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Alfonso Hernández-Montes and Lino Feliciano Reséndis Ocampo

AREA DISTORTION UNDER THE BEURLING-ALHFORS EXTENSION OF MONOMIALS

Summary

The Beurling-Ahlfors extension of a quasisymmetric function is a quasi isometry in the hyperbolic metric of the upper half plane $\mathbb{H}$. In this paper is proved that if $h(x) = x^n$ is an increasing monomial, then its Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$ satisfies for each measurable set $E \subset \mathbb{H}$

$$\frac{n(n+1)A_{\mathcal{H}}(E)}{2^{(n+1)^2}} \leq A_{\mathcal{H}}(f(E)) \leq n(n+1)A_{\mathcal{H}}(E)$$

where $A_{\mathcal{H}}(\cdot)$ denotes hyperbolic area.

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1. Introduction

Let $\mathbb{D} \subset \mathbb{C}$ the unit disk with hyperbolic density $\dfrac{2|dz|}{1-|z|^2}$, (see [3], [4]). Then for each measurable set $E \subset \mathbb{D}$, its hyperbolic area is given by

$$A_{\mathcal{H}}(E) = \int_E \frac{4\,dx\,dy}{(1-|z|^2)^2}.$$

Let the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

with hyperbolic density $\dfrac{|dz|}{y}$, (see [4], [3]). Thus

$$A_{\mathcal{H}}(E) = \int_E \frac{dx\,dy}{(\operatorname{Im}(z))^2} = \int_E \frac{dx\,dy}{y^2}$$

for each measurable set $E \subset \mathbb{H}$, where in both cases $A_{\mathcal{H}}(\cdot)$ denotes hyperbolic area.

Let $K \geq 1$ and $\Omega \subset \mathbb{C}$ a domain. We say that a sense preserving homeomorphism $f : \Omega \to \mathbb{C}$ is a K-quasiconformal mapping if f is absolutely continuous by lines in Ω and

$$\max_{z\in\Omega} \frac{|f_z(z)| + |f_{\overline{z}}(z)|}{|f_z(z)| - |f_{\overline{z}}(z)|} \leq K \qquad \text{for a.e. } z \in \Omega,$$

where $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\overline{z}} = \frac{1}{2}(f_x + if_y)$. Very well known and classic references of quasiconformal mappings are the books [1] and [11].

In [15] were proved the following two results concerning to hyperbolic area distorsion of measurable sets under certain classes of K-quasiconformal mappings.

Theorem 1.1. Let $K \geq 1$ and $f : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal harmonic mapping from the unit disk onto itself. If $E \subset \mathbb{D}$ is a measurable set, then

$$\frac{1}{K} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq \left(\frac{K+1}{2}\right)^2 A_{\mathcal{H}}(E).$$

These bounds are asymptotically sharp when $K \to 1^+$.

Theorem 1.2. Let $K \geq 1$ and $f : \mathbb{H} \to \mathbb{H}$ be a K-quasiconformal mapping given by $f(x + iy) = u(x, y) + iv(y)$. Then for each measurable set $E \subset \mathbb{H}$

$$\frac{1}{K^9} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K^9 A_{\mathcal{H}}(E).$$

These inequalities are asymptotically sharp when $K \to 1^+$.

More results about area distortion can be found in [8],[2], [14], [12], [17], [13], [18], [7] and [6].

If $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $h(\infty) = \infty$ is a quasisymmetric funtion, it is well known that the Beurling-Ahlfors (B-A) extension $f : \mathbb{H} \to \mathbb{H}$ (see [5], [10]) of h is a quasi isometry in the hiperbolic metric of the upper half plane $\mathbb{H}$, that is, there exists $C > 0$ such that

$$\frac{1}{C}\frac{|dz|}{\operatorname{Im}(z)} \leq \frac{|df(z)|}{\operatorname{Im}(f(z))} \leq C\frac{|dz|}{\operatorname{Im}(z)}.$$

As a consequence we have immediately.

Corollary 1.1. Let $M \geq 1$. The Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$ from a M-quasisymmetric function is hiperbolically bi-lipchitz.

