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Dedicated to the memory of
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OPTIMAL CONTROL OF A FRICTIONLESS UNILATERAL CONTACT PROBLEM

Summary

We consider a mathematical model which describes a contact between a nonlinear elastic body and a foundation. The contact is frictionless with Signorini's conditions with a gap. The goal of this paper is to study an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. We state an optimal control problem which has at least one solution. Also we prove a convergence result of a penalized control problem.

Keywords and phrases: optimal control, nonlinear elastic, unilateral contact

1. Introduction

In our daily life and same in industry, the problem of contact between deformable bodies plays an important role in structural and mechanical systems, contact models are of great interest .At now considerable efforts have been made in its modelling and numerical simulations, see [8, 10]. The theory of the optimal control of variational inequalities is very elaborated, see [2, 4, 9, 12, 15, 16, 17]. Despite their mechanical importance, optimal control problems for contact models are not too much developed and represents a difficult task, see [1, 3, 5, 6, 7, 11, 13, 14, 19] . In [13, 14], the authors have studied respectively an optimal control problem of a three-dimensional elastic body in frictional contact with normal compliance with a deformable fondation and an optimal control problem for a nonlinear antiplane of an elastic cylindrical body in frictional contact with a rigid fondation. The optimal control for a bilateral contact between an elastic body and a rigid foundation was studied in [19]. Here, as in [19], the goal of this paper is to study an optimal control of contact problem. Indeed, we consider a nonlinear elastic body wich is in unilateral contact with a rigid fondation. The contact is frictionless associated with Signorini's conditions with a gap. We establish a variational formulation of the mechanic problem and prove the existence and uniqueness result. The optimal control problem concerning this model is denoted respectively by **C1**. This problem consists of minimizing a cost functional wich is convex and continuous. However, we are interested to led the stress tensor field as close as possible to a given target when we act with a control on a part of the boundary.

The paper is structured as follows. In section 2, we describe the mechanical model, list the assumptions on the data and derive the variational formulation. Then we prove an existence and uniqueness result, Theorem 2.1. In section 3, we state and prove the solvability of the optimal control problem **C1**, Theorem 3.1. In section 4, we prove the convergence results of the penalized problem, Theorem 4.2. In section 5, we prove that the penalized control problem C1 has at least one solution Theorem 5.1, then we obtain convergence results, Theorem 5.2.

2. Problem statement and variational formulation

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a domain occupied by a nonlinear elastic body. Ω is supposed to be open,bounded, with a sufficiently regular boundary partitioned into three measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_0 on Ω and a surface traction of density φ_2 on Γ_2. The body is clamped on Γ_1 and, so, the displacement vector u vanishes here. The contact is frictionless associated to signorini's conditions. Thus, the classical formulation in terms of displacement of the mechanical problem is the following.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$\operatorname{div} \sigma(u) = -\varphi_0 \quad \text{in } \Omega, \tag{2.1}$$

$$\sigma(u) = F\varepsilon(u) \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = \varphi \text{ on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \leq g,\, \sigma_\nu \leq 0,\ \sigma_\nu (u_\nu - g) = 0 \\ \sigma_\tau = 0 \end{array}\right\} \text{ on } \Gamma_3. \tag{2.5}$$

(2.1) represents the equilibrium equation such that $\sigma = \sigma(u)$ denotes the stress field and $\varepsilon(u)$ the strain tensor. Equation (2.2) represents the elastic constitutive law where F is a given nonlinear function while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. In condition (2.5) σ_ν denotes the normal stress, u_ν is the normal displacement which is limited by $g \geq 0$; $\sigma_\tau = 0$ defines the frictionless contact. Recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S_d are given by

$$u.v = u_i.v_i, \qquad |v| = (v,v)^{\frac{1}{2}}, \ \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}.\tau_{ij} \qquad |\tau| = (\tau,\tau)^{\frac{1}{2}}, \ \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$). Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d \quad H_1 = \left(H^1(\Omega)\right)^d \quad \sigma = \left\{\sigma = (\sigma_{ij}); \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products:

