DOI: 10.26485/978-83-60655-92-4/2 ISBN 978-83-60655-92-4

(HYPER)COMPLEX SEMINAR 2021
IN MEMORIAM OF
PROF. JULIAN ŁAWRYNOWICZ

*Maria Ławrynowicz, Małgorzata Nowak-Kępczyk, and Osamu Suzuki*

# BINARY AND TERNARY STRUCTURES OF THE EVOLUTIONS IN THE UNIVERSE (2×3×2×.... WORLD) V
## Non-commutative Galois theory of evolution

**Summary**

In a series of papers [19, 20, 21] and [11], a concept of non-commutative Galois theory is introduced and the evolutional system in physics, cosmology, biology and language are described in terms of the theory in a unified manner under the condition that the Galois group is solvable. Then in the evolutions the hierarchy structure can be realized by the following successive extensions of binary and ternary extensions: Namely, we have the following BTBB-structure:

$$B \Rightarrow T \Rightarrow B \Rightarrow B \Rightarrow B,$$

where $B$ (resp. $T$) is the binary (resp. ternary) extension and $\Rightarrow$ means the successive extensions. In this part a mathematical theory on non-commutative Galois extension is presented and the mathematical foundations are given. The BTBB-structure and its complexity system are discussed mathematically.

*Keywords and phrases:* binary structure, ternary structure, complexity system, fractal structure, Galois extension, Galois group.

## Contents

# Introduction

In Part I and II, we have given the definition of concepts of non-commutative binary and ternary extensions for several basic evolutions in this world and we have introduced concepts of a BTBB-structure and its complexity system. In part III, we have given a description of Chomsky theory in terms of non-commutative Galois theory. In Part IV we have given the description of complexity systems in terms of Galois theory. But till now we have not given its mathematical theory.

In this Part V, we introduce a concept of non-commutative Galois extension and give the BTBB-structure and its complexity system in a mathematical manner.

In Section 1, we recall basic facts on the classical Galois theory for an algebraic equation. In Section 2 we introduce a concept of non-commutative Galois theory and give the fundamental theory on the non-commutative Galois extensions. In Section 3 we give several examples of construction of non-commutative extensions. In Section 4 we restrict ourselves to the condition that the Galois group is solvable, we will determine its hierarchy structure. Namely we obtain the BTBB-structure and give its realization scheme. In Section 5, we will give examples of BTBB-structures in mathematics. The first one is just the classical Galois theory and the second one is the formal language theory. Here we will be concerned with with the evolution theory of the knot theory. In Section 6, we proceed to the complexity system of the BTBB-structure. Its Galois extension structure is quite simple and it is given by the successive binary extensions. We give several types of the basic types of binary extensions and show that the evolutionary system generates a fractal structure. We notice that we can see the table of the BTBB-structures and the complexity systems in Parts III, IV.

# 1. The classical Galois theory

We recall some basic facts on the classical Galois theory [36].

**The classical Galois theory**
We start with the solutions of an algebraic equation. We take a polynomial of degree $n$:

$$f_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$

We take the basic field $\mathbb{Q}$ of rational numbers. Namely, we assume that $a_0$, $a_1$,...,$a_n$ are rational numbers. We consider the solution:

$$f_n(\alpha_i) = 0 \text{ for } i = 1, 2, \ldots, n.$$

At first, we notice that we may assume that $a_1 = 0$ by use of Tschimhaus transformation. We omit the case $n = 1$.

($\mathbf{n = 2}$) We consider the equation: $f_2(x) = 0$, $f_2(x) = a_0x^2 + a_1x + a_2$. The solutions of the equation can be given as follows:

$$\therefore x + \frac{a_1}{2a_0} = \pm\frac{1}{2a_0}\sqrt{a_1^2 - 4a_0a_2}.$$

Here we assume that $a_0 = 1$, $a_1 = 0$. Putting $D_2 = a_1^2 - 4a_2$, we have the solutions:

$$x_1 = \sqrt{D_2}/2 \quad \text{and} \quad x_2 = -\sqrt{D_2}/2.$$

Hence we see that the solutions can be found in the binary extension $\mathbb{Q}[\sqrt{D_2}]$ and the both solutions introduce the Galois group of the binary extension by $\sqrt{D_2} \Leftrightarrow -\sqrt{D_2}$.

**($\mathbf{n = 3}$) Cardano's method:** Next we consider the equation: $f_3(x) = 0$, $f_3(x) = a_0x^3 + a_1x^2 + a_2x + a_3$. We solve the equation by Cardano's method: At first we notice that we may assume $a_1 = 0$, by use of the Tschirnhaus transformation. Putting $x = y - a/3y$, we have

$$y^3 - \frac{a^3}{27y^3} + b = 0$$

Hence we have

$$y^3 = \frac{-b \pm \sqrt{b^2 + 4a^3/27}}{2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a^2}{3}\right)^3}.$$

Putting

$$u = \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}, \quad v = \frac{-a}{3y} = \sqrt[3]{-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}},$$

we have the desired solutions:

$$x_1 = u + v, \quad x_2 = ju + j^2v, \quad x_3 = j^2u + jv, \quad (j^3 = 1).$$

Then we can summarize how we can find the solutions in the following manner:
**(1)** At first we make

$$\sqrt{D_3}, \quad D_3 = 4a^3 + 27b^2.$$

**(2)** Next we make

$$\sqrt[3]{-\frac{b}{2} \pm \frac{1}{b\sqrt{3}}\sqrt{D_3}}.$$

This implies that the solution can be obtained by adding $\sqrt{D_3}$ to $\mathbb{Q}$, and making the extension field:

$$F_1 = \left\{x_1 + x_2\sqrt{D_3} \mid x_1, x_2 \in \mathbb{Q}\right\}$$

at first. Then adding $\sqrt[3]{-\frac{b}{2} \pm \frac{1}{b\sqrt{3}}\sqrt{D_3}}$ and making

$$F_2 = \left\{x_1\sqrt[3]{-\frac{b}{2} + \frac{1}{b\sqrt{3}}\sqrt{D_3}} + x_2j\sqrt[3]{-\frac{b}{2} + \frac{1}{b\sqrt{3}}\sqrt{D_3}} + x_3j^2\sqrt[3]{-\frac{b}{2} + \frac{1}{b\sqrt{3}}\sqrt{D_3}}\right\}$$

we can find the solutions in this field.

**(n = 4) Ferrari's method:** We consider the following equation:

$$f_4(x) = 0, \;\; f_4(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4.$$

We may assume that $a_0 = 1$ and $a_1 = 0$ and we consider

$$f_4(x) = 0, \;\; f_4(x) = x^4 - ax^2 - 2bx - c.$$

We solve the equation by use of Ferrari's method: Rewriting $x^4 = ax^2 + 2bx + c$ and adding $2px^2 + p^2$ on the both sides, where $p$ will be determined later, we have

$$(x^2 + p)^2 = (a + 2p)x^2 + 2bx + (c + p^2) \qquad (*)$$

Here we assume that $b^2 - (a + 2p)(c + p)^2 = 0$. Then we have

$$2p^3 + ap^2 + 2cp + (ac - p^2) = 0.$$

Solving the equation by Cardano's method, we have the solutions of $p$. From (*), we have

$$(2 + 2p)x^2 + -2bx + (c + p^2) = (\sqrt{a + p}x + \sqrt{c + p^2})^2.$$

Hence, we have $x^2 + p = \pm(\sqrt{a + 2p}x + \sqrt{c + p^2})^2$. Hence we can obtain the desired solutions:

$$x = \begin{cases} \frac{1}{2}\left(\sqrt{a+2p} \pm \sqrt{a - 2p + 4\sqrt{p^2 + c}}\right), \\ \frac{1}{2}\left(-\sqrt{a+2p} \pm \sqrt{a - 2p - 4\sqrt{p^2 + c}}\right) \end{cases} \qquad \text{(Ferrari's formula)}$$

From this we can see that the solution can be obtained in the following process:

**(1)** At first we make a field of binary extension by adding $\sqrt{D_4}$ to $\mathbb{Q}$; where

$$D_4 = 16p^4x - 4p^3q^2 - 128p^2r^2 + 144pq^2r^{\smile}27q^4 + 256r^2,$$

where we have written $f_4(x) = x^4 + px^3 + qx^2 + r$.

**(2)** Next we add the $\sqrt[3]{*}$ to the field $\mathbb{Q}[\sqrt{D_4}, \sqrt[3]{*}]$, where * is defined by the solution of the intermediate equation of $3^{rd}$ degree (see: (*)).

**(3)** Finally we make two times binary extensions and can get the extension fields. Then we can find the desired solutions in this field.

**(n > 4) Abel's theorem**. In this case we see that the solutions can not be obtained in terms of the successive extension fields:

$$\sqrt{\;}, \quad \sqrt[3]{\;}, \quad \ldots, \quad \sqrt[m]{\;},$$

which is called Abel's theorem [36].

**Galois theory.** Galois analyzed the structure of the solutions and arrived at Galois theory. We consider the equation:

$$f_n(x) = 0, \;\; f_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n.$$

We denote the solutions by $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$. The solutions can be obtained in the complex number field $\mathbb{C}$ which is guaranteed by the famous Gauss theorem. Then putting $f_n(x) = a_0(x-\alpha_1)(x\,\check{}\,\alpha_2)\cdots(x-\alpha_n)$, we see that

$$\alpha_1+\alpha_2+\ldots+\alpha_n = -a_1/a_0, \quad \alpha_1\alpha_2+\ldots+\alpha_{n-1}\alpha_n = a_2/a_0, \ldots,$$

$$\ldots, \quad \alpha_1\alpha_2\cdots\alpha_n = (-1)^n a_n/a_0.$$

**(I)** The first step to the solutions is to make the determinant $D_n$ of the equation

$$D_n = \prod_{i<j}(\alpha_i\,\check{}\,\alpha_j)$$

and to construct the binary extension by adding the $D_n$ to $\mathbb{Q}$. Namely, we construct $\mathbb{Q}[D_n]$.

**(II)** We consider the group which is defined by the permutation of $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of the solution $f(x)=0$, which is denoted by $S_n$. We denote the group of the solutions by $Aut_0(\mathbb{Q}[\alpha_1,\alpha_2,\ldots,\alpha_n])$ $(=S_n)$. Here $Aut_0$ implies the automorphism group of the field fixing the base field $\mathbb{Q}$.