Proof. Let $z_1,\ z_2 \in \mathbb{H}$ and l the hyperbolic segment that joints z_1 and z_2, then there exists $C = C(M) > 0$ such that

$$d_{\mathcal{H}}\left(f(z_1), f(z_2)\right) = \int_{f(l)} \frac{|df(z)|}{\operatorname{Im}(f(z))} \leq \int_l C\frac{|dz|}{\operatorname{Im}(z)} \leq C d_{\mathcal{H}}\left(z_1, z_2\right).$$

In a similar way we get $\frac{1}{C} d_{\mathcal{H}}\left(z_1, z_2\right) \leq d_{\mathcal{H}}\left(f(z_1), f(z_2)\right)$. □

Corollary 1.2. The Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$ of a M-quasisymmetric function satisfies for each measurable set $E \subset \mathbb{H}$

$$\frac{A_{\mathcal{H}}(E)}{C^2} \leq A_{\mathcal{H}}(f(E)) \leq C^2 A_{\mathcal{H}}(E) \ ,$$

where $C = C(M)$.

Proof. Let $E \subset \mathbb{H}$ be a measurable set, thus

$$A_{\mathcal{H}}(f(E)) = \int_{f(E)} \frac{|df(z)|}{(\mathrm{Im}(f(z))^2} \leq \int_E C^2 \frac{|dz|}{\mathrm{Im}^2(z)}] = C^2 A_{\mathcal{H}}(E).$$

We get the other inequality in a similar way. □

The main result in this article is to give lower and upper estimation of the previous inequalities when f is the B-A extension of the family of increasing monomials of degree n. These bounds depend only of the odd degree n. More precisely (see Theorem 4.2).

Theorem 1.3. Let $n \in \mathbb{N}$ an odd number and $h : \mathbb{R} \to \mathbb{R}$ the increasing monomial $h(x) = x^n$. The Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$ of h satisfies for each measurable set $E \subset \mathbb{H}$

$$\frac{n(n+1)A_{\mathcal{H}}(E)}{2^{(n+1)^2}} \leq A_{\mathcal{H}}(f(E)) \leq n(n+1)A_{\mathcal{H}}(E) \ .$$

Moreover we give another new properties about the B-A extension, some of them depend only of the extension formulas, neither of the hypothesis of quasisymmetry.

Observe that some numerical calculus in this article are performed with Wolfram Mathetmatica.

2. Quasisymmetric functions

A quasiconformal mapping $f : \Omega \to D$ between two Jordan domains can always be extended to a homeomorphism between the respective closures. Thus we can define the boundary homeomorphism $f : \partial\Omega \to \partial D$.

Now let $h : \partial\Omega \to \partial D$ be a given homeomorphism under which positive orientation of the boundaries with respect to the Jordan domains Ω and D correspond to each other. The boundary value problem is to find a necessary and sufficient conditions for h to be the boundary function of a quasiconformal mapping $f : \Omega \to D$. If we restrict to the normalized case $\Omega = D = \mathbb{H}$, we have that the boundary values h of a K quasiconformal mapping $f : \mathbb{H} \to \mathbb{H}$, $f(\infty) = \infty$, satisfy the double inequality (see [5])

$$\frac{1}{\lambda(K)} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq \lambda(K) \tag{1}$$

for all x and all $t > 0$ where λ is certain distortion function (see [10]). The inequality is sharp for any given K, x and t.

The previous necessary condition (1) motivates the definition of quasisymmetric function.

Let $M \geq 1$. An increasing homeomorphism $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ with $h(\infty) = \infty$ is said to be M-quasisymmetric if

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \tag{2}$$

for all $x \in \mathbb{R}$ and $t > 0$. The extremal value

$$\rho(h) = \inf\{ \ M \ : \ h \quad \text{is } M\text{-quasisymmetric} \ \}$$

is called the quasisymmetric constant of h.

In [5], Beurling and Ahlfors studied the quasisymmetry of the generalizad monomial

$$h_\alpha(x) = x|x|^{\alpha-1}, \qquad \alpha > 1$$

and obtained that the constant of quasisymetry is

$$\rho(h_\alpha) = 2(1 + t_\alpha)^{\alpha-1} - 1$$

where $1 < t_\alpha < 2$.

Let $p : \mathbb{R} \to \mathbb{R}$ be an increasing polynomial of degree n. Let

$$\mathcal{Q}_n = \sup\{ \ \rho(p) \ : \ \ p \text{ is an increasing polynomial of degree } n \ \}$$

In [17] was proved that if $M_1 = 1$, $M_3 = 7 + 4\sqrt{3}$ and for odd $n \geq 5$,

$$M_n = \frac{3^n(n+1)^n}{4},$$

then $\mathcal{Q}_n \leq M_n$ and $\mathcal{Q}_n \leq \mathcal{Q}_{n+2}$. In particular the increasing polynomials are quasisymmetric.

