https://doi.org/10.26485/0459-6854/2019/69.1/8 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2019 Vol. LXIX

Recherches sur les déformations no. 1

pp. 109–116

Dedicated to the memory of
Professor Leszek Wojtczak

Mikołaj Widzibor

ON THE SEPARATION AXIOMS OF TOPOLOGIES GENERATED BY REGULAR SEQUENCES OF MEASURABLE SETS

Summary

In this paper we study separation axioms for $\mathcal{S}$-density topology, which is a generalization of the classical density topology. Namely, we prove that if the sequence of sets is regular, then the topology generated by it is completely regular, but is not normal.

Keywords and phrases: lower density operator, topology generated by lower density operator, density topology

1. Introduction

Let $\mathbb{R}$ be the set of reals fg and $\mathbb{N}$ be the set of natural numbers. The symbol λ stands for the Lebesgue measure. We use the symbols $\mathcal{L}$ and $\mathbb{L}$ to denote the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and the σ-ideal of Lebesgue null sets, respectively. For any set $A \subset \mathbb{R}$ we denote by $\overline{A}$ its closure in the natural topology on $\mathbb{R}$.

In this paper we study a generalization of density points, called $\mathcal{S}$-density points, introduced by F. Strobin and R. Wiertelak in the paper [5].

Definition 1.1. Let $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}}$, where $S_n \in \mathcal{L}$ for every $n \in \mathbb{N}$. We shall say that the sequence $\mathcal{S}$ *is converging to zero*, if

$$\lim_{n\to\infty} \operatorname{diam}(\{0\} \cup S_n) = 0.$$

Let us denote by $\mathbb{S}$ the family of all sequences of bounded sets with positive measure convergent to zero.

Definition 1.2. Let $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}} \in \mathbb{S}$. We shall say that $x \in \mathbb{R}$ is an $\mathcal{S}$-*density point* of the set $A \in \mathcal{L}$, if

$$\lim_{n\to\infty} \frac{\lambda\left((A-x)\cap S_n\right)}{\lambda(S_n)} = 1.$$

Observe, that for the sequence $\mathcal{S} = \left\{\left[-\frac{1}{n}, \frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ we get the ordinary density points (see [6]).

However, in this paper we shall focus on a special case of sequences of sets, so called regular sequences.

Define for every sequence $\mathcal{S} \in \mathbb{S}$ and for every set $A \in \mathcal{L}$ an operator

$$\Phi_{\mathcal{S}}(A) = \{A \in \mathcal{L} : x \text{ is a } \mathcal{S}\text{-density point of } A\}$$

.

In order to correctly define the regular sequences of sets, we use the following property from the paper [5].

Proposition 1.3. *Let $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}} \in \mathbb{S}$. There exists a convergent to zero sequence $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}} \in \mathcal{I}_{<\omega}$, such that $\Phi_{\mathcal{S}} = \Phi_{\mathcal{I}}$, where $\mathcal{I}_{<\omega}$ stands for the family of all sequences $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$, such that each I_n is a finite sum of pairwise disjoint closed intervals.*

Let us define for every sequence $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}} \in \mathbb{S}$

$$\alpha(\mathcal{S}) = \limsup_{n\to\infty} \frac{\operatorname{diam}\left(\{0\}\cup S_n\right)}{\lambda(S_n)}.$$

Definition 1.4. We shall say that a sequence $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}} \in \mathbb{S}$ is *regular*, if there exists a sequence $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}} \in \mathcal{I}_{<\omega}$ such that $\alpha(\mathcal{I}) < \infty$ and $\Phi_{\mathcal{S}} = \Phi_{\mathcal{I}}$. Otherwise we say that sequence $\mathcal{S} = \{S_n\}_{n\in\mathbb{N}}$ is *irregular*.

The regular sequences of sets had been studied in paper [6]. Now we recall some properties of such sequences, involved in further investigation in this paper.

