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Dedicated to the memory of
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GRAVITATIONAL TOMOGRAPHY AS THE INVERSE PROBLEM IN FIELD THEORY

Summary

At the beginning of the article, the simple and inverse problem in field theory is described and the term tomography is explained. There are also examples of the use of gravitational tomography and its significance in Earth sciences. The main purpose of the article is to present two practical calculation procedures intended to determine the spatial distribution of density in objects with spherical-symmetrical mass distribution. In the first procedure, the object is divided into concentric spheres of equal thickness. In the second procedure, the object is divided into concentric spheres of equal volume, which provides more precise information about the distribution of density in the outer layers of the object. The density values are obtained by solving a system of linear equations with introduced results of measurements of gravity acceleration performed with a gravimeter outside the object.

Keywords and phrases: gravity, tomography, acceleration, spatial distribution, density, computation

1. Introduction

A simple problem in the field theory is that the given data are the values characterizing the sources of this field, and while using them, we need to determine the values describing the considered field in the space that surrounds its sources. In turn, the inverse problem in the theory emerges when having values that describe the field in space we need to calculate the values characterizing the sources of this field [1]. In the case of the gravitational field, the inverse problem is that knowing the gravitational potential values or gravity acceleration values and directions, we need to determine the position of masses or the spatial distribution of densities of this mass [2]. The word tomography comes from Greek. It was created by combining two words: *thomos* and *grapho*, meaning respectively: *cutting* and *image*. Accordingly, tomography mans a method of obtaining images from the internal structure of a studied object as a result of its intersecting with selected planes. Nowadays, tomography is most often associated with a method of the so-called medical imaging diagnosis, which consists in obtaining cross-sectional images of selected parts of a body using e.g. X-rays emitted from a moving lamp or magnetic resonance [3].

However, as a research method, tomography has a much broader application, also in Earth sciences. In geophysics, geology and surveying information about spatial distribution of mass density inside Earth is crucial. In the case of geophysics, this information is critical to understand the phenomena taking place inside our planet [4, 5]. In the case of geology, it allows to detect some useful fossil raw materials. And in the case of surveying, it allows to determine the deviation value and direction from the vertical line of the gravity acceleration in a particular place on Earth, and thus to develop a model of its surface (a geoid) and to establish gravimetric surveying points [6, 7]. It is essential to ensure the necessary accuracy of almost all measurements carried out by surveyors [8, 9]. Specialists that deal with the so-called higher surveying measure the gravitational anomalies, among others of anthropogenic origin, e.g. arising from closed mine pavements, tunnels, shelters [10]. Results of those measurements are broadly used, also for military purposes. Thus, the gravitation tomography becomes useful in each of these sciences. Hence, this method is currently the object of interest of numerous researchers [11, 12].

This paper is intended to provide practical calculation procedures of gravitation tomography, applicable to determine the spatial distribution of mass density for spherical-symmetrical objects. In such an object, the density depends solely on the distance from its center. Two variants of object division will be considered. The first one is a division into spheres of equal thickness, and the second one is the division into spheres of equal volume.

2. Division of the object into spheres of equal thickness

A given object is a sphere with external radius of r_0, and a fragment of its cross-section in presented on Fig. 1. This object will be divided into m of concentric spheres of equal thickness Δr, meeting the condition $\Delta r \ll r$. Then it can be assumed that the density of matter d_j in each of these spheres is constant (j is the number indicator of the sphere and it fulfills the condition ($j = 1, 2, \ldots m$). According to the accepted assumptions, the thickness of each sphere Δr and its radii: internal r_{wj}, external r_{zj}

and average r_{sj} are expressed in the following formulas:

$$\Delta r = \frac{r_0}{m}, \tag{1}$$

$$r_{wj} = \frac{r_0}{m}(j-1), \tag{2}$$

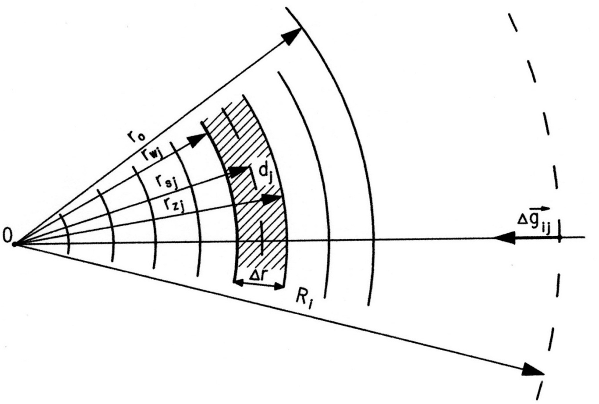

Fig. 1. Scheme of division of an object with spherical-symmetrical distribution of mass into spheres of equal thickness; r_0 – object radius, r_{wj}, r_{sj}, r_{zj} – radii of j-th sphere respectively: internal, average and external, Δr – thickness of each sphere, d_j – mean mass density in j-th sphere, Δg_j – contribution to gravity acceleration produced by mass contained in j-th sphere at distance R_i from the centre of object O.