**(III)** The solvability condition can be given in terms of the subgroup structure of $S_n$ (see Section 2). We notice that the biggest subgroup of $S_n$ is the subgroup which keeps $D_n$ invariant. We call the subgroup the alternative group and denote it by $A_n$.

# 2. Non-commutative Galois theory

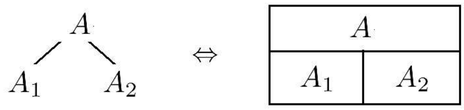
Here we give a general theory of non-commutative Galois extension. At first we give its definition and then we give the fundamental theorem.

### Binary non-commutative extension

We choose an algebra $A$ $(=A_0[x_1,\ldots,x_n])$ which is generated by $x_1,\ldots,x_n$ over an algebra $A_0$. For $A$ we introduce algebras $A_1$ $(=A[x_1^{(1)},x_2^{(1)},\ldots,x_n^{(1)}])$ and $A_2$ $(=A[x_1^{(2)},x_2^{(2)},\ldots,x_n^{(2)}])$. The pair $\{A_1,\ A_2\}$ is called a binary non-commutative Galois extension of $A$, when they satisfy the following conditions:

**(B-1):** There exists an algebraic homomorphism $\sigma_i : A \to A_i : \sigma_i(x_k) = x_k^{(i)}$, $(i = 1,2,\ k=1,2,\ldots,n)$. When it is an isomorphism, we call the extension symmetric. When it is not symmetric, the kernel of $\sigma_i$ is called symmetry breaking elements. When $A$ is generated by $\sigma^* = (\sigma_i^{(1)*},\sigma_i^{(2)*})$ with $\sigma_i^{(1)*}(*) = x_i^{(1)}$, $\sigma_i^{(2)*}(*) = x_i^{(2)}$, $(i=1,2,\ldots,n)$ from a single element * ("origin"), we call the $\sigma^* : \{*\} \to A_1,\ A_2$ the generation operator from the origin *.

**(B-2):** There exists an involution $\gamma^{(2)} :\ A_1 \to A_2$ with $(\gamma^{(2)})^2 = 1$.
In the following we denote the extension by the following diagram which we have used in Part III [11]:



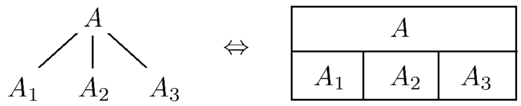
box representation

**Ternary non-commutative extension**

In a completely analogous manner, we can introduce the concept of the ternary extension: We choose an algebra $A$ $(= A_0[x_1, \dots, x_m])$ which is generated by $x_1, \dots, x_m$ over an algebra $A_0$. To $A$ we are choosing three algebras: $A_1$, $A_2$ and $A_3$. We introduce algebra $A_i$ $(= A_i[x_1^{(i)}, \dots, x_m^{(i)}])$, $(i = 1, 2, 3)$. The triple $\{A_1,\ A_2,\ A_3\}$ is called a ternary non-commutative Galois extension of $A$, when they satisfy the following conditions:

**(T-1):** There exists an algebraic homomorphism $\sigma_i\ :\ A \rightarrow A_i\ :\ \sigma_i(x_k) = x_k^{(i)}$, $(i = 1, 2, 3,\ k = 1, 2, \dots, n)$. When it is an isomorphism, we call the extension symmetric. When it is not symmetric, the kernel of $\sigma_i$ is called a symmetry breaking element. When $A$ is generated by $\sigma^* = (\sigma_i^{(1)*}, \sigma_i^{(2)*}, \sigma_i^{(3)*})$ with $\sigma_i^{(1)*}(*) = x^{(1)}$, $\sigma_i^{(2)*}(*) = x^{(2)}$, $\sigma_i^{(3)*}(*) = x^{(3)}$, $(i = 1, 2, \dots, n)$ from a single element * ("origin"), we call the $\sigma^*:\ \{*\} \rightarrow A_1, A_2, A_3$ the generation operator from the origin *.

**(T-2):** There exists a ternary involution $\gamma^{(3)}:\ A_1 \rightarrow A_2$, $\gamma^{(3)2}:\ A_2 \rightarrow A_3$, with $\gamma^{(3)3} = 1$.



box representation

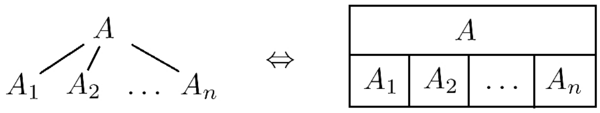
We can also introduce symmetry breaking elements. The succesive extension of binary/ternary extension can be introduced. We give several examples of constructions in Section 3.

**General non-commutative extension**

We choose an algebra $A$ $(= A_0[x_1, \dots, x_m])$ which is generated by $x_1, \dots, x_m$ over an algebra $A_0$. For $A$, choosing algebras: $A_1$, $A_2$, ... and $A_n$ we introduce algebra $A_1$ $(= A_0[x_1^{(1)}, x_2^{(1)}, \dots, x_m^{(1)}])$, ..., and $A_n (= A_0[x^{(n)_1}, x_2^{(n)}, \dots, x_m^{(n)}])$. The $n$-ple $\{A_1, \dots, A_n\}$ is called a $n$-nary non-commutative Galois extension of $A$, when they satisfy the following conditions:

**(G-1):** There exists an algebraic homomorphism $\sigma^{(i)} : A \rightarrow A_i$, $(i = 1, 2, \dots, n)$. When it is an isomorphism, we call the extension symmetric. When it is not symmetric, the kernel of $\sigma^{(i)}$ is called a symmetry breaking element. When $A$ is generated by $\sigma^{(n)*} = (\sigma_i^{(1)}, \dots, \sigma_i^{(n)})$ with $\sigma_1^{(1)}(*) = x_1^{(1)}$, ..., $\sigma_m^{(n)}(*) = x_m^{(n)}$ from a single element * ("origin"), we call the $\sigma^{(i)}:\ A \rightarrow A_i$ the generation operator of seeds.

**(G-2):** There exists an involution $\gamma^{(i)}:\ A_i \rightarrow A_{i+1}$ with $\gamma^{(1)} \cdots \gamma^{(n)} = 1$.
In the following we denote the extension by the following diagram which we have used in Part III.




box representation

**Fundamental theorem of non-commutative Galois extension**

We proceed to the hierarchy structure of the extensions. We consider the succesive extensions adding $Q_i$ $(i = 1, 2, \dots, n)$. Putting $A_k = A[Q_1, Q_2, \dots, Q_k]$ $(k = 1, 2, \dots, n)$, we consider the sequence:

$$(A =)A_0 \ A_1 \ A_2 \ \dots \ A_k \ \dots \ A_n \ (= A[Q_1, Q_2, \dots, Q_n]).$$

Then we have the sequence of Galois groups $\{G_j : \ j = n, n-1, \dots, 0\}$ which is called the tower of subgroups of $S_n$:

$$(S_n) = G_n \sqsupset G_{n-1} \sqsupset G_{n-2} \sqsupset \dots \sqsupset G_k \sqsupset \dots \sqsupset G_0(= E)$$

when the following conditions are satisfied: (1) $G_k A_k \subset A_k$ (2) $g|A_{k-1} =$ identity $(g \in G_k)$. We denote the duality between the above structures as follows:

$$\{A_0 : A_1 : A_2 : \dots : A_k : \dots : A_n\} \ \leftrightarrow \ \{G_n : G_{n-1} : G_{n-2} : \dots : G_k : \dots : G_0\}$$

$$A_1/I(A_0) \Leftrightarrow G_n/G_{n-1}, \ \dots \ , A_{k+1}/I(A_k) \Leftrightarrow G_k/G_{k-1}, \ \dots, A_n/I(A_{n-1}) \Leftrightarrow G_1/G_0,$$

where $I(A_k)$ is an ideal generated by $A_k$.

**Remark** (Classical vs non-commutative Galois extension)
At first we notice that a non-commutative Galois extension can be reduced to a classical one, when $Q_1 = x_1 \otimes E_n$ and the solution of $F(x) = f(x) \otimes E_n$. Only because of this reason, we call our theory non-commutative Galois theory. We see that the characters are quite different from each other.

**The fundamental theorem of non-commutative Galois theory**

We can state the fundamental theorem in our Galois theory which can describe the hierarchy structures not only in mathematics, but also in other fields in the real world.

**Fundamental theorem**
We can prove the following assertions:

(i) If $n > 5$, then $S_n$ has only non trivial subgroup $A_n$, the alternating subgroup.

(ii) If $n = 4$, we have the following sequence of subgroups:

$$S_4 \sqsupset A_4 \sqsupset V_4 \sqsupset Z_2 \sqsupset E,$$

where $V_4$ is the Klein group and $Z_2$ is the cyclic group of order 2

(iii) If $n = 3$, we have the following sequence:

$$S_3 \sqsupset A_3 \sqsupset E$$

(iv) If $n = 2$, we have the following sequence: $S_2(= A_2) \sqsupset E$

# 3. Several constructions of non commutative Galois theories

Here we give several constructions of non-commutative extensions.

**Construction 1 (Extension of polynomial type)**
We take an algebra $A$ and consider a polynomial with coefficients in $A$:

$$F_n^*(x) = F_0x^n + F_1x^{n-1} + F_2x^{n-2} + \ldots + F_{n-1}x + F_n \quad (F_0, F_1, \ldots.., F_n \in A)$$

We assume the following decomposition holds:

$$F_n^*(x) = F_0(x \otimes E - Q_1)(x \otimes E\check{}Q_2)\ldots(x \otimes E\check{}Q_n).$$

Here we assume that there exists at least one $Q_i$ with $Q_i \notin A$. We call the algebra $A[Q_1, \ldots, Q_m]$ which is generated by $A$ and $Q_1$, $Q_2$, …, $Q_n$ non-commutative Galois extension of $A$ of polynomial type. We consider the substitution: $Q_i \leftrightarrow Q_j$. Then we can introduce the concept of Galois group for the non-commutative Galois extension. In order to make our idea clear, we propose the following table:

**Comparisons of commutative vs non-commutative Galois theory**

| Classical Galois theory | Non-commutative Galois theory |
|---|---|
| $a_0, a_1, \ldots, a_n \in Q$ | $F_0, F_1, \ldots, F_n \in A(\subset M_n(\mathbb{R})$ |
| $f(x) = a_0 + a_1x + a_2x^2a + \cdots + a_nx^n$ | $F^*(x) = F_0 + F_1x + F_2x^2 + \ldots + F_nx^n$, $(\lvert F_0\rvert \neq 0)$ |
| $f(x) = a_n(x - \alpha_1)(x\check{}\alpha_2)\cdots(x - \alpha_n)$ | $F(x) = A_n(x \otimes E - Q_1)(x \otimes E - Q_2)\cdots$ $\cdots(x \otimes E - Q_n)$. |
| $S_n = \{\alpha_i \to \alpha_j\}$ | $S_n = \{Q_i \to Q_j\}$ |

The group $S_n$ is called the Galois group of the extension.