Theorem 1.5 ([5],[6])**.** *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and let $A, B \in \mathcal{L}$. Then the operator $\Phi_{\mathcal{S}}$ has following properties:*

1. $\Phi_{\mathcal{S}}(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$;
2. *if* $\lambda(A \triangle B) = 0$, *then* $\Phi_{\mathcal{S}} = \Phi_{\mathcal{S}}$;
3. $\Phi_{\mathcal{S}}(A\cap B) = \Phi_{\mathcal{S}}(A)\cap\Phi_{\mathcal{S}}(B)$;
4. $\lambda(\Phi_{\mathcal{S}}(A) \triangle A) = 0$.

Recall, that an operator satisfying above conditions is called the *lower density operator*. In this case (see [4]) the family

$$\mathcal{T}_{\mathcal{S}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{S}}(A)\}$$

is a topology on $\mathbb{R}$, called $\mathcal{S}$-*density topology*(or the topology generated by the sequence $\mathcal{S}$). Moreover, the paper [6] contains a theorem that there exists a regular sequence of measurable sets such that the topology generated by it contains the ordinary density topology and is essentially stronger than the ordinary density topology.

2. An analogue of Lusin-Menchoff theorem

In this section we will prove an analogue of Lusin-Menchoff theorem in the case of the $\mathcal{S}$-density topology. The ideas of proofs are taken from the book [1] (see also [3]), however they need to be modified to the case of $\mathcal{S}$-density.

Lemma 2.1. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and B be a Borel set. Then for every $x \in B \cap \Phi_{\mathcal{S}}(B)$ there exists a perfect set K such that $x \in K \subset B$.*

Proof. Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ and B be a Borel set. Let $x \in B \cap \Phi_{\mathcal{S}}(B)$. Because x is an $\mathcal{S}$-density point of the set B, the equality $\lambda(B \cap S_n) = 0$ is possible for only finite amount of $n \in \mathbb{N}$. Hence we can assume that $\lambda(B \cap S_n) \neq 0$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists nonempty perfect set $K_n \subset B \cap S_n$. Let

$$K = \{x\} \cup \bigcup_{n \in \mathbb{N}} K_n.$$

Then $x \in K \subset B$ and the set K is perfect. □

Lemma 2.2. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and B be a Borel set. Then for every countable set $C = \{x_i : i \in \mathbb{N}\}$ such that $\overline{C} \subset B \cap \Phi_{\mathcal{S}}(B)$ there exists a perfect set K such that $C \subset K \subset B$.*

Proof. Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ and B be a Borel set. Moreover, let $C = \{x_i : i \in \mathbb{N}\}$ be a countable set such that $\overline{C} \subset B \cap \Phi_{\mathcal{S}}(B)$. Define for every $i \in \mathbb{N}$ sets $B_i = B \cap \left[x_i - \frac{1}{n}, x_i + \frac{1}{n}\right]$. Observe that for every $i \in \mathbb{N}$ we have $x \in \Phi_{\mathcal{S}}(B_i)$. hence, by previous lemma for every $i \in \mathbb{N}$ there exists a perfect set K_i such that $x \in K_i \subset B_i$. Let

$$K = \overline{C} \cup \bigcup_{i \in \mathbb{N}} K_i.$$

Then the set K is perfect and $C \subset K \subset B$. □

Lemma 2.3. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and $E \in \mathcal{L}$. Then for every $\mathcal{T}_{nat}$-closed set X such that $X \subset E \cap \Phi_{\mathcal{S}}(E)$ there exists a perfect set K such that $X \subset K \subset E$.*

Proof. Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ and E be a Lebesgue measurable set and let X be a $\mathcal{T}_{\text{nat}}$-closed set such that $X \subset E \cap \Phi_{\mathcal{S}}(E)$. There exists an F_σ set $A \subset E$ such that $\lambda(E \setminus A) = 0$. Hence $\Phi_{\mathcal{S}}(E) = \Phi_{\mathcal{S}}(A)$. Let $B = A \cup X$. Then B is a Borel set and $X \subset B \cap \Phi_{\mathcal{S}}(B)$. By Cantor-Bendixon Theorem $X = K_1 \cup C$, where K_1 is perfect and

