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MONOGENIC FUNCTIONS WITH VALUES IN ALGEBRAS OF THE SECOND RANK OVER COMPLEX FIELD AND GENERALIZED BIHARMONIC EQUATION WITH SOME CHARACTERISTIC OF THE THIRD ORDER

Dedicated to the memory of prof. J. Ławrynowicz

Abstract

Among all two-dimensional algebras of the second rank with unit over the field of complex numbers, we find the biharmonic algebra $\mathbb{B}$, containing bases $\{e_1, e_2\}$ such that the $\mathbb{B}$-valued "analytic" functions $\Phi(xe_1 + ye_2)$, where x and y are real variables, satisfy a homogeneous partial differential equation of the fourth order with complex coefficients, which has some characteristic of the third order. The set of pairs $(\{e_1, e_2\}, \Phi)$ is described in the explicit form. Particular solutions of this PDE are found.

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1 Statement of the problems

Let D be a domain of the Cartesian plane xOy, $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{C}^+ := \mathbb{C} \setminus \{0\}$.

Consider the equation

$$Lu(x,y) := \left(b_1 \frac{\partial^4}{\partial y^4} + b_2 \frac{\partial^4}{\partial x \partial y^3} + b_3 \frac{\partial^4}{\partial x^2 \partial y^2} + \right.$$
$$\left. + b_4 \frac{\partial^4}{\partial x^3 \partial y} + b_5 \frac{\partial^4}{\partial x^4} \right) u(x,y) = 0 \quad \forall (x,y) \in D, \tag{1}$$

where $b_1 := 1$, $b_2 := -(1+3i)$, $b_3 := 3(i-1)$, $b_4 := 3+i$, $b_5 := -i$, i is the imaginary complex unit, a solution (complex-valued) $u\colon D \longrightarrow \mathbb{C}$ is assumed be such that functions

$$u_1(x,y) := \operatorname{Re} u(x,y), \quad u_2(x,y) := \operatorname{Im} u(x,y) \quad \forall (x,y) \in D, \tag{2}$$

have continuous partial derivatives up to the fourth order.

The characteristic equation of (1) has the form

$$l(s) := b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0 \quad \forall s \in \mathbb{C}, \tag{3}$$

Take into account the equality $l(s) = (s-i)^3\,(s-1)$ for all $s \in \mathbb{C}$, we go to decision that all roots of Eq. (3) are

$$\{b_1, -b_5\} := \ker l, \tag{4}$$

here the root $i = -b_5$ has the third multiplicity.

We call Equation (1) (a special case of) the *generalized biharmonic equation* (a motivation of this name is given in [6, Sect. 4]).

Obvious, a function $u(x,y) = u_1(x,y) + iu_2(x,y)$, $u_k\colon D \longrightarrow \mathbb{R}$, $k = 1,2$, is a solution of Equation (1) if and only if a pair (u_1, u_2) is a solution of the system of partial differential equations

$$\begin{cases} L_R\, u_1(x,y) = L_I\, u_2(x,y), \\ L_I\, u_1(x,y) = -L_R\, u_2(x,y) \quad \forall (x,y) \in D, \end{cases} \tag{5}$$

where L_R and L_I are operators of the type as in the left part in (1) with $b_k := \operatorname{Re} b_k$ and $b_k := \operatorname{Im} b_k$, $k = \overline{1,4}$, respectively.

By B_* we denote an associative algebra of the second rank with unit e and commutative over the field of complex numbers $\mathbb{C}$. Let $\{e_1, e_2\}$ be a basis of B_* satisfying the relation

$$\mathcal{L}(e_1,e_2) := b_1(e_2)^4 + b_2 e_1 (e_2)^3 + b_3 (e_1)^2 (e_2)^2 + b_4 (e_1)^3 e_2 + b_5 (e_1)^4 = 0. \tag{6}$$

We introduce the notation: $\mu_{e_1,e_2} := \{xe_1 + ye_2 : x, y \in \mathbb{R}\}$ (the linear span of the vectors e_1 and e_2 over the field of real numbers $\mathbb{R}$).

In what follows,

$$(x,y) \in D,\ \zeta := x + iy \in D_\zeta := \{\zeta = xe_1 + ye_2 : (x,y) \in D\} \subset \mu_{e_1,e_2},$$

$$z := x + iy \in D_z := \{z = x + iy : (x,y) \in D\} \subset \mathbb{C}.$$

In addition to conditions (6), we assume that the basis $\{e_1, e_2\}$ also satisfies the condition:

($*$) each nonzero element $h \in \mu_{e_1,e_2}$ is invertible (i.e., there exists an inverse element $h^{-1} \in \mathbb{B}_*$ such that $hh^{-1} = e$).