**Binary and ternary extensions**
Next we proceed to binary and ternary extensions which will play the basic roles in the applications. We take an algebra $A = A_0[a_1, a_2, \ldots, a_n]$ and consider binary and ternary extensions.

**(1) Binary extension:** We consider the binary extension:

$$(x\check{}Q_1)(x\check{}Q_2) = x^2\check{}x(Q_1 + Q_2) + Q_1Q_2,$$

where $Q_1 + Q_2 \in A$, $Q_1Q_2 \in A$. Here we write $x \otimes 1$ as $x$ simply and we denote $Q_2$ by $Q_1^*$.
In this case the Galois group is generated by $\gamma^{(2)} :\ A \to A$, $\gamma^{(2)}(Q_1) = Q_1^*$.

**(2) Ternary extension:** Next we proceed to the ternary extension:

$$(x\check{}Q_1)(x\check{}Q_2)(x\check{}Q_3) =$$
$$= x^3\check{}x^2(Q_1 + Q_2 + Q_3) + x(Q_1Q_2 + Q_1Q_3 + Q_2Q_3) - Q_1Q_2Q_3,$$

where $Q_1 + Q_2 + Q_3 \in A$, $Q_1Q_2 + Q_1Q_3 + Q_2Q_3 \in A$, and $Q_1Q_2Q_3 \in A$. Then: Putting $Q_2 = Q_1^*$, $Q_3 = Q_2^*(= Q_1^{**})$, $Q_1 = Q_3^*$, we can define the ternary involution:

$$\gamma^{(3)} : A \to A, \;\; \gamma^{(3)}(Q_1) = Q_1^*, \;\; \gamma^{(3)}(Q_1^*) = Q_1^{**}, \;\; \gamma^{(3)}(Q_1^{**}) = Q_1, \;\; \text{and} \;\; \gamma^{(3)3} = 1.$$

**Examples**
We give examples of simple binary and ternary non-commutative extensions of polynomial type: We choose

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have
(1) $s_i^2 = 1$, $(i = 1, 2, 3)$,
(2) $s_1s_2 = -s_2s_1 = s_3, \;\; s_1s_3 = -s_3s_1 = s_2, \;\; s_2s_3 = -s_3s_2 = s_1.$

**(1) A binary extension:** We consider the following non-commutative polynomial:

$$F_2(x) = (x^{\vee} s_1)(x^{\vee} s_2) = x^2{}^{\vee} x(s_1 + s_2) + s_1s_2.$$

We put

$$T = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : \quad a, b, c \in \mathbb{R} \right\}.$$

Since

$$s_1 + s_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad s_1s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we see that $F_2(x)$ is a polynomial on $T$. We put

$$T_1 = s_1T \;\text{ and }\; T_2 = s_2T \left( = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} \right).$$

Hence putting $Q_1 = s_1$ and $Q_2 = s_2$, we have a binary extension:

$$\begin{array}{ccccccc} & E & & & & T & \\ \diagup & & \diagdown & \Rightarrow & \diagup & & \diagdown \\ Q_1 & & Q_2 & & T_1 & & T_2 \end{array}$$

The Galois group is generated by Ad $s_3$.

**(2) Ternary extension:** We consider the following non-commutative polynomial:

$$F_3(x) = (x^{\vee} s_1)(x^{\vee} s_2)(x^{\vee} s_3) = x^3 I_2{}^{\vee} x^2(s_1+s_2+s_3) + x(s_1s_2 + s_1s_3 + s_2s_3) + s_1s_2s_3.$$

Since

$$s_1+s_2+s_3 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad s_1s_2+s_1s_3+s_2s_3 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad s_1s_2s_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that each element is in $T$. Putting $Q_1 = s_1$, $Q_2 = s_2$ and $Q_3 = s_3$, where $T_i = s_iT$, $(i = 1, 2, 3)$, we have the following ternary extension:

$$\begin{array}{ccc} & T & \\ \swarrow & \downarrow & \searrow \\ T_1 & T_2 & T_3 \end{array}$$

The Galois group is given by Ad $s_3 : T_1 \to T_2$, Ad $s_1 : T_2 \to T_3$, Ad $s_2 : T_3 \to T_1$.

**Construction 2 (Extension of Clifford algebra type)**
We show that the usual Clifford algebra, which is called binary Clifford algebra here and the ternary Clifford algebra, which is introduced by J. Sylvester and rediscovered by R. Kerner give examples of non-commutative Galois extensions.

**(1) Binary Clifford algebra:**
We take the Quaternion numbers $\mathbb{H}$:

$$\mathbb{H} = \{x_0 1 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} : \ \ x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

satisfying

(1) $\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \ \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \ \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j},$
(2) $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$

We have the following matrix representation:

$$\mathbf{i} = \left(\begin{array}{cc|cc} & 1 & \multicolumn{2}{c}{\bigcirc} \\ -1 & & & \\ \hline \multicolumn{2}{c|}{\bigcirc} & & -1 \\ & & 1 & \end{array}\right), \quad \mathbf{j} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\bigcirc} & 1 & \\ & & & 1 \\ \hline -1 & & \multicolumn{2}{c}{\bigcirc} \\ & -1 & & \end{array}\right), \quad \mathbf{k} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\bigcirc} & & 1 \\ & & -1 & \\ \hline & 1 & \multicolumn{2}{c}{\bigcirc} \\ -1 & & & \end{array}\right)$$

Putting

$$Q = \{x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : \ x_1, x_2, x_3 \in \mathbb{R}\}, \ \ Q^* = \{-x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k} : \ x_1, x_2, x_3 \in \mathbb{R}\}$$

we consider the following non-commutative polynomial:

$$(x_0 \breve{\ } Q)(x_0 \breve{\ } Q^*) = x_0^2 + x_0(Q + Q^*) + QQ^* = (x_0^2 + x_1^2 + x_2^2 + x_3^2).$$

Then we have the extension of binary Clifford type over the polynomial algebra $\mathbb{R}[x_1, x_2, x_3]$

$$\begin{array}{ccc} & \mathbb{R}[x_1, x_2, x_3] & \\ \swarrow & & \searrow \\ Q & & Q^* \end{array}$$

The Galois group becomes $Q^* = -Q$. Hence we have an example of binary non-commutative Galois extensions.

**Remark: Binary Dirac operator and Klein-Gordon operators**

$$D = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}, \quad \overline{D} = \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2} - \mathbf{k}\frac{\partial}{\partial x_3}$$

and

$$D\overline{D} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \otimes I_4.$$

**(2) Ternary Clifford algebra:** The algebra, which is generated by

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}$$

is called ternary Clifford algebra or Sylvester-Kerner algebra. We have

$$Q_1^3 + Q_2^3 + Q_3^3\,\check{}\,3\mathbf{j}^2 Q_1 Q_2 Q_3 = 0.$$

Putting

$$\begin{aligned} R_1 &= \{x_1Q_1 + x_2Q_2 + x_3Q_3 : x_1, x_2, x_3 \in \mathbb{R}\}, \\ R_2 &= \{x_1Q_1 + x_2\mathbf{j}^2Q_2 + x_3\mathbf{j}Q_3 : x_1, x_2, x_3 \in \mathbb{R}\}, \\ R_3 &= \{x_1Q_1 + x_2\mathbf{j}Q_2 + x_3\mathbf{j}^2Q_3 : x_1, x_2, x_3 \in \mathbb{R}\}, \end{aligned}$$

we can obtain an example of ternary non-commutative Galois extension: we have

$$\begin{tikzcd} & \mathbb{R}[x_1,x_2,x_3] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ R_1 & R_2 & R_3 \end{tikzcd} \qquad R_1R_2R_3 = x_1^3 + x_2^3 + x_1^3\,\check{}\,3x_1x_2x_3$$

The Galois group is denoted by

$$\gamma^{(1)} : R_1 \to R_2, \quad \gamma^{(2)} : R_2 \to R_3, \quad \gamma^{(3)} : R_3 \to R_1.$$

**Remark: Ternary Dirac and Klein-Gordon operators**
Here we make a short comment on the ternary Dirac operator and ternary Klein-Gordon operator. Making the Fourier transform, we have the operators:

$$\begin{aligned} D &= \quad Q_1\frac{\partial}{\partial x_1} + Q_2\frac{\partial}{\partial x_2} + Q_3\frac{\partial}{\partial x_3} \\ \overline{D} &= \quad Q_1\frac{\partial}{\partial x_1} + \mathbf{j}^2Q_2\frac{\partial}{\partial x_2} + \mathbf{j}Q_3\frac{\partial}{\partial x_3} \\ \overline{\overline{D}} &= \quad Q_1\frac{\partial}{\partial x_1} + \mathbf{j}Q_2\frac{\partial}{\partial x_2} + \mathbf{j}^2Q_3\frac{\partial}{\partial x_3} \end{aligned}$$

and

$$D\,\overline{D}\,\overline{\overline{D}} = \left[\frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3} + \frac{\partial^3}{\partial x_3^3} - 3\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\right] \otimes I_3.$$

**Construction 3 (Primitive construction)**
In order to apply our non commutative Galois theory to evolutions in this real world, we will introduce a concept of primitive construction.