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad (\sigma,\tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is defined as

$$\varepsilon(u) = (\varepsilon_{ij}(u)),\ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$, we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by $v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu\nu$. Also, for a regular function (say C^1) $\sigma \in Q$, we define its normal and tangential components by $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$ and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v) = \int_\Gamma \sigma\nu.v da \ \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\}.$$

Since meas $(\Gamma_1) > 0$, the following Korn's inequality holds [8],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V. \tag{2.6}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We endow V with the inner product

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_Q \quad \forall u, v \in V,$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.6) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. In te next, we introduce the convex of admissible displacements defined as

$$K = \{v \in V;\ v_\nu \le g \ \ a.e. \text{ on } \Gamma_3\}.$$

g is assumed to satisfy

$$g \in L^2(\Gamma_3) \text{ and } g \ge 0 \ a.e. \text{ on } \Gamma_3. \tag{2.7}$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_0 \in H, \quad \varphi \in (L^2(\Gamma_2))^d. \tag{2.8}$$

Next, In the study of Problem P_1 we assume that the nonlinear elasticity operator F satisfies the following assumptions.

$$\begin{cases} (a) \ F : \Omega \times S_d \to S_d; \\ (b) \ \text{there exists } M > 0 \text{ such that} \\ \quad |F(x, \varepsilon_1) - F(x, \varepsilon_2)| \le M |\varepsilon_1 - \varepsilon_2| \ \ \forall \varepsilon_1, \varepsilon_2 \in S_d, \ a.e. \ x \in \Omega; \\ (c) \ \text{there exists } m > 0 \text{ such that} \\ \quad (F(x, \varepsilon_1) - F(x, \varepsilon_2)) . (\varepsilon_1 - \varepsilon_2) \ge m |\varepsilon_1 - \varepsilon_2|^2 \\ \forall \varepsilon_1, \varepsilon_2 \in S_d, \ a.e. x \in \Omega; \\ (d) \ \text{the mapping: } x \to F(x, \varepsilon) \ \text{ is Lebesgue mesurable on } \Omega \\ \text{for any } \varepsilon \in S_d, \\ (e) \ F(x, 0_{S_d}) = 0 \text{ for } a.e. \ x \in \Omega. \end{cases} \tag{2.9}$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula that the problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u \in K$ such that

$$\begin{aligned} &(F\varepsilon(u), \varepsilon(v - u))_Q \\ &\qquad \ge (\varphi_0, v - u)_H + (\varphi, v - u)_{(L^2(\Gamma_2))^d} \ \ \forall v \in K. \end{aligned} \tag{2.10}$$

Theorem 2.1. *Assume* (2.7), (2.8) *and* (2.9). *Then there exists a unique solution of Problem P_2.*

Proof. We define the operator $A : V \to V$ by $(Au, v)_V = (F\varepsilon(u), \varepsilon(v))_Q$, $\forall u, v \in V$. By (2.9), it follows that the operator A is Lipschitz continuous and strongly monotone; (2.7) and the definition of K imply that K is a nonempty, closed and convex subset of V, then moreover by using (2.8), it follows from [27] that the inequality (2.10) has a unique solution. $\square$

3. The optimal control problem

We now suppose that $\varphi_0 \in H$ is fixed and consider the following state variational problem.

Problem Q1. For a given $\varphi \in (L^2(\Gamma_2))^d$ (called control), find $u \in K$ such that

$$(F\varepsilon(u), \varepsilon(v-u))_Q \\ \geq (\varphi_0, v-u)_H + (\varphi, v-u)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \tag{3.1}$$

Following the existence and uniqueness of Problem P_2, we deduce that for every control $\varphi \in (L^2(\Gamma_2))^d$, the state variational problem **Q1** has a unique solution $u \in K$, $u = u(\varphi)$.