$$r_{zj} = \frac{r_0}{m} j, \tag{3}$$

$$r_{sj} = \frac{r_0}{m}\left(j - \frac{1}{2}\right). \tag{4}$$

In order to determine the density of d_j, the values of gravity acceleration g_i at distances R_i from the center of the object (i is an indicator that numbers these distances and at the same time the measurement points $i = 1, 2, \ldots n$, and also $n = m$) were measured. Due to the spherical-symmetrical distribution of density in the object, the g_i vectors are of a radial direction. For an unambiguous determination of d_j it is necessary to take $n = m$ measurements at different points. According to the Newton's law of gravitation, the j-th sphere provides contribution to acceleration Δg_{ij} at a selected distance R_i from the center of the object, expressed with the following formula [2]

$$\Delta g_{ij} = \frac{\frac{4}{3}\pi G(r_{zj}^3 - r_{wj}^3)d_j}{R_i^2}, \tag{5}$$

where G is the gravity constant ($G = 6.67 \cdot 10^{-11}$ $(\mathrm{Nm}^2)/\mathrm{kg}^2$). The resultant acceleration g_i at each point is the sum of these contributions and is given by the following formula

$$g_i = \sum_{j=1}^{j=m} \Delta G_{ij}. \tag{6}$$

The proportionality coefficient k_{ij} may be entered into formula (5) and written down in the following form

$$\Delta g_{ij} = k_{ij} d_j, \tag{7}$$

where the k_{ij} is expressed by the formula

$$k_{ij} = \frac{\frac{4}{3}\pi G \left(r_{zj}^3 - r_{wj}^3\right)}{R_i^2}. \tag{8}$$

Using formulas (6) and (7) for each of the points of measurement of the resultant acceleration g_i the following system of equations is obtained

$$\begin{cases} k_{11}d_1 + k_{12}d_2 + \ldots + k_{1j}d_j = g_1, \\ k_{21}d_1 + k_{22}d_2 + \ldots + k_{2j}d_j = g_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_{i1}d_1 + k_{i2}d_2 + \ldots + k_{ij}d_j = g_i. \end{cases} \tag{9}$$

Since it is a system of linear equations, it can be written down in a matrix form

$$\mathbf{KD} = \mathbf{G}. \tag{10}$$

The matrices $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{G}$ in the system (10) are in the form of

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1j} \\ k_{21} & k_{22} & \ldots & k_{2j} \\ \ldots & \ldots & \ldots & \ldots \\ k_{i1} & k_{i2} & \ldots & k_{ij} \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_j \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \\ \ldots \\ d_i \end{bmatrix}. \tag{11}$$

The purpose of further proceedings in this case of gravitation tomography is to calculate the density of masses of d_i in individual spheres, which is brought down to solving the system of equations (10) and determining the elements of the single-column matrix $\mathbf{D}$. Standard procedure, employed in the theory of linear equations provides the following formula

$$\mathbf{D} = \frac{1}{\det \mathbf{K}} (\mathbf{K}^s)^T \mathbf{G}, \tag{12}$$

where $\det \mathbf{K}$ is the determinant of matrix $\mathbf{K}$ and $(\mathbf{K}^s)^T$ is the matrix transposed to the complement matrix $\mathbf{K}$.

An example of applying the described procedure will be the conduct where the spherical-symmetrical area is divided into 8 spheres. In such a case, the system of equations (9) takes the following form:

$$\begin{cases} k_{11}d_1 + k_{12}d_2 + \ldots + k_{18}d_8 = g_1, \\ k_{21}d_1 + k_{22}d_2 + \ldots + k_{28}d_8 = g_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_{81}d_1 + k_{82}d_2 + \ldots + k_{88}d_8 = g_8. \end{cases} \tag{13}$$

and the contributions to the acceleration Δg_{ij} and the coefficients of proportionality $k-ij$ are calculated from the following formulas respectively:

$$\Delta g_{ij} = \frac{\frac{4}{3}\pi G\left[\left(\frac{r_0}{m}j\right)^3 - \left(\frac{r_0}{m}(j-1)\right)^3\right]d_j}{R_i^2}, \tag{14}$$

$$k_{ij} = \frac{\frac{4}{3}\pi G\left[\left(\frac{r_0}{m}j\right)^3 - \left(\frac{r_0}{m}(j-1)\right)^3\right]}{R_i^2}. \tag{15}$$