C is countable set. Moreover, the set C consists only of isolated points, so $\overline{C} = C$. From the previous lemma there exists a perfect set K_2 such that $C \subset K_2 \subset B$. Put $K = K_1 \cup K_2$. Then K is a perfect set such that $X \subset K \subset B \subset E$. □

Theorem 2.4. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and $E \in \mathcal{L}$. Then for every $\mathcal{T}_{nat}$-closed set X such that $X \subset E \cap \Phi_{\mathcal{S}}(E)$ there exists a perfect set P such that $X \subset P \subset E$ and $X \subset \Phi_{\mathcal{S}}(P)$.*

Proof. Because the sequence $\mathcal{S}$ is regular, there exists a sequence $\mathcal{I} = \{I_n\} \in \mathcal{I}_{<\omega}$ such that $\alpha(\mathcal{I}) = \limsup_{n\to\infty} \frac{\operatorname{diam}(I_n \cup \{0\})}{\lambda(I_n)} < \infty$ and $\Phi_{\mathcal{S}} = \Phi_{\mathcal{I}}$.

We can assume that $I_n \subset [-1, 1]$ for every $n \in \mathbb{N}$. Let $F = E \cap \bigcup_{x\in X}[x-1, x+1]$. Then $X \subset F \cap \Phi_{\mathcal{I}}(F)$. By Lemma 2.3 there exist a perfect set K such that $X \subset K \subset F$. Let us define for every $n \in \mathbb{N}$

$$R_n = \left\{ x \in F : \frac{1}{n+1} < \operatorname{dist}(x, X) \le \frac{1}{n} \right\}.$$

Observe that $F = X \cup \bigcup_{n\in\mathbb{N}} R_n$. For every $n \in \mathbb{N}$, let P_n be a perfect subset of R_n such that $\lambda(R_n \setminus P_n) < \frac{1}{2^{n+1}}$. Let

$$P = K \cup \bigcup_{n\in\mathbb{N}} P_n.$$

Observe that P is a nonempty perfect set such that $X \subset P \subset F$.

We will show now, that $X \subset \Phi_{\mathcal{I}}(P)$. Let $x \in X$.

Case 1. There exists $n_0 \in \mathbb{N}$ such that $x + I_n \subset P$ for every $n > n_0$. Then we have $\lambda(E \cap (x + I_n)) = \lambda(P \cap (x + I_n))$ for every $n > n_0$. Because $x \in \Phi_{\mathcal{I}}(E)$, so $x \in \Phi_{\mathcal{I}}(P)$.

Case 2. Assume that for every $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $(x+I_n)\cap R_j \neq \emptyset$. Define

$$r(n) = \min\{j \in \mathbb{N} : (x + I_n) \cap R_j \neq \emptyset\}.$$

Since $\lim_{n\to\infty} \operatorname{diam}(\{0\} \cup I_n) = 0$, we have $\lim_{n\to\infty} r(n) = \infty$. It is also true that

$$(F \setminus P) \cap (x + I_n) \subset \bigcup_{j\ge r(n)} (R_j \setminus P_j).$$

We have

$$\lambda((F \setminus P) \cap (x + I_n)) \le \lambda\left(\bigcup_{j\ge r(n)} (R_j \setminus P_j)\right) \le \sum_{j\ge r(n)} \lambda(R_j \setminus P_j) < \frac{1}{2^{r(n)}}.$$

Hence

$$\begin{aligned} \lambda(F \cap (x + I_n)) &\le \lambda(P \cap (x + I_n)) + \lambda((F \setminus P) \cap (x + I_n)) \\ &< \lambda(P \cap (x + I_n)) + \frac{1}{2^{r(n)}} \end{aligned}$$

Since $\alpha(\mathcal{I}) = \limsup_{n\to\infty} \frac{\operatorname{diam}(I_n\cup\{0\})}{\lambda(I_n)} < \infty$, we can assume that for every $k \in \mathbb{N}$ there is $\operatorname{diam}(I_k \cup \{0\}) \le 2\alpha(\mathcal{I})\lambda(I_k)$. By the definition of the sets R_n we have that $\operatorname{diam}(I_n \cup \{0\}) \ge \frac{1}{r(n)+1}$. Hence

$$\lambda(I_n) \ge \frac{1}{2\alpha(\mathcal{I})(r(n)+1)}.$$

Finally

$$\frac{\lambda(F\cap(x+I_n))}{\lambda(I_n)} \le \frac{\lambda(P\cap(x+I_n))}{\lambda(I_n)} + \frac{2\alpha(\mathcal{I})(r(n)+1)}{2^{r(n)}}.$$