3. The Beurling-Alhfors extension

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. The Beurling-Ahlfors extension of h is the function $f : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$, $f(x + iy) = u(x, y) + iv(x, y)$ is given by

$$\begin{aligned} u(x,y) &= \frac{1}{2y} \int_{-y}^{y} h(x+t)dt, \\ v(x,y) &= \frac{1}{2y} \int_{0}^{y} \left(h(x+t) - h(x-t)\right) dt \end{aligned} \tag{3}$$

for $y > 0$ and $f(x + i0) = h(x)$. Observe that the previous definition does not require the hypothesis of quasisymmetry.

Example 3.1. Let $n \in \mathbb{N}$. The B-A extension of the monomial $h(x) = x^n$, is given by

$$f(x,y) = \frac{(x+y)^{n+1} - (x-y)^{n+1}}{2(n+1)y} + i\frac{(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}}{2(n+1)y}. \quad (4)$$

Observe that

$$f(0,y) = \frac{y^{n+1} - (-y)^{n+1}}{2(n+1)y} + i\frac{y^{n+1} + (-y)^{n+1}}{2(n+1)y}.$$

If n is even $f(0,y) = \frac{y^n}{n+1}$ and if n is odd $f(0,y) = i\frac{y^n}{n+1}$. In particular if n is an even number then f is not an inyective function.

Let $M \geq 1$ and $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a M-quasisymetric function. Then the B-A extension of h (see [5]) is a $K(M)$-quasiconformal mapping from $\mathbb{H}$ onto itself. Moreover

$$\lim_{M \to 1^+} K(M) = 1.$$

Thus the solution to the boundary value problem in $\mathbb{H}$ is precissely the B-A extension.

Example 3.2. Let $h(x) = x$. Then h is a 1-quasisymmetric function and its B-A extension is 2-quasiconformal and is given by $f(x+iy) = x + \dfrac{iy}{2}$. For each measurable set $E \subset \mathbb{H}$ hold

$$A_e(f(E)) = \frac{1}{2}A_e(E) \quad \text{and} \quad A_{\mathbb{H}}(f(E)) = 2A_{\mathbb{H}}(E).$$

Example 3.3. Let $g(x) = x^3$. Then g is a $7 + 4\sqrt{3}$ - quasisymmetric function and its B-A extension is a $20.7872\ldots$-quasiconformal mapping given by

$$f(x+iy) = x^3 + xy^2 + \frac{i}{4}(6x^2y + y^3).$$

Moreover

$$\frac{y^2 J(x+iy)}{v(x,y)^2} = \frac{12(6x^4 - 3x^2y^2 + y^4)}{(6x^2 + y^2)^2}$$

If $y = mx$ the right hand side is

$$r(m) = \frac{12(6 - 3m^2 + m^4)}{(6+m^2)^2}.$$

Thus

$$\frac{3}{4} \leq r(m) < 12 \qquad \text{for each } m \in \mathbb{R}.$$

Thus given any measurable set $E \subset \mathbb{H}$ we have

$$\frac{3}{4}A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq 12A_{\mathcal{H}}(E).$$

If

$$E = \left\{ (x,y) \in \mathbb{H} \ : \ x \in [1,\infty), \ 0 < y \leq \frac{1}{x^2} \right\},$$

its Euclidean measure is $A_e(E) = 1$. Since $J_f(x+iy) = \frac{3}{4}(6x^4 - 3x^2y^2 + y^4)$ then $A_e(f(E)) = \infty$. That is f explodes the Euclidean area.

Example 3.4. Let $g(x+iy) = 2x + \sin(x+y) + iy$ the $\frac{11+\sqrt{85}}{6}$-quasiconformal mapping, then its boundary function is given by $h(x) = 2x + \sin x$ that is a $2.2\ldots$-quasysimmetric function with B-A extension $3.2\ldots$-quasiconformal given by

$$f(x+iy) = 2x + \frac{\sin x \sin y}{y} + \frac{i}{y}\left(y^2 + \cos x - \frac{\cos(x+y)}{2} - \frac{\cos(x-y)}{2}\right) .$$

Observe that when we apply the B-A extension of h we do not recover the original function g.