**(1)** We begin with the algebraic treatment of elements/ words. We make the algebra without relations, which is called the free algebra $W$: $W = \{a_1, a_2, \ldots, a_n\}$. Next we proceed to the algebraic description of an evolution. At first we give the algebraic description of seeds from the origin $*$ as the free algebra: $\{*\} = \{*_1, *_2, \ldots *_n\}(= W_0)$. Next we describe the evolution in terms of creation operator $\gamma_j$ and its annihilation operators $\gamma_j^*$ as follows:

$$\gamma_j(*) = a_j \quad (j = 1, 2 \ldots, n), \quad (or \;\; \gamma_j(*_k) = \delta_{ik} a_j)$$
$$\gamma_j^*(a_k) = \begin{cases} *_k & (j = k) \\ 0 & (otherwise) \end{cases}$$

Then we can introduce elements of evolutions $W = \{a_1, a_2, \ldots, a_n\}$ which are generated by

$$\gamma_{jk} \circ \gamma_{jk-1} \circ \ldots \circ \gamma_{j1}(*) = a_{jk} \ldots a_{j1} \; (= \gamma).$$

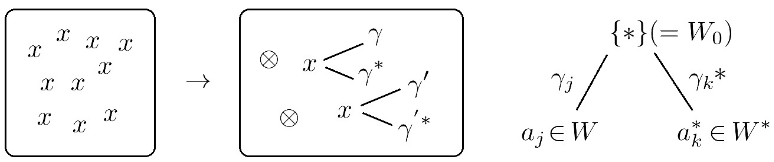
**(2)** Next we proceed to the binary extension over $W$. At first we introduce a conjugate algebra $W^* = \{a_1^*, a_2^*, \ldots, a_n^*\}$ of $W$. The conjugation is denoted by $s : W \to W^*$ and $s^* : W^* \to W$. In order for our scheme to include many examples, we need not assume that the conjugation is necessary involutive. When they are involutive, i.e., $ss^* = s^*s = E$ we call the system symmetric. If not, the elements in $W$ (resp. $W^*$) are called symmetry breaking elements. We extend the operation to the conjugate algebra $W^*$ in an analogous manner: creation operator $\gamma_j^*$ and annihilation operators $\gamma_j$ as follows:

$$\gamma_j^*(*) = a_j^* \;\; (j = 1, 2, \ldots, n), \;\; (or \; \gamma_j^*(*_k) = \delta_{ik} a_j^*)$$
$$\gamma_j(a_k^*) = \begin{cases} *_k & (j = k) \\ 0 & (otherwise). \end{cases}$$

Then $W^*$ is generated by

$$\gamma_{jk}^* \circ \gamma_{jk-1}^* \circ \ldots \circ \gamma_{j1}^*(*) = a_{jk}^* \cdots a_{j1}^* \; (= \gamma^*)$$

and $s(a_j) = a_j^*$, $s^*(a_j^*) = a_j$ for symmetric elements. Then we can create the binary extension as follows:




where $\otimes$ is a symmetry breaking element. We denote the extension by $W_0[\gamma, \gamma^*]$.

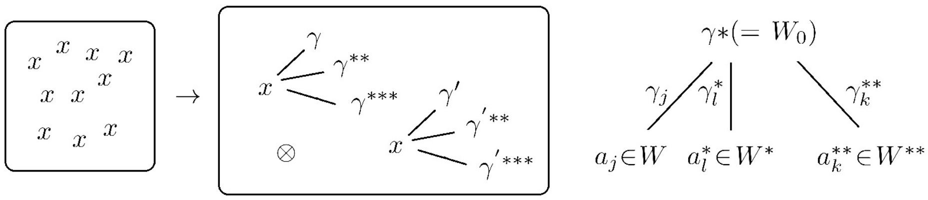
**The characterization of elements of the evolution**: We write $a_j$ or $a_j^*$ as $\alpha_j$. Then $\alpha_m \ldots\ldots \alpha_i \ldots\ldots \alpha_k \ldots\ldots \alpha_1$ is an acceptable element if and only if $(i)$. At any step of the generation, $\#(a_j) > \#(a_k^*)$, where $\#$ is the number of elements and $(ii)$. At the final step, $\#(a_j) = \#(a_k^*)$.

**Examples:** $a_1 a_2^*$ and $a_1 a_2 a_3^* a_4^*$ are acceptable and $a_1^* a_2$, $a_1^* a_2^* a_3 a_4$ are not acceptable.

**(3)** In a completely analogous manner we can introduce the ternary element and construct ternary extensions. We introduce $W_0$, $W$, $W^*$, $W^{**}$:

$$W_0 = \{*_1, *_2, \ldots, *_n\},$$
$$W = \{\gamma = \gamma_{jk} \circ \gamma_{jk-1} \circ \ldots \circ \gamma_{j1}(*) = a_{jk} \ldots\ldots a_{j1}\},$$
$$W^* = \{\gamma^* = \gamma_{jk}^* \circ \gamma_{jk-1}^* \circ \ldots \circ \gamma_{j1}^*(*) = a_{jk}^* \ldots\ldots a_{j1}^*\},$$
$$W^{**} = \{\gamma^{**} = \gamma_{jk}^{**} \circ \gamma_{jk-1}^{**} \circ \ldots \circ \gamma_{j1}^{**}(*) = a_{jk}^{**} \ldots\ldots a_{j1}^{**}\}.$$

Then we can create ternary extensions as follows:




We denote the extension by $W_0[\gamma, \gamma^*, \gamma^{**}]$. The characterization of acceptable elements can be given in an analogous manner.

**(4)** In a completely analogous manner, we can define the successive extension of the binary and ternary extension: $W_0[\gamma, \gamma^*] \to W_1[\gamma', \gamma'^*, \gamma'^{**}]$, where $W_1 = W_0[\gamma, \gamma^*]$. For general extensions can be considered in an analogous manner. The extensions and their Galois group will be given in the next Section.

**Remark (1)** Putting $* = E_2$, $\{*\} = T$, $\gamma_1(E) = s_1$, $\gamma_2(E) = s_2$, $\gamma_3(E) = s_3$, $T_k = s_k(T)$, $(k = 1, 2, 3)$, we can realize the examples in constructions 1. **(2)** Putting $* = E_4$, $\{*\} = R^3$, $\gamma_1(E_4) = \mathbf{i}$, $\gamma_2(E_4) = \mathbf{j}$, $\gamma_3(E_4) = \mathbf{k}$, we can realize the examples in construction (2).

# 4. BTBB-structure

In this Section we take an algebra, mainly free algebra $A[a_1, \ldots, a_n]$, which is generated by $n$-elements $a_1, \ldots, a_n$ and consider a non-commutative Galois extension. Here we formulate the following hypothesis:

"The Galois group of the extension is a solvable group."

Then we have the following hierarchy structure:

**(I:** $n = 2$**)** $\quad S_2(= A_2) \sqsupset E_2$

**(II:** $n = 3$**)** $\quad S_3 \sqsupset A_3 \sqsupset E_3$

**(III:** $n = 4$**)** $\quad S_4 \sqsupset A_4 \sqsupset V_4 \sqsupset Z_2 \sqsupset E_4$

**(IV:** $n > 5$**)** $\quad S_n \sqsupset A_n \sqsupset E_n$

**The construction of the BTBB-structure.** Following the hierarchy structure, we make the BTBB-structure:

**(0)** The trivial extension: The algebra $A[a_1, \ldots, a_n]$ itself gives the trivial extension.

**(1) (I: n= 2)** The binary extension:

$$\begin{array}{ccc} & A & \\ / & & \backslash \\ Q_1 & & Q_1^* \\ & \textbf{[I]} & \end{array}$$

We take a binary extension $A_1$ of $A[a_1, \ldots, a_n]$ by $Q_1, Q_1^* :\ A_1 = A[Q_1, Q_1^*]$. We denote the extension as in the left side configuration. The Galois group of the hierarchy structure is $[A : A_1] \leftrightarrow [G_2 : G_1]$ where $G_2 = S_2 (= A_2)$, $G_1 = E$. Hence we have $G_1/G_0 = S_2$.

**Remark.** Choosing several binary extensions: $A[Q_i, Q_i^*]$ $(i = 1, 2, \ldots, m)$ we make the algebra which is generated by these extensions, which is denoted by $A[Q_1, Q_2, \ldots, Q_m, Q_1^*, Q_2^*, \ldots, Q_m^*]$ which is denoted as follows:

$$\begin{array}{ccc} & A & \\ / & & \backslash \\ Q_1 & & Q_1^* \end{array} + \begin{array}{ccc} & A & \\ / & & \backslash \\ Q_2 & & Q_2^* \end{array} + \cdots\cdots + \begin{array}{ccc} & A & \\ / & & \backslash \\ Q_m & & Q_m^* \end{array}$$

**(2) (II: n= 3)** We proceed to the ternary extension.

$$\begin{array}{ccc} & A & \\ / & | & \backslash \\ Q & Q^* & Q^{**} \end{array}$$

We take a ternary extension of $A[a_1, \ldots, a_n]$ by $Q, Q^*, Q^{**} :\ A_3 = A[Q_1, Q_2, Q_3]$. We express the extension as in the left side configuration. The Galois group is $S_3$.

We consider the successive extension of binary extension $\to$ ternary extensions:

$$A \ \to \ A[Q_1, Q_1^*] \ \to \ A_1[Q_2, Q_2^*, Q_2^*](A_1 = A[Q_1, Q_1^*])$$

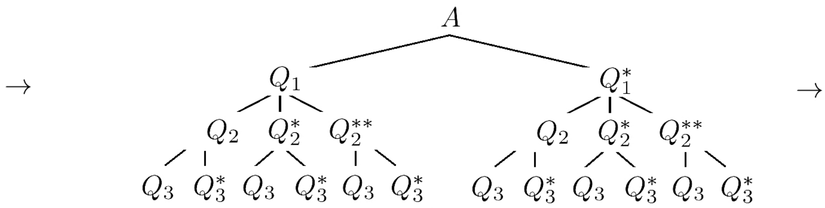
$$\begin{array}{ccc} & A & \\ / & & \backslash \\ Q_1 & & Q_1^* \\ & [\mathrm{I}] & \end{array} \ \to \ \begin{array}{ccc} & A & \\ / & | & \backslash \\ Q_2 & Q_2^* & Q_2^{**} \\ & [\mathrm{II}] & \end{array} \ \to \ \begin{array}{ccccccc} & & & A & & & \\ & & Q_1 & & Q_1^* & & \\ Q_2 & Q_2^* & Q_2^{**} & [\mathrm{III}] & Q_2 & Q_2^* & Q_2^{**} \end{array}$$

The hierarchy structure of Galois groups of each extension can be described as follows:

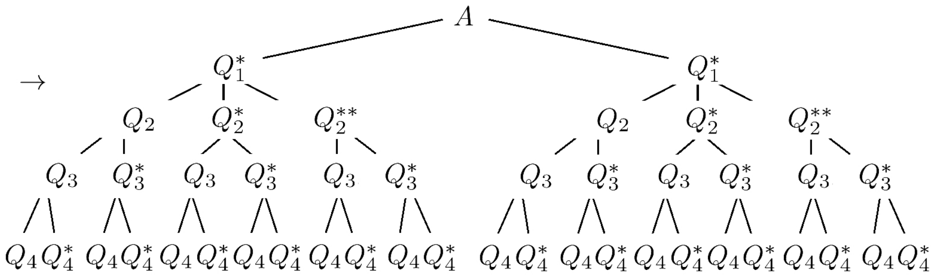
$$\{A \ : \ A_1 \ : \ A_2\} \ \Leftrightarrow \ \{G_3 \ : \ G_2 \ : \ G_1\}$$

where $A_1 = A[Q_1, Q_1^*]$ and $A_2 = A_1[Q_2, Q_2^*, Q_2^{**}]$ and $G_3 = S_3$, $G_2 = A_3$, $G_1 = E$. Then we see that $G_3/G_2 = S_2$ and $G_2/G_1 = S_3$.

