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Dedicated to the memory of
Professor Leszek Wojtczak

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ON SOME KIND OF NUMBERS OF THE FIBONACCI TYPE AND THEIR APPLICATIONS FOR BICOMPLEX NUMBERS

Summary

In this paper we introduce a special kind of numbers of the Fibonacci type and we show some their applications in the theory of bicomplex numbers.

Keywords and phrases: Fibonacci numbers, recurrence relations, bicomplex numbers

1. Introduction and preliminary results

Numbers defined recursively by the pth order linear recurrence relation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \ldots + b_p a_{n-p}, \text{ for n} \geq \text{p}$$

where $p \geq 2$, $b_l \geq 0$, $l = 1, 2, \ldots, p$ are integers with given nonnegative integers $a_0, \ldots, a_{p-1}$ are named as the numbers of the Fibonacci type.

For special values of p, b_l, $l = 1, 2, \ldots, p$ and $a_0, \ldots, a_{p-1}$ we obtain the well-known recurrences which define the numbers of the Fibonacci type. We list some of them

(i) Fibonacci numbers F_n,
$F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with $F_0 = F_1 = 1$,

(ii) Lucas numbers L_n,
$L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$,

(iii) Pell numbers P_n,
$P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$,

(iv) Jacobsthal numbers J_n,
$J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$,

(v) Tribonacci numbers of the first kind t_n,
$t_n = t_{n-1} + t_{n-2} + t_{n-3}$, for $n \geq 3$ with $t_0 = t_1 = t_2 = 1$,

(vi) Tribonacci numbers of the second kind T_n,
$T_n = T_{n-1} + T_{n-2} + T_{n-3}$, for $n \geq 3$ with $T_0 = T_1 = 1$, $T_2 = 2$.

There are some versions of Tribonacci numbers defined by the same linear recurrence relation as T_n but with different initial conditions. For special value of initial conditions we obtain different kinds of Tribonacci numbers. Apart Tribonacci numbers t_n and T_n we define other kinds of Tribonacci numbers namely numbers R_n, S_n and U_n. For $n \geq 0$ we define three types of Tribonacci numbers as follows

$R_0 = 3$, $R_1 = 1$, $R_2 = 3$ and $R_n = R_{n-1} + R_{n-2} + R_{n-3}$ for $n \geq 3$,

$S_0 = 3$, $S_1 = 2$, $S_2 = 5$ and $S_n = S_{n-1} + S_{n-2} + S_{n-3}$ for $n \geq 3$,

$U_0 = 0$, $U_1 = 1$, $U_2 = 2$ and $U_n = U_{n-1} + U_{n-2} + U_{n-3}$ for $n \geq 3$.

The Table 1 presents values of these Tribonacci numbers for $n = 0, 1, ..., 10$.

Table 1.

n	0	1	2	3	4	5	6	7	8	9	10
t_n	1	1	1	3	5	9	17	31	57	105	193
T_n	1	1	2	4	7	13	24	44	81	149	274
R_n	3	1	3	7	11	21	39	71	131	241	443
S_n	3	2	5	10	17	32	59	108	199	366	673
U_n	0	1	2	3	6	11	20	37	68	125	230

Numbers of the Fibonacci type are intensively studied in the literature and they have many interesting generalizations and applications in distinct areas of mathematics, see for example [12], [13], [14], [15]. In this paper we introduce a special kind of numbers of the Fibonacci type and we show some their applications in the theory of bicomplex numbers.

Let $p \geq 2$, $n \geq 3$ be integers. Let consider numbers of the Fibonacci type defined recursively by the pth order recurrence relation of the form

$$P_p(n) = P_p(n-p+1) + P_p(n-1) + P_p(n-p) \text{ for } n \geq p \tag{1}$$

with $P_p(l) = a_l$, $l = 0, 1, \ldots, p-1$.

For special values of a_l and p we obtain the well-known numbers of the Fibonacci type. If $p = 2$, $a_0 = 0$, $a_1 = 1$ then $P_2(n)$ is the nth Pell number. Moreover, for $p \geq 2$ and special values of a_l numbers $P_p(n)$ were introduced and studied in [16].

For $p = 3$ and special a_l, $l = 0, 1, 2$ we obtain Tribonacci numbers t_n, T_n, R_n, S_n and U_n.