Theorem 3.1. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function in a neighborhood of $x \in \mathbb{R}$ with $h'(x) \neq 0$ and f its Beurling-Ahlfors extension. Then

$$\lim_{y \to 0^+} \frac{y^2 J_f(x,y)}{v^2(x,y)} = 2 \quad .$$

Proof. Let $f = u + iv$ the B-A extension of h, where u and v are given by (3). Thus

$$\begin{aligned}
u_x(x,y) = u_x &= \frac{1}{2y}(h(x+y) - h(x-y)),\\
u_y(x,y) = u_y &= -\frac{1}{2y^2}\int_{-y}^{y} h(x+t)\,dt + \frac{1}{2y}(h(x+y) + h(x-y)),\\
v_x(x,y) = v_x &= \frac{1}{2y}\left(h(x+y) + h(x-y) - 2h(x)\right),\\
v_y(x,y) = v_y &= -\frac{1}{2y^2}\int_{0}^{y} (h(x+t) - h(x-y))\,dt + \frac{1}{2y}(h(x+y) - h(x-y)).
\end{aligned}$$

Consider the notation

$$I_1 = I_1(x,y) = \int_{-y}^{y} h(x+t)\,dt, \qquad I_2 = I_2(x,y) = \int_0^y (h(x+t) - h(x-t))\,dt.$$

In this part all the derivatives are with respect to the variable y, that is $I_i' = (I_i)_y$, $I_i'' = (I_i)_{yy}$. Thus

$$\begin{aligned}
I_1' &= h(x+y) + h(x-y), & I_2' &= h(x+y) - h(x-y),\\
I_1'' &= h'(x+y) - h'(x-y), & I_2'' &= h'(x+y) + h'(x-y),
\end{aligned}$$

if y is enough small. Moreover, by continuity of h and h' at x we have

$$\lim_{y\to 0^+} I_1 = \lim_{y\to 0^+} I_2 = \lim_{y\to 0^+} I_1'' = \lim_{y\to 0^+} I_2' = 0$$

$$\lim_{y\to 0^+} I_1' = 2h(x) \qquad \text{and} \qquad \lim_{y\to 0^+} I_2'' = 2h'(x).$$

With this notation

$$v = \frac{I_2}{2y}, \qquad u_x = \frac{I_2'}{2y}, \qquad u_y = -\frac{I_1}{2y^2} + \frac{I_1'}{2y},$$

$$v_x = \frac{I_1' - 2h(x)}{2y}, \qquad v_y = -\frac{I_2}{2y^2} + \frac{I_2'}{2y}.$$

Therefore

$$\begin{aligned} J_f &= u_x v_y - v_x u_y \\ &= \frac{-1}{4y^3} I_2 I_2' + \frac{1}{4y^2}(I_2')^2 - \left(\frac{-1}{4y^3}\big(I_1 I_1' - 2I_1 h(x)\big) + \frac{1}{4y^2}\big((I_1')^2 - 2I_1 h(x)\big) \right). \end{aligned}$$

Then

$$4y^2 J_f = \frac{-I_2 I_2'}{y} + (I_2')^2 + \frac{I_1 I_1'}{y^2} + \frac{4I_1 h(x)}{y} - (I_1')^2. \tag{5}$$

Since h is a C^1 function in a neigborhood of x, we can write for fixed x and y near to 0

$$\begin{aligned} I_1(y) &= I_1(0) + I_1'(0)y + I_1''(0)\frac{y^2}{2} + \circ(y^2) = 2h(x)y + \circ(y^2) \\ I_1'(y) &= I_1'(0) + I_1''(0)y + \circ(y) = 2h(x) + \circ(y) \end{aligned}$$

and

$$\begin{aligned} I_2(y) &= I_2(0) + I_2'(0)y + I_2''(0)\frac{y^2}{2} + \circ(y^2) = h'(x)y^2 + \circ(y^2) \\ I_2'(y) &= I_2'(0) + I_2''(0)y + \circ(y) = 2h'(x)y + \circ(y). \end{aligned}$$

Using these approximations in (5) we get

$$\lim_{y\to 0^+} \frac{y^2 J_f(x,y)}{v^2(x,y)} = 2.$$

□

The B-A extension of an increasing monomial x^n has other properties.