After applying the formula (15) for the distance R_i and taking into account the relation (1), the following sequence of formulas for the proportionality coefficients k_{ij} is obtained:

$$k_{i1} = \frac{4\pi G\Delta r^3}{3R_i^2} = \frac{0.0026\pi G r_0^3}{R_i^2}, \tag{16}$$

$$k_{i2} = \frac{28\pi G\Delta r^3}{3R_i^2} = \frac{0.0182\pi G r_0^3}{R_i^2}, \tag{17}$$

$$k_{i3} = \frac{76\pi G\Delta r^3}{3R_i^2} = \frac{0.0494\pi G r_0^3}{R_i^2}, \tag{18}$$

$$k_{i4} = \frac{148\pi G\Delta r^3}{3R_i^2} = \frac{0.0969\pi G r_0^3}{R_i^2}, \tag{19}$$

$$k_{i5} = \frac{244\pi G\Delta r^3}{3R_i^2} = \frac{0.1589\pi G r_0^3}{R_i^2}, \tag{20}$$

$$k_{i6} = \frac{364\pi G\Delta r^3}{3R_i^2} = \frac{0.2370\pi G r_0^3}{R_i^2}, \tag{21}$$

$$k_{i7} = \frac{508\pi G\Delta r^3}{3R_i^2} = \frac{0.3307\pi G r_0^3}{R_i^2}, \tag{22}$$

$$k_{i8} = \frac{676\pi G\Delta r^3}{3R_i^2} = \frac{0.4401\pi G r_0^3}{R_i^2}. \tag{23}$$

In the case of spheres located at a large distance from the center of the object, i.e. for $j \gg 1$, it is possible to calculate the approximate values of proportionality coefficients k_{ij} from simplified formulas. For this purpose, the volume of a sphere will be calculated not as the difference in volume of spheres of adjacent r_{wj}, r_{zj} radii but as the product of the surface area of a sphere of medium r_{sj} radius and its thickness Δr. The following is obtained then

$$k_{ij} \approx \frac{4\pi G r_{si}^2 \Delta r}{R_i^2}. \tag{24}$$

After substituting the formula (4) to the formula (24), we can write down what follows

$$k_{ij} \approx \frac{4\pi G r_0^3}{R_i^2 m^3}\left(j - \frac{1}{2}\right)^2. \tag{25}$$

Because $j \gg 1$, formula (25) can be simplified even more and then

$$k_{ij} \approx 4\pi G \frac{r_0^3 j^2}{R_i^2 m^3}. \tag{26}$$

To check the difference between the approximate and the exact value, formula (26) will be applied for $j = m = 8$. In such a case we will obtain what follows from the formula (26)

$$k_{i8} = \frac{0.4688\pi G r_0^3}{R_i^2}. \tag{27}$$

Comparing this value with the exact value k_{i8} calculated from formula (23) leads to the conclusion that the relative error of this approximation does not exceed 0.03.

A flaw of the described procedure, which is based on dividing the sphere of equal thickness Δr, is that the volume of spheres that are increasingly more distant from the center of the object grows rapidly. This is indicated by the sequence of values of proportionality coefficients k_{ij}, expressed in formulas (16–23). According to formula (26) this growth is approximately proportional to the square of index j, meaning the sphere. As a result, the calculated density d_j is assigned to areas with increasing volume, and the structure of external layers of the objects is identified with decreasing resolution.