2. Identities for $P_p(n)$–numbers

Theorem 2.1. *Let $p \geq 2$, $t \geq 0$ be integers. Then for $n \geq 2p$ holds*

$$\begin{aligned}\sum_{l=0}^{t} P_p(n+l) &= \sum_{l=0}^{t} \left(P_p(n+l-1) + 2P_p(n+l-p)\right) + \\ &+P_p(n+t-p+1) - P_p(n-p).\end{aligned} \tag{2}$$

Proof. Using the definition of $P_p(n)$ in $t+1$ steps we have

$$\begin{aligned}P_p(n) &= P_p(n-1) + P_p(n-p+1) + P_p(n-p)\\ P_p(n+1) &= P_p(n) + P_p(n-p+2) + P_p(n-p+1)\\ P_p(n+2) &= P_p(n+1) + P_p(n-p+3) + P_p(n-p+2)\\ &\vdots\\ P_p(n+t-1) &= P_p(n+t-2) + P_p(n+t-p) + P_p(n+t-p-1)\\ P_p(n+t) &= P_p(n+t-1) + P_p(n+t-p+1) + P_p(n+t-p).\end{aligned}$$

By simple calculations the result follows. □

Theorem 2.2. *Let $p \geq 2$, $t \geq 2$ be integers. Then for $n \geq 2p$ and even t holds*

$$\begin{aligned}&2\sum_{l=0}^{\frac{t}{2}-1} \left(P_p(n+2l) - P_p(n+2l+1)\right) =\\ &= P_p(n-1) + P_p(n-p) - P_p(n+t-1) - P_p(n+t-p).\end{aligned} \tag{3}$$

Proof. Using the definition of $P_p(n)$ we have

$$\begin{aligned}P_p(n) &= P_p(n-1) + P_p(n-p+1) + P_p(n-p)\\ -P_p(n+1) &= -P_p(n) - P_p(n-p+2) - P_p(n-p+1)\\ P_p(n+2) &= P_p(n+1) + P_p(n-p+3) + P_p(n-p+2)\\ -P_p(n+3) &= -P_p(n+2) - P_p(n-p+4) - P_p(n-p+3)\\ &\vdots\\ P_p(n+t-2) &= P_p(n+t-3) + P_p(n+t-p-1) + P_p(n+t-p-2)\\ -P_p(n+t-1) &= -P_p(n+t-2) - P_p(n+t-p) - P_p(n+t-p-1).\end{aligned}$$

By simple calculations the result follows. □

Theorem 2.3. *Let $p \geq 2$, $t \geq 1$ be integers. Then for $n \geq 2p$ and odd t holds*

$$\begin{aligned}&2\sum_{l=0}^{\frac{t-1}{2}} \left(P_p(n+2l) - P_p(n+2l+1)\right) =\\ &= P_p(n-1) + P_p(n-p) - P_p(n+t) - P_p(n+t-p+1).\end{aligned} \tag{4}$$

Proof. For odd $t \geq 1$ we have

$$\begin{aligned}
P_p(n) &= P_p(n-1) + P_p(n-p+1) + P_p(n-p) \\
-P_p(n+1) &= -P_p(n) - P_p(n-p+2) - P_p(n-p+1) \\
P_p(n+2) &= P_p(n+1) + P_p(n-p+3) + P_p(n-p+2) \\
-P_p(n+3) &= -P_p(n+2) - P_p(n-p+4) - P_p(n-p+3) \\
&\vdots \\
P_p(n+t-1) &= P_p(n+t-2) + P_p(n+t-p) + P_p(n+t-p-1) \\
-P_p(n+t) &= -P_p(n+t-1) - P_p(n+t-p+1) - P_p(n+t-p),
\end{aligned}$$

and by simple calculations the result follows. □

Recall that Tribonacci numbers T_n have been firstly defined by Feinberg in 1963, see [3]. The characteristic equation has the form $x^3 - x^2 - x - 1 = 0$ and it has roots

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\beta = \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1 + \epsilon\sqrt{3}}{2},\ \epsilon^2 = -1.$$

Hence the Binet formula for the Tribonacci number T_n has the form

$$T_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}. \tag{5}$$

Tribonacci numbers R_n, S_n, T_n and U_n were considered in [2], [3], [4], [6], [11] where among other Binet formulas for them were found. Moreover in [6] some relations between Tribonacci numbers were given. We recall these dependences