It means that $x \in \Phi_{\mathcal{I}}(P)$. □

3. The main result

In this section we will show that the topology generated by a regular sequence of sets is completely regular, but it is not normal. First, we need the definition of $\mathcal{S}$-approximately continuous function and some its properties.

Definition 3.1. Let $\mathcal{S} \in \mathbb{S}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately continuous* at a point $x_0 \in \mathbb{R}$, if there exists a set $U_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi(U_{x_0})$ and $\lim_{x\to x_0, x\in U_{x_0}} f(x) = f(x_0)$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately continuous*, if it is approximately continuous at every point $x \in \mathbb{R}$.

Similarly as in paper [3] we can prove the following

Proposition 3.2. *Let $\mathcal{S} \in \mathbb{S}$ and let $f, g : \mathbb{R} \to \mathbb{R}$ be $\mathcal{S}$-approximately continuous functions. Then the functions $f + g$ and $f \cdot g$ are $\mathcal{S}$-approximately continuous functions. Moreover, if $f(x) \ne 0$ for every $x \in \mathbb{R}$, the function $\frac{1}{f}$ is also $\mathcal{S}$-approximately continuous.*

Definition 3.3. Let $\mathcal{S} \in \mathbb{S}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately upper semi-continuous* at a point $x_0 \in \mathbb{R}$, if for every $a > f(x_0)$ there exists a set $U_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi(U_{x_0})$ and $f(x) > a$ for every $x \in U_{x_0}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately upper semi-continuous*, if it is approximately upper semi-continuous at every point $x \in \mathbb{R}$.

Let $\mathcal{S} \in \mathbb{S}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately lower semi-continuous* at a point $x_0 \in \mathbb{R}$, if for every $a < f(x_0)$ there exists a set $U_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi(U_{x_0})$ and $f(x) < a$ for every $x \in U_{x_0}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-*approximately lower semi-continuous*, if it is approximately upper semi-continuous at every point $x \in \mathbb{R}$.

The proof of the next proposition is analogous to the case of $\mathcal{J}$-density (see [3]).

Proposition 3.4. *Let $\mathcal{S} \in \mathbb{S}$. A function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}$-approximately continuous if and only if it is $\mathcal{S}$-approximately upper and $\mathcal{S}$-approximately lower semi-continuous.*

Theorem 3.5. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and E be a F_σ set such that $E \subset \Phi_{\mathcal{S}}(E)$. Then there exists a $\mathcal{S}$-approximately continuous function f such that*

1. $0 < f(x) \leq 1$ *for* $x \in E$,
2. $f(x) = 0$ *for* $x \notin E$.

Proof. Let $\mathcal{S} \in \mathbb{S}$ be a regular sequence of sets and E be a F_σ set such that $E \subset \Phi_{\mathcal{S}}(E)$. Assume that $E = \bigcup_{n=1}^{\infty} F_n$, where each of the sets F_n is a $\mathcal{T}_{\text{nat}}$-closed set. Theorem 2.4 implies (see [1]) the existence of the family $\{P_\alpha : \alpha \in [1, \infty)\}$ of $\mathcal{T}_{\text{nat}}$-closed sets such that

1. if $\alpha_1 < \alpha_2$, then $P_{\alpha_1} \subset P_{\alpha_2} \cap \Phi_{\mathcal{S}}(P_{\alpha_2})$,
2. $\bigcup_{n=1}^{\infty} P_n = E$.

Let

$$f(x) = \begin{cases} \frac{1}{\inf\{\alpha : x \in P_\alpha\}} & \text{for } x \in E \\ 0 & \text{for } x \notin E \end{cases}.$$

Observe that the function f fulfils the conditions (1), (2).

