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THE PROBLEM OF EXTREMAL DECOMPOSITION OF A COMPLEX PLANE WITH FREE POLES

Summary

Although much research (f. e. [1], [3], [5], [7–15]) has been devoted to the extremal problems of a geometric function theory associated with estimates of functionals defined on systems of non-overlapping domains, however, in the general case the problems remain unsolved.

The paper describes the problem of finding the maximum of a functional. This problem is to find a maximum of the product of inner radii of mutually non-overlapping symmetric domains with respect to a unit circle and the inner radius in some positive certain degree of the domain with respect to zero and description of extreme configurations.

The topic of the paper is devoted to the study of the problem of the classical direction of the geometric theory of complex variable functions, namely, the extremal problems for non-overlapping domains.

Keywords and phrases: inner radius of domain, non-overlapping domains, radial system of points, separating transformation, quadratic differential, Green's function

In geometric function theory of a complex variable problems maximizing the product of inner radii of non-overlapping domains are well known [1–15]. One of the such problems is considered in the article.

Let $\mathbb{N}$, $\mathbb{R}$ be a set of natural and real numbers, respectively, $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be an expanded complex plain or a sphere of Riemann. Let $r(B,a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$ (see, f.e. [1–5]). The inner radius of the domain B is associated with the generalized Green function $g_B(z,a)$ of the domain B by the relations

$$g_B(z,a) = -\ln|z-a| + \ln r(B,a) + o(1), \quad z \to a,$$
$$g_B(z,\infty) = \ln|z| + \ln r(B,\infty) + o(1), \quad z \to \infty.$$

The system of non-overlapping domains is called a finite set of arbitrary domains $\{B_k\}_{k=0}^n$, $n \in \mathbb{N}$, $n \geq 2$ such that $B_k \subset \overline{\mathbb{C}}$, $B_k \cap B_m = \emptyset$, $k \neq m$, $k, m = \overline{0, n}$.

Further we consider the following system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1, n}\}$, $n \in \mathbb{N}$, $n \geq 2$, satisfying the conditions $|a_k| \in \mathbb{R}^+$, $k = \overline{1, n}$ and $0 = \arg a_1 < \arg a_2 < \cdots < \arg a_n < 2\pi$.

Denote by $P_k = P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\}$, $a_{n+1} := a_1$, $\alpha_k := \frac{1}{\pi} \arg \frac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k = \overline{1, n}$, $\sum\limits_{k=1}^{n} \alpha_k = 2$.

Consider the following problem.

Problem. Let $\gamma \in (0, n]$, $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $|a_1| = \ldots = |a_n| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$, where $B_0, \ldots, B_n$ are pairwise non-overlapping domains and $B_1, \ldots, B_n$ are symmetric domains with respect to the unit circle. Find the exact upper bound of the product

$$I_n(\gamma) = r^\gamma(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k).$$

For $\gamma = 1$ the problem was formulated as an open problem in the paper [1]. L.V. Kovalev solved the problem for $n \geq 2$ and $\gamma = 1$ [3, 4]. In publication [9] for any positive $\gamma > 1$ it was proved, that exist some unknown number for which the problem has solution. For $\gamma \in (0,\, 1]$ and $n \geq 2$ the problem was solved in the paper [8]. For $\gamma \in (1; n^{\frac{1}{3}}]$ and $n \geq 14$ the problem was solved in the paper [10]. In the paper [15] it was obtained some result in more general problem in contrast to this problem. The previous result was proved for $\gamma \in (1, \frac{3}{2}]$ and $n \geqslant 9$. In [7] the problem was solved for $\gamma \in (1, \sqrt{n}]$ and $n \geqslant 8$. The investigation of this problem shows that the greatest difficulties arise at the initial values of the parameter n, namely $n = 2$ and $n = 3$. The following theorem substantially complements the results of the papers [2–4].