Theorem 3.2. Let $n \in \mathbb{N}$ be an odd number. If $h : \mathbb{R} \to \mathbb{R}$ is given by $h(x) = x^n$ and $f : \mathbb{H} \to \mathbb{H}$ its Beurling-Ahlfors extension, then

a) $\displaystyle\lim_{y\to 0^+} \frac{y^2 J_f(x,y)}{v^2(x,y)} = 2$ for each $0 \neq x$ fixed.

b) $\displaystyle\frac{y^2 J_f(0,y)}{v^2(0,y)} = n(n+1)$ for each $y > 0$.

c) $\displaystyle\lim_{y\to\infty} \frac{y^2 J_f(x,y)}{v^2(x,y)} = n(n+1)$ for each fixed x.

Proof. a) follows from Theorem 3.1. To prove b) we have from (4)

$$\begin{aligned}
u_x(x,y) = u_x &= \frac{(x+y)^n - (x-y)^n}{2y} \\
u_y(x,y) = u_y &= -\frac{(x+y)^{n+1} - (x-y)^{n+1}}{2(n+1)y^2} + \frac{(x+y)^n + (x-y)^n}{2y} \\
v_x(x,y) = v_x &= \frac{(x+y)^n + (x-y)^n - 2x^n}{2y} \\
v_y(x,y) = v_y &= -\frac{(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}}{2(n+1)y^2} + \frac{(x+y)^n - (x-y)^n}{2y}.
\end{aligned}$$

Since n is an odd number

$$\begin{aligned}
J_f(0,y) = {} & \frac{y^n - (-y)^n}{2y}\left(-\frac{y^{n+1} + (-y)^{n+1}}{2(n+1)y^2} + \frac{y^n - (-y)^n}{2y}\right) \\
& - \left(-\frac{y^{n+1} - (-y)^{n+1}}{2(n+1)y^2} + \frac{y^n + (-y)^n}{2y}\right)\left(\frac{y^n + (-y)^n}{2y}\right) \\
& = y^{n-1}\left(-\frac{y^{n-1}}{n+1} + y^{n-1}\right) = \frac{n}{n+1}y^{2n-2}.
\end{aligned}$$

Thus

$$\frac{y^2 J_f(0,y)}{v^2(0,y)} = \frac{\dfrac{ny^{2n}}{n+1}}{\dfrac{y^{2n}}{(n+1)^2}} = n(n+1).$$

To calculate c) we observe that

$$\frac{y^2 J_f(x,y)}{v^2(x,y)} = \frac{4y^4 J_f(x,y)}{\frac{1}{(n+1)^2}\left[(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}\right]^2}.$$

So we calculate first $4y^4 J_f(x,y)$. Since it is not to easy to reach our final expression we write some steps