We shall say that a basis $\{e_1, e_2\}$ *satisfies condition* **(mb)** if it satisfies the condition (6) and ($*$) simultaneously.

For each required basis $\{e_1, e_2\}$ satisfying condition **(mb)**, we consider functions *monogenic* in D_ζ, i.e., functions $\Phi\colon D_\zeta \longrightarrow \mathbb{B}_*$ of the form

$$\Phi(\zeta) = U_1(x,y)\, e_1 + U_2(x,y)\, ie_1 + U_3(x,y)\, e_2 + U_4(x,y)\, ie_2 \quad \forall \zeta \in D_\zeta, \tag{7}$$

having the classical derivative $\Phi'(\zeta)$ at any point ζ in D_ζ:

$$\Phi'(\zeta) := \lim_{h \to 0,\, h \in \mu_{e_1,e_2}} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}.$$

We also denote each component $U_k\colon D \longrightarrow \mathbb{R}$ in (7) by $\mathrm{U}_k\,[\Phi]$, i.e., $\mathrm{U}_k\,[\Phi(\zeta)] := U_k(x,y)$, $k \in \{1,\ldots,4\}$.

If a monogenic function Φ has continuous derivatives $\Phi^{(k)}(\zeta)$ up to the k-th order, inclusively, $k \geq 4$, in the domain D_ζ, then, according to the relations

$$L\Phi(\zeta) = \mathcal{L}(e_1, e_2)\Phi^4(\zeta) = 0 \tag{8}$$

for any $\zeta \in D_\zeta$ (these relations are deduced by analogy with the corresponding relations in [1, Sec. 6] for a special case of the operator L of the type (1)) and equality (7), we conclude that the components U_k, $k = \overline{1,4}$, satisfy the equation (1) in the domain D, i.e, are its real-valued solutions. Thus, functions $u(x,y) \equiv u_{k,m}(x,y) := U_k(x,y) + iU_m(x,y)$, $k \neq m$; $k, m \in \overline{1,4}$, are solutions of the equation (1).

Note that hypercomplex "analytic" functions $\Phi(xe_1 + ye_2)$ with values in finite-dimensional algebras over the field of real (of dimension four) or complex (of dimension two) numbers whose components satisfy equations of the form (1) (mostly of elliptic type) were considered, e.g., in [7, 8, 9, 10, 11, 12].

Despite the availability of numerous works, the complete description of the indicated triples $\mathbb{B}_*$, $\{e_1, e_2\}$, Φ (or similar objects for the other definitions of "monogeneity") associated with Equation (1) has been unknown. Note, that the statement of this problem for the biharmonic equation and its solution has been done by I. P. Mel'nichenko in [13]; for the case $b_1 = b_5 = 1$, $b_2 = b_4 = 0$, $b_3 > 2$ (in (1)) this problem was formulated and solved in [1]; for $b_1 = 1 = b_5 = 1$, $b_2 = b_4 = 0$, $-2 < b_3 < 2$, its partial solution was found in [3]; for $b_1 = 1$, $b_5 = p^2$, $b_2 = b_4 = 0$, $b_3 = p^2 + 1$, $p > 0$, $p \neq 1$, its partial solution was found in [4]; the case when all four roots of the equation (3) are simple is considered in [6].

2 Commutative and associative algebras of the second rank and their bases associated with Equation (1)

It is known (cf., e.g., [14]) that there exist (to within an isomorphism) two associative algebras of the second rank with unit e commutative over the field of complex numbers $\mathbb{C}$:

$$\mathbb{B} := \{c_1 e + c_2 \rho : c_k \in \mathbb{C}, k = 1,2\},\ \rho^2 = 0, \tag{9}$$

$$\mathbb{B}_0 := \{c_1 e + c_2 \omega : c_k \in \mathbb{C}, k = 1,2\},\ \omega^2 = e. \tag{10}$$

The algenra $\mathbb{B}_0$ contains a basis of orthogonal idempotents $\{\mathcal{I}_1, \mathcal{I}_2\}$, where

$$\mathcal{I}_1 = \frac{1}{2}\left(e + \omega\right), \mathcal{I}_2 = \frac{1}{2}\left(e - \omega\right), \mathcal{I}_1\mathcal{I}_2 = 0,\ (\mathcal{I}_k)^2 = \mathcal{I}_k, k = 1,2. \tag{11}$$