**(3) (III: n=4)** We continue the extension and have the hierarchy structure: $E_4 \sqsubset Z_2 \sqsubset V_4 \sqsubset A_4 \sqsubset S_4$.




$[IV]$




$[V]$

Then we have the following correspondence of the hierarchy structure

$$\{A \,:\, A_1 \,:\, A_2 \,:\, A_3 \,:\, A_4\} \Leftrightarrow \{G_4 \,:\, G_3 \,:\, G_2 \,:\, G_1 \,:\, G_0\},$$

where $A_1 = A[Q_1, Q_1]$, $A_2 = A_1[Q_2, Q_2^*, Q_2^{**}]$, $A_3 = A_2[Q_3, Q_3^*]$, $A_4 = A_3[Q_4, Q_4^*]$ and $G_4 = S_4$, $G_3 = A_4$, $G_2 = V_4$, $G_1 = Z_2$, $G_0 = E$. The hierarchy structure of Galois group of each extension can be described as follows:

$$S_2 = G_4/G_3, \;\; S_3 = G_3/G_2, \;\; S_2 = G_2/G_1, \;\; S_2 = G_1/G_0.$$

**The role of the Galois groups in the evolutions**

Finally we give the roles of Galois groups in the evolution. At first we notice that the Galois group in the classical Galois theory tells the structure of solutions in terms of the permutation group independently from the proper form of the solution of the given equation. Here we shall show that the Galois group of BTBB-structure characterizes the evolution independently from deformations and mixing in the evolution. Hence we see that the Galois group governs the evolution. We give several examples:

(1) We can describe the change of sentences in terms of the deformation of the Galois group.

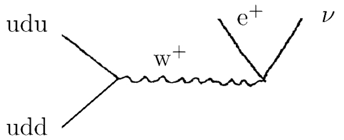
I read a book $\rightarrow$ He reads a book, She is pretty $\rightarrow$ Is she pretty?

Hence we see that the binary and ternary structures are preserved under the changes of sentences.

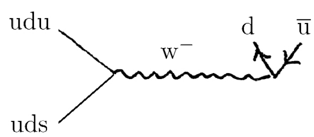
(2) The mutation on proteins in molecular biology, for example, XYZ $\leftrightarrow$ XY'Z can be described in terms of the deformation of the Galois group.

(3) In the theory of quark physics, the roles of leptons and weak bosons are describing the interacting particles, for examples, in (i) the change udu $\leftrightarrow$ udd and (ii) the change of colors. These interactions can be described in the Galois group:

(i) The udu $\leftrightarrow$ udd process




(ii) The change of colors




# 5. BTBB-structures in mathematics

We are going to the BTBB structures in mathematics. We have not treated the evolutions in mathematics in Parts I$\sim$IV in details. Here we treat three topics. The first is the structure in algebraic equation, the second one is in the formal language theory, and the third one is in knot theory. The third topic is essentially new.

**(1) The BTBB-structure in algebraic equation**

As in Section 1 we can find the BTBB-structure in solving an equation. We can get the following sequence of the extension. Hence we can obtain the BTBB-structure.

$$\sqrt{\ } \Rightarrow \sqrt[3]{\ } \Rightarrow \sqrt{\ } \Rightarrow \sqrt{\ }$$

**(2) The BTBB-structure for formal language theory**

Next we will find the BTBB-structure in the formal language theory. (1) The origin of the formal language theory is the finite automaton defined by set of words: $\{a_1, a_2, \ldots, a_m\}$. The acceptable sequences constitute sentences. (2) The first binary extension creates context free sentences. (3) The ternary extension creates context sensitive sentences. (4) The further binary sentences create general 0 type sentences. These sentences are equivalent to Turing machines and finally we may expect to have the universal Turing machines by successive extensions.

**(3) The BTBB-structure for knots theory**

At first we begin with the description of knots in terms of sequence of intersection points. Then we proceed to the Reidemeister generations of knots. This gives a language structure of knots. Finally we shall find the BTBB-structure in the knot theory. Here we give only the outline without proofs. The complete results will be given in another paper.

**(a) The description of knots:**

We choose a knot. Then we have a sequence of $m$ intersection points: $A_1, A_2, \ldots, A_m$: Putting $\alpha_1\alpha_2\cdots\alpha_{2m}$, we choose $\alpha = a_k$ (positive element) when the knot goes over the knot, and $\alpha = a_k^*$ (negative element) when the knot goes under the knot, respectively.

 $\alpha_i = a_i$  $\alpha_i = a_i^*$

We notice that the knot has the following "cyclic symmetry". Namely, we see

$$\alpha_1\alpha_2\cdots\alpha_{2m} = \alpha_{2m}\alpha_1\alpha_2\cdots\alpha_{2m-1}$$

In the following we restrict ourselves to the trivial knot.

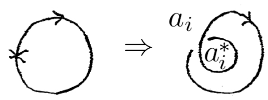
**(b) The generation of knots:**
We can generate knots by the three kinds of Reidemeister's generations. We begin with an introduction of the concept of "$k$-(referenced) knot". A knot is called $k$-knot, if the knot is expressed in terms of $k$ referenced intersection points $\alpha_1\alpha_2\cdots\alpha_{2k}$ and the remained sequence $S_1$, $S_2$,…, $S_m$. We notice that $S_i$ may contain other referenced intersection points. For example, $\alpha_k\alpha_k^*S$, $\alpha_i\alpha_j^*S_1\alpha_i^*\alpha_jS_2$, are 1-knot, 2-knot, respectively.

**(1) Type I Reidemeister's generations (1-referenced knot)**
At first we will obtain 1-indices knots by the successive type I Reidemeister's generations:

$$(i) \quad S \to \alpha_k\alpha_k^*S \ \ (\Leftrightarrow S \to \alpha_kS\alpha_k^* \Leftrightarrow S \to S\alpha_k\alpha_k^*)$$

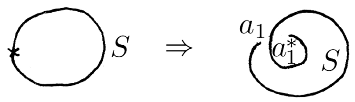
We give examples:

**Example 1 (Basic 1-knot:** $\alpha_k\alpha_k^*$**)**




$\{\emptyset\} \Rightarrow a_ia_i^*$

**Example 2 (General basic binary 1-knot)**

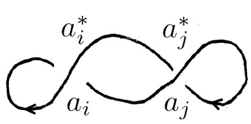



$S \Rightarrow a_1Sa_1^*$

**(2) Type II Reidemeister's generations (2-referenced knot)**
Next we are concerned with 2-knot. The knot $\alpha_i\alpha_i^*\alpha_j\alpha_j^*$ is called the basic 2-knot. For example:

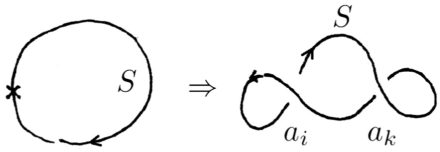
$$\text{(ii)} \quad S \to \alpha_i\alpha_i^*S\alpha_k\alpha_k^*,\ S_1, S_2 \to \alpha_iS_1\alpha_i^*\alpha_kS_2\alpha_k^*, \ldots$$

**Example 1 (2-knot:)** $a_ia_i^*a_ja_j^*$

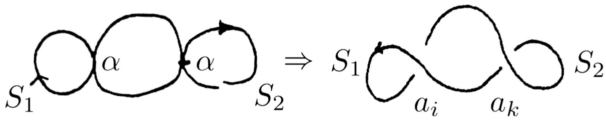



$\{\emptyset\} \Rightarrow a_ia_i^*a_ja_j^*$

**Example 2 (The 2-knots)** in $(ii)$



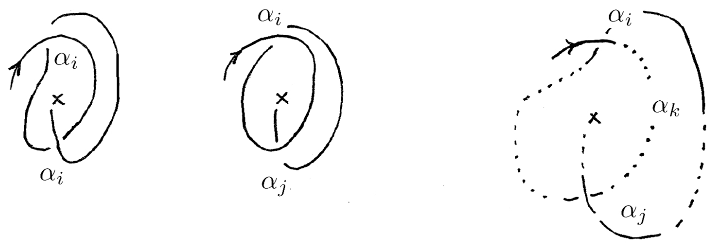

$S \Rightarrow a_i a_i^* S a_k a_k^*$




$S_1, S_2 \Rightarrow a_i S_1 a_i^* \, a_k S_2 a_k^*$

**Example 3 (Non real binary 2-knots)**
We will give a criterion whether the given knots of binary type are real knot or not. Here we give several examples and see the basic idea on the criterion. We begin with examples

(1) $\alpha_i\alpha_j^*\alpha_i^*\alpha_j$ (2) $\alpha_i\alpha_j\alpha_i^*\alpha_j^*$ (3) $\ldots\alpha_i\ldots\alpha_j^*\ldots\alpha_k\ldots\alpha_i^*\ldots\alpha_j\ldots$

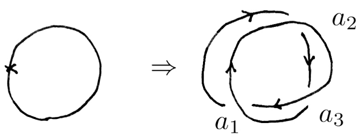



These conditions are called "The entangle knot condition" which describes the non-existence of real knots. This will be discussed in terms of the ordered brackets in the context free sentences.