Now, for a given $u_d \in K$, we define the cost functional

$$\mathcal{L}\colon K \times (L^2(\Gamma_2))^d \to \mathbb{R}_+,$$

by

$$\mathcal{L}(u, \varphi) = \frac{\alpha}{2} \|u - u_d\|_V^2 + \frac{\beta}{2} \|\varphi\|^2_{(L^2(\Gamma_2))^d},$$

where α, $\beta > 0$. Next, we define the set of admissible pairs U_{ad} by

$$U_{ad} = \left\{(u, \varphi) \in K \times (L^2(\Gamma_2))^d, \text{ such that } (3.1) \text{ is satisfied}\right\},$$

and we consider the optimal control problem below.

Problem C1. Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}(u^*, \varphi^*) = \min_{(u,\varphi)\in U_{ad}} \mathcal{L}(u, \varphi).$$

Theorem 3.1. *Assume* (2.7), (2.8), (2.9) *and* (2.10). *Then Problem **C1** has at least one solution.*

Proof. We put $v = 0_V$ in (3.1), then, using (2.8) and (2.10) (c), we deduce that the solution u of Problem **Q1** is bounded in V as

$$\|u\|_V \leq \frac{\|\varphi_0\|_H + c\,\|\varphi\|_{(L^2(\Gamma_2))^d}}{m},$$

where $c > 0$. This inequality implies that

$$\inf_{(u,\varphi_2)\in U_{ad}} \{\mathcal{L}(u, \varphi)\} \in \mathbb{R}.$$

Now, let us denote

$$\inf_{(u,\varphi_2)\in U_{ad}} \{\mathcal{L}(u, \varphi)\} = \theta. \tag{3.2}$$

Then, there exists a minimizing sequence $(u^n, \varphi^n) \subset U_{ad}$ such that

$$\lim_{n\to\infty} \mathcal{L}(u^n, \varphi^n) = \theta. \tag{3.3}$$

The sequence (u^n, φ^n) is bounded in $V \times (L^2(\Gamma_2))^d$, so there exists an element

$$(u^*, \varphi^*) \in V \times (L^2(\Gamma_2))^d,$$

such that passing to a subsequence still denoted by (u^n, φ^n), we deduce that as $n \to \infty$,

$$u^n \to u^* \text{ weakly in } V, \tag{3.4}$$

$$\varphi^n \to \varphi^* \text{ weakly in } (L^2(\Gamma_2))^d. \tag{3.5}$$

To prove that $u^* \in K$, we have that $u_\nu^n \leq g$ *a.e.* on Γ_3 and as (3.4) implies $u_\nu^n \to u_\nu^*$ strongly in $L^2(\Gamma_3)$, it follows that there exists a subsequence still denoted (u_ν^n) such that $u_\nu^n \to u_\nu^*$ *a.e.*on Γ_3 which implies that $u_\nu^* \leq g$ *a.e.* on Γ_3.

Now to end the proof we need to prove that

$$u^n \to u^* \text{ strongly in } V \text{ as } n \to \infty. \tag{3.6}$$

Indeed, as $(u^n, \varphi^n) \in U_{ad}$, then u^n satisfies the inequality:

$$\begin{aligned} &(F\varepsilon(u^n), \varepsilon(v - u^n))_Q \\ &\qquad \geq (\varphi_0, v - u^n)_H + (\varphi^n, v - u^n)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned} \tag{3.7}$$

Using (2.9) (c) and (3.7), we deduce that

$$\begin{aligned} m\left\|u^n - u^*\right\|_V^2 &\leq (F\varepsilon(u^n) - F\varepsilon(u^*), \varepsilon(u^n - u^*))_Q \\ &\leq (F\varepsilon(u^n), \varepsilon(u^n - u^*))_Q - (F\varepsilon(u^*), \varepsilon(u^n - u^*))_Q \\ &\leq -(F\varepsilon(u^*), \varepsilon(u^n - u^*))_Q + \\ &+ (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} \end{aligned} \tag{3.8}$$