Theorem 1. *Let $\gamma \in (1, \gamma_4]$, $\gamma_4 = 1,37$. Then for any different points of a unit circle and for any different system of non-overlapping domains B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, 4}$, where the domains B_k, $k = \overline{1, 4}$, have symmetry with respect to the unit circle, the following inequality holds*

$$r^\gamma\left(B_0, 0\right) \prod_{k=1}^{4} r\left(B_k, a_k\right) \leqslant \frac{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}}{\left(1 - \frac{\gamma}{8}\right)^{2 + \frac{\gamma}{4}}} \left(\frac{1 - \frac{\sqrt{2\gamma}}{4}}{1 + \frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}. \tag{1}$$

Equality in this inequality is achieved when a_k and B_k, $k = \overline{0, 4}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w) dw^2 = -\frac{\gamma w^8 + 2(16 - \gamma) w^4 + \gamma}{w^2 (w^4 - 1)^2}\, dw^2. \tag{2}$$

In proving the theorem we will use the ideas of some works [3, 5, 10] and properties of separating transformation (see, e.g., [1]).

Proof. The proof of theorem consist of two cases: when $\alpha_0\sqrt{2\gamma} \geqslant 2$ and $\alpha_0\sqrt{2\gamma} < 2$, where $\alpha_0 = \max\limits_k \alpha_k$.

I case.

Let $\alpha_0\sqrt{2\gamma} \geqslant 2$, $\alpha_0 = \max\limits_k \alpha_k$.

Denote

$$\Lambda_4(\gamma) = \frac{I_4(\gamma)}{I_4^0(\gamma)} = \frac{r^\gamma(B_0,0)\prod\limits_{k=1}^{4} r(B_k,a_k)}{r^\gamma(B_0^{(0)},0)\prod\limits_{k=1}^{4} r(B_k^{(0)},a_k^{(0)})}.$$

Using the result of paper [7], we obtain inequality

$$r^\gamma(B_0,0)\prod_{k=1}^{4} r(B_k,a_k) \leqslant 4^{-\frac{\gamma}{2}}\left[2^n 4\alpha_0(2-\alpha_0)^3 3^{-3}\right]^{1-\frac{\gamma}{4}}. \tag{3}$$

In the paper [10] we found the exact value

$$I_4^0(\gamma) = r^\gamma\left(B_0^{(0)},0\right)\prod_{k=1}^{4} r\left(B_k^{(0)},\omega_k^{(0)}\right) = \frac{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}}{\left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}}}\cdot\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}, \tag{4}$$

where $0\cup\{a_k^{(0)}\}_{k=1}^4$ and $\{B_k^{(0)}\}_{k=0}^4$ are, respectively, poles and circular domains of quadratic differential (2).

Taking into account the above mentioned values, we obtain the estimate

$$\begin{gathered}
\Lambda_4(\gamma) \leqslant \frac{4^{-\frac{\gamma}{2}}\left[2^4\cdot\alpha_0(2-\alpha_0)^3 3^{-3}\right]^{1-\frac{\gamma}{4}}}{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}\cdot\left(1-\frac{\gamma}{8}\right)^{-2-\frac{\gamma}{4}}\cdot\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}} \leqslant \\
\leqslant \frac{4^{-\frac{\gamma}{2}}\left[2^4\cdot\frac{2^4}{\sqrt{2\gamma}}\left(1-\frac{1}{\sqrt{2\gamma}}\right)^3 3^{-3}\right]^{1-\frac{\gamma}{4}}}{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}\cdot\left(1-\frac{\gamma}{8}\right)^{-2-\frac{\gamma}{4}}\cdot\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}} \leqslant \\
\leqslant 4^{-\frac{\gamma}{2}}\left[2^8\cdot\frac{1}{\sqrt{2\gamma}}\right]^{1-\frac{\gamma}{4}}\cdot\left[1-\frac{1}{\sqrt{2\gamma}}\right]^{3-\gamma+\frac{\gamma}{4}}\cdot\left(\frac{1}{3}\right)^{3-\gamma+\frac{\gamma}{4}}\times \\
\times\left(\frac{8}{\gamma}\right)^{\frac{\gamma}{4}}\cdot\left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}}\cdot\left(\frac{1+\frac{\sqrt{2\gamma}}{4}}{1-\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}} \leqslant \\
\leqslant 4^{-\frac{\gamma}{2}}\cdot 2^{8-2\gamma}\cdot\left(\frac{1}{\sqrt{2\gamma}}\right)^{1-\frac{\gamma}{4}}\cdot\left[1-\frac{1}{\sqrt{2\gamma}}\right]^{3-\gamma+\frac{\gamma}{4}}\cdot\left(\frac{4}{3}\right)^{3-\gamma+\frac{\gamma}{4}}\times \\
\times 4^{-(3-\gamma+\frac{\gamma}{4})+\frac{\gamma}{2}}\cdot(2\gamma)^{-\frac{\gamma}{4}}\cdot\left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}}\cdot\left(\frac{1+\frac{\sqrt{2\gamma}}{4}}{1-\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}} \leqslant
\end{gathered} \tag{5}$$