3. Division of the object into spheres of equal volume

The flaw described in the above part of the paper can be avoided by dividing the examined object into m concentric spheres of equal volumes V_j (Fig. 2). Accordingly, the volume of each sphere is expressed with the following formula

$$V_j = \frac{V_0}{m} = \text{const.} \tag{28}$$

where V_0 is the volume of the entire spherical object, calculated from the formula

$$V_0 = \frac{4}{3}\pi r_0^3. \tag{29}$$

After substituting formula (29) to formula (28) the volume of each sphere V_j is given by the formula

$$V_j = \frac{4}{3}\frac{\pi r_0^3}{m}. \tag{30}$$

The contributions to the acceleration from each of the spheres Δg_{ij} and the proportionality coefficients Δg_{ij} for each of the spheres will be calculated in the same way as previously by adapting formulas (7) and (8). The following is obtained then

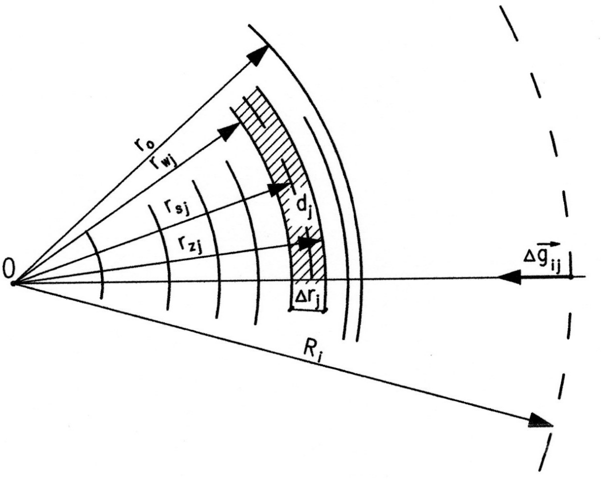

Fig. 2. Scheme of division of an object with spherical-symmetrical distribution of mass into spheres of equal volume; Δr_j – thickness of j-th sphere, the other symbols have the same meaning as given in the description in Fig. 1.

$$\Delta g_{ij} = \frac{G V_j d_j}{R_i^2}, \tag{31}$$

$$k_{ij} = \frac{4\pi G r_0^3}{3 m R_i^2}. \tag{32}$$

Formula (31) shows that the contributions to the acceleration Δg_{ij} from each sphere are the same, while the proportionality coefficients of k_{ij} do not depend on j and are constant for all values of R_i. The procedure is afterwards the same as before; we write down the system of linear equations (9), introducing the proportionality coefficients expressed with formula (29). Then, the system is solved with formulas (10–12).

Division of the object concerned into concentric spheres of equal volumes causes that the radii of the spheres located in the internal part of the object grow slower and slower. The dimensions of these spheres will now be calculated for the mentioned division. The contribution to the acceleration Δg_{ij} from the j-th sphere can be calculated using its radii: internal r_{wj} and external r_{zj}, using the formula

$$\Delta g_{ij} = \frac{\frac{4}{3}\pi G\left(r_{zj}^3 - r_{wj}^3\right) d_j}{R_i^2}. \tag{33}$$

On the other hand, the volume of a sphere with a radius equal to the external radius of the of r_{zj}, fulfills the equation

$$\frac{4}{3}\pi r_{zj}^3 = \frac{4}{3}\pi r_0^3 \frac{j}{m}, \tag{34}$$

which the following formula for this radius is obtained from

$$r_{zj} = \sqrt[3]{\frac{j}{m}} r_0. \tag{35}$$

Similarly, the volume of a sphere with a radius equal to the internal radius of the of r_{wj}, fulfills the equation

$$\frac{4}{3}\pi r_{wj}^3 = \frac{4}{3}\pi r_0^3 \frac{j-1}{m}, \tag{36}$$

which, when transformed, gives the following formula for this radius

$$r_{wj} = \sqrt[3]{\frac{j-1}{m}} r_0. \tag{37}$$

The thickness of the j-th Δr_{ij} sphere is equal to the difference between its outer radius and inner r_{vj} and is given by the formula

$$\Delta r_j = \left(\sqrt[3]{\frac{j}{m}} - \sqrt[3]{\frac{j-1}{m}}\right) r_0. \tag{38}$$

The mean radius of the j-th sphere r_{sj} will be the arithmetic mean of the outer radii of the outer r_{zj} and inner r_{wj}. The mean radius defined in this way is expressed by the formula

$$r_{sj} = \frac{1}{2}\left(\sqrt[3]{\frac{j}{m}} + \sqrt[3]{\frac{j-1}{m}}\right) r_0. \tag{39}$$