It is obvious that

$$\mathcal{I}_1 + \mathcal{I}_2 = e, \mathcal{I}_1 - \mathcal{I}_2 = \omega. \tag{12}$$

An element $A = c_1 e + c_2 \rho$ in $\mathbb{B}$, $c_k \in \mathbb{C}$, $k = 1,2$, is invertible if and only if $c_1 \in \mathbb{C}^+$. If this condition is satisfied, then the following equality is true for the inverse element (see [15]):

$$A^{-1} = \frac{1}{c_1}\, e - \frac{c_2}{(c_1)^2}\, \rho. \tag{13}$$

The element $w = c_1\mathcal{I}_1 + c_2\mathcal{I} \in \mathbb{B}_0$, $c_k \in \mathbb{C}$, $k = 1, 2$, is invertible if and only if $c_k \in \mathbb{C}^+$, $k = 1, 2$. If this condition is satisfied, then the following equality is true for the inverse element (see [16, p. 38]):

$$w^{-1} = \frac{1}{c_1}\, \mathcal{I}_1 + \frac{1}{c_2}\, \mathcal{I}_2. \tag{14}$$

The theorem presented below gives the description of all couples $\mathbb{B}_*$, $\{e_1, e_2\}$, where a basis $\{e_1, e_2\}$ satisfies condition **(mb)**. In particular, it is established that $\mathbb{B}_* = \mathbb{B}$.

Theorem 2.1. *The algebra $\mathbb{B}_0$ does not contain any basis $\{e_1, e_2\}$ satisfying condition* **(mb)**.

All bases of the algebra $\mathbb{B}$ satisfying condition **(mb)** *can be represented in the form*

$$e_1 = \alpha\, e + \beta_1\, \rho, \quad e_2 = i\alpha\, e + \beta_2\, \rho, \tag{15}$$

where α is arbitrary complex number, complex numbers β_1 and β_2 satisfy condition

$$\beta_2 \neq i\beta_1. \tag{16}$$

Proof. We seek pairs of basis elements e_1, e_2 of the form

$$e_k = \alpha_k e + \beta_k \rho,\ k = 1, 2, \tag{17}$$

where the unknown complex coefficients α_k, β_k, $k = 1, 2$, satisfy the relation

$$\Delta_{e_1 e_2} := \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0. \tag{18}$$

It is easy to obtain formulas

$$(e_m)^k = (\alpha_m)^{k-1} (\alpha_m e + k\beta_m \rho)\,, \quad k = \overline{1,4}, m = \overline{1,2}. \tag{19}$$

Substituting (18) into (6) and taking into account (19), we get

$$\mathcal{L}(e_1, e_2) = b_1\alpha_2^3 (\alpha_2 e + 4\beta_2\rho) + b_2 (\alpha_1 e + \beta_1\rho)\, \alpha_2^2 (\alpha_2 e + 3\beta_2\rho) +$$
$$+ b_3\alpha_1\alpha_2 (\alpha_1 e + 2\beta_1\rho) (\alpha_2 e + 2\beta_2\rho) + b_4\alpha_1^2 (\alpha_1 e + 3\beta_1\rho) (\alpha_2 e + \beta_2\rho) +$$
$$+ b_5\alpha_1^3 (\alpha_1 e + 4\beta_1\rho) = A_e\, e + A_\rho\, \rho, \tag{20}$$

where

$$A_e := b_1\alpha_2^4 + b_2\alpha_2^3\alpha_1 + b_3\alpha_2^2\alpha_1^2 + b_4\alpha_2\alpha_1^3 + b_5\alpha_1^4,\ A_\rho := (b_2\beta_1 + 4b_1\beta_2)\, \alpha_2^3 +$$
$$+ (3b_2\beta_2 + 2b_3\beta_1)\, \alpha_1\alpha_2^2 + (2b_3\beta_2 + 3b_4\beta_1)\, \alpha_1^2\alpha_2 + \alpha_1^3 (b_4\beta_2 + 4b_5\beta_1)\,.$$