Using (3.4), we have that

$$\lim_{n\to\infty} (F\varepsilon(u^*), \varepsilon(u^n - u^*))_Q = \lim_{n\to\infty} (Au^*, u^n - u^*)_V = 0.$$

On the other hand, since (φ^n) is bounded in $(L^2(\Gamma_2))^d$, by (3.4) we get

$$\lim_{n\to\infty} (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} = 0.$$

Using (2.8) and (3.4), it follows

$$\lim_{n\to\infty} \left((\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d}\right) = 0.$$

Consequently we deduce that the right hand side of the inequality (3.8) tends to zero as $n \to +\infty$ and then we get (3.6) . Moreover, using (3.5) and (3.6), we pass to the limit as $n \to +\infty$ in (3.7), to obtain that (u^*, φ^*) satisfies the inequality (3.1) . Then

$$(u^*, \varphi^*) \in U_{ad}. \tag{3.9}$$

We now use the weakly lower semicontinuity of $\mathcal{L}$ to deduce that

$$\liminf_{n\to+\infty} \mathcal{L}(u^n, \varphi^n) \geq \mathcal{L}(u^*, \varphi^*). \tag{3.10}$$

It follows now from (3.3) and (3.10) that

$$\theta \geq \mathcal{L}(u^*, \varphi^*). \tag{3.11}$$

In addition (3.2) and (3.9) yield

$$\mathcal{L}(u^*, \varphi^*) \geq \theta. \tag{3.12}$$

Then to end the proof, it suffices to combine the inequalities (3.11) and (3.12). $\square$

4. A penalized problem

In this section, we define the functional $j_\delta : V \times V \to \mathbb{R}$ by

$$j_\delta (u, v) = \int_{\Gamma_3} \frac{1}{\delta} (u_\nu - g)_+ v_\nu da, \ \forall (u, v) \in V \times V,$$

where $\delta > 0$ is the parameter of penalization. Then, the variational formulation of the penalized problem with frictionless contact is the following.

Problem P_δ. Find a displacement $u_\delta \in V$ such that

$$(F\varepsilon (u_\delta), \varepsilon (v))_Q + j_\delta (u_\delta, v) = (f, v)_V, \ \forall v \in V. \tag{4.1}$$

We have the result below.

Theorem 4.1. *Assume that* (2.7), (2.8) *and* (2.9) *hold. Then, Problem P_δ has a unique solution.*

Proof. We define the operator $A : V \to V$ by

$$(Au, v)_V = (F\varepsilon (u), \varepsilon (v))_Q + j_\delta (u, v) \quad \forall u, v \in V.$$

Using (2.7), (2.8), (2.9), and that for $r, s \in \mathbb{R}$, $(r - s)(r_+ - s_+) \geq (r_+ - s_+)^2$ and $|r_+ - s_+| \leq |r - s|$, where we use the notation that for $a \in \mathbb{R}$, $a_+ = \max(a, 0)$, then A is strongly monotone and Lipschitz continuous. So, Problem $P_{2\delta}$ has a unique solution. □

Now, we study the convergence of the sequence (u_δ), as $\delta \to 0$ in the following theorem.