$$\begin{aligned}
4y^4 J_f = {} & \frac{y}{n+1}\left\{[(x+y)^{n+1} - (x-y)^{n+1}][(x+y)^n + (x-y)^n]\right\} - \\
& - \frac{y}{n+1}\left\{[(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}][(x+y)^n - (x-y)^n]\right\} + \\
& + y^2\left\{[(x+y)^n - (x-y)^n]^2 - [(x+y)^n + (x-y)^n]^2\right\} + \\
& + 2yx^n\left\{y[(x+y)^n + (x-y)^n] - \frac{1}{n+1}[(x+y)^{n+1} - (x-y)^{n+1}]\right\} \\
= {} & \frac{y}{n+1}\left[(x+y)^{2n+1} + (x+y)^{n+1}(x-y)^n - (x+y)^n(x-y)^{n+1} - (x-y)^{2n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{y}{n+1}\left[(x+y)^{2n+1} - (x+y)^{n+1}(x-y)^n + (x+y)^n(x-y)^{n+1} - (x-y)^{2n+1}\right] \\
&+ \frac{2yx^{n+1}}{n+1}\left[(x+y)^n - (x-y)^n\right] + 2y^2x^n\left[(x+y)^n + (x-y)^n\right] \\
&+ y^2\left[(x+y)^n - (x-y)^n - (x+y)^n - (x-y)^n\right] \\
&\cdot\left[(x+y)^n - (x-y)^n + (x+y)^n + (x-y)^n\right] - \frac{2yx^n}{n+1}\left[(x+y)^{n+1} - (x-y)^{n+1}\right] \\
&= \frac{2y}{n+1}\left[(x+y)^{n+1}(x-y)^n - (x-y)^{n+1}(x+y)^n\right] \\
&+ \frac{2yx^{n+1}}{n+1}\left[(x+y)^n - (x-y)^n\right] + y^2[-2(x-y)^n][2(x+y)^n] \\
&+ 2y^2x^n[(x+y)^n + (x-y)^n] - \frac{2yx^n}{n+1}\left[(x+y)^{n+1} - (x-y)^{n+1}\right] \\
&= \frac{4y^2}{n+1}[(x+y)^n(x-y)^n] + \frac{2yx^{n+1}}{n+1}[(x+y)^n - (x-y)^n] \\
&- 4y^2[(x+y)^n(x-y)^n] + 2y^2x^n[(x+y)^n + (x-y)^n] \\
&- \frac{2yx^n}{n+1}[(x+y)^{n+1} - (x-y)^{n+1}] \\
&= \frac{-4y^2n}{n+1}(x+y)^n(x-y)^n + (x+y)^n\left[\frac{2yx^{n+1}}{n+1} + 2y^2x^n - \frac{2yx^n(x+y)}{n+1}\right] \\
&+ (x-y)^n\left[\frac{-2yx^{n+1}}{n+1} + 2y^2x^n + \frac{2yx^n(x-y)}{n+1}\right] \\
&= \frac{-4y^2n}{n+1}(x+y)^n(x-y)^n + (x+y)^n\left[2y^2x^n - \frac{2y^2x^n}{n+1}\right] \\
&+ (x-y)^n\left[2y^2x^n - \frac{2y^2x^n}{n+1}\right] \\
&= \frac{-4y^2n}{n+1}(x+y)^n(x-y)^n + \left[2y^2x^n\left(\frac{n}{n+1}\right)\right]\left[(x+y)^n + (x-y)^n\right] \\
&= \frac{n}{n+1}\left(-4y^2(x+y)^n(x-y)^n + 2y^2x^n\left[(x+y)^n + (x-y)^n\right]\right).
\end{aligned}
$$

From here we get

$$
\begin{aligned}
\frac{y^2 J_f(x,y)}{v^2(x,y)} &= \frac{\dfrac{n}{n+1}\left(-4y^2(x+y)^n(x-y)^n + 2y^2x^n\left[(x+y)^n + (x-y)^n\right]\right)}{\dfrac{1}{(n+1)^2}\left[(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}\right]^2} \\
&= n(n+1)\frac{-4y^2(x+y)^n(x-y)^n + 2y^2x^n\left[(x+y)^n + (x-y)^n\right]}{\left[(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}\right]^2}.
\end{aligned} \tag{6}
$$

Now, for $m \neq 0$, sustitute $y = mx$ in (6)

$$\begin{aligned}\frac{y^2 J_f(x,mx)}{v^2(x,mx)} &= \qquad (7)\\ &= n(n+1)\frac{-4(mx)^2(x+mx)^n(x-mx)^n + 2(mx)^2x^n\left[(x+mx)^n + (x-mx)^n\right]}{\left[(x+mx)^{n+1} + (x-mx)^{n+1} - 2x^{n+1}\right]^2}\\ &= \frac{n(n+1)m^2\left[2\big((1+m)^n + (1-m)^n\big) - 4(1-m^2)^n\right]}{\left[(1+m)^{n+1} + (1-m)^{n+1} - 2\right]^2}.\end{aligned}$$

Since $n \in \mathbb{N}$ is odd and this last expression is symmetric in m we have

$$\lim_{m\to\infty} \frac{y^2 J_f(x,mx)}{v^2(x,mx)} = \lim_{m\to\infty} \frac{n(n+1)4m^{2n+2}}{4m^{2n+2}} = n(n+1).$$

□

4. The main result

Lemma 4.1. Let $n \in \mathbb{N}$ an odd number. Then the function $g : [0,\infty) \to \mathbb{R}$ given by

$$g(m) := 2\big((1+m)^n + (1-m)^n\big) - 4(1-m^2)^n.$$

is increasing.