**(3) The Type II Reidemeister's generations (3-referenced knots)**
Next we will be concerned with 3-knots. The 3-knot $\alpha_i\alpha_j\alpha_k\alpha_i^*\alpha_j^*\alpha_k^*$ is called the basic 3 knot. We begin with the Type II Reidemeister generation: For example,

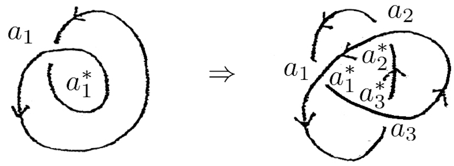
$$\text{(iii)}\quad S \to \alpha_i\alpha_i^* S \to \alpha_i\alpha_j S_1 \alpha_k S_2 \alpha_i^* S_3 \alpha_j^* \alpha_k^*$$

**Example 1 (The basic 3-knot)**



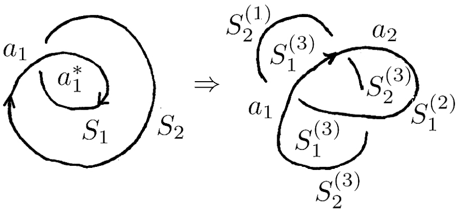

$\{\emptyset\} \;\Rightarrow\; a_1a_2a_3a_1^*a_2^*a_3^*$

**Example 2 (Deformation of the basic 3-knot)**



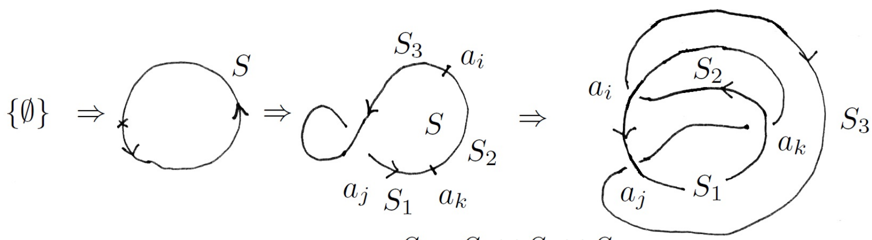

$a_1a_1^* \;\Rightarrow\; a_1a_3^*a_2^*a_1^*a_3a_2$

**Example 3 (General Reidemeister move of type II)**




$$a_1S_1a_1^*S_2 \ \Rightarrow$$
$$a_1S_1^{(1)}a_2S_1^{(2)}a_3S_1^{(3)}a_1^*S_2^{(1)}a_2^*S_2^{(2)}a_3^*S_2^{(3)}$$
$$S_1 = S_1^{(1)} \cup S_2^{(1)} \cup S_3^{(1)}$$
$$S_2 = S_1^{(2)} \cup S_2^{(2)} \cup S_3^{(2)}$$

**Example 4 (The knot in (iii))**

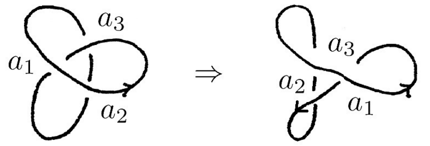



$$S = S_1 \cup S_2 \cup S_3$$
$$\{\emptyset\} \ \Rightarrow \ S \ \Rightarrow \ a_ja_j^*S \ \Rightarrow \ a_ia_jS_1a_kS_2a_i^*S_3a_j^*a_k^*$$

**(4) The Type III Reidemeister's generations of 3-knots**

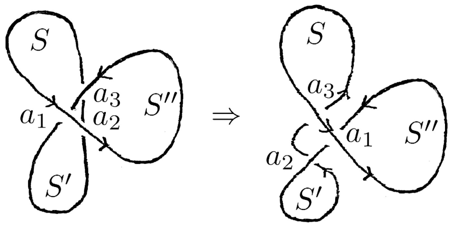
Next we will be concerned with the Reidemeister generation (deformation) of Type III. The following slide of a part of a knot curve is called the Reidemeister generation (deformation) of Type III.

**Example 1 (The basic Reidemeister move of type III)**




$a_1a_2a_3a_1^*a_2^*a_3^* \ \Rightarrow\ a_1a_1^*a_2a_2^*a_3^*a_3$

**Example 2 (General Reidemeister move of type III)**




$a_1a_2S''a_3a_1^*S'a_2^*a_3^*S$
$\Rightarrow\ a_1S''a_1^*a_2S'a_2^*a_3^*Sa_3$

The "reduction of triple knots" can be observed in the deformation: For example,

$$\alpha_i\alpha_j\alpha_k\alpha_i^*\alpha_j^*\alpha_k^* \ \rightarrow\ \alpha_i\alpha_i^*\alpha_j\alpha_j^*\alpha_k^*\alpha_k$$

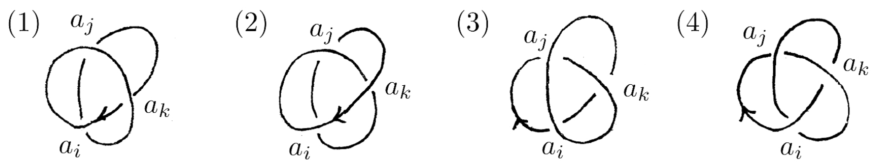
More generally,

$$\begin{aligned}&...\alpha_i\ldots\alpha_j\ldots\alpha_k\ldots\alpha_i^*\ldots\alpha_j^*\ldots.\alpha_k^*\ldots,\\ &\qquad\rightarrow\ \ldots\alpha_i\ldots\alpha_i^*\ldots\alpha_j\ldots\alpha_j^*\ldots\alpha_k^*\ldots\alpha_k\ldots,\end{aligned}$$

Next we proceed to the construction of the list of all the 3-knots.

**(i) The three 3-indices of basic type:**

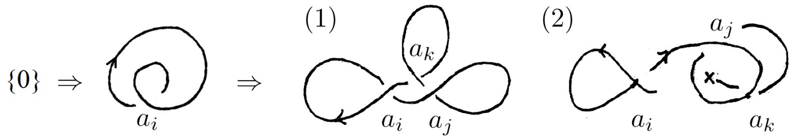
(1) $\alpha_i\alpha_j\alpha_k\alpha_i^*\alpha_j^*\alpha_k^*$ (The basic 3-knot), (2) $\alpha_i\alpha_j\alpha_k^*\alpha_i^*\alpha_j^*\alpha_k$,
(3) $\alpha_i^*\alpha_j^*\alpha_k\alpha_i\alpha_j\alpha_k^*$, (4) $\alpha_i\alpha_j^*\alpha_k\alpha_i^*\alpha_j\alpha_k^*$




We denote the 3-knot of (2) by $\{\alpha_i,\alpha_j,\alpha_k\}\rightarrow\{\alpha_i,\alpha_j,\alpha_k^*\}$, (3) by $\{\alpha_i,\alpha_j,\alpha_k\}\rightarrow\{\alpha_i^*,\alpha_j^*,\alpha_k\}$ (4) by $\{\alpha_i,\alpha_j,\alpha_k\}\rightarrow\{\alpha_i,\alpha_j^*,\alpha_k\}$ simply. The first three knots are equivalent to (1). The final one is a knot but non-trivial knot. We can not find the real knots for the following sequences:
$\alpha_i\alpha_j\alpha_k\alpha_{i'}^*\alpha_{j'}^*\alpha_{k'}^*$ where $\{i',j',k'\}$ are non-trivial permutations of $\{i,j,k\}$. Hence we see that the real trivial knots of 3-knots are essentially unique and equivalent to the basic knot.

**(ii) Degenerate type**
The 3-knot is called of degenerate type, when it is obtained from Type I basic knot.
(1) $\alpha_i\alpha_i^*\alpha_j\alpha_j^*\alpha_k\alpha_k^*$ (The basic degenerate type) (2) $\alpha_i\alpha_i^*\alpha_j\alpha_k\alpha_j^*\alpha_k^*$.

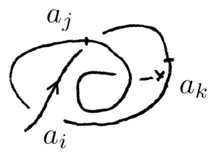
We see that the real knot sequence is only (1). The other knots including (2) have the entanglements. Hence they are not real knots.



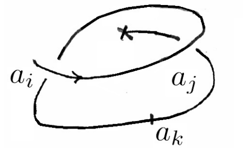

### (iii) Intermediate type

The remained knots are called intermediate knots. We give examples.

(1) $\alpha_i\alpha_j^*\alpha_i^*\alpha_k\alpha_j\alpha_k^*$




(2) $\alpha_i\alpha_j\alpha_i^*\alpha_k\alpha_j^*\alpha_k^*$




Then we see that there do not exist real knots in intermediate type. We will discuss them in the context of languages given by knots.

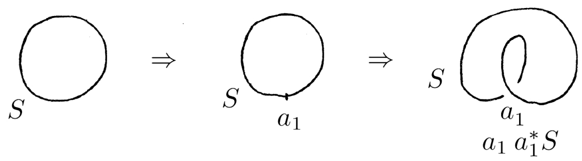
### (3) The generation of general trivial knots (Reidemeister theorem)

**The Reidemeister theorem:** *Every trivial knot can be generated from the circle knot by the following three kinds of Reidemeister generations successively:*

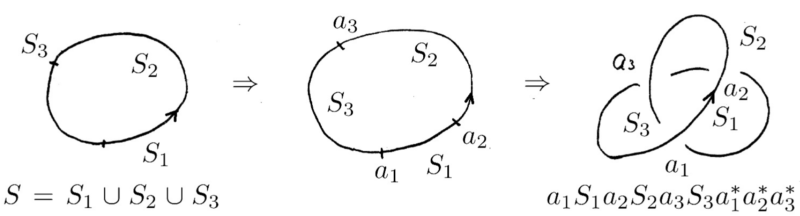
*(1) The basic binary generation (Reidemeister generation of type I)*
*(2) The ternary generation (Reidemeister generation of type II and III)*
*(3) The deformation (by the Reidemeister move).*

We give several examples of generations. Here we give a successive binary and ternary generations of a given a knot $S$.