Hence, the required α_k, β_k, $k = 1, 2$, must satisfy the following system:

$$A_e = 0,\ A_\rho = 0,\ \Delta_{e_1 e_2} \neq 0. \tag{21}$$

Consider the first equation in (21). According to inequality $b_1 \neq 0$, we get that $\alpha_1 \neq 0$ (otherwise, $\alpha_1 = \alpha_2 = 0$, which contradicts the third relation in (21)), and the equality

$$\frac{\alpha_2}{\alpha_1} = b_* \quad \forall b_* \in \ker l. \tag{22}$$

holds. Dividing both sides of the second equation in (21) by α_1^3 and using (22), we get

$$-\, l_\circ(b_*)\beta_1 + l'\left(b_*\right)\beta_2 = 0, \tag{23}$$

where

$$l_\circ(s) := -\left(b_2 s^3 + 2b_3 s^2 + 3b_4 s + 4b_5\right) \quad \forall s \in \mathbb{C},$$

$l'(b_*)$ is the value of the derivative of the polynomial $l(s)$ from (3) for $s = b_*$. Taking into account that $b_* \in \ker l$, we have two cases: 1) $b_* = b_1$, 2) $b_* = -b_5$.

Let $b_* = b_1$. Then (22) turns onto

$$\alpha_2 = \alpha_1 = \alpha \in \mathbb{C}^+. \tag{24}$$

Simple computation shows that

$$l_\circ\left(b_1\right) \equiv l'\left(b_1\right) = -2\left(1 + i\right). \tag{25}$$

Therefore, $e_1 = \alpha\, e + \beta\, \rho \equiv e_2$ $(\alpha, \beta \in \mathbb{C}^+)$, which contradicts the linear independence of the pair e_1 and e_2 over the field of complex numbers. Thus, we arrive at the conclusion, that if $b_* = b_1$, then there is no required bases $\{e_1, e_2\}$ in $\mathbb{B}$.

Let $b_* = -b_5$. Then (22) turns onto

$$\alpha_2 = i\alpha_1 \equiv i\alpha,\ \alpha \in \mathbb{C}^+. \tag{26}$$

Due to the fact that the root $s = -b_5$ has the third multiplicity, we get $l'\left(-b_5\right) = 0$. Then it easy to show that $l_\circ\left(-b_5\right) = 0$. Therefore, the equation (23) is true for any $\beta_k \in \mathbb{C}$, $k = 1, 2$.

Among the obtained couples $\{e_1, e_2\}$, it is necessary to select the set of linearly independent couples. To this end, we check the validity of the third relation in system (21). Substituting (26) to (17), and then the letter to (18), we obtain $\Delta_{e_1, e_2} = \alpha\left(\beta_2 - i\beta_1\right) \neq 0$, $\alpha \in \mathbb{C}^+$, if and only if $\beta_2 \neq i\beta_1$.

Joining results of cases 1) and 2) we obtain the formulas (15).

Let us find necessary bases in the algebra $\mathbb{B}_0$. It easy to show that elements

$$e_k = \alpha_k\, \mathcal{I}_1 + \beta_k\, \mathcal{I}_2, k = 1, 2, \tag{27}$$

satisfy the equalities

$$e_k^n = \alpha_k^n\, \mathcal{I}_1 + \beta_k^n\, \mathcal{I}_2, n = \overline{1, 4}, k = 1, 2 \tag{28}$$

Denote $(e_k)^0 := 1, k = 1, 2,\ \lambda^0 := 1$ for all real λ. Then

$$\mathcal{L}(e_1, e_2) = \sum_{k=1}^{5} b_k \left(\alpha_2^{5-k}\mathcal{I}_1 + \beta_2^{5-k}\mathcal{I}_2\right)\left(\alpha_1^{k-1}\mathcal{I}_1 + \beta_1^{k-1}\mathcal{I}_2\right) =$$

$$= \sum_{k=1}^{5} b_k \left(\alpha_2^{5-k}\alpha_1^{k-1}\mathcal{I}_1 + \beta_2^{5-k}\beta_1^{k-1}\mathcal{I}_2\right).$$

Thus, the required system for the coefficients of the basis elements e_k, $k = 1, 2$, has the form

$$A_e \equiv \sum_{k=1}^{5} b_k \alpha_2^{5-k}\alpha_1^{k-1} = 0,\ \sum_{k=1}^{5} b_k \beta_2^{5-k}\beta_1^{k-1} = 0,\ \Delta_{e_1e_2} \equiv \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0. \quad (29)$$

