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Dedicated to the memory of
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OPTION PRICING IN CRR MODEL WITH TIME DEPENDENT PARAMETERS FOR TWO PERIODS OF TIME - PART I

Summary

In this paper we present some generalization of the Cox-Ross-Rubinstein (CRR) option pricing model. We assume that two parameters of the model (an interest rate of a bank account and a volatility of the logarithm of the stock price's changes) are different in each of two analyzed periods of time.

Keywords and phrases: Cox-Ross-Rubinstein model (CRR model), Black-Scholes formula, option pricing

1. Introduction

A European call option is a security that gives the holder the right to buy a stock on a particular date T (expiry date) for a predetermined price K (the strike price). The holder has to pay a premium for getting this right (the option price). The difference between the market price of a stock and the exercise price comprises the profit on this option investment. That is why the holder will exercise the option if the market price of a stock is greater than the strike price K fixed at the moment 0. In another case he will not do it because an exercising option will not give him the profit.

Formally, a European call option is defined as follows.

Definition 1.1. A European call option is a pair (T, C_T) where $T > 0$ and $C_T(\cdot)$:

$\Re_+ \to \Re$ is the function

$$C_T(s) = (s-K)^+ = \begin{cases} s-K, & \text{if} \quad s > K \\ 0 & \text{if} \quad s \le K, \end{cases} \quad \text{for some } K \in \Re_+.$$

We recall the Black-Scholes formula and the CRR model.

In 1973 Fischer Black and Myron Scholes presented the following formula for the option pricing for continuous-time market.

In the Black-Scholes model a stock price $S(t)$ at time t is defined as

$$S(t) = s_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad t \ge 0,$$

where W_t is a Wiener process, $s_0 > 0$, $\sigma > 0$, and μ are constants, s_0 is the stock price observed at time 0 and σ is volatility. Moreover, the instantaneous interest rate $r > 0$ of a bank account is also assumed to be constant, i.e. $\exp(rt)$ is the value of the one unit of money in a bank account at time $t \ge 0$ (see e.g. [7] for details).

Theorem 1.1. (Black-Scholes option pricing, 1973 [2]). *The time* 0 *fair price* $C_0^{BS}(s_0)$ *of a European call option with strike price* K *and expiry date* τ *in the Black-Scholes model is given by*

$$C_0^{BS}(s_0) = s_0 \phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right) - \frac{K}{e^{r\tau}} \phi\left(\frac{\ln\frac{s_0}{K} + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}\right),$$

where $\phi(\cdot)$ *is the cumulative normal distribution function,* σ *is volatility of the logarithm of the stock price's changes (the volatility of the random variable* $ln\frac{S(n)}{s_0}$*, where* $S(n)$ *is a stock price after* n *moments portfolio's change).*

Next, in 1979 Cox, Ross and Rubinstein presented a discrete-time option pricing formula (CRR model). They assumed that in each step the upper and lower stock prices' changes are the same. So they got the following possible changes of the stock price:

$$S_0 = s_0, \quad S_t = u \cdot S_{t-1} \quad \text{or} \quad S_t = d \cdot S_{t-1}, \quad 1 \le t \le T, \quad t \in N,$$

where

T is a fixed natural number of short periods (the expiry date),
t is the number of the present step,
s_0 is a positive constant (the stock price at moment 0),
u is the upper stock price's change during one short period,
d is the lower stock price's change during one short period
(u and d are the only possible changes of stock prices during one short period).

Cox, Ross and Rubinstein proved the following theorem describing option pricing:

Theorem 1.2. (CRR option pricing, 1979 [4]). *In the CRR model the fair price $C_0(s_0)$ at moment 0 of a European call option with expiry date T and strike price $K = s_0 u^{k_0} d^{T-k_0}$ for a certain $k_0 = 0, 1, \ldots, T$, is given by:*

$$C_0(s_0) = s_0 \bar{D} - \frac{K}{\hat{r}^T} D^*,$$

where

$$\bar{D} = \sum_{k=k_0}^{T} \binom{T}{k} \cdot \bar{p}^k \cdot \bar{q}^{T-k}, \quad D^* = \sum_{k=k_0}^{T} \binom{T}{k} \cdot p^{*K} \cdot q^{*T-k},$$

$$k_0 = \frac{\ln \frac{K}{S_0} - T \cdot \ln d}{\ln\left(\frac{u}{d}\right)}, \quad p^* = \frac{\hat{r} - d}{u - d}, \quad q^* = \frac{u - \hat{r}}{u - d}, \quad \bar{p} = p^* \cdot \frac{u}{\hat{r}}, \quad \bar{q} = q^* \cdot \frac{d}{\hat{r}},$$

$\hat{r}$ is a bank account and $0 < d < \hat{r} < u$.