Now we will show that f is continuous at every point $x \notin E$. Let $x_0 \notin E$ and $n \in \mathbb{N}$. From (2) we have that $x \notin P_n$ for every $n \in \mathbb{N}$. Since the sets P_n are $\mathcal{T}_{\text{nat}}$-closed, there exists a number $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap P_n = \emptyset$. Moreover, because of (1), we have that $(x_0 - \delta, x_0 + \delta) \cap P_\alpha = \emptyset$ for every $\alpha \leq n$. Hence, if $x \in (x_0 - \delta, x_0 + \delta)$ then $\inf\{\alpha : x \in P_\alpha\} \geq n$. Consequently $f(x) \leq \frac{1}{n}$ for every $x \in (x_0 - \delta, x_0 + \delta)$. Since $f(x_0) = 0$, the function f is continuous at point x_0. Similarly one can show that f upper semi-continuous at every point $x \in E$.

In order to end the proof, we will show that f is $\mathcal{S}$-approximately lower semi-continuous at every point $x \in E$. Let $x_0 \in E$ and $a < f(x_0)$ be chosen arbitrarily. Then $f(x_0) = \frac{1}{\inf\{\alpha : x_0 \in P_\alpha\}} = \frac{1}{M}$ and $a < \frac{1}{M+\epsilon}$ for some $\epsilon > 0$. From (1) we have that

$$x_0 \in P_{M+\frac{\epsilon}{4}} \subset P_{M+\frac{\epsilon}{2}} \cap \Phi_{\mathcal{S}}(P_{M+\frac{\epsilon}{2}}).$$

Moreover, if $x \in P_{M+\frac{\epsilon}{2}}$, then

$$f(x) \geq \frac{1}{M + \frac{\epsilon}{2}} > \frac{1}{M + \epsilon} > f(x_0) > a.$$

Hence $P_{M+\frac{\epsilon}{2}} \subset \{x \in \mathbb{R} : f(x) > a\}$ and $x_0 \in \Phi_{\mathcal{S}}\left(\{x \in \mathbb{R} : f(x) > a\}\right)$. Denote $U_{x_0} = \{x \in \mathbb{R} : f(x) > a\}$. Then $x_0 \in \Phi_{\mathcal{S}}\left(U_{x_0}\right)$ and for $x \in U_{x_0}$ we have $f(x) > a$. It means that f is $\mathcal{S}$-approximately lower semi-continuous at the point x_0. ☐

Corollary 3.6. *Let $\mathcal{S} = \{S_n\} \in \mathbb{S}$ be a regular sequence of measurable sets and $E_1, E_2, H \subset \mathbb{R}$ be pairwise disjoint sets such that $E_1 \cup E_2 \cup H = \mathbb{R}$. Moreover, assume that $E_1 \cup H$ and $E_2 \cup H$ are F_σ sets and $E_1 \cup H \subset \Phi_{\mathcal{S}}(E_1 \cup H)$ and $E_2 \cup H \subset \Phi_{\mathcal{S}}(E_2 \cup H)$. Then there exists a $\mathcal{S}$-approximately continuous function f such that*

1. $0 < f(x) < 1$ *for* $x \in H$,
2. $f(x) = 0$ *for* $x \notin E_1$,
3. $f(x) = 1$ *for* $x \notin E_2$.

Proof. By Theorem 3.5 there exist $\mathcal{S}$-approximately continuous functions g, h such that

1. $0 < g(x) \leq 1$ for $x \notin E_1$ and $g(x) = 0$ for $x \in E_1$,
2. $0 < h(x) \leq 1$ for $x \notin E_2$ and $h(x) = 0$ for $x \in E_2$.