As in (21), we see that $\alpha_1 \neq 0$. In a similar way, we consider the second equation in (29) and the relation $\Delta_{e_1e_2} \neq 0$, as a result, we obtain that $\beta_1 \neq 0$. With the use of elementary transformations we arrive at the conclusion that the system (29) is equivalent to the system:

$$l\left(\frac{\alpha_2}{\alpha_1}\right) = 0,\ l\left(\frac{\beta_2}{\beta_1}\right) = 0,\ \Delta_{e_1e_2} \neq 0. \quad (30)$$

All the solutions of the system (30) have the form:

$$\frac{\alpha_2}{\alpha_1} = \widetilde{s}_1,\ \frac{\beta_2}{\beta_1} = \widetilde{s}_2 \quad \forall\, \widetilde{s}_k \in \ker l, k = 1, 2, \widetilde{s}_1 \neq \widetilde{s}_2. \quad (31)$$

Then (27) takes the following form

$$e_1 = \alpha_1\,\mathcal{I}_1 + \beta_1\mathcal{I}_2 \quad e_2 = \widetilde{s}_1\alpha_1\,\mathcal{I}_1 + \widetilde{s}_2\beta_1\,\mathcal{I}_2.$$

Consider possible cases: 1) $\widetilde{s}_1 = -b_5 \equiv i$; 2) $\widetilde{s}_1 = b_1 \equiv 1$. In the first case we have $\zeta = xe_1 + ye_2 \equiv \alpha_1\,(x + iy)\,\mathcal{I}_1 + \beta_1\,(x + y)\,\mathcal{I}_2$. By the formula (14), we see that the inverse element ζ^{-1} does not exist for $y = -x$. Thus this basis $\{e_1, e_2\}$ does not satisfy the condition **(mb)**. By the similar arguments we can prove that ζ^{-1} does not satisfy the condition **(mb)** in the second case. This enables us to conclude that the required bases do not exist in the algebra (10).

The Theorem is proved. □

The equalities (15) yield relations

$$e = \frac{\beta_2\,e_1 - \beta_1\,e_2}{\alpha\,(\beta_2 - i\beta_1)}, \quad \rho = \frac{e_2 - ie_1}{\beta_2 - i\beta_1}, \quad (32)$$

and the both equalities follow the multiplication table for the basis $\{e_1, e_2\}$ in (15):

$$e_1^2 = \frac{\alpha}{\beta_2 - i\beta_1}\left((\beta_2 - 2i\beta_1)\,e_1 + \beta_1 e_2\right),$$

$$e_2^2 = \frac{\alpha}{\beta_2 - i\beta_1}\left(\beta_2 e_1 + (\beta_1 + 2i\beta_2)\,e_2\right),$$

$$e_1e_2 = \frac{\alpha}{\beta_2 - i\beta_1}\left(\beta_1 e_1 + \beta_2 e_2\right).$$

3 Monogenic functions related to Equation (1)

We consider monogenic functions functions $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$, where a basis $\{e_1, e_2\}$ has the form of (15). Each such function has four real components-functions U_k, $k = \overline{1,4}$, in (7).

As in the case where a biharmonic operator is considered instead of the operator L (cf., e.g., [17, 18]), we establish the following theorem:

Theorem 3.1. *function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is monogenic in the domain D_ζ if and only if its components $U_k\colon D \longrightarrow \mathbb{R}$, $k = 1,4$, in decomposition* (7) *are differentiable in the domain D and the following analog of the Cauchy–Riemann conditions is true*:

$$\frac{\partial\Phi(\zeta)}{\partial y}\, e_1 = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2 \quad \forall\, \zeta = xe_1 + ye_2 \in D_\zeta. \tag{33}$$

Using (15) we have

$$\zeta = xe_1 + ye_2 \equiv \alpha z\, e + (\beta_1 x + \beta_2 y)\, \rho \quad \forall\zeta \in \mu_{e_1,e_2}. \tag{34}$$

Similar to the proof of Theorems 1 and 2 in [18] with use of Theorem 3.1 and the equality (34), we obtain an expression of monogenic functions via holomorphic functions of complex variable z in the domain D_z.

Theorem 3.2. *A function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is monogenic in D_ζ if and only if the following equality is fulfilled*

$$\Phi(\zeta) = F(z)\, e + \left(\frac{\beta_1 x + \beta_2 y}{\alpha}\, F'(z) + F_0(z)\right) \rho \quad \forall\zeta \in D_\zeta, \tag{35}$$

where F, F_0 are some holomorphic functions of the complex variable z in the domain D_z, F' is the derivative of F.