$$\leqslant 4^{-\frac{\gamma}{2}} \cdot 2^{2+\frac{\gamma}{2}} \cdot \left(\frac{1}{\sqrt{2\gamma}}\right)^{1+\frac{\gamma}{4}} \cdot \left[1-\frac{1}{\sqrt{2\gamma}}\right]^{3-\gamma+\frac{\gamma}{4}} \cdot \left(\frac{4}{3}\right)^{3-\gamma+\frac{\gamma}{4}} \times$$

$$\times \left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}} \cdot \left(\frac{1+\frac{\sqrt{2\gamma}}{4}}{1-\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}.$$

So,

$$\Lambda_4(\gamma) \leqslant 4^{-\frac{\gamma}{2}} \cdot \left[1-\frac{1}{\sqrt{2\gamma}}\right]^{3-\gamma+\frac{\gamma}{4}} \cdot \left(\frac{4}{\sqrt{2\gamma}}\right)^{1+\frac{\gamma}{4}} \cdot \left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}} \times$$

$$\times \left(\frac{1+\frac{\sqrt{2\gamma}}{4}}{1-\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}} \cdot \left(\frac{4}{3}\right)^{3-\gamma+\frac{\gamma}{4}}.$$

Let

$$\tau_4(\gamma) = 4^{-\frac{\gamma}{2}} \cdot \left[1-\frac{1}{\sqrt{2\gamma}}\right]^{3-\gamma+\frac{\gamma}{4}} \cdot \left(\frac{4}{\sqrt{2\gamma}}\right)^{1+\frac{\gamma}{4}} \cdot \left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}} \times \qquad (6)$$

$$\times \left(\frac{1+\frac{\sqrt{2\gamma}}{4}}{1-\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}} \cdot \left(\frac{4}{3}\right)^{3-\gamma+\frac{\gamma}{4}}.$$

Denote by $m_4(\gamma)$ the numerator of right side the inequality (5)

$$m_4(\gamma) = 4^{-\frac{\gamma}{2}} \left[4^4 \cdot \frac{1}{\sqrt{2\gamma}} \left(1-\frac{1}{\sqrt{2\gamma}}\right)^3 3^{-3}\right]^{1-\frac{\gamma}{4}}.$$

Consider the graph of the function $m_4(\gamma)$.

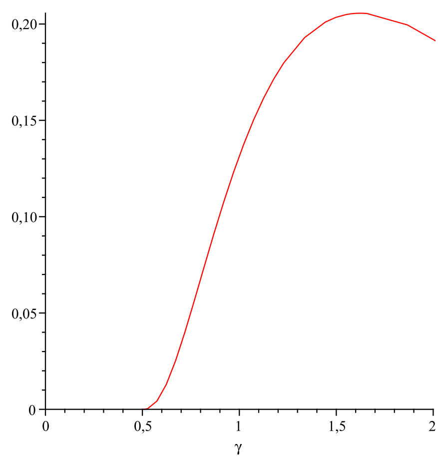

Fig. 1. Graph of the function $y = m_4(\gamma)$

Then the following equality holds

$$\ln(m_4(\gamma)) = -\frac{\gamma}{2}\ln 4+$$

$$+(1-\frac{\gamma}{4})\left[4\ln 4 - \frac{1}{2}\ln 2\gamma + 3\ln\left(1-\frac{1}{\sqrt{2\gamma}}\right) - 3\ln 3\right],$$

$$[\ln(m_4(\gamma))]'_\gamma = \frac{1}{2}\left(-3\ln 4 + \frac{3}{2}\ln 3 + \frac{1}{4}\ln(2\gamma) - \frac{3}{2}\ln\left(1-\frac{1}{\sqrt{2\gamma}}\right)\right)+$$

$$+\left(1-\frac{\gamma}{4}\right)\frac{1}{2\gamma}\left(\frac{3}{2(\sqrt{2\gamma}-1)}-1\right).$$

It is obviously that $[\ln(m_4(\gamma))]'_\gamma > 0$ for $\gamma \in (1, \gamma_4]$.