Theorem 4.2. *Assume* (2.7), (2.8) *and* (2.9). *Then, the following strong convergence holds:*

$$u_\delta \to u \text{ strongly in } V. \tag{4.2}$$

Proof. The proof is carried out in several steps. In the first step, we shall prove that there exists $\bar{u} \in K$ such that after passing to a subsequence still denoted (u_δ), such that we have

$$u_\delta \to \tilde{u} \text{ weakly in } V. \tag{4.3}$$

Indeed, take $v = u_\delta$ in (4.1), we have

$$(F\varepsilon (u_\delta), \varepsilon (u_\delta))_Q + j_\delta (u_\delta, u_\delta) = (f, u_\delta)_V. \tag{4.4}$$

As $j_\delta (u_\delta, u_\delta) \geq 0$, it is easy to deduce by (2.9) (c) that

$$\|u_\delta\|_V \leq \|f\|_V / m. \tag{4.5}$$

Then, there exists an element $\tilde{u} \in V$ and a subsequence still denoted u_δ such that (4.3) holds. On the other hand from (4.4), we get the inequality

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu} - g}{\delta} \right)_+ (u_{\delta\nu} - g) \, da \leq (f, u_\delta)_V,$$

which implies that

$$\left\|(u_{\delta\nu}-g)_+\right\|^2_{L^2(\Gamma_3)} \leq \delta \left\|f\right\|^2_V \; /m.$$

Then, we obtain

$$\left\|(\tilde{u}_{\ \nu}-g)_+\right\|_{L^2(\Gamma_3)} \leq \liminf_{\delta\to 0} \left\|(u_{\delta\nu}-g)_+\right\|_{L^2(\Gamma_3)} = 0. \tag{4.6}$$

Moreover we conclude by (4.6) that $(\tilde{u}_\nu - g)_+ = 0$, i-e. $\tilde{u}_{\ \nu} \leq g$ *a.e.* on Γ_3 and then $\tilde{u} \in K$. Now, in the second step we shall prove that $\tilde{u} = u$.

In fact, Let $v \in K$ and take $v - u_\delta$ in (4.1), we have

$$\begin{aligned} &(F\varepsilon(u_\delta), \varepsilon(v-u_\delta))_Q + j_\delta(u_\delta, v-u_\delta) \\ &= (f, v-u_\delta)_V \quad \forall v \in K. \end{aligned} \tag{4.7}$$

Then, as

$$j_\delta(u_\delta, v-u_\delta) = \textstyle\int_{\Gamma_3} \left(\dfrac{u_{\delta\nu}-g}{\delta}\right)_+ (v_\nu - u_{\delta\nu})\, da \leq 0,$$

we get

$$(F\varepsilon(u_\delta), \varepsilon(v-u_\delta))_Q \geq (f, v-u_\delta)_V \quad \forall v \in K. \tag{4.8}$$

Moreover, take $v = \tilde{u}$ in the inequality above yields

$$(F\varepsilon(u_\delta), \varepsilon(u_\delta-\tilde{u}))_Q \leq (f, u_\delta-\tilde{u})_V\,,$$

so we deduce that

$$\limsup_{\delta\to 0}\,(F\varepsilon(u_\delta), \varepsilon(u_\delta-\tilde{u}))_Q \leq 0. \tag{4.9}$$

Therefore, using (2.9) (c) and the convergence (4.3), we deduce that

$$\liminf_{\delta\to 0}\,(F\varepsilon(u_\delta), \varepsilon(u_\delta-v))_Q \geq (F\varepsilon(\tilde{u}), \varepsilon(\tilde{u}-v))_Q \ \ \forall v \in K. \tag{4.10}$$

On the other hand, using (4.3) and (4.8), yields

$$\limsup_{\delta\to 0}\,(F\varepsilon(u_\delta), \varepsilon(u_\delta-v))_Q \leq (f, \tilde{u}\ -v)_V \quad \forall v \in K. \tag{4.11}$$

Now, by combining (4.10) and (4.11), we see that

$$(F\varepsilon(\tilde{u}\ ), \varepsilon(v-\tilde{u}\ ))_Q \geq (f, v-\tilde{u}\ )_V \quad \forall v \in K. \tag{4.12}$$

Moreover, we take $v = u$ in (4.12) and $v = \tilde{u}$ in (2.11) and we add the resulting inequalities to obtain by using (2.9) (c) that

$$m\left\|\tilde{u}\ -u\right\|^2_V \leq 0, \tag{4.13}$$

and then from (4.13), we get

$$\tilde{u}\ = u. \tag{4.14}$$

In the third step, we shall prove (4.2). For thus, from the arguments used above we see that any weakly convergent subsequence of the sequence $(u_\delta) \subset V$ converges weakly to the unique solution u of Problem P_2. The estimate (4.5) implies that the sequence (u_δ) is bounded in V. Thus, using a standard compactness argument, we conclude that the whole sequence (u_δ) converges weakly to u.