$$R_n = T_{n-1} + 2T_{n-2} + 3T_{n-3},\ for\ n \geq 3 \tag{6}$$

$$S_n = 3T_n - T_{n-1},\ for\ n \geq 1 \tag{7}$$

$$U_n = T_{n-1} + T_{n-2},\ for\ n \geq 2 \tag{8}$$

$$\sum_{l=0}^{n} U_l = T_{n+1} - 1 \tag{9}$$

$$\sum_{l=1}^{n} R_l = 2U_{n+1} + U_{n-1} - 3 \tag{10}$$

$$\sum_{l=0}^{n} S_l = \frac{3U_{n+2} + 2U_{n+1} - U_n - 2}{2} \tag{11}$$

$$\sum_{l=0}^{n} T_l = \frac{U_{n+2} + U_{n+1} - 1}{2}. \tag{12}$$

From the above identities we can obtain other relations

$$2T_n = U_{n+1} + U_{n-1}, \ for \ n \geq 1 \tag{13}$$

$$\sum_{l=1}^{n} R_l = 3T_n + T_{n-1} - 3 \tag{14}$$

$$\sum_{l=0}^{n} S_l = T_{n+2} + 2T_n - 1 \tag{15}$$

$$\sum_{l=0}^{n} T_l = \frac{T_{n+2} + T_n - 1}{2}. \tag{16}$$

3. The $P_p(n)$–bicomplex numbers

Let consider the set $\mathbb{C}$ of complex numbers $a+bi$, where $a, b \in \mathbb{R}$, with the imaginary unit i. Let $\mathbb{B}$ be the set of bicomplex numbers w of the form

$$w = z_1 + z_2 j, \tag{17}$$

where $z_1, z_2 \in \mathbb{C}$. Then i and j are commuting imaginary units, i.e.

$$ij = ji, \ i^2 = j^2 = -1. \tag{18}$$

Let $w_1 = (a_1 + b_1 i) + (c_1 + d_1 i)j$ and $w_2 = (a_2 + b_2 i) + (c_2 + d_2 i)j$ be any arbitrary two bicomplex numbers. Then the equality, the addition, the substraction, the multiplication and the multiplication by scalar are defined in the natural way.
Equality: $w_1 = w_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $w_1 + w_2 = ((a_1 + a_2) + (b_1 + b_2)i) + ((c_1 + c_2) + (d_1 + d_2)i)\, j$,
substraction: $w_1 - w_2 = ((a_1 - a_2) + (b_1 - b_2)i) + ((c_1 - c_2) + (d_1 - d_2)i)\, j$,
multiplication by scalar $s \in \mathbb{R}$: $sw_1 = (sa_1 + sb_1 i) + (sc_1 + sd_1 i)j$,
multiplication:
$w_1 \cdot w_2 = ((a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2) + (a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1)i) +$
$+ ((a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1) + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1)i)\, j$.

Bicomplex numbers were introduced in 1892 by Segre, see [10]. The theory of bicomplex numbers is developed, many of papers concerning this topic are published quite recently, see for example [5], [7], [8], [9]. In this paper we introduce the $P_p(n)$–bicomplex numbers BP_n^p.

For $n \geq 0$ the nth $P_p(n)$–bicomplex number BP_n^p is defined as

$$BP_n^p = (P_p(n) + P_p(n+1)i) + (P_p(n+2) + P_p(n+3)i)j, \tag{19}$$

where $P_p(n)$ is defined by (1).

In the set $\mathbb{C}$, the complex conjugate of $x + yi$ is $\overline{x+yi} = x - yi$. In the set $\mathbb{B}$, for a bicomplex number $w = (a + bi) + (c + di)j$, there are three distinct conjugations. The bicomplex conjugation of BP_n^p with respect to i has the form

$$\begin{aligned}\overline{BP_n^p}^i &= \overline{(P_p(n) + P_p(n+1)i)} + \overline{(P_p(n+2) + P_p(n+3)i)}j = \\ &= (P_p(n) - P_p(n+1)i) + (P_p(n+2) - P_p(n+3)i)j.\end{aligned}$$

The bicomplex conjugation of BP_n^p with respect to j has the form

$$\begin{aligned}\overline{BP_n^p}^j &= (P_p(n) + P_p(n+1)i) - (P_p(n+2) + P_p(n+3)i)j = \\ &= (P_p(n) + P_p(n+1)i) + (-P_p(n+2) - P_p(n+3)i)j.\end{aligned}$$

The third kind of conjugation is a composition of the above two conjugations. Putting $k := ji = ij$ we can define the bicomplex conjugation of BP_n^p with respect to k as follows

$$\begin{aligned}\overline{BP_n^p}^k &= \overline{(P_p(n) + P_p(n+1)i)} - \overline{(P_p(n+2) + P_p(n+3)i)}j = \\ &= (P_p(n) - P_p(n+1)i) + (-P_p(n+2) + P_p(n+3)i)j.\end{aligned}$$