(1) Binary generation of a given knot $S$



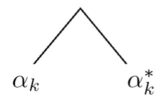

(2) Ternary generation of a given knot $S$

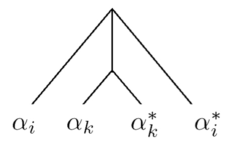


**(c) The language theory of knots**
Next we will give the formal language theory of knots.

**(1) Association of context free sentence to 2-trivial knots**
We can find context free sentences from knots which are generated by the Type I generation: We can associate the following diagrams. We give examples:

(1) $\alpha_k \alpha_k^* \to \{\alpha_k \alpha_k^*\}$ (2) $\alpha_i \alpha_k \alpha_k^* \alpha_i^* \to \{\alpha_i \{\alpha_k \alpha_k^*\} \alpha_i^*\}$



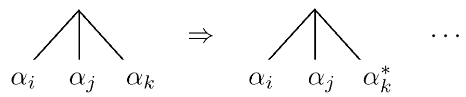

Then we have the following characterization of the acceptable knots, i.e.: The knot is entangle condition free. Namely, for each step of the generation, we have $\#\,(\,) \geq \#(\,\})$ and at the final stage $\#\,\{\,) = \#(\,\})$. Here we calculate it putting $\{\alpha_i \alpha_j^*\} = \delta_{ij}$. When the zero appears, the sentences are not acceptable. We give examples: $\{_{ii}\}$, $\{_{ii}\}\{_{jj}\}$ are acceptable and $\{_{ik}\}$ $(i \neq j)$, $\{_i\{_{ji}\}_j\}$ are non-acceptable.

**(2) Association of context sensitive sentences**
We can associate context sensitive sentences to acceptable basic 3-knot sequences by the generation rules:

$$\{\alpha_i \alpha_j \alpha_k\} \;\to\; \{\alpha_i \alpha_j \alpha_k^*\}, \quad \{\alpha_i \alpha_j \alpha_k\} \;\to\; \{\alpha_i^* \alpha_j \alpha_k\},$$

$$\{\alpha_i \alpha_j \alpha_k\} \to \{\alpha_i \alpha_j^* \alpha_k^*\}.$$

We can associate the following diagram. For general 3-indices knots, we can generate sentences by use of the generation rules for binary sentences: We can associate general sentences following the generation scheme.




**(3) Association of general sentences of knot sequences**

We can identify the 2-knots from the circle knot as the binary extension and the 3-knots from the circle knot as the ternary extension. The successive extension can be identified with the successive generation of knots. Then we can obtain acceptable sentences by Reidemeister theorem.

**(e) The BTBB-structure of knots**
We can find the BTBB- structure of knots.

We can construct the BTBB-structure in the following steps:

**The generation scheme of BTBB-structure of knots**

**(1)** The origin of the evolution is the set of finite points $\{a_1, \dots, a_m\}$

$$\Downarrow$$

**(2)** The first binary extension creates pairs $\{a_i, a_i^*\}$ $(t = 1, 2, \dots, m)$ by (I):

$$\Downarrow$$

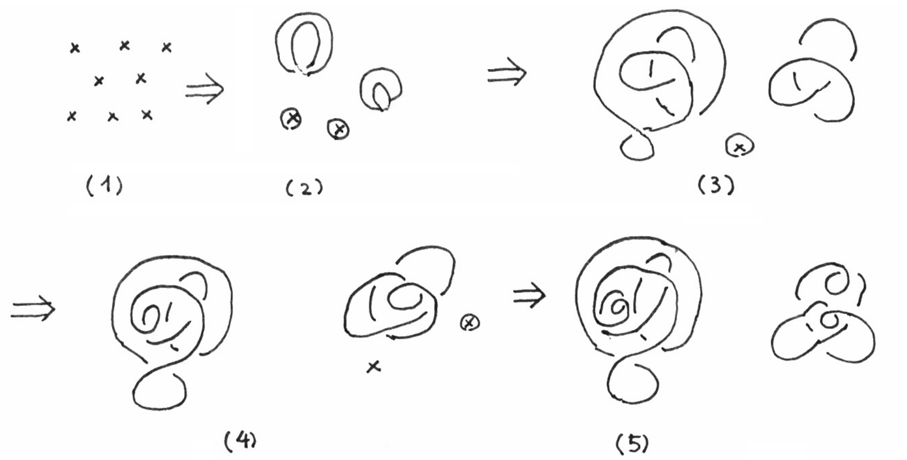
**(3)** The ternary extension. This extension creates a knot of type II and III from a triple of points $\{a_i, a_j, a_k\}$

$$\Downarrow$$

**(4)** The second binary extension

$$\Downarrow$$

**(5)** The final binary extension




**Remark.** As in the case of language, we may consider the total evolution of the trivial knot. Then we may expect that any trivial knots can be obtained by this scheme.

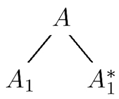
## 6. The complexity system of BTBB-structure

We have given the generations of a complexity system by the successive binary extensions in Part IV [?]. Here we recall the generations

### Basic generation

We begin with the basic generation:

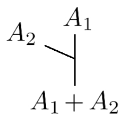
**(1) Generation of tree type**




$\leftrightarrow$

<table>
<tr><td colspan="2">$A$</td></tr>
<tr><td>$A_1$</td><td>$A_1^*$</td></tr>
</table>

**(2) Generation of linear type**

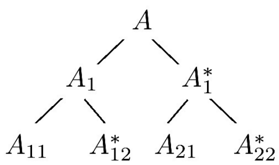



$\leftrightarrow$

<table>
<tr><td>$A_1$</td></tr>
<tr><td>$A_1 + A_2$</td></tr>
</table>

We will show that we can generate complexity systems in the nature can be the successive binary extensions of type (1) and type (2) generations. We demonstrate simple successive generations.

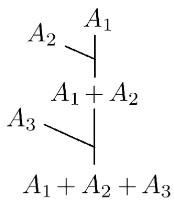
**(1)' Successive extension of tree type**




$\leftrightarrow$

<table>
<tr><td colspan="4">$A$</td></tr>
<tr><td colspan="2">$A_1$</td><td colspan="2">$A_1^*$</td></tr>
<tr><td>$A_{11}$</td><td>$A_{12}^*$</td><td>$A_{21}$</td><td>$A_{22}^*$</td></tr>
</table>

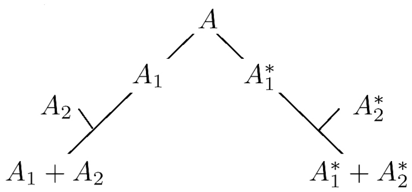
**(2)' Successive extension of linear type**




$\leftrightarrow$

<table>
<tr><td>$A_1$</td></tr>
<tr><td>$A_1 + A_2$</td></tr>
<tr><td>$A_1 + A_2 + A_3$</td></tr>
</table>

**(3) Generation of mixed type (I)**

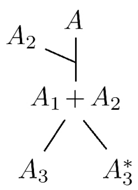



$\leftrightarrow$

<table>
<tr><td colspan="2">$A$</td></tr>
<tr><td colspan="2">$A_1 + A_1^*$</td></tr>
<tr><td>$A_1 + A_2$</td><td>$A_1^* + A_2^*$</td></tr>
</table>

(Tree type $\rightarrow$ Linear type)

**(3)' Generation of mixed type (II)**




$\leftrightarrow$

<table>
<tr><td colspan="2">$A$</td></tr>
<tr><td colspan="2">$A_1 + A_2$</td></tr>
<tr><td>$A_3$</td><td>$A_3^*$</td></tr>
</table>

(Linear type $\rightarrow$ Tree type)

**Generation of fractal sets**

We will show that we can obtain basic fractal structures by generations of above types. At first we notice that the basic fractal sets can be realized by the generation above given.

**(1) Fractal set of flower type:** We choose a compact set $A_0$, for example, a closed interval and a system of contractions $\{s_j : A_0 \to A_0\}$, $i = 1, 2, \ldots, M$. Then we put $\{A_n : n = 1, 2, \ldots M\}$ by

$$A_n = \bigcup_{j=1}^{M} s_j(A_{n-1}), \quad n = 1, 2, \ldots M.$$

Here we assume the separation condition $s_i(A^0_{n-1}) \cap s_j(A^0_{n-1}) = E^o$, $(i \neq j)$, where $E^o$ is the open kernel of $E$. Then we have the sequence:

$$A_0 \to A_1 \to A_2 \to \ldots \to A_n \to \ldots$$

Putting

$$A = \bigcap_{n=0}^{\infty} A_n,$$

we have the fractal set of flower type. We notice that $A$ is $s_j$-invariant, i.e. $s_j(A) = A$, $(j = 1, 2, \ldots, M)$.

$$\boxed{A_0} \to \begin{array}{c}\boxed{s_1(A_0)}\\ \vdots \\ \boxed{s_m(A_0)}\end{array} \to \begin{array}{c}\boxed{s_1s_1(A_0)}\\ \vdots\\ \boxed{s_is_j(A_0)}\\ \vdots\\ \boxed{s_ms_m(A_0)}\end{array} \;\cdots\cdots \to \cdots\cdots \to \begin{array}{c}\vdots\\ \vdots\\ \vdots\\ \vdots\end{array}$$

**(2) The fractal set of branch type (or evolution type):** Let $A_0$ be a compact set and let $s_j$ $(j = 1, 2, \ldots, M)$ be a system of contractions on $A_0$ with the separation condition. Also we prepare a shift $s_o$ with $s_o(A_0) = A$. Putting

$$A_n = A_0 \cup s_o(\bigcup_{j=1}^{M} s_j(A_{n-1})), \quad n = 1, 2, \ldots$$

we have the sequence

$$A_0 \to A_1 \to A_2 \to \ldots \to A_n \to \ldots$$

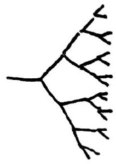
Putting

$$A = \bigcup_{n=0}^{\infty} A_n$$

we obtain a fractal set of branch type.