Next, we use (2.10) (c) to find that

$$m\left\|u_\delta\ -u\right\|_V^2 \le \left(F\varepsilon\left(u_\delta\right), \varepsilon\left(u_\delta-u\right)\right)_Q-\left(F\varepsilon\left(u\right), \varepsilon\left(u_\delta-u\right)\right)_Q .$$

We now pass to the limit in this inequality and use (4.3), (4.9) and (4.14) to deduce (4.2). □

5. A penalized control problem

Now, for $\delta > 0$ and a fixed $\varphi_0 \in H$, we introduce the following penalized state problem.

Problem Q2. For a given $\varphi \in \left(L^2\left(\Gamma_2\right)\right)^d$ (called control), find $u_\delta \in V$ such that

$$\left(F\varepsilon\left(u_\delta\right), \varepsilon\left(v\right)\right)_Q+j_\delta\left(u_\delta, v\right)=\left(\varphi_0, v\right)_H+(\varphi, v)_{\left(L^2\left(\Gamma_2\right)\right)^d},\ \forall v \in V. \tag{5.1}$$

By Theorem 4.1, for every $\varphi \in \left(L^2\left(\Gamma_2\right)\right)^d$, the problem Q2 has a unique solution $u_\delta \in V$, $u_\delta=u_\delta\left(\varphi\right)$. In addition, we have

$$\left\|u_\delta\right\|_V \le \frac{1}{m}\left(\left\|\varphi_0\right\|_H+c\left\|\varphi\right\|_{\left(L^2\left(\Gamma_2\right)\right)^d}\right).$$

Furthermore, we define the set U_{ad}^δ as

$$U_{ad}^\delta=\left\{(u, \varphi) \in V \times\left(L^2\left(\Gamma_2\right)\right)^d,\ \text{such that }\ (5.1)\ \text{ is satisfied}\right\}.$$

Using the functional $\mathcal{L}$, given by (3.2), we introduce the following optimal control problem associated to the penalized contact problem.

Problem C2. Find $(\bar{u}, \bar{\varphi}) \in U_{ad}^\delta$ such that $\mathcal{L}\left(\bar{u}, \bar{\varphi}\right))=\min\limits_{(u, \varphi) \in U_{ad}^\delta}\{\mathcal{L}\left(u, \varphi\right))\}$.

With arguments similar to those used in Theorem 3.1, the following result can be proved.

Theorem 5.1. *Assume that* (2.7), (2.8) *and* (2.9) *hold. Then Problem C2 has at least one solution.*

Now, we have the convergence result below.

Theorem 5.2. *Assume that* (2.7), (2.8) *and* (2.9) *hold, and let* $(\bar{u}_\delta, \bar{\varphi}_\delta)$ *be a solution of Problem C2. Then, there exists a solution* (u^*, φ^*) *of Problem C1 such that passing to a subsequence of* $(\bar{u}_\delta, \bar{\varphi}_\delta)$ *still denoted* $(\bar{u}_\delta, \bar{\varphi}_\delta)$, *the following convergences hold:*

$$\begin{aligned}&\bar{u}_\delta \rightarrow u^* \ \textit{strongly in } V,\ \textit{as } \delta \rightarrow 0,\\ &\bar{\varphi}_\delta \rightarrow \varphi^* \ \textit{weakly in } \left(L^2\left(\Gamma_2\right)\right)^d,\ \textit{as } \delta \rightarrow 0.\end{aligned}$$