Cox, Ross and Rubinstein showed that their formula converges to the Black-Scholes formula if we take into account the large number of moments of the portfolio's change.

In the both presented models all parameters such as an interest rate of a bank account and a volatility of the logarithm of the stock price's changes do not vary from the moment 0 to the expiry date T. This assumption is not realistic, because these parameters change. That is why in this paper we analyse some generalization of CRR model in which these parameters are different in each of two period of time. Next, we demonstrate the convergence of the option price in the presented model to the formula corresponding to the Black-Scholes formula.

2. CRR model with time dependent parameters for two periods of time, notation and formulation of main result

We now present some generalization of CRR model. We consider two periods of time (for example two months) in which we have n moments of the portfolios change and assume that in each period of time the parametres: an interest rate of a bank account and the volatility σ are constant. However these parameters depend on periods of time (these parameters are different in the first and in the second period of time). In the CRR model the authors considered only one period of time with constant parameters.

We denote

K - the strike price,

s_{i-1} - the stock price at the begining i-th month, $i = 1$ or $i = 2$ (s_1 is the random variable), s_0 is a positive constant (the stock price at moment 0),

r_i - an interest rate of a bank account for i-th month,

σ_i - the volatility of the logarithm of the stock price's changes for i-th month,

$\hat{r}_{i,n} = e^{\frac{r_i}{n}}$ - an interest rate in i-th month between the stock price's changes (an interest rate for one short period),
$u_{i,n} = e^{\frac{\sigma_i}{\sqrt{n}}}$ - the upper possible price change in i-th month between the stock price's changes,
$d_{i,n} = e^{-\frac{\sigma_i}{\sqrt{n}}}$ - the lower possible price change in i-th month between the stock price's changes,
$p^*_{i,n} = \frac{\hat{r}_{i,n} - d_{i,n}}{u_{i,n} - d_{i,n}}$ - the martingale probability of the stock price's growth in i-th month between the stock price's changes, $q^*_{i,n} = 1 - p^*_{i,n}$,
where $i = 1$ or $i = 2$.

Let us note that at the beginning of the second period of time the stock price takes the value: $s_0 \cdot u^l_{1,n} \cdot d^{n-l}_{1,n}$, where l is a natural number. In the first period of time the stock price increases l times. When we price a European call option at the beginning of the second period of time we take into account the beginning stock price s_1 for this period, where $s_1 = s_0 \cdot u^l_{1,n} \cdot d^{n-l}_{1,n}$. So the option price at the beginning of the second period of time is given by the formula:

$$(2.1) \qquad C_{1,n}(s_1) = \frac{1}{e^{r_2}} \sum_{l=0}^{n} \binom{n}{l} \cdot p^{*l}_{2,n} \cdot q^{*n-l}_{2,n} \cdot (s_1 \cdot u^l_{2,n} \cdot d^{n-l}_{2,n} - K)^+, s_1 \in (0, \infty)$$

and the option price at the time 0 is given by the formula:

$$(2.2) \qquad C_{0,n}(s_0) = \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p^{*l}_{1,n} \cdot q^{*n-l}_{1,n} \cdot C_{1,n}(s_0 \cdot u^l_{1,n} \cdot d^{n-l}_{1,n}).$$

Our result is the following:

Theorem 2.1. *Under the previous assumptions and notation $p^*_{i,n}$, $q^*_{i,n}$, $u_{i,n}$, $d_{i,n}$, r_i, σ_i, $i = 1, 2, ...$, for a fixed $K > 0$ and $C_{1,n}(s_1)$ given by (2.1) the following convergence holds: for any $s_0 > 0$ the limit option price at the time 0 is given by the formula:*

$$\lim_{n\to\infty} C_{0,n}(s_0) := \lim_{n\to\infty} \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p^{*l}_{1,n} \cdot q^{*n-l}_{1,n} \cdot C_{1,n}(s_0 \cdot u^l_{1,n} \cdot d^{n-l}_{1,n}) =$$

$$= s_0 \cdot \phi(A \cdot \ln s_0 + B) - \frac{K}{e^r_1 \cdot e^r_2} \cdot \phi(A \cdot \ln s_0 + \tilde{B})$$

where $\phi(\cdot)$ is the cumulative normal $N(0,1)$ distribution function, $A = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}}$,

$B = A \cdot (-\ln K + r_1 + r_2 + 0,5 \cdot (\sigma_1^2 + \sigma_2^2)), \tilde{B} = A \cdot (-\ln K + r_1 + r_2 - 0,5 \cdot (\sigma_1^2 + \sigma_2^2)).$

Let us see that the right hand side of the above equality is an analogue of the Black-Scholes formula. In particular, taking $r = r_1 = r_2$ and $\sigma = \sigma_1 = \sigma_2$ we get Black-Scholes option pricing for $\tau = 2$ that is given in Theorem 1.1.

3. Auxiliary facts

In the proof of Theorem 2.1 we use a few lemmas, theorems and remarks.

Lemma 3.1. *Under the previous notation, let for any $s_0 > 0$*

$$C_{0,n}(s_0) := \frac{1}{e^{r_1}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{1,n}^{*l} \cdot q_{1,n}^{*n-l} \cdot \tilde{C}_{1,n}(s_0 \cdot u_{1,n}^{l} \cdot d_{1,n}^{n-l}), \tag{3.1.1}$$

where now $(\tilde{C}_{1,n}(\cdot))_{n\in N}$ is a sequence of nonnegative functions $\tilde{C}_{1,n}(s_1) \geq 0$ for any $s_1 > 0$. If every term of sequence $(\tilde{C}_{1,n}(\cdot))_{n\in N}$ is estimated from below by a step function (independent of n) that takes a finite number of values:

$$\sum_{k=1}^{\infty} x_k \cdot \mathbb{I}_{(s^{`}_{1,k}, s^{``}_{1,k}]}(s_1) \leq \tilde{C}_{1,n}(s_1) \qquad \forall s_1 > 0, \tag{3.1.2}$$

where $x_k \geq 0, k = 0, 1, ...,$ and intervals $(s^{`}_{1,k}, s^{``}_{1,k}]$ are mutually disjoint,
then for any $s_0 > 0$ we have:

$$\underline{\lim}_{n\to\infty} C_{0,n}(s_0) \geq \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \sum_{k=1}^{\infty} x_k \mathbb{I}_{(\frac{1}{\sigma_1}\ln s^{`}_{1,k}, \frac{1}{\sigma_1}\ln s^{``}_{1,k}]}(t)dt,$$

where $f(\cdot)$ is the density of the standard normal distribution.

Lemma 3.2. *Under the previous assumptions and notation let $C_{1,n}(s_1) \geq 0$, $s_1 > 0$, be the functions given by the formula (2.1) and $C_{0,n}(s_0)$, $s_0 > 0$ be the functions defined by (2.2), $n = 1, 2, ...$ Let $0 = s'_{1,0} < s''_{1,0} = s'_{1,1} < s''_{1,1} = ..., \lim_{k\to\infty} s'_{1,k} = \infty$. Additionally, suppose that every function $C_{1,n}$ is estimated from above by a step function independent of n:*

$$\sum_{k=1}^{\infty} \tilde{x_k} \cdot \mathbb{I}_{(s'_{1,k}, s''_{1,k}]}(s_1) \geq C_{1,n}(s_1) \qquad \forall s_1 > 0, \tag{3.2.1}$$

where $\tilde{x_k} \geq 0, k = 0, 1, ...$
Then for any $s_0 > 0$ we have:

$$\overline{\lim}_{n\to\infty} C_{0,n}(s_0) \leq \frac{1}{e^{r_1}} \int_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \sum_{k=1}^{\infty} \tilde{x_k} \mathbb{I}_{(\frac{1}{\sigma_1}\ln s'_{1,k}, \frac{1}{\sigma_1}\ln s''_{1,k}]}(t)dt,$$

where $f(\cdot)$ is the density of the standard normal distribution.