Remark 3.3. *A particular case of Theorem 3.2 for $e_1 = e$ ($\alpha = 1$, $\beta_1 = 0$) follows also from the paper [11], where a "monogeneity" of function Φ is understud as differentiable in the sense of Gateaux and continuous in the domain D_ζ function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$. Note also that the existence of the basis* (15) *is postulated only (coefficients at e and ρ in* (15) *are not found in the explicit form).*

Corollary 3.4. *Every monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ has continuous derivatives $\Phi^{(n)}$ of any order n, $n = 1, 2, \ldots$ in the domain D_ζ. The components $U_k = \mathrm{U}_k\,[\Phi]$, $k = \overline{1,4}$, are infinitely continuously differentiable functions in the domain D and satisfy Equation (1) in this domain.*

It follows from Theorems 3.1 and 3.2 that $U_k = \mathrm{U}_k\,[\Phi]$, $k = \overline{1,4}$ are infinitely continuously differentiable functions in the domain D and, therefore, the equality (8) is valid.

Substituting (32) into (35) and replacing $\frac{F(z)}{\alpha(\beta_2 - i\beta_1)}$ onto $F(z)$ and $\frac{F_0(z)}{(\beta_2 - i\beta_1)}$ onto $F_0(z)$, we obtain an expression of monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ with respect to the basis $\{e_1, e_2\}$ in (15):

$$\Phi(\zeta) = \left(\beta_2 F(z) - i\,(\beta_1 x + \beta_2 y)\, F'(z) - iF_0(z)\right) e_1 +$$

$$+\left(-\beta_1 F(z)+\left(\beta_1 x+\beta_2 y\right) F'(z)+F_0(z)\right) e_2 \equiv$$

$$\equiv \mathrm{U}_{e_1}\left[\Phi(\zeta)\right]\, e_1+\mathrm{U}_{e_2}\left[\Phi(\zeta)\right]\, e_2 \quad \forall \zeta \in D_\zeta. \tag{36}$$

Consider notations:

$$V_1(x,y):=\operatorname{Re}\mathrm{U}_{e_1}\left[\Phi(\zeta)\right],\ V_2(x,y):=\operatorname{Im}\mathrm{U}_{e_1}\left[\Phi(\zeta)\right],$$

$$V_3(x,y):=\operatorname{Re}\mathrm{U}_{e_2}\left[\Phi(\zeta)\right],\ V_4(x,y):=\operatorname{Im}\mathrm{U}_{e_2}\left[\Phi(\zeta)\right].$$

Corollary 3.5. *For any complex numbers c_1 and c_2 functions*

$$u_{c_1,c_2}(x,y):=c_1\mathrm{U}_{e_1}\left[\Phi(\zeta)\right]+c_2\mathrm{U}_{e_2}\left[\Phi(\zeta)\right] \quad \forall (x,y)\in D, \tag{37}$$

are solutions of Equation (1).
For any real numbers a_k, $k=\overline{1,4}$, a function

$$u_{a_1,a_2,a_3,a_4}(x,y):=\sum_{k=0}^{4} a_k V_k(x,y), \tag{38}$$

is a real-valued solution of Equation (1).

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Funkcje monogeniczne o wartościach w algebrach drugiego rzędu nad ciałem zespolonym i uogólnione równanie biharmoniczne z pewną charakterystyką trzeciego rzędu

S t r e s z c z e n i e

Wśród wszystkich dwuwymiarowych algebr drugiego rzędu z jedynką nad ciałem liczb zespolonych znajdujemy algebrę biharmoniczną $\mathbb{B}$ zawierającą bazy $\{e_1, e_2\}$ takie, że $\mathbb{B}$-wartościowe funkcje "analityczne" $\Phi(xe_1+ye_2)$, gdzie x i y są zmiennymi rzeczywistymi, spełniają jednorodne równanie różniczkowe cząstkowe czwartego rzędu o współczynnikach zespolonych, które mają charakterystykę trzeciego rzędu. Zbiór par $(\{e_1, e_2\}, \Phi)$ jest opisany w formie jawnej. Wyznaczono rozwiązania szczególne tego równania różniczkowego cząstkowego.

Słowa kluczowe: przemienna algebra zespolona z jedynką, równania różniczkowe cząstkowe, ze zespolonymi współczynnikami czwartego rzędu, funkcja monogeniczna