Proof. Let $u_{\delta 0} \in V$ be the unique solution of Problem Q2 with $\varphi=0_{\left(L^2\left(\Gamma_2\right)\right)^d}$.

$$\mathcal{L}\left(u_{\delta 0}, 0_{\left(L^2\left(\Gamma_2\right)\right)^d}\right)=\frac{\alpha}{2}\left\|u_{\delta 0}-u_d\right\|_V^2 \le \alpha\left(\left\|u_{\delta 0}\right\|_V^2+\left\|u_d\right\|_V^2\right)$$

and since,

$$\|u_{\delta 0}\|_V \leq \frac{1}{m}\|\varphi_0\|_H$$

we deduce, that there exists a constant $c > 0$ such that

$$\begin{aligned}\mathcal{L}(\bar{u}_\delta, \bar{\varphi}_\delta) &\leq \mathcal{L}\left(u_{\delta 0}, 0_{(L^2(\Gamma_2))^d}\right)\\ &\leq \alpha\left(\|\varphi_0\|_H^2 + \|u_d\|_V^2\right).\end{aligned}$$

Therefore, $(\bar{u}_\delta, \bar{\varphi}_\delta)$ is a bounded sequence in $V \times (L^2(\Gamma_2))^d$.

Consequently, passing to a subsequence still denoted $\left(\bar{u}^\delta, \bar{\varphi}^\delta\right)$, it follows that there exists an element $(u^*, \varphi^*) \in V \times (L^2(\Gamma_2))^d$ such that

$$\begin{aligned}&\bar{u}_\delta \to u^* \text{ weakly in } V \quad \text{as } \delta \to 0,\\ &\bar{\varphi}_\delta \to \varphi^* \text{ weakly in } (L^2(\Gamma_2))^d \text{ as } \delta \to 0.\end{aligned}$$

Since $(\bar{u}_\delta, \bar{\varphi}_\delta) \in U_{ad}^\delta$, arguing as in the proof of Lemma 4.3, we deduce that $u^* \in K$. Moreover, we have

$$\begin{aligned}&m\|\bar{u}_\delta - u^*\|_V^2 \leq (F\varepsilon(u^*) - F\varepsilon(\bar{u}_\delta), \varepsilon(u^* - \bar{u}_\delta))_Q\\ &\leq (F\varepsilon(u^*, \varepsilon(u^* - \bar{u}_\delta))_Q + j_\delta(\bar{u}_\delta, u^*) - j_\delta(\bar{u}_\delta, \bar{u}_\delta) +\\ &+ (\varphi_0, u^* - \bar{u}_\delta)_H + (\bar{\varphi}_\delta, u^* - \bar{u}_\delta)_{(L^2(\Gamma_2))^d}.\\ &\leq (F\varepsilon(u^*, \varepsilon(u^* - \bar{u}_\delta))_Q + (\varphi_0, u^* - \bar{u}_\delta)_H + (\bar{\varphi}_\delta, u^* - \bar{u}_\delta)_{(L^2(\Gamma_2))^d}.\end{aligned} \tag{5.2}$$

On the other hand as $\bar{u}_\delta \to u^*$ weakly in V implies that $\bar{u}^\delta \to u^*$ strongly in $(L^2(\Gamma_2))^d$ and $u_{\delta\nu} \to u_\nu^*$ strongly in $L^2(\Gamma_3)$, then, it follows that the right hand side of (5.2) converges to zero as $\delta \to 0$. Hence we deduce that $\bar{u}^\delta \to u^*$ strongly in V as $\delta \to 0$. Now, we must prove that $(u^*, \varphi^*) \in U_{ad}$. Indeed, let $v \in K$ and put $(v - \bar{u}_\delta)$ in (5.1), then by using (5.2), it follows that when $\delta \to 0$, the following convergences hold:

$$(F\varepsilon(\bar{u}_\delta), \varepsilon(v - \bar{u}_\delta))_Q \to (F\varepsilon(u^*), \varepsilon(v - u^*))_Q,$$

$$(\varphi_0, v - \bar{u}_\delta)_H + \left(\bar{\varphi}^\delta, v - \bar{u}_\delta\right)_{(L^2(\Gamma_2))^d} \to (\varphi_0, v - u^*)_H + (\varphi^*, v - u^*)_{(L^2(\Gamma_2))^d}.$$