Proof. Since $n-1$ is an even number, we have

$$g'(m) = 2n\big((1+m)^{n-1} - (1-m)^{n-1} + 4m(1-m^2)^{n-1}\big)$$

$$\geq (1+m)^{n-1} - (1-m)^{n-1}.$$

In particular $g'(0) = 0$ and $g'(m) > 0$ for each $m \in (0,\infty)$. □

Lemma 4.2. Let $n \in \mathbb{N}$ an odd number. Then

$$h(m) := (1+m)^{n+1} + (1-m)^{n+1} - 2 \leq \begin{cases} 2^{n+1}m^2, & \text{if} \quad 0 \leq m \leq 1 \\ 2^{n+1}m^{n+1}, & \text{if} \quad 1 < m < \infty. \end{cases}$$

Proof. It is a consequence of the equality

$$(1+m)^{n+1} + (1-m)^{n+1} - 2 = 2\sum_{k=1}^{\frac{n+1}{2}} \binom{n+1}{2k} m^{2k}.$$

□

Lemma 4.3. Let $n \in \mathbb{N}$ an odd number. Then

$$g(m) \geq \begin{cases} 4nm^2 & \text{if} \quad 0 \leq m \leq 1, \\ 2^{n+1} & \text{if} \quad 1 < m < \sqrt{2^{n+1}}, \\ \dfrac{m^{2n}}{2} & \text{if} \quad \sqrt{2^{n+1}} < m < \infty. \end{cases}$$

Proof. Consider the developments

$$2\big((1+m)^n + (1-m)^n\big) = 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{2k}.$$

and

$$\begin{aligned} 4(1-m^2)^n &= 4\sum_{k=0}^{n} \binom{n}{k} (-1)^k m^{2k} \\ &= 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} (-1)^{2k} m^{4k} + 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} (-1)^{2k+1} m^{2(2k+1)} \\ &= 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{4k} - 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+2}. \end{aligned}$$

Thus we can rewrite $g(m)$ as

$$g(m) = 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} \big(m^{2k} - m^{4k}\big) + 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+2}$$

and if $0 \leq m \leq 1$, then $m^{2k} - m^{4k} \geq 0$ for $k = 0,\ 1, \ldots, \frac{n-1}{2}$ and therefore

$$g(m) \geq 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+2} \geq 4nm^2.$$

If $1 \leq m \leq \sqrt{2^{n+1}}$, by Lemma 4.1

$$g(m) \geq g(1) = 2^{n+1}.$$

If $\sqrt{2^{n+1}} < m < \infty$, we have

$$\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{4k} \leq m^{2n-2} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} \leq 2^n m^{2n-2} \leq \frac{m^{2n}}{2}$$

and we now rewrite $g(m)$ as

$$g(m) = 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{2k} - 4\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{4k} + \frac{m^{2n}}{2} + 4\sum_{k=0}^{\frac{n-3}{2}} \binom{n}{2k+1} m^{4k+2} + \frac{m^{2n}}{2}$$

$$\geq \frac{m^{2n}}{2}$$

that concludes the proof. □

Now we can prove the main result. The case $n = 3$ was treated in Example 3.3.

Theorem 4.1. Let $n \in \mathbb{N}$, $n \geq 5$ be an odd number. If $h : \mathbb{R} \to \mathbb{R}$ is given by $h(x) = x^n$ and $f : \mathbb{H} \to \mathbb{H}$ its Beurling-Ahlfors extension, then

$$\frac{n(n+1)}{2^{(n+1)^2}} \leq \frac{y^2 J_f(x,y)}{v^2(x,y)} \leq n(n+1)$$

where

Proof. The left hand inequality is a direct consequence of the formula (7) and Lemmas 4.2 and 4.3. The right hand inequality is proved if we prove that $m^2 g(m) \leq h(m)^2$, that is

$$m^2[2\big((1+m)^n + (1-m)^n\big) - 4(1-m^2)^n] \leq [(1+m)^{n+1} + (1-m)^{n+1} - 2]^2$$

or equivalently

$$\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} \left(m^{2k+2} - m^{4k+2}\right) + \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+4} \leq \left[\sum_{k=1}^{\frac{n+1}{2}} \binom{n+1}{2k} m^{2k}\right]^2.$$

Since $m^{2(0)+2} - m^{4(0)+2} = 0$ we rewrite the previous inequality as

$$\sum_{k=1}^{\frac{n-1}{2}} \binom{n}{2k} \left(m^{2k+2} - m^{4k+2}\right) + \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+4}$$
$$\leq \sum_{k=1}^{\frac{n+1}{2}} \sum_{j=1}^{\frac{n+1}{2}} \binom{n+1}{2k} \binom{n+1}{2j} m^{2(k+j)}.$$