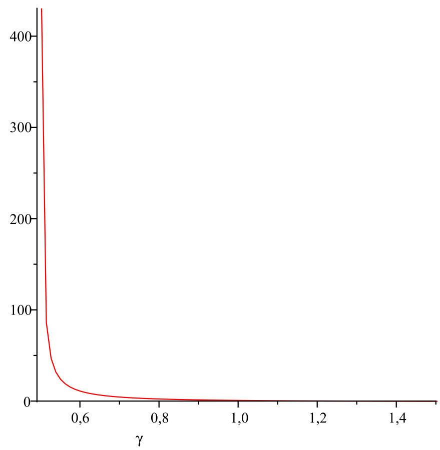

Fig. 2. Graph of the function $y = [\ln(m_4(\gamma))]'_\gamma$

It is easy to proof (see f. e. [10])

$$(\ln I_4^0(\gamma))' = \frac{1}{4}\ln\frac{\gamma}{8-\gamma} + \frac{\sqrt{2}}{2\sqrt{\gamma}}\ln\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right) < 0.$$

In our case, the function $m_4(\gamma)$ is strictly ascending and $I_4^0(\gamma)$ is strictly degression for $\gamma \in (1;\gamma_4]$, and exist $\gamma_4 \approx 1,376031658$ such than $\tau_4(\gamma_4) = 1$. It means that $\tau_4(\gamma) < 1$, and $I_4(\gamma) < I_4^0(\gamma)$, $\alpha_0\sqrt{2\gamma} \geqslant 2$.

II case.

Now consider the case when $\alpha_0\sqrt{\gamma} < 2$. In this case, the theorem is easily obtained from the result of the paper [11]. The inequality $1,37 \leqslant 0,25n^2$ is hold for $n = 4$, then using the result of the paper [11] the inequality holds

$$r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{4} r\left(B_k,a_k\right)\leqslant \frac{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}}{\left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}}}\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}.$$

The equality sign is verified directly. The theorem 1 is proved. □

Let us consider $f_0, f_1, ..., f_4$ is system of univalent conformal mappings of a unit circle on the system of non-overlapping domains B_k, $k=\overline{0,4}$, (besides, domains B_k, $k=\overline{1,4}$ have symmetry with respect to a unit circle) such that $f_k(U)=B_k$, and $f_0(0)=0$, $|f_k(0)|=1$, $k=\overline{1,4}$.

Let us denote by P_4 the class of all systems of function with above consider conditions.

The following statement holds as an application of the above result.

Corollary. For any system of mapping which belongs to P_4, the following inequality holds

$$|f_0'(0)|^{\gamma}\prod_{k=1}^{4}|f_k'(0)|\leqslant \frac{\left(\frac{\gamma}{8}\right)^{\frac{\gamma}{4}}}{\left(1-\frac{\gamma}{8}\right)^{2+\frac{\gamma}{4}}}\left(\frac{1-\frac{\sqrt{2\gamma}}{4}}{1+\frac{\sqrt{2\gamma}}{4}}\right)^{\sqrt{2\gamma}}.$$

Equality is achieved for system of function which univalent conformal map of a unit circle on the system of circular domains of the quadratic differential (2).

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PROBLEM EKSTREMALNEGO ROZKŁADU PŁASZCZYZNY ZESPOLONEJ Z DOWOLNYMI BIEGUNAMI

S t r e s z c z e n i e

Chociaż wiele badań (zob. [1], [3], [5], [7–15]) zostało poświęconych problemom ekstremalnym geometrycznej teorii funkcji, związanych z oszacowaniem funkcjonałów zdefiniowanych na układach rozłącznych obszarów, to jednak w ogólnym przypadku problemy te pozostają nadal otwarte.

Praca opisuje problem znalezienia maksimum pewnego funkcjonału. Ten problem, to znalezienie maksimum iloczynu wewnętrznych promieni wzajemnie rozłącznych symetrycznych obszarów (względem okręgu jednostkowego) i wewnętrznego promienia w pewnej dodatniej potędze w obszarze względem zera oraz opis tych ekstremalnych konfiguracji.

Tematem pracy jest badanie problemu z klasycznych zagadnień geometrycznej teorii funkcji, a mianowicie problemów ekstremalnych w wzajemnie rozłącznych obszarach.

Słowa kluczowe: wewnętrzny promień obszaru, rozłączne obszary, promieniowy system punktów, transformacja rozdzielająca, różniczka kwadratowa, funkcja Greena