On the other hand, as

$$j_\delta\left(\bar{u}_\delta, v - \bar{u}^\delta\right) \leq 0,$$

then, (u^*, φ^*) satisfies (3.1) and $(u^*, \varphi^*) \in U_{ad}$.

Now, let $(\bar{u}, \bar{\varphi})$ be a solution of Problem C1 and let us consider the sequence (u_δ) such that, for each $\delta > 0, u_\delta$ is the unique solution of Problem Q2 with the data $\varphi_0 \in H$ and $\bar{\varphi} \in (L^2(\Gamma_2))^d$.Obviously, for every $\delta > 0, (u_\delta, \bar{\varphi}) \in U_{ad}^\delta$. Using Theorem 5.1 we deduce that

$$(u_\delta, \bar{\varphi}) \to (\bar{u}, \bar{\varphi}) \text{ strongly in } V \times (L^2(\Gamma_2))^d \text{ as } \delta \to 0. \tag{5.3}$$

Since the functional $\mathcal{L}$ is convex and continuous, we have

$$\mathcal{L}\left(u^{*},\varphi^{*}\right)\leq\lim_{\delta\to 0}\inf\mathcal{L}\left(\bar{u}_{\delta},\bar{\varphi}_{\delta}\right). \tag{5.4}$$

Also, since $(\bar{u}_{\delta},\bar{\varphi}_{\delta})$ is a solution of Problem C2, we have that

$$\lim_{\delta\to 0}\sup\mathcal{L}\left(\bar{u}_{\delta},\bar{\varphi}_{\delta}\right)\leq\lim_{\delta\to 0}\sup\mathcal{L}\left(u_{\delta},\bar{\varphi}\right) \tag{5.5}$$

and using (5.3), it follows that

$$\lim_{\delta\to 0}\sup\mathcal{L}\left(u_{\delta},\bar{\varphi}\right)=\mathcal{L}\left(\bar{u},\bar{\varphi}\right). \tag{5.6}$$

Now, taking into account that $(\bar{u},\bar{\varphi})$ is a solution of Problem C1, then have that

$$\mathcal{L}\left(\bar{u},\bar{\varphi}\right)\leq\mathcal{L}\left(u^{*},\varphi^{*}\right). \tag{5.7}$$

Consequently, from (5.4)-(5.7) , one obtains

$$\mathcal{L}\left(\bar{u},\bar{\varphi}\right)=\mathcal{L}\left(u^{*},\varphi^{*}\right).$$

□

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OPTYMALNE STEROWANIE W ZAGADNIENIU BEZPOŚREDNIEJ STYCZNOŚCI CIAŁ STAŁYCH PRZESUWANYCH JEDNOKIERUNKOWO

S t r e s z c z e n i e

Rozpatrujemy model matematyczny określający styczność poruszającego się ciała elastycznego względem podłoża. Zakładamy, że zmiany wzajemnego położenia następują bez tarcia z zachowaniem warunków Signoriniego odnośnie do luk. Podjęte jest zagadnienie optymalnego sterowania tensorem naprężenia ciała względem podłoża. Sterowanie działa jednokierunkowo w odniesieniu do brzegu elastycznej tarczy.

Słowa kluczowe: sterowanie optymalne, zagadnienie nieliniowe jednokierunkowego kontaktu elastycznego