Then the function $f(x) = \frac{g(x)}{g(x)+h(x)}$ fulfils conditions (1)-(3) and by virtue of Proposition 3.2 is $\mathcal{S}$-approximately continuous. □

Theorem 3.7. *Let* $\mathcal{S} = \{S_n\} \in \mathbb{S}$ *be a regular sequence of measurable sets. Then the topological space* $(\mathbb{R}, \mathcal{T}_{\mathcal{S}})$ *is completely regular.*

Proof. Let F be a $\mathcal{T}_{\mathcal{S}}$-closed set and let $x_0 \notin F$. There exists a G_δ set K such that $F \subset K$ and $\lambda(F) = \lambda(K)$ and $x_0 \notin K$. Put $E_1 = \{x_0\}$, $E_2 = K$ and $H = \mathbb{R} \setminus (K \cup \{x_0\})$. Then by Corollary 3.6 there exists a $\mathcal{S}$-approximately continuous function $f : \mathbb{R} \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in F$. Consequently, the topological space $(\mathbb{R}, \mathcal{T}_{\mathcal{S}})$ is completely regular. □

Finally, we will show that for every regular sequence $\mathcal{S} \in \mathbb{S}$ the topology $\mathcal{T}_{\mathcal{S}}$ is not normal. In the paper [2] authors proved a following theorem:

Theorem 3.8. *Let* $(X, \mathcal{S}, \mathcal{J})$ *be a measurable space, where* $\mathcal{S}$ *is a* σ*-algebra of subsets of* X *and* $\mathcal{J} \subset \mathcal{S}$ *is a proper* σ*-ideal. Moreover, let* $\Phi : X \to 2^X$ *be a semi-lower density operator on* $(X, \mathcal{S}, \mathcal{J})$ *(i. e. an operator which satisfies only conditions (1)-(3) from the Theorem 1.5) generating topology* $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$*. If* $\mathcal{J}$ *contains a set* B *such that*

$$\operatorname{card}\left(2^B\right) > \operatorname{card}\{B \cap \Phi(A) : A \in \mathcal{S}\}$$

then $(X, \mathcal{T}_\Phi)$ *is not normal.*

For every $A \in \mathcal{L}$ the set $\Phi_{\mathcal{S}}(A)$ is Borel (see [5]). Moreover, for any $A \in \mathcal{L}$, there exists a Borel set $C \subset A$ such that $\Phi_{\mathcal{S}}(A) = \Phi_{\mathcal{S}}(C)$. Hence, the assumptions of the above theorem are satisfied for the measure space $(\mathbb{R}, \mathcal{L}, \mathbb{L})$, the operator $\Phi_{\mathcal{S}}$ and the set B equal to the Cantor set. Thus we have

Corollary 3.9. *The topological space* $(\mathbb{R}, \mathcal{T}_{\mathcal{S}})$ *is not normal.*

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lecture Notes in Math. **695**, Springer-Verlag (1978).

[2] J. Hejduk, A. Loranty, *On abstract and almost-abstract density topologies*, Acta Math. Hungar. **155(2)** (2018), 228–240.

[3] J. Hejduk, R. Wiertelak, *On some properties of $\mathcal{J}$-approximately continuous functions*, Math. Slovaca **67(6)** (2017), 1–10.

[4] J. Hejduk, R. Wiertelak, *On the abstract density topologies generated by lower and almost lower density operators*, Traditional and present-day topics in real analysis, Łodź University Press, 2013.

[5] F. Strobin, R. Wiertelak, *On a generalization of density topologies on the real line*, Topology Appl. **199** (2016), 1–16.

[6] M. Widzibor, R. Wiertelak, *On properties of operators generated by regular sequences of measurable sets*, submitted, (2018).

Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22, PL-90-238 Łódź
Poland
E-mail: widzibor@gmail.com

Presented by Jacek Hejduk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on January 15, 2019.

O AKSJOMATACH ODDZIELANIA TOPOLOGII GENEROWANYCH PRZEZ REGULARNE CIĄGI ZBIORÓW MIERZALNYCH

S t r e s z c z e n i e

W przedstawionym artykule badamy aksjomaty oddzielania dla topologii $\mathcal{S}$-gęstości, które są uogólnieniem klasycznej topologii gęstości. Głównym wynikiem jest całkowita regularność topologii generowanej przez regularny ciąg zbiorów zbieżny do zera. Pokazujemy też, że tego typu topologie nie są normalne.

Słowa kluczowe: operator dolnej gęstości, topologia generowana przez operator dolnej gęstości, topologia gęstości