We will finish the proof if we can show that

$$\left[\binom{n}{1} + \binom{n}{2}\right] m^4 + \sum_{k=2}^{\frac{n-1}{2}} \binom{n}{2k} m^{2k+2} \leq \sum_{k=1}^{\frac{n+1}{2}} \binom{n+1}{2k} \binom{n+1}{2} m^{2(k+1)} \tag{8}$$

and

$$\sum_{k=1}^{\frac{n-1}{2}} \binom{n}{2k+1} m^{4k+4} \leq \sum_{k=2}^{\frac{n+1}{2}} \binom{n+1}{2k} \binom{n+1}{2k} m^{4k} \tag{9}$$

It is necessary to study only the coefficients in the previous inequalities. We recall the identity

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{10}$$

In particular

$$\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2} \leq \binom{n+1}{2}\binom{n+1}{2}$$

and is easy to see that

$$\binom{n}{2k} < \binom{n+1}{2k}\binom{n+1}{2}$$

thus (8) is proved. From (10) we have

$$\binom{n+1}{2k+2} = \binom{n}{2k+2} + \binom{n}{2k+1}$$

and thus

$$\binom{n}{2k+1} < \binom{n+1}{2k+2} < \binom{n+1}{2k+2}\binom{n+1}{2k+2}$$

and (9) is proved comparing coefficients. □

As an immediate consequence of this result we get

Theorem 4.2. Let $n \in \mathbb{N}$ and odd number and $h : \mathbb{R} \to \mathbb{R}$ the increasing monomial $h(x) = x^n$. The Beurling-Ahlfors extension $f : \mathbb{H} \to \mathbb{H}$, of the monomial h is given by

$$f(x, y) = \frac{(x+y)^{n+1} - (x-y)^{n+1}}{2(n+1)y} + i\,\frac{(x+y)^{n+1} + (x-y)^{n+1} - 2x^{n+1}}{2(n+1)y}$$

and satisfies for each measurable set $E \subset \mathbb{H}$

$$\frac{n(n+1)A_{\mathcal{H}}(E)}{2^{(n+1)^2}} \leq A_{\mathcal{H}}(f(E)) \leq n(n+1)A_{\mathcal{H}}(E)\ .$$

By Theorem 3.2 the right bound is sharp, however the left bound is far to be sharp. Using Wolfram Mathematica we get the following example.

Example 4.1. Let $h(x) = x^5$. Then h is a $129.569151\ldots$-quasisymmetric functions and its B-A extensiion is a $197.\,2675\ldots$-quasiconformal mapping that satisfies for each measurable set $E \subset \mathbb{H}$

$$1.\,71646 \cdots A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq 30A_{\mathcal{H}}(E).$$

The left hand bound in the Theorem for $n = 5$ is $\frac{5(6)}{2^{36}} = \frac{15}{2^{35}}$.

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Alfonso Hernández-Montes
Escuela Superior de Ingeniería Mecánica y Eléctrica
Instituto Politécnico Nacional
Unidad Adolfo López Mateos
Avenida Luis Enrique Erro s/n, 07738
E-mail: alhernandezmon@ipn.mx

Lino Feliciano Reséndis Ocampo
Universidad Autónoma Metropolitana Azcapotzalco
Av. San Pablo No. 180, Col. Reynosa Tamaulipas, Azcapotzalco, 02200, México.
E-mail: lfro@azc.uam.mx

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ZNIEKSZTAŁCENIA POLA W ROZSZERZENIU BEURLINGA-AHLFORSA JEDNOMIANÓW

S t r e s z c z e n i e

Rozszerzenie Beurlinga-Ahlforsa funkcji quasisymetrycznej jest quasi izometrią górnej półpłaszczyzny $\mathbb{H}$. W pracy udowodniono, że jeśli $h(x) = x^n$ jest rosnącym jednomianem, to dla jego rozszerzenia Beurlinga-Ahlforsa $f : \mathbb{H} \to \mathbb{H}$ spełniony jest warunek $E \subset \mathbb{H}$

$$\frac{n(n+1)A_{\mathcal{H}}(E)}{2^{(n+1)^2}} \leq A_{\mathcal{H}}(f(E)) \leq n(n+1)A_{\mathcal{H}}(E)$$

dla dowolnego zbioru mierzalnego $E \subset \mathbb{H}$ gdzie $A_{\mathcal{H}}(\cdot)$ oznacza pole hiperboliczne.

Słowa kluczowe: pole hiperboliczne, odwzorowanie quasikonforemne, rozszerzenie Beurlinga-Ahlforsa