We have the following relation between these fractals: We call the boundary of the set: $A \setminus \bigcup_{j=0}^{\infty} A_j$ main boundary of $A$ which is denoted by $bA$. Then we can obtain the fractal of flower type $bA$ of $A$. For the understanding the relationship between these two kinds of fractal sets, we give well known fractals of Cauliflower type. We may say that the boundary of the Cauliflower is just the flower of the Cauliflower.

$$\boxed{A_0} \to \boxed{A_0} + \begin{array}{c}\boxed{s_1(A_0)}\\ \vdots \\ \boxed{s_m(A_0)}\end{array} \to \boxed{A_0} + \begin{array}{c}\boxed{s_1(A_0)}\\ \vdots \\ \boxed{s_m(A_0)}\end{array} + \begin{array}{c}\boxed{s_1 s_1(A_0)}\\ \vdots \\ \boxed{s_i s_j(A_0)}\\ \vdots \\ \boxed{s_m s_m(A_0)}\end{array} \to \cdots\cdots$$

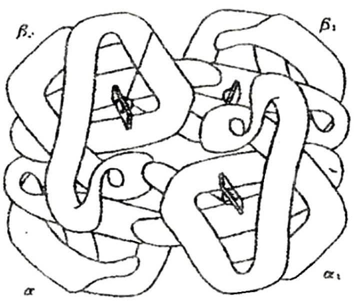


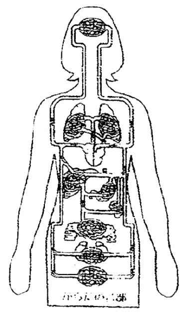
$A_0$ $\quad$ $A_1$ $\quad$ $A_2$ $\quad$ $A_n$ $\quad$ $A$

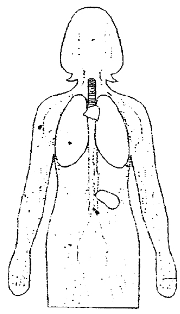
(Linear type $\to$ Tree type)

**(3) Complexity type**

We see that the generation body $A$ which is generated by the successive binary extensions is dense in the total fractal set $\hat{A}$, which is the closure of $A$. We can find many fractal sets which have the following properties: The boundary $b\hat{A}$ is also dense in $\hat{A}$. The fractal set with these conditions is called a fractal set of Hilbert type. The typical mathematical example is the generation of the rational numbers in the real numbers. In the animal body, the recurrence system of blood or neural flows makes a Hilbert system. We give other examples of 3th and 4th cells.





We can describe the evolution system in terms of fractals

**Fractal description of the evolution**

**(1)** The seeds of the evolution are given

$\Downarrow$

**(2)** The complexity system is created by the BTBB-structure and successive binary extensions. It becomes a fractal set of branch type

$\Downarrow$

**(3)** The border of the complexity system admits a simple (fractal) random walk (We may use the Tsallis entropy here)

$\Downarrow$

**(4)** A successive evolution begins from the seeds

$\Downarrow$

......

# References


[1] C. C. Adams, The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots, American Mathematical Society Providence, Rhode Island 2004.

[2] J. Barrow, Theories of everything, Oxford University Press, 1991.

[3] H. Bergson, L'evolution creatrice, Presses Universitaires de France, (1907).

[4] S. B. Carroll, J. K. Grenier, S. D. Weatherbee, From DNA to diversity (Molecular genetics and the evolution of animal design), Blackwell Publishing 2001.

[5] N. Chomsky, Three models of the description of languages, PGIT, 2:3, 113-124, 1956.

[6] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, John Wiley and Sons (1990).

[7] W. Feller, An introduction to probability theory and its applications (Vol. 1) Springer Verlag, New York, Inc. (1988), John Wiley and Sons 1957.

[8] P. G. de Gennes, Scaling Concepts in Polymer Physics, Cornel University Press 1979.

[9] S. W. Hawking, A brief history of time (From The Big Bang to Black Holes), Bantam Press, London 1988.

[10] N. Iwabe et al., *Evolutional relationship of archaebacteria, eubacteria, and eukaryotes interred from phylogenetic trees of duplicated genes*, Proc. Natl. Acad. Sci. USA, 86, 9355-9359 (1989).

[11] J. Ławrynowicz, M. Nowak-Kępczyk, O. Suzuki, in collaboration with M. Othman, *Binary and ternary structures in the evolution of the universe III. The entropy description of the evolution*, Bull. Soc. Sci. Lettres Łódź Sér. Rech.

Déform. **70** no. 1, (2020) 11–41.
DOI: 10.26485/0459-6854/2020/70.1/1.

[12] PartIV Julian Ławrynowicz, Małgorzata Nowak-Kępczyk, Osamu Suzuki, *Binary and ternary structures of the evolutions in the universe (2x3x2x...-world) IV. The entropy description of evolution.*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **70** no. 1, (2020) 43–81.
DOI: 10.26485/0459-6854/2020/70.1/2

[13] S. Kaufman, Investigations, Oxford University Press, 2000.

[14] M. Kimura, The neutral theory of molecular evolution, Cambridge University Press, 1983.

[15] P. Lévy, *Sur certains processus stochastiques homogoénes*, Compositio Mathematica, 1939.

[16] J. Ławrynowicz, M. Nowak-Kępczyk, and M. Zubert, *Mathematics behind two related Nobel Prizes 2016: in Physics - topology governing physics of phase transitions, in Chemistry - geometry of molecular nanoengines*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **69** no. 1, (2019).

[17] J. Ławrynowicz, O. Suzuki, A. Niemczynowicz, & M. Nowak-Kępczyk, "Fractals and chaos related to Ising-Onsager-Zhang lattices. Ternary approach vs. binary approach", Int. J. of Geom. Meth. in Modern Physics, Vol. 15, No. 11, 1850187 (2018). 01 Nov 2018,
https://doi.org/10.1142/S0219887818501876 .

[18] J. Ławrynowicz, M. Nowak-Kępczyk, A. Valianti, and M. Zubert, *Physics of Complex Alloys - One Dimensional Relaxation Problem*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **65** no. 3 (2015), 27–48.

[19] J. Ławrynowicz, K. Nôno, D. Nagayama, and O. Suzuki, *Non-commutative Galois theory on Nonion algebra and* su(3) *and its application to construction of quark models*, Proc. of the Annual Meeting of the Yukawa Inst. Kyoto "The Hierarchy Structure in Physics and Information Theory" Soryuusironnkennkyuu, Yukawa Institute, Kyoto 2011, pp. 145–157 [http://www2.yukawa.kyoto-u.ac.jp].

[20] J. Ławrynowicz, K. Nôno, D. Nagayama, and O. Suzuki, *A method of noncommutative Galois theory for binary and ternary Clifford Analysis*, Proc. ICMPEA (Internat. Conf. on Math. Probl. in Eng. Aerospace, and Sciences) Wien 2012, AIP (Amer. Inst. of Phys.) Conf. 1493 (2012), 1007–1014.

[21] J. Ławrynowicz, K. Nôno, and O. Suzuki, *Binary and ternary Clifford analysis vs. non-commutative Galois extensions. I. Basics of the comparison*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **62**, no. 1 (2012), 33–42.

[22] S. Ohno, Evolution by gene duplication, Springer Verlag, New York Inc. 1970.

[23] L. E. Orgel, Prebiotic chemistry and the origin of RNA world, Crit. Rev. Biochem. Mol. Biol. **39** (2004) 99–123.

[24] R. Penrose, Casuality, Quantum theory and The universe in On space and time, Cambridge Univ. Press 2008.

[25] D. H. Perkin, Introduction to High Energy Physics, Cambridge Univ. Press 2000.

[26] Particle Data Group, *Review of Particle Physics,* in: Phys. Letter B **667** (2008) 1–6.

[27] I. Prigogine, From being to becoming, W. H. Friedman and Company, San Francisco 1997.

[28] J. Silk, A short history of the universe, Scientific American Library (Book 53), W. H. Freeman, Reprint edition 1997.

[29] L. Smolin, The life of the cosmos, Oxford Univ. Press, Oxford, New York 1997.

[30] O. Suzuki, *Mathematical theory of symmetry breaking in binary and ternary structures*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **70**, no. 1 (2020), to appear.

[31] O. Suzuki, *Binary and ternary structure in the evolutions of the universe (*$2 \times 3 \times 2 \times 2 \times \cdots$ *world). From space-time to molecular biology*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **69**, no. 1 (2019), 13–26.

[32] O. Suzuki, *Binary and ternary structure in the evolutions of the universe (*$2 \times 3 \times 2 \times 2 \times \cdots$ *world) II. The description of further stages of the evolutions (Polymers, molecular biology, and natural language)*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **69**, no. 1 (2019), 27–34.

[33] O. Suzuki, J. Ławrynowicz, M. Nowak-Kępczyk and M. Zubert, *Some geometrical aspects of binary, ternary, quaternary, quinary and senary structures in physics*, Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **68** no. 2, (2018) 109–122. DOI: 10.26485/0459-6854/2018/68.2/11

[34] C. Tsallis, Introduction to nonextensive statistical mechanics (Approaching a complex world), Springer Verlag, New York, Inc. 2009.

[35] H. Umezawa, Advanced field theory (Micro, Macro and Thermal Physics), American Institute of Physics 1993.

[36] B. L. van der Waerden, Moderne algebra 2, Berlin Verlag von Julius Springer 1937.

Presented by Osamu Suzuki, online
during the Hypercomplex Seminar, Nov. 11th, 2021

Osamu Suzuki, Department of Information Sciences, College of Humanities and Sciences, Nihon University, Sakurajosui 3-25-40, Setagaya-ku, 156-8550 Tokyo, Japan
e-mail: 1944butterfly@gmail.com

Małgorzata Nowak-Kępczyk, Institute of Mathematics, Informatics and Landscape Architecture, The John Paul II Catholic University of Lublin, Al. Racławickie 14, P.O. Box 129, PL-20-950 Lublin, Poland, e-mail: gosianmk@wp.pl

Maria Ławrynowicz, Faculty of Biology and Environmental Protection, University of Łódź, ul. Józefa Pilarskiego nr 14/16, 90-231 Łódź, Poland
e-mail: maria.lawrynowicz@biol.uni.lodz.pl

**STRUKTURY BINARNE I TERNARNE W EWOLUCJI WSZECHŚWIATA (świat 2x3x2x... wymiarowy) (V)**
**Nierzemienna teoria Galois w ewolucji**

S t r e s z c z e n i e

Wprowadzamy koncepcję nieprzemiennej teorii Galois, a system ewolucyjny w fizyce, kosmologii, biologii i języku opisano w kategoriach teorii w ujednolicony sposób w przypadku, gdy grupa Galois jest rozwiązalna. W tym przypadku strukturę hierarchii można zrealizować przez następujące kolejne rozszerzenia rozszerzeń binarnych i ternarnych: Mianowicie,

$$B \Rightarrow T \Rightarrow B \Rightarrow B \Rightarrow B$$

struktura, gdzie $B$ (odp. $T$) jest binarnym (lub trójskładnikowym) rozszerzeniem, a $\Rightarrow$ oznacza kolejne rozszerzenia (struktura BTBB). W tej części przedstawiono matematyczną teorię dotyczącą nieprzemiennych rozszerzeń Galois oraz podano podstawy matematyczne. Struktura BTBB i jej system złożoności są omówione matematycznie.

*Słowa kluczowe*: struktura binarna, struktura ternarna, system złożony, rozszerzenie Galois, grupa Galois.