Corollary 3.1. *If we take for fixed $K > 0$*

$$C_{1,n}(s_1) = \frac{1}{e^{r_2}} \sum_{l=0}^{n} \binom{n}{l} \cdot p_{2,n}^{*l} \cdot q_{2,n}^{*n-l} \cdot (s_1 \cdot u_{2,n}^{l} \cdot d_{2,n}^{n-l} - K)^{+}, s_1 > 0 \tag{3.1}$$

for $\tilde{C}_{1,n}(\cdot)$ in Lemma 3.1,
then we have the following lower bound:

$$\underline{\lim}_{n\to\infty} C_{0,n}(s_0) \geq \sup_{\varepsilon>0,\varphi\in\Phi_\varepsilon} \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \varphi(t)dt$$

where, for fixed $\varepsilon > 0, \Phi_\varepsilon(\cdot)$ is the set of step functions defined on $\mathbb{R}$ that take a finite number of values and bound from below the function $(\hat{c}_1(\cdot) - \varepsilon)^+$, where

$$\hat{c}_1(t) := e^{\sigma_1 t}\phi\left(\frac{\sigma_1 t - \ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{K}{e^{r_2}}\phi\left(\frac{\sigma_1 t - \ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right), \tag{3.2}$$

$t \in \mathbb{R}, f(\cdot)$ is the density of the standard normal distribution, $\phi(\cdot)$ is the cumulative normal distribution function.

Corollary 3.2. *The following upper bound is true*

$$\overline{\lim}_{n\to\infty} C_{0,n}(s_0) \leq \inf_{\varepsilon>0,\psi\in\Psi_\varepsilon} \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \psi(t)dt,$$

where, for fixed $\varepsilon > 0, \Psi_\varepsilon(\cdot)$ is the set of step functions defined on disjoint intervals in $\mathbb{R}$, $(t^{'}_{1,k}, t^{''}_{1,k}]$ which satisfy the assumptions of Lemma 3.2 with $t^{'}_{1,0} = -\infty$ and bound from above the function $\hat{c}_1(\cdot) + \varepsilon$, where $\hat{c}_1(\dot{})$ is given by (3.2).

Theorem 3.1. *Under the previous notation and assumptions, for a fixed $K > 0$ and $C_{1,n}(s_1)$ given by (3.1), for any $s_0 > 0$ we have the following equality:*

$$\lim_{n\to\infty} C_{0,n}(s_0) = \frac{1}{e^{r_1}} \int\limits_{-\infty}^{\infty} f\left(t - \frac{\ln s_0}{\sigma_1} - \frac{r_1}{\sigma_1} + \frac{\sigma_1}{2}\right) \cdot$$

$$\cdot \left[e^{\sigma_1 t}\phi\left(\frac{\sigma_1 t - \ln K + r_2 + \frac{\sigma_2^2}{2}}{\sigma_2}\right) - \frac{K}{e^{r_2}}\phi\left(\frac{\sigma_1 t - \ln K + r_2 - \frac{\sigma_2^2}{2}}{\sigma_2}\right)\right] dt,$$

where $f(\cdot)$ is the density of the standard normal distribution, $\phi(\cdot)$ is the cumulative normal distribution function.

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WYCENA OPCJI W MODELU CRR Z PARAMETRAMI ZALEŻNYMI OD CZASU DLA DWÓCH JEDNOSTEK CZASU - CZĘŚĆ I

S t r e s z c z e n i e

W pracy przedstawiono pewien uogólniony model Coxa-Rossa-Rubinsteina na wycenę opcji. Założono, że dwa parametry modelu (stopa procentowa oraz współczynnik zmienności logartymu cen akcji-volatility) zmieniają się w każdej z dwóch jednostek czasu.

Słowa kluczowe: model Coxa-Rossa-Rubinsteina, model CRR, wzór Blacka-Scholesa, wycena opcji