Moreover, between the following radii: the inner j-th r_{vj} sphere and the outer r_{wj} sphere with the number $(j-1)$ there is equality, i.e

$$r_{wj} = r_{z(j-1)}. \tag{40}$$

To give an example of how to apply the derived dependencies, the division of the object into $m = 8$ spheres will be considered. Using formulas (39) and (38) for this example, the following sequence of equations is obtained, expressing the outer radii of the spheres of r_{zj} and their thicknesses Δr_j:

$$r_{z1} = \sqrt[3]{\frac{1}{8}} r_0 = 0.5000 r_0 \quad \Delta r_j = 0.5000 r_0, \tag{41}$$

$$r_{z2} = \sqrt[3]{\frac{2}{8}} r_0 = 0.6300 r_0 \quad \Delta r_j = 0.1300 r_0, \tag{42}$$

$$r_{z3} = \sqrt[3]{\frac{3}{8}} r_0 = 0.7211 r_0 \quad \Delta r_j = 0.0911 r_0, \tag{43}$$

$$r_{z4} = \sqrt[3]{\frac{4}{8}} r_0 = 0.7937 r_0 \quad \Delta r_j = 0.0726 r_0, \tag{44}$$

$$r_{z5} = \sqrt[3]{\frac{5}{8}} r_0 = 0.8550 r_0 \quad \Delta r_j = 0.0613 r_0, \tag{45}$$

$$r_{z6} = \sqrt[3]{\frac{6}{8}} r_0 = 0.9086 r_0 \quad \Delta r_j = 0.0536 r_0, \tag{46}$$

$$r_{z7} = \sqrt[3]{\frac{7}{8}} r_0 = 0.9565 r_0 \quad \Delta r_j = 0.0479 r_0, \tag{47}$$

$$r_{z8} = \sqrt[3]{\frac{8}{8}} r_0 = 1.0000 r_0 \quad \Delta r_j = 0.0435 r_0. \tag{48}$$

While analyzing the results obtained, it is easy to notice that the outer radius of the first sphere of r_{z1}, equals half the radius of the entire area of r_0 and also equals the thickness of Δr_1 of this sphere (see equations (41)). Furthermore, thicknesses of subsequent spheres decrease quickly.

4. Conclusions

In order to obtain the most accurate information on the internal structure of the examined object of a spherical-symmetrical mass distribution, it is necessary to divide it into as many concentric spheres as possible ($m \gg 1$). The obtained spatial distribution of mass thickness in the object will be characterized with higher resolution then. It will be also justified to use the approximation consisting in replacement of its distribution of mass density with average density d_j ascribed to the distance from the center of the object equal to the average radius of this sphere r_{sj}. Dividing the object into a large number of concentric spheres causes an increase in the number of linear equations in the system (9). To ensure solvability of this system it is necessary to perform gravimetric measurements of gravity acceleration g_i also in sufficiently large and the same number of points ($n = m$). The simplest division of the object into concentric spheres of equal thickness Δr causes the masses of external spheres to increase rapidly and to give more and more contributions to the gravity acceleration. In situation where it is necessary to investigate the mass distribution more accurately in the external part of the examined object, e.g. in the case of Earth, a better solution is to make a division into spheres of equal volume. Then, regarding the decreasing thicknesses of external spheres, we can obtain greater resolution for the density distribution in this part of the object. The spherical-symmetrical distribution of mass densities is a special but often encountered case of distribution. With a good approximation, such a distribution can be found e.g. in astronomical objects, including Earth. The most general case of mass distribution, where density in a given point of the object depends both on the distance of this point from the center of the object (radius r_j) and the polar and azimuthal angles, will be a subject of another paper.

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TOMOGRAFIA GRAWITACYJNA JAKO PROBLEM ODWROTNY W TEORII POLA

S t r e s z c z e n i e

Na wstępie artykułu opisano, na czym polega problem prosty i odwrotny w teorii pola oraz wyjaśniono znaczenie terminu tomografia. Podano też przykłady zastosowania tomografii grawitacyjnej i jej znaczenie w naukach o Ziemi. Głównym celem artykułu jest przedstawienie dwóch praktycznych procedur obliczeniowych, przeznaczonych do wyznaczania rozkładu przestrzennego gestości w obiektach o kulisto-symetrycznym rozkładzie masy. W pierwszej procedurze obiekt jest dzielony na współśrodkowe sfery o równej grubości. W drugiej procedurze następuje podział obiektu na współśrodkowe sfery o równej objętości, co daje dokładniejszą informację o rozkładzie gęstości w zewnętrznych warstwach obiektu. Wartości gęstości są otrzymywane w wyniku rozwiązania układu równań liniowych, do którego zostały wprowadzone wyniki pomiarów przyspieszenia siły ciężkości, wykonane grawimetrem na zewnątrz obiektu.

Słowa kluczowe: grawitacja, tomografia, przyspieszenie, rozkład przestrzenny, gęstość, obliczanie