the tetrahedron A_{15}. Notice that an intersection of relative boundaries of the polyhedra $\mathcal{U}'_G$ and $\mathcal{U}_G$ is the union of four triangular bipyramids $\operatorname{conv}\{h^1, h^3, h^5, h^8, h^{12}\}$, $\operatorname{conv}\{h^1, h^2, h^6, h^7, h^{11}\}$, $\operatorname{conv}\{h^2, h^3, h^4, h^9, h^{13}\}$ and $\operatorname{conv}\{h^4, h^5, h^6, h^{10}, h^{14}\}$. The center of the octahedron is the vector $h = (1, 1, 1, 1, 1, 1, 1, 1)$. A corresponding G-polyhedron A_h is an octahedron. Since $h = \frac{1}{2}(h^1 + h^4) = \frac{1}{2}(h^2 + h^5) = \frac{1}{2}(h^3 + h^6) = \frac{1}{2}(h^{15} + h^{16})$, we obtain $A_h = \frac{1}{2}(A_1 \overset{G}{+}$

$A_4) = \frac{1}{2}(A_2 \overset{G}{+} A_5) = \frac{1}{2}(A_3 \overset{G}{+} A_6) = \frac{1}{2}(A_{15} \overset{G}{+} A_{16})$. Let us notice that in the semigroup of compact convex sets $\mathcal{B}(\mathbb{R}^3)$ an octahedron is indecomposable. It follows from indecomposability of its triangular faces. However, in $\mathcal{B}_G(\mathbb{R}^3)$ an octahedron A_h is a G-sum of two of its edges in three different ways. Minkowski sums $A_1 + A_4$, $A_2 + A_5$ and $A_3 + A_6$ are three mutually perpendicular squares. Also, an octahedron A_h is a G-sum of two tetrahedrons. However, a Minkowski sum $A_{15} + A_{16}$ is a couboctahedron.

Figure 7 is similar to Figure 5. We have $A_{11} = -A_7$, that is triangles A_{11} and A_7 are homothetic with a scale factor -1. In a similar way, $A_{12} = -A_8$, $A_{13} = -A_9$ and $A_{14} = -A_{10}$. Figure 8 is similar to Figure 4.

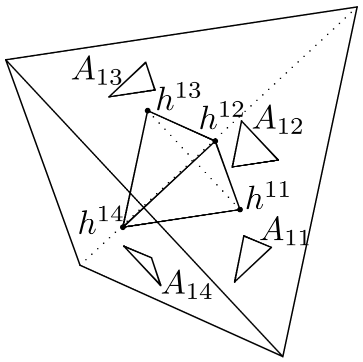

Fig. 7. Intersection of $\mathcal{U}'_G$ and the hyperplane $\{h | \sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = -\frac{8}{3}\}$.

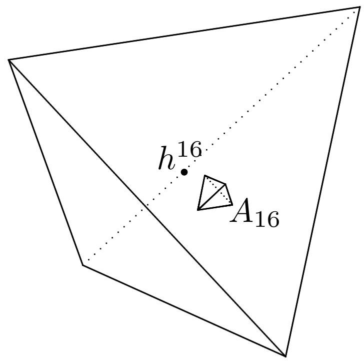

Fig. 8. Intersection of $\mathcal{U}'_G$ and the hyperplane $\{h | \sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = -4\}$.

We also have $A_{16} = -A_{15}$, and tetrahedrons A_{16} and A_{15} are also homothetic with a scale factor -1. Figure 9 is similar to Figure 3.

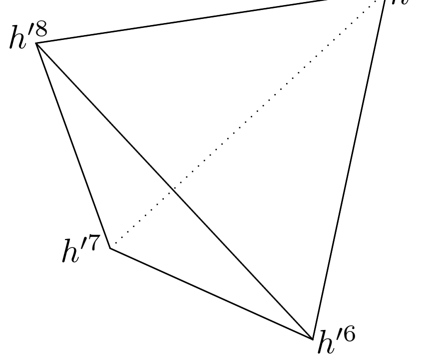

Fig. 9. Second base of $\mathcal{U}'_G$ contained in the hyperplane $\{h | \sum_{i=1}^4 h_i - \sum_{i=5}^8 h_i = -8\}$.

5. Conclusions

In [4] authors study families of polyhedra homothetic to summands of one fixed polyhedron. In this paper we replaced the restriction of "homothetic to summands of one fixed polyhedron" with "faces parallel to fixed set of planes", specifically, we study the cone of G-polyhedra "with faces parallel to the planes $x_1 \pm x_2 \pm x_3 = 0$". In [4] the authors do not modify Minkowski addition. Here we modify it and this G-addition is more adequate to the physical nature of crystal growth.

In this paper we show that a bounded slice of the cone of G-polyhedra is an intersection of a prism with tetrahedral base and a bipyramid with inverted tetrahedral base. The base of the prism and the base of the pyramid are homothetic with scale -1. The height of the prism is double height of the bipyramid. The intersection has 16 vertices corresponding to 2 tetrahedra, 8 faces of these tetrahedra and 6 edges. It means that exactly these 16 of G-polyhedra are indecomposable and any G-polyhedron is a positive combination of at most 5 of these indecomposable G-polyhedra.

We believe that understanding of the geometry of the cone of all G-polyhedra will enable us to reduce pairs of G-polyhedra to minimal pairs. Recently, Tomasz Stroiński communicated that he obtained an effective method of calculating a minimal pair contained in a given pair of G-polyhedra which is being prepared for publication.

Let us observe that the grid G that was studied in this paper is very small. It contains only 8 vectors. In spite of this the geometry of the cone of all G-polyhedra is quite complex. However, we hope that future research of larger grids will give us better tools for description, prediction and design of crystal shapes.

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Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87, PL-61-614 Poznań
Poland
E-mail: jgrz@amu.edu.pl
rich@amu.edu.pl

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MODELOWANIE WZROSTU KRYSZTAŁÓW: WIELOŚCIANY WYPUKŁE O ŚCIANACH RÓWNOLEGŁYCH DO PŁASZCZYZN $X_1 \pm X_2 \pm X_3 = 0$

S t r e s z c z e n i e

Rosnący kryształ można interpretować jako szereg wielościanów o ścianach równoległych do skończonego zbioru płaszczyzn. Taki szereg tworzy krzywą, na ogół linię łamaną, w wektorowej przestrzeni wielościanów wirtualnych. W tym artykule badamy geometrię stożka wszystkich wielościanów o ścianach równoległych do ścian ośmiościanu foremnego.

Słowa kluczowe: wzrost kryształu, abstrakcyjny stożek wielościanów wypukłych, zmodyfikowana suma Minkowskiego