Using the bicomplex conjugation of BP_n^p with respect to i, j, k respectively and (19) we can write

$$\begin{aligned}BP_n^p \cdot \overline{BP_n^p}^i &= \left(|P_p(n) + P_p(n+1)i|^2 - |P_p(n+2) + P_p(n+3)i|^2\right) + \\ &\quad +2\Re\left((P_p(n) + P_p(n+1)i) \cdot \overline{(P_p(n+2) + P_p(n+3)i)}\right) j = \\ &= P_p^2(n) + P_p^2(n+1) - P_p^2(n+2) - P_p^2(n+3) + \\ &\quad +2\left(P_p(n)P_p(n+2) + P_p(n+1)P_p(n+3)\right) j.\end{aligned}$$

$$\begin{aligned}BP_n^p \cdot \overline{BP_n^p}^j &= (P_p(n) + P_p(n+1)i)^2 + (P_p(n+2) + P_p(n+3)i)^2 = \\ &= P_p^2(n) - P_p^2(n+1) + P_p^2(n+2) - P_p^2(n+3) + \\ &\quad +2\left(P_p(n)P_p(n+1) + P_p(n+2)P_p(n+3)\right) i.\end{aligned}$$

$$\begin{aligned}BP_n^p \cdot \overline{BP_n^p}^k &= \left(|P_p(n) + P_p(n+1)i|^2 + |P_p(n+2) + P_p(n+3)i|^2\right) + \\ &\quad -2\Im\left((P_p(n) + P_p(n+1)i) \cdot \overline{(P_p(n+2) + P_p(n+3)i)}\right) k = \\ &= P_p^2(n) + P_p^2(n+1) + P_p^2(n+2) + P_p^2(n+3) + \\ &\quad -2\left(P_p(n+1)P_p(n+2) - P_p(n)P_p(n+3)\right) k.\end{aligned}$$

In the set $\mathbb{C}$, the modulus of $x + yi$ is $|x + yi| = \sqrt{(x + yi) \cdot \overline{(x + yi)}} = \sqrt{x^2 + y^2}$. In the set $\mathbb{B}$ there are four different moduli, named: real modulus $|BP_n^p|$, i−modulus $|BP_n^p|_i$, j−modulus $|BP_n^p|_j$ and k−modulus $|BP_n^p|_k$. We give the formulae of the squares of these modules:

$$\begin{aligned}|BP_n^p|^2 &= |P_p(n) + P_p(n+1)i|^2 + |P_p(n+2) + P_p(n+3)i|^2 = \\ &= P_p^2(n) + P_p^2(n+1) + P_p^2(n+2) + P_p^2(n+3),\end{aligned}$$

$$|BP_n^p|_i^2 = BP_n^p \cdot \overline{BP_n^p}^i,$$

$$|BP_n^p|_j^2 = BP_n^p \cdot \overline{BP_n^p}^j,$$

$$|BP_n^p|_k^2 = BP_n^p \cdot \overline{BP_n^p}^k.$$

For $p = 2$, $P_2(0) = 0$ and $P_2(1) = 1$ we obtain bicomplex Pell numbers. The properties of this numbers can be found in [1].

For $p = 3$ we obtain, in particular, distinct Tribonacci bicomplex numbers. Using presented earlier Tribonacci numbers T_n, S_n, R_n and U_n we have four types of Tribonacci bicomplex numbers. Then

$$BTT_n = (T_n + T_{n+1}i) + (T_{n+2} + T_{n+3}i)j \tag{20}$$

$$BTR_n = (R_n + R_{n+1}i) + (R_{n+2} + R_{n+3}i)j \tag{21}$$

$$BTS_n = (S_n + S_{n+1}i) + (S_{n+2} + S_{n+3}i)j \tag{22}$$

$$BTU_n = (U_n + U_{n+1}i) + (U_{n+2} + U_{n+3}i)j \tag{23}$$

Using the above definitions we can write initial bicomplex Tribonacci numbers, e.g. BTT_n

$$\begin{aligned}BTT_0 &= (1 + i) + (2 + 4i)j,\\ BTT_1 &= (1 + 2i) + (4 + 7i)j,\\ BTT_2 &= (2 + 4i) + (7 + 13i)j,\\ BTT_3 &= (4 + 7i) + (13 + 24i)j.\end{aligned}$$

4. Properties of Tribonacci bicomplex numbers

In this section we give relations between Tribonacci bicomplex numbers.

Theorem 4.1. *Let n be an integer. Then*

(i) $BTR_n = BTT_{n-1} + 2BTT_{n-2} + 3BTT_{n-3},\ for\ n \geq 3$,
(ii) $BTS_n = 3BTT_n - BTT_{n-1},\ for\ n \geq 1$,
(iii) $BTU_n = BTT_{n-1} + BTT_{n-2},\ for\ n \geq 2$,
(iv) $2BTT_n = BTU_{n+1} + BTU_{n-1},\ for\ n \geq 1$.

Proof. (i) Using (21) and (6) we have

$$\begin{aligned}
BTR_n &= (R_n + R_{n+1}i) + (R_{n+2} + R_{n+3}i)j = \\
&= ((T_{n-1} + 2T_{n-2} + 3T_{n-3}) + (T_n + 2T_{n-1} + 3T_{n-2})\, i) + \\
&\ + ((T_{n+1} + 2T_n + 3T_{n-1}) + (T_{n+2} + 2T_{n+1} + 3T_n)\, i)\, j = \\
&= ((T_{n-1} + T_n i) + (T_{n+1} + T_{n+2} i) j) + \\
&\ +2\, ((T_{n-2} + T_{n-1} i) + (T_n + T_{n+1} i) j) + \\
&\ +3\, ((T_{n-3} + T_{n-2} i) + (T_{n-1} + T_n i) j) = \\
&= BTT_{n-1} + 2BTT_{n-2} + 3BTT_{n-3}.
\end{aligned}$$

In the same way, using (7), (8) and (13) one can easily prove identities (ii)-(iv). □

The next theorem gives formulas for sums of Tribonacci bicomplex numbers.

Theorem 4.2. *Let n be an integer. Then*

(i) $\sum_{l=0}^{n} BTU_l = BTT_{n+1} - BTT_0$,

(ii) $\sum_{l=1}^{n} BTR_l = 2BTU_{n+1} + BTU_{n-1} - ((3+4i) + (7+14i)j)$,

(iii) $\sum_{l=0}^{n} BTS_l = \dfrac{3BTU_{n+2} + 2BTU_{n+1} - BTU_n}{2} - ((1+4i) + (6+11i)j)$,

(iv) $\sum_{l=0}^{n} BTT_l = \dfrac{BTU_{n+2} + BTU_{n+1} - ((1+3i) + (5+9i)j)}{2}$,

(v) $\sum_{l=1}^{n} BTR_l = 3BTT_n + BTT_{n-1} - ((3+4i) + (7+14i)j)$,

(vi) $\sum_{l=0}^{n} BTS_l = BTT_{n+2} + 2BTT_n - ((1+4i) + (6+11i)j)$,

(vii) $\sum_{l=0}^{n} BTT_l = \dfrac{BTT_{n+2} + BTT_n - ((1+2i) + (4+8i)j)}{2}$.

Proof. (i) Using (23) and (9) we have

$$\begin{aligned}
\sum_{l=0}^{n} BTU_l &= BTU_0 + BTU_1 + \ldots + BTU_n = \\
&= (U_0 + U_1 i) + (U_2 + U_3 i)j + \\
&\ +(U_1 + U_2 i) + (U_3 + U_4 i)j + \ldots + \\
&\ +(U_n + U_{n+1} i) + (U_{n+2} + U_{n+3} i)j = \\
&= (U_0 + U_1 + \ldots + U_n) + (U_1 + U_2 + \ldots + U_{n+1})\, i + \\
&\ + ((U_2 + U_3 + \ldots + U_{n+2}) + (U_3 + U_4 + \ldots + U_{n+3} i))\, j = \\
&= (T_{n+1} - 1) + (T_{n+2} - 1 - U_0) i + \\
&\ +((T_{n+3} - 1 - U_0 - U_1) + (T_{n+4} - 1 - U_0 - U_1 - U_2) i) j = \\
&= (T_{n+1} + T_{n+2} i) + (T_{n+3} + T_{n+4} i) j - ((1+i) + (2+4i)j),
\end{aligned}$$

which ends the proof. In the same way one can easily prove (ii)-(vii). □

Using identities for $P_p(n)$–numbers more properties of the $P_p(n)$–bicomplex numbers can be given.

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O PEWNYM RODZAJU LICZB TYPU FIBONACCIEGO I ICH ZASTOSOWANIACH W TEORII LICZB DWUZESPOLONYCH

S t r e s z c z e n i e

W pracy pokazane zostało zastosowanie pewnego rodzaju liczb typu Fibonacciego w teorii liczb dwuzespolonych.

Słowa kluczowe: liczby Fibonacciego, zależności rekurencyjne, liczby dwuzespolone